

XVI. NETWORK SYNTHESIS

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A. DETERMINATION OF ROOT LOCI FROM SADDLEPOINT CONSIDERATIONS

A problem of major importance in the synthesis of active networks is the location of the poles of the system function as a pertinent gain is varied. This report deals with a method for finding the motion of poles in the complex s -plane as the gain is varied. The proposed analysis stems from results expressed by R. F. Baum (1) in his investigation of the behavior of network functions in the vicinity of saddlepoints, poles, and zeros.

Evans' method of root locus (2) was developed in the field of servomechanisms, where the problem of system stability is of primary importance. However, except for the asymptotic location of poles, this method is essentially graphical in nature. The locus of poles is found by determining which values of s satisfy the condition $KG(s) = -1 + j0$ in a feedback system described by $F(s) = KG(s)/1 + KG(s)$. The root locus obtained by this method clearly yields the values of s for which $G(s)$ is purely real. It is known from complex variable theory that loci of constant real and imaginary parts can be constructed in the s -plane (3). In particular, since the locus for which the imaginary part is zero must yield the values of s for which $G(s)$ is purely real, it must correspond to the desired locus of the poles of $F(s)$.

In addition, a family of imaginary loci which yields additional information on the character of the rational function $F(s)$ can be found; hence, an investigation of both the real and imaginary loci in the s -plane should be fruitful in root loci determinations.

Baum's work on this aspect of the problem entails the study of saddlepoints, poles, and zeros of a rational function $F(s)$ through the construction of these two sets of curves in the s -plane: the real locus and the imaginary locus. The real locus (L_R) is defined by the values of s for which $F(s)$ is purely real (imaginary part of $F(s) = 0$). The imaginary locus (L_I) is defined by the values of s for which $F(s)$ is purely imaginary (real part of $F(s) = 0$). It is found that where L_R and L_I intersect, either a pole or zero of $F(s)$ is located, since both conditions cannot otherwise be satisfied simultaneously; and where these loci cross (L_R intersecting L_I) a saddlepoint is located. Then, in any physical system, all poles, zeros, and saddlepoints must be interconnected by either L_R or L_I , or both, thereby yielding information on the location of these critical points and on the behavior of the function in the vicinity of these points.

The results of Baum's work, which should prove useful for the determination of root loci, are contained in four theorems. They are summarized below:

Given a rational function $Z(s) = R + jX$ of a complex variable s on any path on which $Z(s)$ is either real or imaginary:

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I. The derivative $dR/d|s|$ or $dX/d|s|$ changes sign at a saddlepoint of uneven order.

II. The derivative $dR/d|s|$ or $dX/d|s|$ is zero but does not change sign at a saddlepoint of even order.

III. A zero follows a zero, or a pole follows a pole, if between the two there is an uneven number of saddlepoints of uneven order.

IV. A zero follows a pole (or vice versa) if between the two there is an even number of saddlepoints of uneven order.

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References

1. R. F. Baum, Unpublished article (1947).
2. W. R. Evans, Progress Report (Project MX-770), North American Aviation Inc., Downey, California (June 1953).
3. E. A. Guillemin, The Mathematics of Circuit Analysis (John Wiley and Sons, Inc., New York, 1949), p. 298.

B. COMPLEXITY OF GROUNDED NETWORKS HAVING ZEROS OF TRANSMISSION IN THE RIGHT HALF-PLANE

It is well known that two terminal-pair networks can have zeros of transmission in two different ways: (a) when the input of the network is virtually isolated from the output because a certain impedance within the network has a pole or a zero and (b) when two signals traveling along different paths arrive at the output in the proper phase so that they cancel and give no output. This process of cancellation is especially important when zeros are desired in the right half-plane. If the configuration to be used in the synthesis is an ungrounded two terminal-pair network, no particular problem is involved, since the cancellation can be achieved by a bridge arrangement (as in the lattice); but if a grounded two terminal-pair network (without mutual coupling) is to be used, the cancellation must be caused by parallel grounded paths; and it is found that in order to realize zeros in the right half-plane the network must have a certain minimum complexity that is related to the angle between the positive real axis and the line joining the zero with the origin. In this work some quantitative results concerning this relationship are derived.

Theorem 1: The transfer function (voltage ratio, transfer impedance, or admittance) of a grounded two terminal-pair network having no mutual coupling has all positive coefficients. This refers to the actual transfer function before common factors in the numerator and denominator are canceled. If the transfer function is expressible in terms of determinants as Δ_{ik}/Δ , both Δ_{ik} and Δ have all positive coefficients.

Proof*: Consider making a star-delta transformation at all interior nodes of the network, thus reducing the circuit to two nodes, one at the input and one at the output. Such a transformation has the property that while it may yield nonphysical elements it always yields elements with positive coefficients. The theorem follows immediately.

Thus in order to synthesize a function having zeros in the right half-plane, it may be necessary to multiply the numerator and the denominator of the given function by some Hurwitz polynomial in order to make all of the coefficients in the numerator positive. This, in turn, will make the resulting network more complicated. It is therefore of interest to predict the minimum degree of a polynomial having all positive coefficients and having a zero at some specified location in the right half-plane.

Theorem 2: A polynomial of degree n with all positive coefficients cannot have a zero closer to the positive real axis than an angle of $\pm\pi/n$ radians.

Proof*: Consider a Nyquist plot taken along the pie-shaped wedge in the s -plane formed by the two rays from the origin at the angles $\pm\pi/n$ radians. Since along these rays the value of the given polynomial

$$P(s) = \sum_{k=0}^n a_k s^k$$

is represented by the vector sum

$$P(\rho e^{\pi/n}) = \sum_{k=0}^n a_k \rho^k e^{k(\pi/n)}$$

and since the angle of each of these vectors is less than 180° (except for the n^{th} , which is equal to 180°) it is clear that the Nyquist plot does not encircle the origin; hence no zeros can lie inside the pie-shaped wedge. If all of the coefficients except a_0 and a_n are zero, the locus can just touch the origin, and there will be a zero on the boundary.

Thus the positive coefficient polynomials of least order that contain a zero at a specified angle θ with the positive real axis are the Butterworth functions, $s^n + 1$, and the order of the required polynomial is $n \geq \pi/\theta$. Unfortunately, for n greater than 4 (or θ less than 45°) these functions contain more than one zero in the right half-plane and are not suitable polynomials with which to multiply the numerator and the denominator of a given transfer function. However, the theorem does place a lower bound on the complexity of networks having right half-plane zeros.

*Proofs for theorems 1 and 2 were suggested by Professor S. J. Mason.

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The question then arises: Can this same objective (all positive coefficients) be accomplished by using only Hurwitz polynomials or negative real root polynomials (RC) and what is the best that can be done under these restrictions? The following theorem is a partial answer to that question.

Theorem 3: The function $(s+1)^n (s^2 - 2 \cos \theta s + 1)$ has all positive coefficients for $n \geq 2 \cos \theta / 1 - \cos \theta$, for n even; and $n + 1 \geq 2 \cos \theta / 1 - \cos \theta$, for n odd.

The proof follows from the actual multiplication of the two polynomials in terms of the binomial coefficients and it will not be repeated here.

It is strongly suspected that this represents the best that can be done with negative real root polynomials. If the polynomial is constrained only to be Hurwitz, a value of n somewhere between that given by theorems 2 and 3 is expected, although an optimum polynomial is yet to be found for this case.

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