XVII. NONLINEAR CIRCUITS

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A. TOPOLOGY OF DYNAMIC SYSTEMS

1. Formulation

A dynamic system, that is, a physical system whose behavior is a function of the real time variable t, can be characterized mathematically by means of certain excitation and response quantities and their time derivatives in relation to the physical "constants" of its components. Thus the formulation results in a system of differential equations that involve functional equalities of a number of variables and their time derivatives. In the case of nonlinear systems of higher order, very little is revealed, through this individual formulation, about the nature of the problem in general. Thus it may be that by introducing new variables we shall arrive at a standard formulation that characterizes any general problem.

Consider a set of functional equalities of the variables $x, y, \ldots$, and so on, and their time derivatives $dx/dt, d^2x/dt^2, \ldots, dy/dt, d^2y/dt^2, \ldots$. (We shall use the standard notation $\dot{x} = dx/dt$ for time derivatives.)

$$p_i(x, \dot{x}, \ddot{x}, \ldots, y, \dot{y}, \ddot{y}, \ldots; t) = q_i(x, \dot{x}, \ddot{x}, \ldots, y, \dot{y}, \ddot{y}, \ldots; t)$$

$$i = 1, 2, 3, \ldots, k \qquad (1)$$

with their initial conditions $x(0) = C_1$, $\dot{x}(0) = C_2$, and so on. The total number of dependent variables in Eq. 1, that is, x, y, and so on (not their time derivatives) is k. Each variable involves $m_i (i = 1, 2, \ldots, k)$ of its time derivatives. Obviously, the total number of initial conditions for a particular solution is

$$\sum_{i=1}^{k} m_i + k$$

Now let us introduce a set of new variables that involve only the first-order time derivative. In particular, let

$$\left.\begin{array}{ll} x = z_1 & y = z_{m_1+1} \\ \dot{x} = z_2 & \dot{y} = z_{m_1+2} \\ \dot{z}_2 = z_3 & \cdots\cdots\cdots \\ \cdots\cdots\cdots & \\ \dot{z}_{m_1-1} = z_{m_1} & \end{array}\right\} \qquad (2)$$

Solving the set of Eq. 1 for the highest order time derivative of each of the dependent variables and substituting the new variables (Eq. 2), we obtain the set

$$\dot{z}_i = f_i(z_1, z_2, \ldots, z_n; t) \qquad i = 1, 2, \ldots, n \tag{3}$$

with the initial conditions

$$z_i(0) = c_i \qquad i = 1, 2, \ldots, n$$

where

$$n = k + \sum_{i=1}^{k} m_i \tag{4}$$

Now if we let $t = z_{n+1}$, we can eliminate the explicit time dependence in Eq. 3, but we have to add two more equations to our system:

$$\dot{z}_{n+1} = 1$$

$$z_{n+1}(0) = 0$$

Thus we have

$$\left. \begin{array}{l} \dot{z}_i = f_i(z_1, z_2, \ldots, z_{n+1}) \\ \\ z_i(0) = c_i \end{array} \right\} \quad i = 1, 2, \ldots, n+1 \tag{5}$$

With this formulation we can express any dynamic physical system by means of a set of autonomous differential equations (the right-hand side of Eq. 5 does not explicitly involve the time variable t). Of course

$$z_i = z_i(t) \tag{6}$$

To make the formulation more compact we shall introduce the notion of functional vectors. Let $\mathfrak{Z}$ be a vector whose components are z_i, $\mathfrak{F}$ a vector with components $f_i(\mathfrak{Z})$, and $\mathfrak{C}$ the vector defined by the component c_i. Then we can write Eq. 5 in vector form:

$$\dot{\mathfrak{Z}} = \mathfrak{F}(\mathfrak{Z})$$

$$\mathfrak{Z}(0) = \mathfrak{C} \tag{7}$$

We can now derive the condition that ensures that this vector equation has a unique solution. In particular, we shall state a Lipschitz condition in vector form. Let a region G exist in the space defined by the vector $\mathfrak{Z}$, that is, a set of points with a particular $\mathfrak{Z}_G$ value. Then, if $\mathfrak{Z}_G^{(1)}$ and $\mathfrak{Z}_G^{(2)}$ are any two points of G, Eq. 7 will have a unique solution

if

$$\left\| \mathfrak{F}\left(\mathfrak{Z}_G^{(1)}\right) - \mathfrak{F}\left(\mathfrak{Z}_G^{(2)}\right) \right\| \leq \left\| \mathfrak{Z}_G^{(1)} - \mathfrak{Z}_G^{(2)} \right\| K \tag{8}$$

where K is a constant characteristic of the entire region G. This condition implies that $\mathfrak{F}(\mathfrak{Z})$ is continuous and bounded in G, and that it has uniformly bounded partial derivatives with respect to z_i in G. A sufficient, but not necessary, condition is that

$$\left\| \mathfrak{F}(\mathfrak{Z}) \right\| \leq M \left\| \mathfrak{Z} \right\| \qquad\qquad \mathfrak{Z} \text{ in } G \tag{8'}$$

where

$$\left\| \mathfrak{Z} \right\| = \sum_{i=1}^{n+1} \left| z_i \right| \tag{9}$$

2. An Iterative Method of Solution

We now proceed to solve Eq. 7, subject to the condition stated previously. Equation 7 can be written in the integral form

$$\mathfrak{Z}(t) = \mathfrak{C} + \int_0^t \mathfrak{F}(\mathfrak{Z}) \, d\xi \tag{10}$$

and its solution can be found by the successive construction of the functions

$$\Phi_1(t) = \mathfrak{C} + \int_0^t \mathfrak{C} \, d\xi$$

$$\cdots\cdots\cdots\cdots\cdots \tag{11}$$

$$\Phi_{n+1} = \mathfrak{C} + \int_0^t \Phi_n(\xi) \, d\xi$$

That this process will converge to the final solution is obvious, since

$$\Phi_n = \Phi_1 + (\Phi_2 - \Phi_1) + (\Phi_3 - \Phi_2) + \ldots + (\Phi_n - \Phi_{n-1}) \tag{12}$$

and $(\Phi_n - \Phi_{n-1})$ vanishes in the limit (as imposed by the Lipschitz condition).

3. The Theorem of Successive Approximations

In the preceding paragraph we presented a method of successive approximations for the solution of a set of first-order differential equations. It is of interest to clarify, if

possible, in a broad fashion, the conditions of its applicability. Then it will not be necessary in each separate case to trace the whole method, but it will be sufficient to show that it fulfills the conditions of its applicability.

THEOREM: Let the family $\{\phi\}$ of functions be defined on one and the same set (it makes no difference what kind) Φ and possessing the following properties:

1. Each function ϕ is limited

$$|\phi| \leqslant M_\phi \tag{13}$$

2. The limit of each uniform decreasing successive function family is also a function of the same family.

3. For the family $\{\phi\}$ there exists an operator $A(\phi)$ by which each function of this family is transformed into another function of the same family.

4. For any two functions ϕ_1 and ϕ_2 of this family there exists the relation

$$|A(\phi_2) - A(\phi_1)| \leqslant m \text{ upper-bound } |\phi_2 - \phi_1|$$

where $0 \leqslant m < 1$. Here by the term upper-bound we understand the upper-limit value $|\phi_2 - \phi_1|$ on the set Φ.

Then the equation

$$\phi = A(\phi) \tag{14}$$

has one and only one solution among the functions of the analyzed families.

Proof: Let us take some function ϕ_o of the given family and let us construct the function

$$\phi_1 = A(\phi_o)$$

which we shall name the "first approximation" of the solution of Eq. 14. According to property 3, ϕ_1 belongs in $\{\phi\}$, and, therefore, one may construct the "second approximation" obtaining

$$\phi_2 = A(\phi_1)$$

The function ϕ_2 also belongs in the family $\{\phi\}$. Consequently, this process can continue indefinitely, and thus we can obtain the successive functions

$$\phi_o, \phi_1, \phi_2, \ldots, \phi_n \tag{15}$$

where

$$\phi_k = A(\phi_{k-1}) \qquad \text{for} \quad n \geqslant k \geqslant 1$$

We shall prove that this sequence of functions decreases uniformly. For this we shall examine the series

$$\phi_0 + (\phi_1 - \phi_0) + (\phi_2 - \phi_1) + \ldots \tag{16}$$

If $|\phi_0| \leq M_0$ and $|\phi_1| \leq M_1$, which follows from property 1, then

$$|\phi_1 - \phi_0| \leq M_0 + M_1 = M$$

From property 4 for the operator $A(\phi)$ we obtain

$$|\phi_{n+1} - \phi_n| = |A(\phi_n) - A(\phi_{n-1})| \leq m \text{ upper-bound } |\phi_n - \phi_{n-1}|$$

Therefore, taken term by term, the absolute magnitude of each part of the series of Eq. 16 does not exceed the corresponding part of the convergent series

$$M_0 + M + Mm + Mm^2 + \ldots \tag{17}$$

each term of which is a positive constant. Therefore, the sequence of Eq. 15, the parts of which are summed by the series of Eq. 16, uniformly approaches some function ϕ. According to property 2 this limit function also belongs in the family $\{\phi\}$. Let us notice further that

$$|A(\phi) - A(\phi_{n-1})| \leq m \text{ upper-bound } |\phi - \phi_{n-1}|$$

But, because $|\phi - \phi_{n-1}| \to 0$ uniformly at $n \to \infty$, $A(\phi_{n-1})$ also uniformly "descends" to $A(\phi)$. Therefore in the equality

$$\phi_n = A(\phi_{n-1})$$

one may pass in the limit as $n \to \infty$ to

$$\phi = A(\phi)$$

Now if Eq. 14 had two solutions $\phi^{(1)}$ and $\phi^{(2)}$ in the family $\{\phi\}$, then, necessarily,

$$|\phi^{(2)} - \phi^{(1)}| = |A[\phi^{(2)}] - A[\phi^{(1)}]| \leq m \text{ upper-bound } |\phi^{(2)} - \phi^{(1)}|$$

and consequently

$$\text{upper-bound } |\phi^{(2)} - \phi^{(1)}| \leq m \text{ upper-bound } |\phi^{(2)} - \phi^{(1)}|$$

This is only possible for $\phi^{(2)} \equiv \phi^{(1)}$ because $m < 1$.

4. Examples

In this paragraph we shall apply the previously stated theorem in some well-known problems of analysis and will show that conditions 1 to 4 are necessarily met.

a. It is well known that the equation

$$x = f(x) \tag{18}$$

has a unique solution if f(x) is definite and differentiable in all values of real x and if everywhere

$$f'(x) = \left| \frac{\partial f(x)}{\partial x} \right| \leq T < 1 \tag{19}$$

The conditions given above follow from the theorem in this manner. Let us choose a set Φ consisting of only one point. Then, each function accepts on M only one value. Consequently, the family $\{\phi\}$ will consist of all real numbers. Therefore, it is evident that conditions 1 and 2 are fulfilled. Now for the operator $A(\phi)$ let us assume the function $f(\phi)$. Since f(x) is defined for all positive values of the argument x, each real number is transformed by means of $f(\phi)$ to some other real number; consequently, condition 3 is fulfilled. Condition 4 is also fulfilled because

$$\left| f(x_2) - f(x_1) \right| = \left| f'[x_1 + \Delta(x_2 - x_1)] \right| \left| x_2 - x_1 \right| \leq T \left| x_2 - x_1 \right|$$

b. Let the function f(x, y) be defined for $a \leq x \leq b$ and positive values of y, continuous on x, and have everywhere bounded derivatives on y, which always succeed some constant $T > 0$. Then the equation

$$f(x, y) = 0 \tag{20}$$

has only one solution y(x) which is continuous and falls on the inside of the interval $a \leq x \leq b$.

In order to apply the conditions of the theorem for the set Φ let us assume the closed interval $[a, b]$, and for the family $\{\phi\}$ let us choose all the families of functions which are continuous in this interval $[a, b]$. Then it is evident, that conditions 1 and 2 are fulfilled. Let us further assume the operator

$$A(\phi) = \phi - \frac{1}{M} f(x, \phi) \tag{21}$$

where M is the upper limit of the quantity

$$f'_y(x, y) = \frac{\partial f(x, y)}{\partial y} \tag{22}$$

It is evident that this operator satisfies condition 3. From the other side, because

$$\left| A(\phi_2) - A(\phi_1) \right| = \left| \phi_2 - \frac{1}{M} f(x, \phi_2) - \phi_1 - \frac{1}{M} f(x, \phi_1) \right|$$

$$= \left| (\phi_2 - \phi_1) - \frac{f'_\phi(x, \phi_1 + \Delta(\phi_2 - \phi_1))}{M} \left| \phi_2 - \phi_1 \right| \right|$$

$$\leq \left| \phi_2 - \phi_1 \right| \left(1 - \frac{T}{M} \right)$$

condition 4 is also fulfilled. It should be recognized that what was just shown is that

the equation

$$\phi = \phi - \frac{1}{M} f(x, \phi) \tag{23}$$

has a unique solution for the stated conditions. However, Eq. 23 is everywhere identically equal to

$$f(x, \phi) = 0$$

5. On the Degree of Smoothness

Before we proceed to the presentation of a very important theorem concerning the solutions of dynamic systems, perhaps we should clarify what is meant by "degree of smoothness." Since all responses in the dynamic system ultimately must be explicit functions of time (real variable t), the derivatives of these functions with respect to t acquire a defined physical sense. Thus a function $y(t)$ which possesses continuous derivatives up to the pth order ($p \geq 0$) is said to possess a pth degree of smoothness, where under 0 degree is meant the function itself.

THEOREM: If $f(y, t)$ is of nth degree of smoothness, then any solution of the equation

$$\dot{y} = f(y, t) \tag{24}$$

is of $n + 1$ degree of smoothness.

Proof: Let $y(t)$ be any solution of Eq. 24. Then we have the identity

$$y'(t) = f[y(t), t] \tag{25}$$

Since the function $y(t)$ satisfies Eq. 24, then it has everywhere derivatives on t. Therefore, if $f(y, t)$ is continuous on t, then the right-hand part of Eq. 25 is continuous and it means that $y'(x)$ is also continuous. This is for n = 0.

Let us now assume $n \geq 1$. Then the right-hand part of Eq. 25 has continuous derivatives in t, and thus in turn the left part of this identity has continuous derivatives in t. It means that the function $y(t)$ has continuous derivatives up to the second order. We now differentiate Eq. 25, with respect ot t. Thus

$$y''(t) = f'_t(y, t) + f'(y, t) y'(t) \tag{26}$$

Applying to this identity the same reasoning which we used with Eq. 25, we find that if n = 2, then $y(t)$ has continuous derivatives up to the third order. We can differentiate again and proceed with the same reasoning up to the nth degree.

N. DeClaris

B. SUMMARY OF PIECEWISE LINEAR TRANSFER FUNCTION SYNTHESIS PROCEDURES

Over the past year, investigations were conducted in the general area of piecewise linear network analysis and synthesis. One phase of these investigations was the synthesis, by the use of diode networks, of piecewise linear resistive transfer functions of one or more variables. The synthesis procedures were based on an "algebra of inequalities" (see ref. 1 for explanation of the notation used below). The synthesis problem can be resolved into two parts: (a) arbitrary function synthesis (general-purpose function generation) and (b) particular function synthesis (special-purpose function generation).

1. Arbitrary Function Synthesis

Here the problem is as follows: Given an arbitrary function,

$$y = f(x_1, x_2, \ldots, x_n)$$

tabulated at regular intervals in the n-variable space, construct a diode network which will produce a piecewise linear and continuous function taking on the prescribed values at the tabulated points.

In general, the arbitrary function can be built up as the summation of a number of simple "unit functions" of varying heights, each unit function being centered over a different point in the grid of tabulation (2). Figure XVII-1 is a table of three different types of unit functions which can be superposed to obtain a general piecewise linear function of a single variable. The functions pictured are the "ramp," "step," and "triangle." Next to each is its algebraic representation; an expression for the required height of the kth unit, A_k, in terms of the prescribed ordinates of the desired function, $a_k, a_{k-1}, \ldots$; and a realization of the unit with a diode network. In practice, many of these units would be fed to a summing amplifier to obtain the over-all function. Figure XVII-2 is a similar table illustrating the extension of these unit functions to a second independent variable. (Not all of these functions are new developments, but they are included for purposes of comparison.) It will be noted that the unit functions become more difficult to realize as one reads down the table. However, this difficulty is accompanied by a simplification of the expression for the height of the ij^{th} function: in fact, when one uses triangle or pyramid* functions, the actual ordinate of the desired function at a given point equals the height of the unit function centered over that point. This is a decided advantage insofar as propagation of errors and ease in "programing" an arbitrary function are concerned. Generalization of this procedure to more variables is fairly straightforward.

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*Note that the two types of pyramids shown in Fig. XVII-2 must be alternated with each other over the grid of tabulation to obtain the correct over-all function.
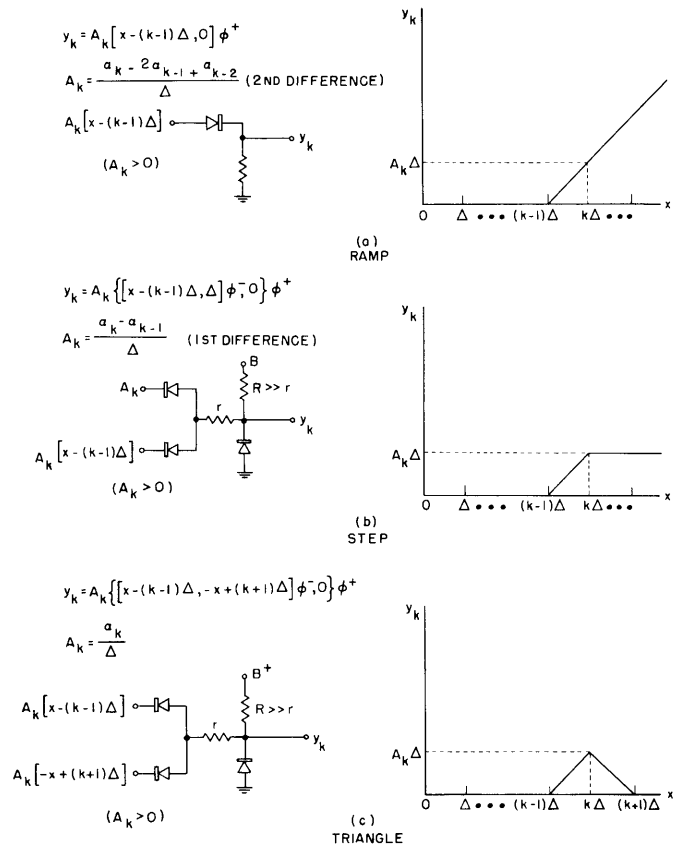
$y_k = A_k \left[x - (k-1)\Delta, 0 \right] \phi^+$

$A_k = \dfrac{a_k - 2a_{k-1} + a_{k-2}}{\Delta}$ (2ND DIFFERENCE)

$A_k \left[x - (k-1)\Delta \right]$

$(A_k > 0)$

(a)
RAMP

$y_k = A_k \left\{ \left[x - (k-1)\Delta, \Delta \right] \phi^-, 0 \right\} \phi^+$

$A_k = \dfrac{a_k - a_{k-1}}{\Delta}$ (1ST DIFFERENCE)

A_k

$A_k \left[x - (k-1)\Delta \right]$

$(A_k > 0)$

(b)
STEP

$y_k = A_k \left\{ \left[x - (k-1)\Delta, -x + (k+1)\Delta \right] \phi^-, 0 \right\} \phi^+$

$A_k = \dfrac{a_k}{\Delta}$

$A_k \left[x - (k-1)\Delta \right]$

$A_k \left[-x + (k+1)\Delta \right]$

$(A_k > 0)$

(c)
TRIANGLE

Fig. XVII-1

Unit functions of one variable.

$y_{ij} = A_{ij} \left\{ \left[x_1 - (i-1)\Delta, x_2 - (j-1)\Delta \right] \phi^-, 0 \right\} \phi^+$

$A_{ij} = \dfrac{1}{\Delta} \left[a_{ij} - a_{i-2,j-2} + a_{i-1,j-2} + a_{i-2,j-1} - a_{i-1,j-1} \right]$

$A_{ij} \left[x_1 - (i-1)\Delta \right]$

$A_{ij} \left[x_2 - (j-1)\Delta \right]$

$(A_{ij} > 0)$

(a)
RAMP

$y_{ij} = A_{ij} \left\{ \left[x_1 - (i-1)\Delta, x_2 - (j-1)\Delta, \Delta \right] \phi^-, 0 \right\} \phi^+$

$A_{ij} = \dfrac{1}{\Delta} \left[a_{ij} + a_{i-1,j-1} - a_{i,j-1} - a_{i-1,j} \right]$

$A_{ij} \left[x_1 - (i-1)\Delta \right]$

$A_{ij} \left[x_2 - (j-1)\Delta \right]$

$A_{ij}\Delta$

$(A_{ij} > 0)$

(b)
STEP

FOR (A) PYRAMID

$y_{ij} = A_{ij} \left\{ \left[x_1 - (i-1)\Delta, x_2 - (j-1)\Delta, -x_1 + (i+1)\Delta, -x_2 + (j+1)\Delta \right] \phi^-, 0 \right\} \phi^+$

$A_{ij} = \dfrac{a_{ij}}{\Delta}$

$A_{ij} \left[x_1 - (i-1)\Delta \right]$

$A_{ij} \left[x_2 - (j-1)\Delta \right]$

$A_{ij} \left[-x_1 + (i+1)\Delta \right]$

$A_{ij} \left[-x_2 + (j+1)\Delta \right]$

$(A_{ij} > 0)$

(c)
PYRAMIDS
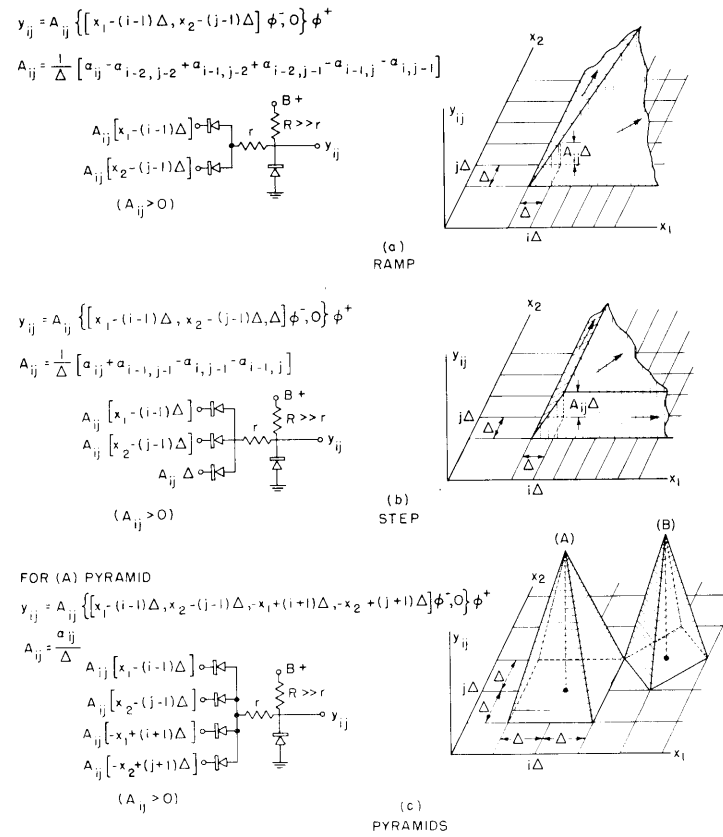
Fig. XVII-2

Unit functions of two variables.

2. Particular Function Synthesis

In the designing of a network to realize one specific function, the preceding methods would not necessarily prove the most efficient ones. Usually, a function will have certain geometrical properties that will suggest special types of piecewise linear approximation and physical realization. This is especially true of functions given in algebraic form.

When one considers functions of two independent variables, the surface

$$y = f(x_1, x_2)$$

can be easily approximated and realized if it is (a) completely convex or concave (3) and/or (b) a ruled surface. Examples of functions of type (a) are

$$y = \left| \left(x_1^2 + x_2^2 \right)^{1/2} \right| \qquad \text{(cone)}$$

$$y = x_1^2 + x_2^2 \qquad \text{(paraboloid of revolution)}$$

$$y = \left| \left(x_1^2 + x_2^2 + a^2 \right)^{1/2} \right| \qquad \text{(hyperboloid of revolution)}$$

Examples of type (b) functions are

$$y = \left| \left(x_1^2 + x_2^2 \right)^{1/2} \right|$$

$$y = x_1 x_2 \qquad \text{(multiplier)}$$

$$y = \frac{x_1 x_2}{a^2 + x_2^2}$$

$$y = \tan^{-1} \frac{x_2}{x_1}$$

Type (a) functions are easily realizable because they may be approximated piecewise-linearly to any degree of accuracy by an expression of the form

$$y = (a_1 x_1 + a_2 x_2 + a_3, \quad b_1 x_1 + b_2 x_2 + b_3, \quad \ldots) \phi^{\pm}$$

This type of expression can be realized with the most elementary type of diode network. (Realization of the cone is discussed in ref. 4.)

Type (b) functions (ruled surfaces) can be approximated by segments of planes which intersect along lines coinciding with the generatrix of the surface. The approximation is exact along these intersections. In general, additional diagonal intersections must be

$$y \approx \frac{Kx_1}{a^2 + x_2^2}$$
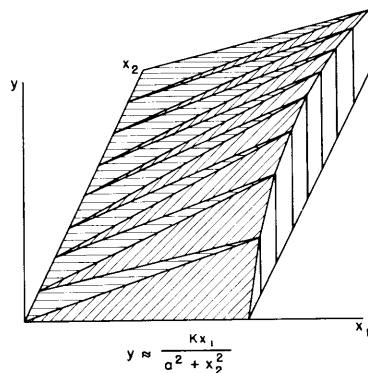
Fig. XVII-3

Approximation of a ruled surface.

added so that the surface is divided into triangular segments of planes. (Figure XVII-3 illustrates such an approximation.) Once the approximation has been made, the resultant intersections are classified as either convex or concave, and the surface is divided into a linear combination of two functions: one containing only the convex intersections, and the other containing only the concave intersections. These latter functions, being of type (a), are readily realizable.

T. E. Stern

References

1. Quarterly Progress Report, Research Laboratory of Electronics, M. I. T., Jan. 15, 1955, pp. 108-112.

2. Quarterly Progress Report, Research Laboratory of Electronics, M. I. T., July 15, 1954, pp. 85-88.

3. J. L. W. V. Jensen, Sur les fonctions convexes et les inégalités entre les valeurs moyennes, Acta Mathematica 30, 175-193 (1906).

4. Quarterly Progress Report, Research Laboratory of Electronics, M. I. T., April 15, 1955, pp. 89-93.

C. PIECEWISE-LINEAR DRIVING-POINT CONDUCTANCES

It has been stated previously (1) that a necessary condition for the realizability of a driving-point conductance of 2^n linear regions with n diodes is that the conductance be describable by a Gray code. A Gray code is one whose successive states can be obtained by changing one variable only. The object of this report is to establish whether or not this is a sufficient condition and to evolve a synthesis procedure for the realization of the realizable conductances.

To facilitate the discussion, a few definitions are made.

Definition IV (see ref. 1, Definitions I-III): A "Gray state" is one which could be a successive state of a given state in a Gray code.

THEOREM V (see ref. 1, Theorems I-IV): A state has as many Gray states as there are variables. This is seen from the fact that changing one variable at a time, we can produce only as many new states as there are variables.

1. Gray Code Sufficiency for Realizability

One of the difficult problems so far has been to determine whether or not a given driving-point conductance picture can be represented by a Gray code. By use of the ideas of switching circuits, a method was found which, upon the application of a suitable mapping procedure, would determine if an arbitrary but possible code assigned to the conductance picture can be replaced by a Gray code. Although this mapping procedure works in all cases, it gets rather awkward as the number of variables n is increased. Also, its usefulness is severely limited by the fact that its application requires the assignment of a possible code to the driving-point conductance picture.

A more useful method would be one which would immediately yield a realizable Gray code when we know in what order decreases or increases in slope of the given driving-point conductance take place. Such a method has been developed and is described below.

2. The Code Graph

Nodes of the graph represent states of the network and transmissions represent permissible changes of state. The graph of n variables has n + 1 levels, each corresponding to the introduction of a different number of 1's into the states of the network. Figure XVII-4(a, b) shows the skeleton code graphs for n equals 2 and n equals 3. The various transmissions indicated are all the possible transmissions consistent with the requirement that the resultant code be a Gray code. Note that the number of transmissions from each node is n in agreement with Theorem V. There can be no transmission between nodes on the same level for that would necessitate multiple switching which is contradictory to the definition of a Gray code. A change in level upward in the code graph (the introduction of an additional 1 into the new state of the network) corresponds to an increase in slope of the driving-point conductance — in agreement with Theorem IV (1). With these simple properties of the code graph in mind it is possible to go from the given conductance picture directly to a realizable Gray code or to the determination of the network's unrealizability by the method illustrated below.

3. Example 1

Problem: The given driving-point conductance is shown in Fig. XVII-5(a). Find out if it is realizable with two diodes (can be represented by a Gray code of 2 variables), and
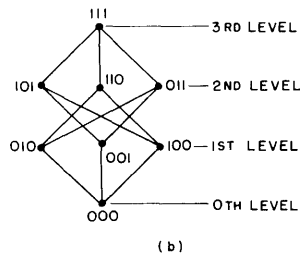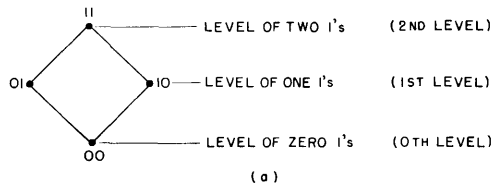
Fig. XVII-4

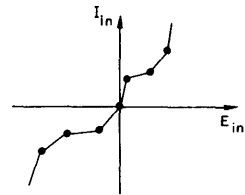(a) Code graph for n = 2; (b) code graph for n = 3.

Fig. XVII-6
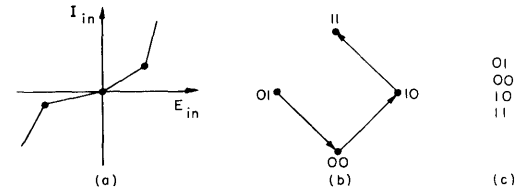
Given driving point conductance.

Fig. XVII-5

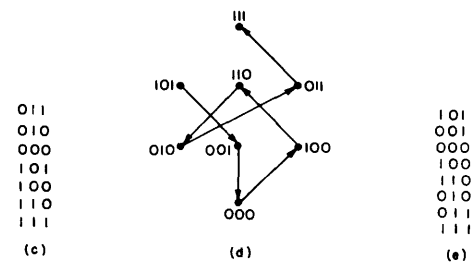(a) Given driving point conductance; (b) code
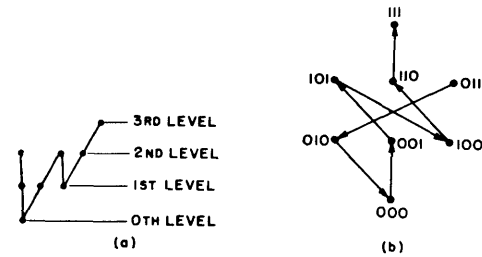graph for Fig. XVII-5(a); (c) code.

Fig. XVII-7

(a) Auxiliary sketch to establish starting level; (b) code graph for
Fig. XVII-6; (c) code; (d) alternate code graph; (e) code.

if so, what is the code that describes it.

Solution: First, one quickly establishes the correct level of the starting point, which in this case can be done by inspection. Then, if one starts anywhere on the correct level (01 in this example) and follows only the permissible transmissions as indicated on the skeleton code graph of Fig. XVII-4(a), remembering not to traverse the same node more than once, the succession of slope increases, and decreases are translated into increases and decreases of level of the code graph, resulting in Fig. XVII-5(b). The code thus obtained is readily seen to be that of Fig. XVII-5(c) which is recognized as code g of reference 1, where the realization of this code is given in Fig. XV-6.

4. Example 2

Consider the driving-point conductance given in Fig. XVII-6. A quick sketch of increases and decreases in slopes such as that shown in Fig. XVII-7(a) establishes the "dynamic range" of the code and shows that the given eight linear region conductances, if realizable with the theoretically minimum number of three diodes, must correspond to a code graph which starts on the second level. Starting at the arbitrary point 011 on the second level, we obtain the code graph of Fig. XVII-7(b). Since the graph could be drawn by our traversing only transmissions existing on the skeleton code graph of three variables, the conductance is known to be described by a Gray code. This code is given in Fig. XVII-7(c). This graph is not unique. Figure XVII-7(d) shows an alternate code graph with the code of Fig. XVII-7(e).

5. Properties of Code Graphs

In the following, some of the properties of code graphs will be briefly summarized without proof.

The maximum number of successive increases or decreases in slope of the input conductance is n where n is the number of diodes. The maximum difference in the sum of increases and decreases of slopes cannot exceed n.

If a graph is started at a certain level and a specified order of increases and decreases in slopes (levels) is observed, the resulting graphs, if they can be drawn at all, will all be alike except that the columns will be interchanged (see Fig. XVII-7(c, e)). Column 1 changed to column 2, column 2 to column 3, and column 3 to column 1. It does not make any difference, therefore, which code graph is used as a basis for synthesis. It is only the numbering (designation) of the diodes that is different. An important thing to notice is that in Example 2 as well as in Example 1, diode one switches once, diode two switches twice, and diode three, four times. In this type of switching sequence the nth diode switches 2^{n-1} times. Consider the conductance, however, shown in Fig. XVII-8. It is described by a Gray code as seen from the code graph of Fig. XVII-8. The first diode switches three times, the second switches three times and the third once.
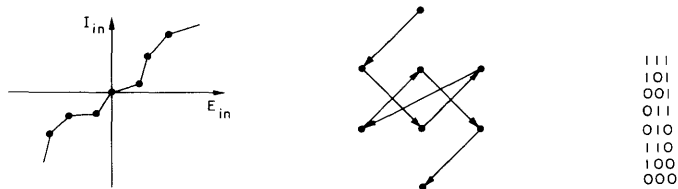
Fig. XVII-8

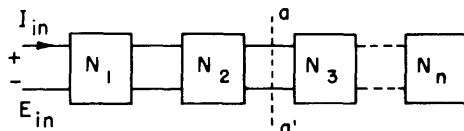Driving point conductance with 331 switching sequence.

Fig. XVII-9

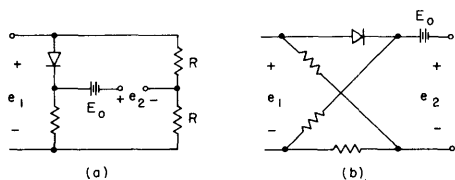Cascade of two terminal pair networks.
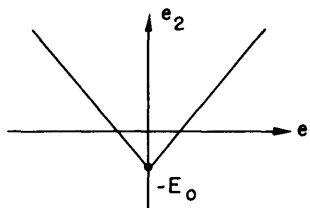

Fig. XVII-10

Canonic two terminal pair network.

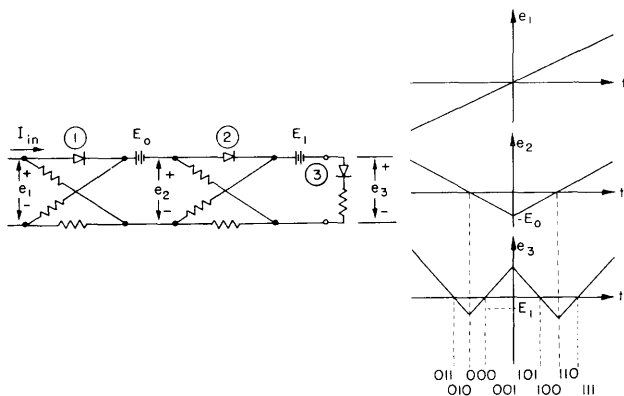Fig. XVII-11

Transfer function of network of Fig. XVII-10.

Fig. XVII-12

Three diode network with eight states.

Fig. XVII-13

Code of Fig. XVII-12.

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This poses the question of the type of network that can be used to realize the different switching sequences.

6. Network Realization

The usual Foster or Cauer type of series or parallel connection of simple canonic network forms will realize any driving-point conductance not containing negative resistance regions. This type is discussed by Stern (2). The disadvantage of this realization is that n diodes will yield only n + 1 linear regions. Since this investigation stresses the importance of economy of elements, the Foster and Cauer type of realization will not be discussed. The Darlington type of cascaded two terminal pair canonic network forms seems to lend itself more readily to minimization techniques. Consider a cascade of two terminal pair networks, each containing a single diode as shown in Fig. XVII-9. This type of network is capable of realizing the switching sequence where the nth diode switches 2^{n-1} times, but cannot realize any other switching sequence. A code such as that of Fig. XVII-8 cannot be realized with the type of network shown in Fig. XVII-9. The reason is that if we have a cascade of two terminal pair networks, each section acts as a load on the previous section; or, in other words, the network to the right of any arbitrary dividing line such as aa' in Fig. XVII-9 just acts as a load on the network to the left of the dividing line and does not actively influence the conditions imposed on the diodes contained in the left-hand part of the network. Therefore, if the network of Fig. XVII-9 is claimed to realize a switching sequence such as 331 of Fig. XVII-8, by disconnecting the last section, we would be left with a two-diode network exhibiting six breakpoint — that is, seven states. This contradicts the fact that the maximum number of states a two-diode network can have is four. If we insist on the requirement that the second diode switches three times, this can be done only as a result of the direct, active influence of the third diode on the conditions imposed on the second diode. This cannot be done with a simple cascade, nonfeedback network. The network of Fig. XVII-9, however, can realize the 1, 2, 4, 8, . . . switchings for the 1st, 2nd, 3rd, . . . diodes, because if an arbitrary number of sections are disconnected, we are always left with a remaining network which has m diodes and exhibits 2^m states.

Work is now under way to determine what type of network, if any, can be used to realize these other switching sequences. Is it possible to do it with a feedback network without amplifiers?

7. Networks Not Exhibiting All 2^n States

So far we have considered only the realization of those networks which exhibited all the 2^n states. If a given conductance has less than that many states, other states have to be added either in between existing states or at the end as terminating states in such a way that the resulting new conductance is completed to have all 2^n states and is

described by a Gray code of the $1, 2, 4, \ldots$ switching variety. This new conductance may now be constructed by means of a cascade of two terminal pair networks and the breakpoints bounding the undesired states be made to converge by a suitable choice of parameters. This method is called the method of "completing the network."

8. A Canonic Two Terminal Pair Network Form

If a network containing a single diode and behaving like a biased absolute value device can be built, a circuit exhibiting all 2^n states can be constructed.

Consider the network shown in Fig. XVII-10(a, b) and having the transfer function shown in Fig. XVII-11. Figure XVII-11 shows that if e_1 is a ramp, e_2 will go from positive to negative to positive. If a diode, or another canonic network is connected across the terminals of e_2, the resulting two-diode network will exhibit four states. As an illustration, consider the network of Fig. XVII-12. The code of Fig. XVII-12 is shown in Fig. XVII-13. It is assumed, to make the analysis easier, that each successive section is a negligible load on the previous one. The network behavior does not lose any of its generality as a result of this assumption.

An interesting property of this type of network is that n diodes, $n - 1$ batteries, and 3n resistors can give a network capable of acting as a nonbandwidth limited frequency multiplier, multiplying the input frequency by $2^n, 2^{n-1}, 2^{n-2}$, and so on. To be rigorous, this device is a frequency multiplier for only a certain limited number of waveshapes such as triangular and sawtooth waveforms. For an arbitrary waveform not containing any major discontinuities and having a sufficiently large amplitude, this cascade of two terminal pair networks multiplies the time average zero crossings by 2^n. An investigation utilizing this property is in progress.

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References

1. Quarterly Progress Report, Research Laboratory of Electronics, M.I.T., April 15, 1955, p. 85.
2. Quarterly Progress Report, Research Laboratory of Electronics, M.I T., April 15, 1955, p. 92.

D. GRAPHICAL PHASE PLANE ANALYSIS

In many types of nonlinear problems the characteristics of the nonlinear components are expressed in graphical rather than analytical form, and it is most convenient to analyze circuit behavior from a graphical point of view. With this in mind, we have developed some techniques which are suitable for the graphical solution of the general problem of two nonlinear energy storage elements coupled with an arbitrary dissipative device.

To facilitate analysis, it has been found convenient to make stepwise linear approximations to the dissipative characteristics; thereby, one can construct paths of operation and limit cycles with a simple compass and straight edge. The accuracy of this approximate graphical method is limited only by the patience of the human operator and, in most cases, slide rule accuracy is possible with considerably less effort than conventional difference equation methods require.

Figure XVII-14 illustrates the use of stepwise linear approximations for the analysis of a simple series RLC circuit with $2R = (L/C)^{1/2}$ and initial conditions $i^2 L = v^2 C = 1$. If time is eliminated from the equations of motion we arrive at the relation $(di/de) = C(v-e)/Li$, and by defining new variables $i' = i(L)^{1/2}$, $e' = e(C)^{1/2}$, $v' = v(C)^{1/2}$, we can normalize this equation to read $(di'/de') = (v' - e'/i')$. The stepwise linear approximation makes v' or i' constant over small regions of operation and thus, within these regions, the path of operation can be traced out with a compass or, in some cases, with a straight edge. In Fig. XVII-14 as the path of operation moves from a to b to c and so on, the compass center is assumed to move in jumps from A to B to C, etc., instead of moving continuously as it would for an exact solution. Note that although only a few steps are used, the approximation is amazingly close to the exact solution. Moreover, the time scale can be computed easily by observing that the compass always rotates with an angular frequency of $\omega = (LC)^{1/2}$ rad/sec. For purpose of comparison, the time and voltage coordinates of point $f(i' = 0)$ have been computed by the approximate method and by the exact analytical solution.

	Approximate	Exact
$\dfrac{t_f}{(LC)^{1/2}}$	$138° \ 28.2' = 2.31204$	$\left(\dfrac{16}{15}\right)^{1/2} \tan^{-1}\left(\dfrac{5}{3}\right)^{1/2} = 2.30209$
$e_f C^{1/2}$	0.67894	$\left(\dfrac{3}{2}\right)^{1/2} \exp\left(-\dfrac{t_f}{4}\right) = 0.68897$

The close accuracy of this apparently crude approximation can be attributed to the fact that an average compass center is assumed; the slope of the piecewise-circular approximate curve contains discontinuities since it is based on average rather than initial slopes, but the end points of the arc are accurately determined. By limiting the piecewise circular arcs to about 10° each and by observing some of the more subtle points of this approximate method, it is possible to obtain slide rule accuracy with surprisingly little effort.

Figure XVII-15 shows the application of the stepwise linear analysis to a somewhat more complicated circuit. It is the same circuit as shown in Fig. XVII-14 except for the addition of a resistor in parallel with the capacitor. Note that now the compass center is not restricted to move along the $i' = 0$ axis, but may be located anywhere in
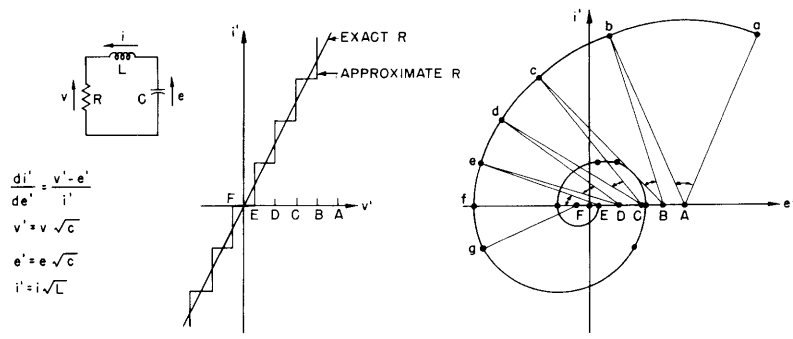
Fig. XVII-14
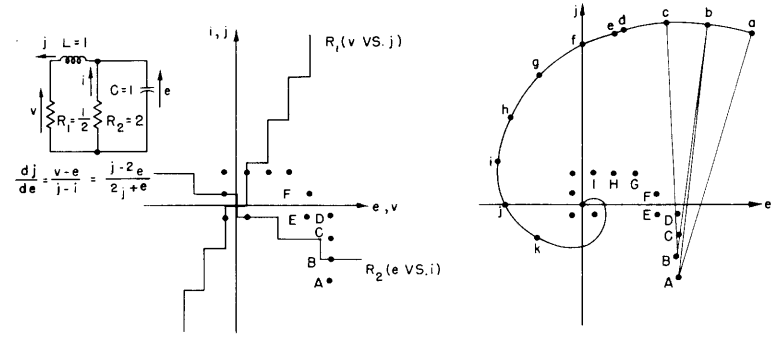
Piecewise circular solution of a series RLC circuit.

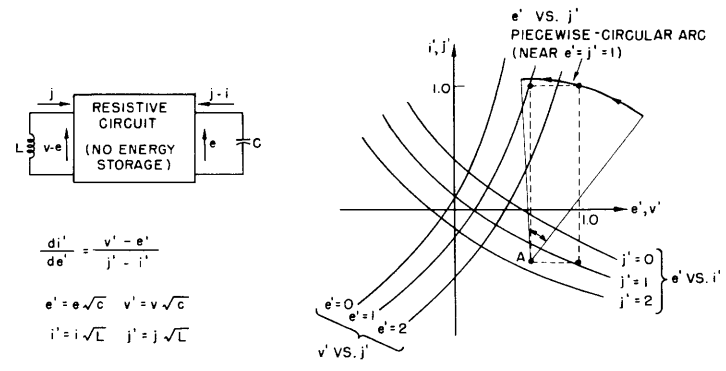Fig. XVII-15

Piecewise circular solution of an RLCG circuit.

Fig. XVII-16

Graphical construction for general second-order system.

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the plane. Here again the approximation is very good as long as the circular arcs are less than about 30° and the compass center moves in relatively small jumps. It is, of course, not necessary to make a formal stepwise linear approximation and one can vary the size of the steps at will; the important point is to locate the compass center on an average-over-a-region basis rather than on an initial-slope basis.

The most general two energy storage case, and perhaps the one of greatest interest, is an extension of that of Fig. XVII-16 with the additional possibility that the energy storage elements may be nonlinear. Fortunately, however, we can define new variables e', v', i', j', such that the analysis is the same as for the linear reactance case. This redefining of variables may, of course, radically change the nature of the solution and alter the time scale. Present work is being devoted to a study of this class of nonlinear circuits with the following questions in mind:

1. Can circuits that will not oscillate with any two linear energy storage elements oscillate with two nonlinear reactances?

2. Can a circuit containing no negative resistances or dependent generators be made to oscillate with nonlinear reactances? (A recent paper by Duffin (1) proves this to be impossible on an energy basis.)

3. If the nonlinear reactances are allowed to have negative incremental values how does this affect the stability conditions?

4. In what way does hysteresis or memory in the nonlinear reactance affect the stability conditions?

It is worth noting that the graphical solution described above can be used to solve several classes of nonlinear differential equations. Several examples are:

$$f(x, y)dx + g(x, y)dy = 0$$

$$\ddot{x} = f(x, \dot{x})$$

$$\dot{x} = f(x, y), \qquad \dot{y} = g(x, y)$$

The classical van der Pol equation $\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0$, for example, is easily solved by this graphical method.

These methods are readily applicable to circuits containing two lumped reactances but, in many cases, such as microwave filter design, it is more convenient to approximate the energy storage elements by lossless transmission lines. Even with distributed energy storage, however, it is still possible to use simple graphical techniques. In particular, problems involving a lossless transmission line with arbitrary resistive termination are readily solved, and the method may be extended to circuits containing more than one line. The method of graphical construction is indicated in Fig. XVII-17 for the simple case of a step of voltage applied through a small resistor to a diode-terminated line. The construction hinges on the fact that a transmission line insists that all changes in terminal voltage and current be related by $(\Delta v/\Delta i) = R_o$ where R_o is
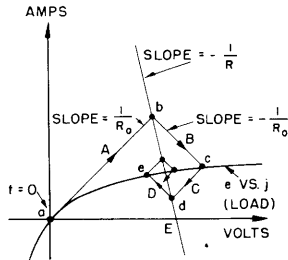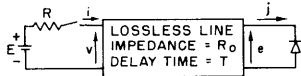
Fig. XVII-17

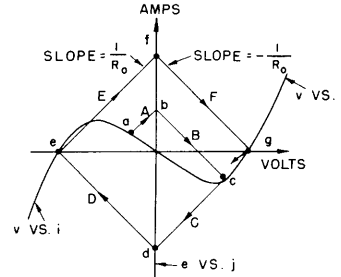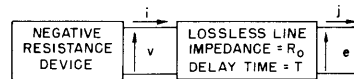Graphical transmission line analysis.

Fig. XVII-18

Transmission line oscillator.

the characteristic resistance of the line. The traveling waves are represented by vectors with slopes of $\pm R_0$ (plus or minus depending on polarity definitions of voltage and current) and with magnitude proportional to the amplitude of the wave.

Referring to Fig. XVII-17, we see that when the step is first applied, the voltage v and current i must increase together so that $(\Delta v/\Delta i)$ equals R_0, and also so that the input circuit satisfies Ohm's law. Thus the wave that travels down the line can be represented by the vector A with a slope R_0 and a length just sufficient to intersect the input volt-ampere load line. This wave of amplitude A travels down the line and T seconds later reaches the load. To satisfy the e, j load line a reflection must be sent back down the line; this reflection can be represented graphically by a vector B with slope $-R_0$. The reflections will continue, C, D, E, and so on, as the line approaches the equilibrium operating point e = v, i = j.

This graphical representation is especially convenient for transmission line oscillators. Consider, for example, the behavior of a transmission line terminated at one end by a negative resistance and at the other by a short circuit; Fig. XVII-18 shows the "vector" path of operation for this oscillator. Note that we arrive at an eventual limit cycle much as we would for a parallel inductance-capacitance circuit replacing the shorted line. The steady-state waveforms are, of course, square waves of frequency f = 1/4T where T is the transmission line one-way delay time. The coordinates of the ends of the vectors correspond to the terminal voltage and currents, and the ideal path of operation would be a series of points a, b, c, and so forth.

In the hope of developing appropriate graphical methods of analysis, we are working on the more general case of lossy and nonlinear transmission lines. Also, we believe

that, besides the lumped and distributed energy storage elements, there may be other approximations, more closely approximating existing components, that will lend themselves to graphical analysis.

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References

1. R. J. Duffin, J. Appl. Phys. 26, 603 (1955).