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STEEPEST ASCENT DECOMPOSITION METHODS FOR MATHEMATICAL PROGRAMMING/ECONOMIC EQUILIBRIUM ENERGY PLANNING MODELS

by

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ABSTRACT

A number of energy planning models have been proposed for combining econometric submodels which forecast the supply and demand for energy commodities with a linear programming submodel which optimizes the processing and transportation of these commodities. We show how convex analysis can be used to decompose these planning models into their econometric and linear programming components. Steepest ascent methods are given for optimizing the decomposition, or equivalently, for computing economic equilibria for the planning models.

1. Introduction

A number of energy planning models have recently been implemented or proposed which combine (1) econometric submodels for forecasting supply and demand for energy commodities as functions of the prices on these commodities with (2) a linear programming submodel for optimizing the processing and transportation of the commodities. Specific models include, for example, the FEA Project Independence Evaluation System (Hogan (1974)), the world oil market model of Kennedy (1974), and a proposed integration of the Brookhaven Energy System Optimization Model (Hoffman (1973)), with econometric models developed by Data Resources, Inc. (Jorgenson (1975)). The models are equilibrium models because prices, commodities supplied and demanded, and process and transporation activity levels are all variables to be determined simultaneously in a generic time period in equilibrium. The equilibrium conditions can be interpreted as necessary and sufficient Kuhn-Tucker optimality conditions for a related concave programming problem which has its own interpretation.

The purpose of this paper is to discuss how mathematical programming methods can be used to decompose and solve the concave programming problem, and thereby the equilibrium model, into its linear programming and econometric parts. The linear programming submodel communicates with the econometric submodels by passing to them vectors of shadow prices on the energy commodities. The shadow prices are optimal for the linear programming submodel with fixed commodity levels. The econometric submodels compare the shadow prices with the vector of commodity prices required to produce the fixed commodity levels assumed in the linear programming solution. If these two price vectors are equal, then an equilibrium solution has been reached.[1] Equivalently, the equilibrium conditions establish optimality of the prices, commodity levels and processing and distribution levels in the implied concave programming problem.

Although we will focus our attention on the analysis and solution of mathematical programming/economic equilibrium models arising in energy planning, the approach is appropriate to similar models in other areas. Included are agriculture models such as the U.S. energy sector model of Hall et. al. (1975), the world wheat market model of Schmitz and Bawden (1973), and the water resources planning model of Flinn and Guise (1971).

The plan of this paper is the following. Section two contains a statement of the basic concave programming problem to be analyzed, plus a discussion of how it has been used in energy modeling. The following section contains the Kuhn-Tucker optimality conditions for the mathematical programming problem which we interpret as economic equilibrium conditions. Two of these optimality conditions constitute the interface between econometric forecasting of supply and demand for energy commodities and optimization of processing and transporting of these commodities. Section four discusses decomposition methods, based on the optimality conditions, for computing an optimal solution to the concave programming problem, or equivalently, for computing an economic equilibrium. The final section, section five, discusses a number of future areas of research.

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[1] Strict equality is not required between the shadow and commodity price for a commodity at a zero level.

2. Mathematical Programming/Economic Equilibrium Models

In its mathematical programming form, the basic problem we wish to analyze and solve is

$$\phi^* = \max\{f(d) - g(s) - cx\} \tag{1a}$$

$$\text{s.t. } A^1x - s \leq 0 \tag{1b}$$

$$A^2x - d \geq 0 \tag{1c}$$

$$s \geq 0,\ d \geq 0,\ x \geq 0 \tag{1d}$$

where f and -g are concave differentiable functions. It is assumed that (1) has an optimal solution. The vector d is the demand for energy commodities and the vector s is the supply of these commodities. For reasons that will become clear later, we assume that the inverse functions ∇f^{-1} and ∇g^{-1} exist on the non-negative orthant. According to the inverse function theorem (e.g. Apostol, 1957; p. 144), ∇f^{-1} and ∇g^{-1} will exist on the non-negative orthant if ∇f and ∇g have continuous first partials and non-vanishing Jacobians on that region. These assumptions appear reasonable for our model. As we shall see in the following section, the econometric specification of f and g will actually be given by ∇f^{-1} and ∇g^{-1}. For the moment, the intuitive justification that f is concave is that the social benefit f(d) due to satisfied demand d increases monotonically, but at a decreasing rate. Conversely, the function g is convex because the cost g(s) of delivering the supply s increases monotonically, but at an increasing rate since the less expensive quantities are supplied first.

The Project Independence Evaluation System Integrating Model of the FEA (Hogan (1974)) is a U.S. energy sector model for the year 1985 very similar to problem (1). The supply commodities in that model are coal, oil, gas, synthetics and imports in different regions of the

United States. The commodities demanded are the same physical commodities for industrial, commercial and residential use, again in different regions of the United States. The FEA model also considers, at least implicitly, cross cut constraints of the form $Bx \leq b$ involving scarce national resources such as steel and capital availability. The world oil market model of Kennedy (1974) is an equilibrium model derived from a problem of the form (1) where the functions f and g are quadratic. The Brookhaven Energy System Optimization Model (Hoffman (1973)) of the U.S. energy sector assumes supply and demand in problem (1) are exogeneously set, and the objective is to minimize the cost of processing and transportation. There is a project underway to combine this model with the interindustry economic model developed by Data Resources, Inc. (Jorgenson (1975)). It is hoped that the methodology discussed here will aid in that integration.

3. Optimality/Equilibrium Conditions

The interpretation of the Kuhn-Tucker optimality conditions for a variety of economic models as the embodiment of market equilibrium conditions has long been recognized (e.g., see Karlin (1959), Intrilligator (1971)). These models are generally theoretical and the optimality conditions are used to study existence, uniqueness and stability of the equilibrium solution. The difference with the energy planning models discussed in the previous section is that they are <u>empirical</u> models consisting of two distinctly different types of submodels which need to be hooked together; namely, econometric and linear programming submodels. In this context, the Kuhn-Tucker optimality conditions provide a practical mechanism for integrating these diverse models. Moreover, the purpose of an implemented energy model similar to

(1) is to provide numerical answers. The optimality conditions are used in the following section to derive decomposition solution methods for numerically optimizing problem (1).

Let p and q be vectors of shadow prices on the constraints (1b) and (1c), respectively. The optimality conditions are: The solution $\bar{s}$, $\bar{d}$, $\bar{x}$ is optimal in problem (1) if and only if there exist shadow prices $\bar{p}$, $\bar{q}$ satisfying

$$\nabla g(\bar{s}) - \bar{p} \geq 0 \quad \text{with equality if } \bar{s}_i > 0 \tag{2a}$$

$$\nabla f(\bar{d}) - \bar{q} \leq 0 \quad \text{with equality if } \bar{d}_j > 0 \tag{2b}$$

$$c + \bar{p}A^1 - \bar{q}A^2 \geq 0 \text{ with equality if } \bar{x}_k > 0 \tag{2c}$$

$$\bar{p}(A^1\bar{x} - \bar{s}) = 0 \tag{3a}$$

$$\bar{q}(A^2\bar{x} - \bar{d}) = 0 \tag{3b}$$

$$A^1\bar{x} - \bar{s} \leq 0 \tag{4a}$$

$$A^2\bar{x} - \bar{d} \geq 0 \tag{4b}$$

$$\bar{s} \geq 0,\ \bar{d} \geq 0,\ \bar{x} \geq 0,\ \bar{p} \geq 0,\ \bar{q} \geq 0 \tag{4c}$$

The connection between the econometric forecasting submodels and the linear programming submodel is effected by the conditions (2a) and (2b). To see this, let $u = \nabla g(s)$ and $v = \nabla f(d)$ denote vectors of commodity prices on supply and demand, respectively. Then if $\bar{s}_i > 0$, condition (2a) states that $\bar{u}_i = \bar{p}_i$; that is, the commodity price for supply commodity i equals the shadow price for that commodity and they are in equilibrium. If $\bar{s}_i = 0$, then we permit $\bar{u}_i \geq \bar{p}_i$ because a further lowering of the supply price on commodity i would not induce the supply to increase from 0. A similar argument holds for the optimality condition (2b) on the equilibrium between prices on demand commodities and the relevant shadow prices. An equilibrium interpretation of the other optimality conditions is well known and straightforward and is therefore omitted. Note, however, that this interpretation does not depend on the sufficiency of the Kuhn-Tucker conditions due to the concavity of f and -g. If for some reason these functions were not concave, then some solutions to the optimality conditions might not be optimal for problem (1) although they could still be interpreted as equilibrium solutions.

Thus far we have not considered the computational and empirical consequences of trying to establish the optimality conditions. Before entering into a discussion about solution methods, it is important to emphasize that typical econometric submodels are designed to compute s from u and d from v, rather than the inverse relation as we have stated it in (2a) and (2b). In other words, the econometric submodels consist of the functions ∇g^{-1} and ∇f^{-1} which are used to compute $s = \nabla g^{-1}(u)$ and $d = \nabla f^{-1}(v)$. This implies that in order to hook up the econometric submodels with the linear programming submodel, we must assume that the econometric mappings $G = \nabla g^{-1}$ and $F = \nabla f^{-1}$ can be inverted at various points to give us the values of $\nabla g = G^{-1}$ and $\nabla f = F^{-1}$ at these points for use in testing the optimality conditions. This might be done functionally, or by some iterative procedure which exploits the monotonicity and continuity of ∇g^{-1} and ∇f^{-1}.

4. Steepest Ascent Decomposition Methods

In this section, we discuss how problem (1) can be solved by decomposing it into econometric and linear programming submodels. For $s \geq 0$, $d \geq 0$, define the function

$$\phi(s,d) = f(d) - g(s) + \max - cx \quad \text{s.t.} \quad A^1x \leq s, \quad A^2x \geq d, \quad x \geq 0. \tag{5}$$

It can easily be shown that $\phi(s,d)$ is a concave function. Moreover, it is continuous, but not everywhere differentiable on the convex subset of the non-negative orthant where it is finite. By linear programming duality (ruling out the case that $\phi(s,d) = +\infty$ since (1) is assumed to have an optimal solution, but permitting $\phi(s,d) = -\infty$),

$$\phi(s,d) = f(d) - g(s) + \min ps - qd \quad \text{s.t.} \quad c - pA^1 + qA^2 \geq 0, \quad p \geq 0,\ q \geq 0. \tag{6}$$

We assume the convex polyhedral set

$$\Pi = \{(p,q) \mid c - pA^1 + qA^2 \geq 0, \quad p \geq 0,\ q \geq 0\} \tag{7}$$

is nonempty. In general, Π will be unbounded because we expect there to be s,d combinations in (5) which do not admit feasible linear programming solutions. The issue of infeasible s,d combinations could and probably should be handled directly in our subsequent development by the generation and use of constraints of the form $p^r s - q^r d \geq 0$ for rays (p^r, q^r) of the polyhedron Π. For expositional reasons, however, we choose to eliminate the possibility that Π is unbounded by assuming that we know a value $M > 0$ such that all p,q satisfying the optimality conditions (2), (3), (4) also satisfy

$$\sum_i p_i + \sum_j q_j \leq M. \tag{8}$$

The addition of the constraint (8) to (7) bounds the dual feasible region and implies that for all $s \geq 0$, $d \geq 0$,

$$\phi(s,d) = f(d) - g(s) + \min_{t=1,\ldots,T} p^t s - q^t d \tag{9}$$

where the (p^t, q^t) are the dual extreme points. Of course, the addition of the contraint (8) to (6) is equivalent to the addition of an activity in (5) that permits a feasible linear programming solution to always be found, but possibly at a very high cost.

The original mathematical programming problem (1) is equivalent to

$$\begin{aligned} \phi^* &= \max \phi(s,d) \\ &\text{s.t. } s \geq 0,\ d \geq 0, \end{aligned} \tag{10}$$

where $\phi(s,d)$ is given by (9). The solution of (1) by solving (10) is a decomposition approach which is illustrated schematically in figure 1. The computation alternates between the linear programming submodel and the supply and demand submodels. A feasible solution s,d,x, to (1) is generated each time the LP submodel is solved. As mentioned above, the

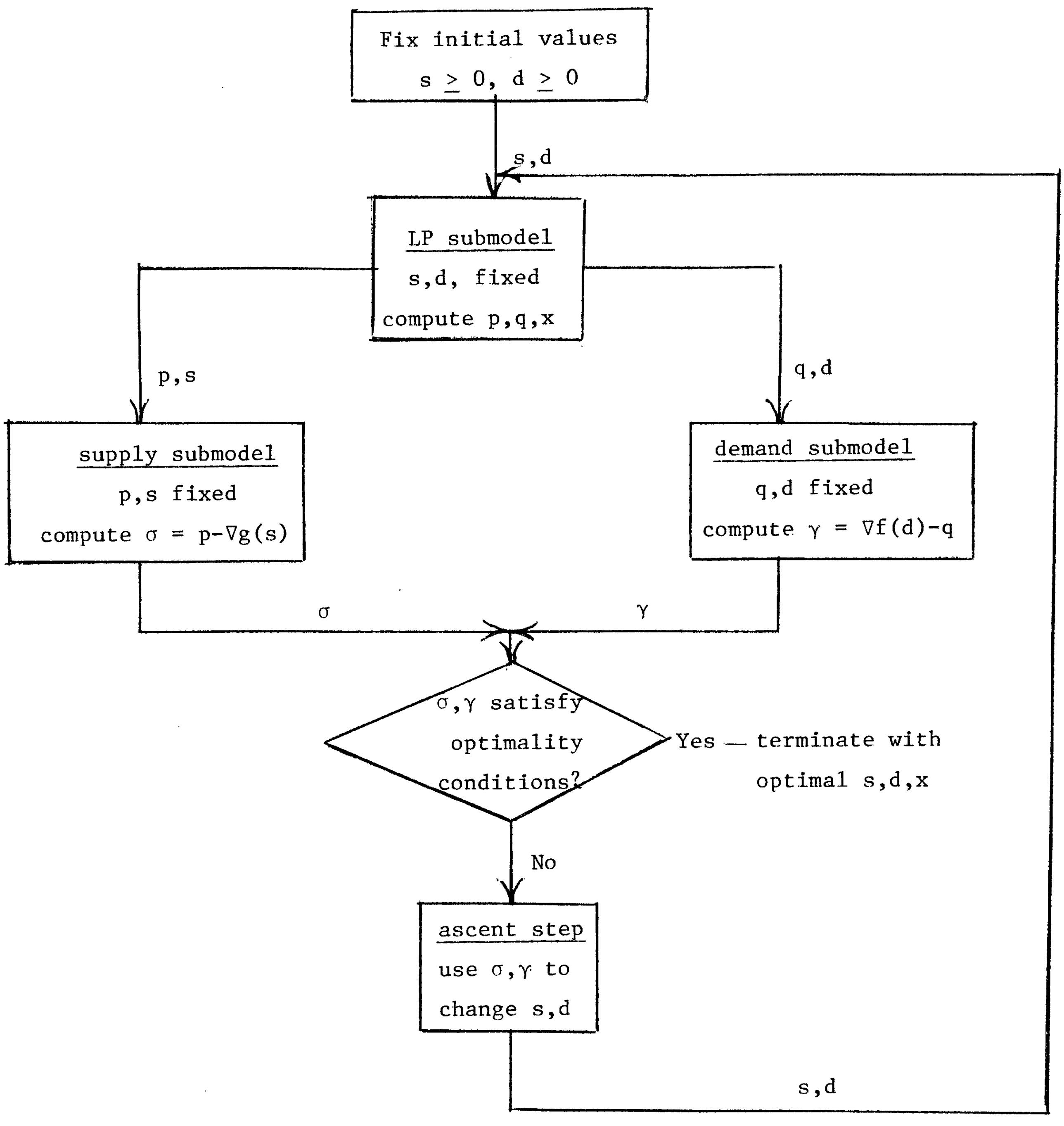
Fix initial values
s ≥ 0, d ≥ 0
s,d
LP submodel
s,d, fixed
compute p,q,x
p,s
q,d
supply submodel
p,s fixed
compute σ = p-∇g(s)
demand submodel
q,d fixed
compute γ = ∇f(d)-q
σ
γ
σ,γ satisfy
optimality
conditions?
Yes — terminate with
optimal s,d,x
No
ascent step
use σ,γ to
change s,d
s,d

Figure 1

manner of computing σ and γ in the supply and demand submodels, respectively, depends upon their structure. If (σ,γ) does not satisfy the optimality conditions for problem (1) (equivalent to and derivable from the optimality conditions (2), (3), (4)), the s and d in the LP submodel are changed by taking an ascent step with respect to the function $\phi(s,d)$. The steepest ascent approach to large scale linear programming decomposition is discussed by Grinold (1972); see also Fisher, Northup and Shapiro (1975). The algorithmic approaches to be discussed below use the same ideas, but our main reason for decomposing (1) is to overcome the incompatabilities of econometric and linear programming submodels and their realizations as computer systems.

Steepest ascent decomposition methods use concepts of convex analysis which we briefly review. Rockafellar (1970) gives a thorough mathematical treatment of convex analysis. Its relation to decomposition methods is developed in detail in Shapiro (1976). A subgradient (σ,γ) of ϕ at (s,d) is a vector satisfying

$$\phi(\bar{s},\bar{d}) \leq \phi(s,d) + \sigma(\bar{s}-s) + \gamma(\bar{d}-d) \quad \text{for all } \bar{s},\bar{d} \tag{11}$$

If there is a unique subgradient of ϕ at s,d, then it is the gradient of ϕ. Any subgradient at (s,d) can be tried as a direction of ascent in maximizing $\phi(s,d)$ because it points into the half space containing all optimal solutions. The difficulty with this approach is that ϕ may not acutally increase in a subgradient direction from (s,d) although (s,d) is not optimal and the function does increase in another subgradient direction.

The difficulty due to multiple subgradients can be overcome by procedures capable of generating, if necessary, the set $\partial\phi(s,d)$ of all subgradients, called the <u>subdifferential</u>. Define the index set

$$T(s,d) = \{t \mid \phi(s,d) = f(d) - g(s) + p^t s - q^t d\}.$$

Then it can be shown that $\partial\phi(s,d)$ is a bounded convex polyhedron with extreme points $(\sigma^t,\gamma^t) = (-\nabla g(s) + p^t, \nabla f(d) - q^t)$ for some of the $t \in T(s,d)$. The Kuhn-Tucker optimality conditions (2), (3), (4) can be restated as follows: The solution $(\bar{s},\bar{d}) \geq 0$ is optimal in (10) if and only if there exists λ_t, $t \in T(\bar{s},\bar{d})$ (equivalently $(\bar{\sigma},\bar{\gamma}) \in \partial\phi(\bar{s},\bar{d})$) satisfying

$$\bar{\sigma}_i = -\frac{\partial g(\bar{s})}{\partial s_i} + \sum_{t\in T(\bar{s},\bar{d})} p_i^t \lambda_t \begin{cases} = 0 & \text{if } \bar{s}_i > 0 \\ \leq 0 & \text{if } \bar{s}_i = 0 \end{cases}$$

$$\bar{\gamma}_j = \frac{\partial f_j(\bar{d})}{\partial dj} - \sum_{t\in T(\bar{s},\bar{d})} q_j^t \lambda_t \begin{cases} = 0 & \text{if } \bar{d}_j > 0 \\ \leq 0 & \text{if } \bar{d}_j = 0 \end{cases} \tag{12}$$

$$\sum_{t\in T(\bar{s},\bar{d})} \lambda_t = 1$$

$$\lambda_t \geq 0, \quad t \in T(\bar{s},\bar{d})$$

The optimality conditions (12) for problem (10) are the basis for solution methods including

(a) subgradient optimization

(b) primal-dual ascent algorithm

(c) simplicial approximation.

These methods are not mutually exclusive but complementary, and they could be integrated, at least conceptually, into a hybrid algorithm. Space does not permit us to give a great deal of detail about the application of these methods to (10). Reference is given to more detailed treatments of the methods.

(a) subgradient optimization

This is the simplest to implement but it can require considerable experimentation with parameter settings and could require knowledge about (10) which we do not have. It has worked well for nondifferentiable concave programming problems closely related to (1) the traveling salesman problem (Held and Karp (1971)) and (2) machine scheduling problems (Fisher (1976)).

The idea is to generate a sequence of non-negative solutions $\{(s^\ell, d^\ell)\}_{\ell=1}^{\infty}$ to (10) by the rule

$$\begin{aligned} s_i^{\ell+1} &= \max\{s_i^\ell + \theta^\ell \sigma_i^\ell,\ 0\} \quad \text{for all } i \\ d_j^{\ell+1} &= \max\{d_j^\ell + \theta^\ell \gamma_j^\ell,\ 0\} \quad \text{for all } j \end{aligned} \tag{13}$$

where $(\sigma^\ell, \gamma^\ell)$ is any subgradient and the scalars θ^ℓ satisfy $\sum_{\ell=1}^{\infty} \theta^\ell = +\infty$ but $\theta^\ell \to 0$. Note that no attempt is made to guarantee that the function ϕ actually increases from point to point. Polyak (1967) shows that if $\nabla g(s^\ell)$ and $\nabla f(d^\ell)$ are uniformly bounded, then the (s^ℓ, d^ℓ) given by (13) will converge to an optimal solution to (10). The theoretical and practical rates of convergence may be slow, however. Thus, Polyak (1969) suggests the rule

$$\theta^{\ell} = \rho^{\ell} \frac{(\phi^* - \phi(s^{\ell}, d^{\ell}))}{||(\sigma^{\ell}, \gamma^{\ell})||^2}, \tag{14}$$

where $0 < \varepsilon_1 < \rho^{\ell} < 2 - \varepsilon_2 < 2$ which has proven superior. Note that the formula (14) involves knowledge of the maximal value ϕ^*, which we do not know, and the functional value $\phi(s^{\ell}, d^{\ell})$, which we do not know explicitly but may be able to compute. Figure 1 is an accurate description of how subgradient optimization would work on problem (10).

(b) primal-dual ascent algorithm

This algorithm is given for the piecewise linear case by Fisher and Shapiro (1974) and Fisher, Northup and Shapiro (1975), and in the general case by Lemarechal (1974). In order to construct a convergent algorithm, we must settle for an ε-optimal solution ($\varepsilon > 0$) which is any $(\bar{s}, \bar{d}) \geq 0$ such that $\phi^* \leq \phi(\bar{s}, \bar{d}) + \varepsilon$. The algorithm of Lemarechal (1974) about to be described converges finitely, and ε can be successively reduced if necessary. The algorithm works with ε-subgradients of ϕ which are any vectors (σ, γ) at (s, d) satisfying

$$\phi(\bar{s}, \bar{d}) \leq \phi(s, d) + \sigma(\bar{s} - s) + \gamma(\bar{d} - d) + \varepsilon \quad \text{for all } \bar{s}, \bar{d}.$$

The set of all ε-subgradients is denoted by $\partial\phi_{\varepsilon}(s, d)$ and it is a convex polyhedron. If we let $T_{\varepsilon}(s, d) = \{t \mid f(d) - g(s) + p^t s - q^t d \leq \phi(s, d) + \varepsilon\}$, then the extreme points of $\partial\phi_\varepsilon(s,d)$ are included among the points $(\sigma^t,\gamma^t) = (-\nabla g(s) + p^t, \nabla f(d) - q^t)$ for $t \in T_\varepsilon(s,d)$. The conditions (12) with $T(\bar{s},\bar{d})$ replaced by $T_\varepsilon(\bar{s},\bar{d})$ are necessary and sufficient for ε-optimality.

The idea of the algorithm is to try at each point (s,d) to establish the optimality conditions by solving a phase one linear programming problem. Since the set $T_\varepsilon(s,d)$ can be quite large, the procedure begins with a small subset. If the optimality conditions are not established, then a direction of possible ascent is indicated. If this direction contains a solution (s',d') such that $\phi(s',d') > \phi(s,d) + \varepsilon$, then a step is taken. Otherwise, the subset of $T_\varepsilon(s,d)$ is augmented by an ε-subgradient and the phase one linear programming is reoptimized.

The primal-dual ascent algorithm has the advantage over subgradient optimization that it does not require knowledge of ϕ^*, and the sequential values of $\phi(s,d)$ increase by at least ε at each step. It has the disadvantage that it does more work at each point (s,d), and it is more complex to program. In terms of figure 1, if the ε-subgradient (σ,γ) does not satisfy the optimality conditions, then the LP submodel may be resolved, perhaps several times, before an ascent step is taken.

(c) simplicial approximation

This method has been applied to related types of economic equilibrium problems by Scarf and Hansen (1973). In effect, the method performs a very special type of search over a compact set of non-negative (s,d) known to contain an optimal solution to (10). The idea is to approximate (12) by subgradients calculated at distinct, but close together points (s,d). Space does not permit a fuller development of this method. Complete details are given by Fisher, Northup and Shapiro (1975) for a mathematical programming problem that is sufficiently similar to (10) for the approach there to be applicable here. In terms of figure 1, the simplicial approximation test for termination is the indicated approximation of the optimality conditions. If these conditions are not satisfied, then instead of the ascent step, we have the exchange of one of the current points in the approximating set for a new point (s,d) for which a subgradient (σ,γ) is calculated as shown. The number of commodities which can be efficiently handled by simplicial approximation is not yet known. For the moment, this number appears to be less than 100, perhaps substantially so.

5. Conclusions and Areas for Future Research

The proposed decompostion scheme for mathematical programming/ economic equilibrium energy planning models is conceptual but fully implementable. At the M.I.T. Energy Lab, we are currently considering an integration of the Brookhaven Energy System Optimization Model with some of the econometric models developed at M.I.T. This integration should provide the ideas given above with a rigorous test.

On the other hand, there remain a number of conceptual questions to be studied in greater detail. Hogan (1974) is concerned with the integrability of the functions ∇f and ∇g, a property which we have assumed throughout. This point needs further. A possibly related construct which might provide some insight is the Legendre transform (Rockafellar (1970; chapter 26)) which relates convex properties of a function to the inverse of its gradient.

An important area of future research is the identification, analysis and solution of dynamic models derived from (1) whose solutions converge to an optimal solution to (1). The econometric supply and demand models are naturally dynamic, and dynamic mathematical programming submodels can also be constructed (see Shapiro (1975) for some ideas about how to do this). In terms of the decomposition approach, Grinold (1972) gives an ascent algorithm for solving dynamic linear programming problems as they would arise in this context. The idea would be to fix supply and demand levels over the planning horizon, solve the dynamic linear programming problem, and then adjust the supply and demand levels in the same spirit as given above. The dynamic linear programming energy model of Nordhaus (1973) which has fixed supply and demand levels could be a candidate for this type of extension.

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