The "Price of Anarchy" under Nonlinear and Asymmetric Costs

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Abstract

In this paper we characterize the "price of anarchy", i.e., the inefficiency between user and system optimal solutions, when costs are non-separable, asymmetric and nonlinear, generalizing earlier work that has addressed "the price of anarchy" under separable costs. This generalization models traffic equilibria, competitive multi-period pricing and competitive supply chains. The bounds established in this paper are tight and explicitly account for the degree of asymmetry and nonlinearity of the cost function. We introduce an alternate proof method for providing bounds that uses ideas from semidefinite optimization. Finally, in the context of multi-period pricing our analysis establishes that user and system optimal solutions coincide.

Keywords: System and User-Optimization, Traffic Equilibrium, Price of Anarchy.

1 Introduction

There has been an increasing literature in the recent years trying to quantify the inefficiency of Nash equilibrium problems (user-optimization) in non-cooperative games. The fact that there is not full efficiency in the system is well known both in the economics but also in the transportation literature (see [1], [15]). This inefficiency of user-optimization was first quantified by Papadimitriou and Koutsoupias [27] in the context of a load balancing game. They coined the term "the price of anarchy" for characterizing the degree of efficiency loss. Subsequently, Roughgarden and Tardos [36]

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and Roughgarden [35] applied this idea to the classical network equilibrium problem in transportation with arc cost functions that are separable in terms of the arc flows. They established worst case bounds for measuring this inefficiency for affine separable cost functions and subsequently for special classes of separable nonlinear ones (such as polynomials). Recently Johari and Tsitsiklis [25] also studied this problem in the context of resource allocation between users sharing a common resource. In their case the problem also reduces to one where each player has a separable payoff function. Correa, Schulz and Stier Moses [7] have also studied "the price of anarchy" in the context of transportation for capacitated networks. The cost functions they consider are also separable.

Wardrop [39] was perhaps the first to state the equilibrium principles in the context of transportation. Dafermos and Sparrow [8] coined the terms "user-optimized" and "system-optimized" in order to distinguish between Nash equilibrium where users act unilaterally in their own self interest versus when users are forced to select the routes that optimize the total network efficiency. Smith [37] and Dafermos [10] recognized that this problem can be cast as a variational inequality. In [9] Dafermos considered how the decentralized "user-optimized" problem can become a centralized "system optimization" problem through the imposition of tolls. Recently, Hearn and co-authors [23], [24], have studied the problem of imposing tolls in order to induce a behavior to users so that their route choices are optimizing the overall system. They study a variety of criteria for imposing tolls. Finally, Cole, Dodis and Roughgarden [6] have also studied the notion of introducing tolls (taxes) in order to make the decentralized problem efficient in a centralized manner. The review paper by Florian and Hearn [18], the book by Nagurney [29], and the references therein summarize the relevant literature in traffic equilibrium problems. Traffic equilibrium problems are typically modeled through variational inequalities (see Section 2 for details). The books by Facchinei and Pang [16] summarizes the developments in the area of variational inequalities.

Nash equilibrium problems arise in a variety of settings and model competitive and non-cooperative behavior (see Section 2 for more details). In this paper we study the inefficiency of equilibrium by comparing how the presence of competition affects the total profit in the system in a decentralized (user-optimized) versus a centralized optimization (system-optimized) setting. We establish a bound on the ratio of the overall profit of the system in these two settings. This work is the first to consider non-separable, asymmetric cost functions and is important since, as we discuss in Section 2, it allows modeling more realistic situations. Furthermore, it allows a unifying framework which naturally extends results in the current literature. In particular, our contributions versus the existing literature are the following.

1. We consider **non-separable** functions in the sense that cost functions also depend on the strategies of the competitors. Furthermore, cost functions can be **asymmetric** in the sense that different competitors' strategies affect their cost functions differently. This generalization is important since the strategies of one's competitors will influence his/her own cost in an asymmetric way. In particular, we introduce a measure of asymmetry (denoted by c^2 in Section 3) which quantifies **the degree of asymmetry** of the competitors' cost functions. We establish that the ratio of the total cost in the system operating in a user-optimized setting versus the total cost in a system optimized setting is bounded by

$$\begin{cases} \dfrac{4}{4-c^2} & \text{if } c^2 \leq 2 \\ c^2 & \text{if } c^2 > 2. \end{cases}$$

   We illustrate how our results are a natural generalization of the bound established by [35], [36] since when the problem functions are separable and affine the bound becomes 4/3. Furthermore, when the functions are non-separable but symmetric we establish that the bound is still 4/3, since $c^2 = 1$. The results in the affine case allow the feasible region to be a non-convex set.

2. When the cost function has no constant term then the bound becomes c^2. An important implication of this result arises in the context of multi-period pricing. In this setting our analysis establishes that user and system optimal solutions coincide.

3. We generalize our results to **nonlinear** functions. We introduce a measure which quantifies **the degree of nonlinearity** of the problem function (denoted by A in Section 4). We establish that the bound naturally extends to involve the nonlinearity parameter A, i.e.,

$$\begin{cases} \dfrac{4}{4-c^2 A} & \text{if } c^2 \leq \frac{2}{A} \\ c^2A^2 - 2(A-1) & \text{if } c^2 > \frac{2}{A}. \end{cases}$$

4. We establish that the bounds above are **tight** for affine as well as for some nonlinear problems.

5. We introduce an alternative **semidefinite optimization** formulation for deriving these bounds. This approach does not require positive definiteness of the Jacobian matrix (i.e., it does not need to be invertible). Therefore, the solution does not need to be unique. We illustrate that this approach gives rise to the same bound when the Jacobian matrix is positive definite.

The remainder of this paper is structured as follows. In Section 2, we present some application areas where quantifying the loss of efficiency due to competition is important. Furthermore, we illustrate that in these applications the cost functions involved are non-separable and asymmetric. In Section 3, we establish a bound for affine asymmetric problems and establish that this bound is tight. In Section 4, we generalize the bound in Section 3, to general nonlinear asymmetric functions and show that the bound is tight. In Section 5, we introduce an alternative way for determining bounds using a semidefinite programming (SDP) formulation. We illustrate that the SDP approach yields the same bound in the case of a positive definite Jacobian matrix. In Section 6, we discuss our conclusions.

2 Application Areas

In this section, we present several application areas where addressing the question of what is the loss of efficiency due to decentralization is important. Furthermore, we illustrate that in these applications it is more realistic to consider costs functions that are non-separable in terms of the competitors' strategies.

2.1 The Network Equilibrium Problem

A fundamental principle that is widely used to establish how traffic is distributed in transportation systems, that is, how travelers choose their routes, is the following user-optimizing principle (UO) (introduced by Wardrop [39]): Given a transportation network consisting of roads and highways, the traffic flow pattern in the network is established assuming that travelers behave as follows: no traveler can decrease his/her cost (typically travel time) by making a unilateral decision to change his/her route. "The application of network equilibrium models, in various transportation planning contexts, has increased dramatically over the past 25 years. All cities in developed countries carry out quantitative transportation planning activities" (see Florian, [17]). In particular, several software packages exist in the market for these purposes (such as EMME/2) and are used by more than 700 organizations in countries on five continents (see [17]). This principle has generated a large body of literature in the area of transportation planning.

In this problem setting we assume a network (N, A), where N is the set of nodes and A the set of arcs in the network. P denotes the set of paths and W the set of origin-destination (O-D) pairs in the network. P_w denotes the set of paths connecting the O-D pair w, with paths $p_w = 1_w, ..., n_w \in P_w$. There is a total of n paths in the overall network and a total of n_w paths connecting O-D pair w. $C_p(F)$ denotes the cost function of a traveler on path p and is a function of the path flow vector F. $c_a(f)$ denotes the cost function of a traveler on arc a and is a function of the arc flow vector f.

The travelers in the network operate under Wardrop's user-optimizing principle. The traffic planner wishes to find a way of distributing the demand d_w of each O-D pair w among the set of paths P_w connecting this O-D pair, so that (perhaps after re-ordering the path indices)

$$C_{1_w}(F^*) = C_{2_w}(F^*) = ... = C_{s_w}(F^*) \leq C_{(s+1)_w}(F^*) \leq ... \leq C_{n_w}(F^*), \tag{1}$$

for all O-D pairs w, where $F^*_{1_w}, ..., F^*_{s_w} > 0$, while $F^*_{(s+1)_w} = ... = F^*_{n_w} = 0$, and $\sum_{p=1_w}^{n_w} F^*_p = d_w$, for all O-D pairs w in the network. Therefore, the user-equilibrium problem seeks to determine the values $F^*_{1_w}, ..., F^*_{n_w}$ that satisfy (1).

The flow f_a on an arc a connects with the path flow F_p on a path p through the equation

$$f_a = \sum_{\{p:\ a \in p\}} F_p$$

and that the cost function C_p on a path p and the cost function c_a on an arc a connect through

$$C_p(F) = \sum_{\{a \in p\}} c_a(f).$$

These give rise to the following equivalent variational inequality formulation,

Find a feasible flow pattern (f^*, F^*), satisfying $c(f^*)^t(f - f^*) \geq 0$, for all feasible (f, F).

Notice that the variational inequality formulation we consider involves a problem function c representing the arc cost as a function of the flow. The presence of congestion in a large transportation network where paths share arcs and there are several intersections, suggests that modeling these cost functions (often representing travel times) through separable functions (i.e., arc cost functions that depend only on the flow on that arc) may not be as realistic. In a large congested transportation network the travel time to traverse an arc will be affected by traffic congestion at its neighboring arcs. For example, the presence of a bottleneck or an accident at an arc ahead will slow traffic down at neighboring arcs as well. Furthermore, travel times are not affected by the flow on neighboring arcs in a symmetric way. For example, consider two consecutive arcs, then the travel time to traverse the first arc is not affected the same way by the flow of the arc ahead as the travel time of the arc ahead is affected by the flow of the arc behind. This discussion leads us to conclude that it is more realistic to consider asymmetric cost functions in terms of the flow (see [26] for a discussion on how these travel times may be determined).

There is an analogous system optimization principle (SO), which is appropriate only when some central authority fully controls the traffic pattern. In this case the central authority minimizes the total cost in the system. That is,

$$\begin{aligned} \text{minimize} \quad & \sum_a c_a(f)^t f_a \\ \text{subject to} \quad & f_a = \textstyle\sum_{p:\ a\in p} F_p, \quad \forall a \in A \\ & \textstyle\sum_{p=1_w}^{n_w} F_p = d_w, \quad \forall w \in W \\ & F_p \geq 0,\ \forall p \in P, \quad f_a \geq 0, \quad \forall a \in A. \end{aligned} \tag{2}$$

During the recent years, researchers have extensively studied network equilibrium problems and have developed theory and solution algorithms. The review paper by Florian and Hearn [18], the book by Nagurney [29] and the references they cite provide a thorough review of the problem.

2.2 Competitive Multi-period Pricing

Consider a market of a single homogeneous perishable product (a flight or a hotel stay can be such examples), a set of sellers I (of cardinality n) with fixed total inventories $C = (C_1, ..., C_n)$ to allocate for this product over a finite time horizon T. Sellers compete through the prices they set at each period. Each seller i sets his/her price levels $p_i = (p_i^1, ..., p_i^T)$ over the time horizon optimally, as best response prices to the competitors' prices. Furthermore, seller i in any period will see the maximum share of demand that corresponds to the prices set by all sellers at time t, $h_i^t(p^t)$. Each seller i decides on the prices p_i^t as well as on the allocations d_i^t of the inventory to set at each period t for the product. Given a seller i, the set of his/her competitors is denoted by $-i$. In this setting we assume that demand is deterministic.

The best response policy of seller i, given all competitors' pricing policies $\bar{p}_{-i}$, is the solution of optimization problem:

$$\mathcal{BR}(\bar{p}_{-i}, C) = \quad \text{argmax}_{d_i, p_i} \sum_{t=1}^{T} d_i^t \cdot p_i^t \tag{3}$$

$$\text{such that } d_i^t \leq h_i^t(p_i^t, \bar{p}_{-i}^t), \quad \sum_{t=1}^{T} d_i^t \leq C_i, \quad p_i^{\min} \leq p_i \leq p_i^{\max}, \quad d_i^t \geq 0, \quad t = 1, ..., T.$$

In this setting, $p_i^{\min} > 0$ and $p_i^{\max} > 0$ are the minimum and maximum threshold prices of each seller i. Note that in this optimization problem, given the competitors' pricing strategies, $x_{-i} = (d_{-i}, p_{-i})$, seller i maximizes the objective function $J_i(x_i) = J_i(d_i, p_i) = \sum_{t=1}^{T} d_i^t \cdot p_i^t$ within the feasible region

$$\mathcal{K}_i(\bar{x}_{-i}) \quad = \quad \left\{ (d_i, p_i) | d_i^t \leq h_i^t(p_i^t, \bar{p}_{-i}^t), \sum_{t=1}^{T} d_i^t \leq C_i, p_i^{\min} \leq p_i \leq p_i^{\max},\ d_i^t \geq 0,\ t = 1, ..., T \right\}.$$

Perakis and Sood study this problem in [33]. They impose the assumption that the demand function $h_i^t(p^t)$ for seller i is decreasing with respect to his/her own price p_i^t. Furthermore, for all $i \in I$, $t =$

$1, ..., T$, they assume that the demand functions $h_i^t(p^t)$ are concave functions of the vector of prices p^t over the set of feasible prices. These two assumptions allow them to write the market equilibrium problem for all the sellers competing for this product in the market as the following variational inequality formulation. Find $x_u \in \mathcal{K}$ satisfying

$$F(x_u)^t(x - x_u) \geq 0, \quad \forall x \in \mathcal{K}.$$

Notice that in this formulation vector $x = (d, p)$, problem function $F(x) = -(p, d)$, vector $x_u = (d_u, p_u)$ denotes the variational inequality solution and

$$\mathcal{K} = \{x = (x_1, x_2, \dots, x_n) | x_i \in \mathcal{K}_i(x_{-i}) \forall i \in I\}.$$

In [33], they establish existence of solution for the variational inequality problem and subsequently illustrate that such a solution is also a Nash equilibrium solution. It is important to notice at this point that the variational inequality problem function $F(x) = -Qx$, where matrix $Q = \begin{pmatrix} 0 & \mathcal{I} \\ \mathcal{I} & 0 \end{pmatrix}$ and $\mathcal{I}$ is an identity matrix of appropriate dimension, is non-separable. Furthermore, in [33] they illustrate that the problem function becomes $F(x_u) = -(p_u, h(p_u))$ at the variational inequality solution $x_u = (d_u, p_u)$. Nevertheless, this is also a non-separable function of the price since demand of seller i depends on the prices of his/her competitors as well. This model applies to many application areas such as pricing for airline flights, hotel rooms and others.

The problem described above is decentralized in the sense that each seller optimizes for him/herself. In a centralized version of the problem a central authority is optimizing the total profit of the system (SO) forcing sellers to comply. Although this latter version would yield the highest total profit, it is often impossible (or even illegal) to force sellers to comply with the (SO) pricing scheme. As a result, it is important to determine the "price of anarchy", that is, how much is lost due to lack of coordination. Notice that in this setting as demand for each seller depends not only on his/her prices but also on competitors' prices, the problem is non-separable. Furthermore, if market conditions for the sellers are not the same, the demand functions will be asymmetric, in the sense that the demand for seller i will not be affected by the price seller j sets the same way seller j's demand is affected by the price seller i sets. Finally, notice that the total overall profit in the system is $\sum_{i \in I} J_i(x) = \sum_{i \in I} \sum_{t=1}^T d_i^t \cdot p_i^t$.

Although the analysis we present in this paper (see Section 3) focuses on comparing the decentralized versus the centralized setting in the context of cost minimization, for the multi-period pricing problem we described above, similar results also apply. In particular, as we discuss in Subsection 3.2 in more detail, the asymmetry constant involved in the bound is $c^2 = 1$. This implies that in this setting and for a class of demand functions (i.e., those with a positive definite Jacobian matrix) the centralized solution coincides with the decentralized one.

2.3 Other Applications

Competitive supply chain management is another important application area. In this setting a retailer and several suppliers compete, each optimizing for themselves. The suppliers announce prices to the retailer each optimizing their profits keeping in mind the presence of competing suppliers. Each supplier has a limited capacity and therefore, may not be able to absorb the entire order from the retailer. On the other hand, the retailer announces quantity orders from some (or all) of the suppliers in order to optimize his/her profit. Several researchers have studied variants of this problem, examples include Cachon [2] [3], Lariviere and Cachon [4], Martinez-de-Albeniz and Simchi-Levi [28], Perakis and Zaretsky [34] and the references therein. Notice that the presence of many suppliers competing for the retailer's orders gives rise to payoff functions (in this setting representing their corresponding profit) for the suppliers that depend not only on their prices but also on the competing suppliers' prices. An important issue in this setting is how the overall profit in the decentralized supply chain compares with the overall profit when the supply chain is centrally optimized, that is, a central authority forces the suppliers and the retailer to set their prices and quantity orders respectively in order to maximize the overall profit of the overall system. A key issue in this area is how to coordinate the supply chain, that is, how to introduce a contract between suppliers and the retailer in order to improve the efficiency of the supply chain. As a result, quantifying how much is lost in the overall efficiency due to decentralization and the presence of competition is a key question in this setting.
There are several other application domains where asking the question *"what is the price of anarchy"* (i.e., what is the loss of efficiency due to decentralization) is relevant. Furthermore, in many of these settings it is more realistic to assume that the cost functions involved are not separable and asymmetric. Such examples include Nash Cournot equilibria (see Gabay and Moulin [20]), spatial price and market equilibria (see for example, Dafermos and Nagurney [12], [14], Florian and Los [19], Nagurney [29]), oligopolistic market equilibria (see for example, Dafermos and Nagurney [13], Nagurney [29]), pure exchange economic equilibria (see Dafermos [11], Nagurney [29]), migration equilibria, financial economic equilibria (see for example, Nagurney [29]), competitive supply chains (see for example, Nagurney et al [30]).

3 A Bound for Affine and Asymmetric Cost Functions

In this section, we establish a bound between the user and the system optimization problems in the context of minimizing cost. For the (UO) decentralized problem we will consider the variational inequality problem of finding $x_u \in K$ satisfying

$$F(x_u)^t(x - x_u) \geq 0, \quad \text{for all } x \in K. \tag{4}$$

Let x_u and x_s denote solutions of the user and system optimization problems respectively. Let $Z_u = F(x_u)^t x_u$ be the total cost for the user-optimized problem (UO) and $Z_s = F(x_s)^t x_s = \min_{x \in K} F(x)^t x$ be the total cost for the system-optimized problem (SO). In this section, we provide a bound on Z_u/Z_s for cost functions $F(x) = Gx + b$, with $G \succ 0$ (i.e., positive definite) and asymmetric matrix, $b^t x \geq 0$ for all $x \in K$ (notice that this follows when constant vector $b \geq 0$ and $K \subseteq R^n_+$). In this case, the system optimization problem involves the minimization of a strictly convex quadratic function over the set K.

3.1 A Measure of Asymmetry of a Matrix

For a matrix G, we consider the symmetrized matrix

$$S = \frac{G + G^t}{2}$$

and introduce the following measure c^2 of the degree of asymmetry of matrix G:

Definition 1

$$c^2 \equiv ||S^{-1}G||_S^2 = \sup_{w \neq 0} \frac{||S^{-1}Gw||_S^2}{||w||_S^2} = \sup_{w \neq 0} \frac{w^t G^t S^{-1} G w}{w^t S w}.$$

Note that by setting $l = S^{1/2}w$, the previous definition of c^2 becomes

$$c^2 = \sup_{l \neq 0} \frac{l^t S^{-1/2} G^t S^{-1} G S^{-1/2} l}{||l||^2} = \lambda_{max}(S^{-1/2} G^t S^{-1} G S^{-1/2}).$$

When the matrix G is positive definite and symmetric, that is, $G = G^t$ (and therefore, $S = G$), then $c^2 = 1$. As an example, consider

$$G = \begin{bmatrix} 1 & a \\ -a & 1 \end{bmatrix}.$$

Since $S = I$, it easily follows that $c^2 = 1 + a^2$. The quantity c^2 in this case quantifies the observation that as $|a|$ increases, the degree of asymmetry of G increases as well.

This constant was originally used in Hammond [21] in order to prove the convergence of a Newton-type method for solving variational inequality problems. The next lemma indicates that if $G^2 \succeq 0$, then the degree of asymmetry of G is bounded.

Lemma 1 *If G^2 is a positive semidefinite matrix then $c^2 = \|S^{-1}G\|_S^2 \leq 2$.*

For a proof see Hammond [21].

Notice that this condition connects with the degree of asymmetry of matrix G as follows. When $G^2 \succeq 0$ then for all $w \in R^n$, $w^t G^2 w = (G^t w)^t G w \geq 0$. This suggests that the angle between vectors $G^t w$ and Gw is less than 90 degrees, restricting the asymmetry of matrix G.

3.2 The Bound

We are now ready to establish the main result of this section.

Theorem 1 *For an affine variational inequality problem with problem function $F(x) = Gx + b$, with $G \succ 0$, $b^t x \geq 0$ for all $x \in K$, we have:*

$$\frac{Z_u}{Z_s} \leq \begin{cases} \dfrac{4}{4 - c^2} & \text{if } c^2 \leq 2 \\ c^2 & \text{if } c^2 > 2. \end{cases}$$

Proof:

We first observe that since x_u solves variational inequality (4) and $x_s \in K$ then the variational inequality formulation implies that

$$\begin{aligned} Z_u &= F(x_u)^t x_u \\ &\leq F(x_u)^t x_s \\ &= x_u^t G^t x_s + b^t x_s \\ &= x_u^t G^t S^{-1} S x_s + b^t x_s && \text{(multiplying with } S \text{ and } S^{-1}) \\ &\leq \|(x_u)^t G^t S^{-1}\|_S \, \|x_s\|_S + b^t x_s && \text{(from Cauchy's inequality, and } S \succ 0) \\ &\leq \|x_u\|_S \, \|S^{-1}G\|_S \, \|x_s\|_S + b^t x_s && \text{(from the norm inequality)} \\ &= c \, \|x_u\|_S \, \|x_s\|_S + b^t x_s. && \text{(Definition 1)} \end{aligned}$$

For every $a_1,\ a_2 \geq 0$, we have

$$0 \leq (\sqrt{a_1}\|x_u\|_S - \sqrt{a_2}\|x_s\|_S)^2 = a_1\|x_u\|_S^2 + a_2\|x_s\|_S^2 - 2\sqrt{a_1 a_2}\|x_u\|_S\|x_s\|_S$$

leading to

$$2\sqrt{a_1 a_2}\|x_u\|_S\|x_s\|_S \le a_1\|x_u\|_S^2 + a_2\|x_s\|_S^2.$$

This in turn implies that if we choose $2\sqrt{a_1 a_2} \ge c$, then

$$c\|x_u\|_S\|x_s\|_S \le a_1\|x_u\|_S^2 + a_2\|x_s\|_S^2.$$

We thus obtain that for all $a_1, a_2 \ge 0$ and $a_1 a_2 \ge c^2/4$:

$$\begin{aligned} Z_u &\le a_1\|x_u\|_S^2 + a_2\|x_s\|_S^2 + b^t x_s \\ &= a_1 x_u^t S x_u + a_2 x_s^t S x_s + b^t x_s \\ &= a_1(x_u^t G^t x_u + b^t x_u) + a_2(x_s^t G^t x_s + b^t x_s) - a_1 b^t x_u - (a_2 - 1) b^t x_s. \end{aligned}$$

If we further select $a_2 \ge 1$ and since $b^t x_u,\ b^t x_s \ge 0$, we obtain

$$Z_u \le a_1 Z_u + a_2 Z_s.$$

If we further impose the condition $a_1 \le 1$, we obtain the bound:

$$\frac{Z_u}{Z_s} \le \frac{a_2}{1 - a_1}.$$

Given that we have freedom to select a_1 and a_2, we find the best upper bound by solving

$$\begin{aligned} \text{minimize} \quad & \frac{a_2}{1 - a_1} \\ \text{subject to} \quad & a_1 a_2 \ge c^2/4 \\ & a_2 \ge 1,\ 0 \le a_1 \le 1. \end{aligned} \tag{5}$$

The optimal solution to Problem (5) is given as follows:
If $c^2 \le 2$, the optimal solution is $a_1 = c^2/4$, $a_2 = 1$ leading to

$$\frac{Z_u}{Z_s} \le \frac{4}{4 - c^2}.$$

If $c^2 > 2$, the optimal solution is $a_1 = 1/2$, $a_2 = c^2/2$ leading to

$$\frac{Z_u}{Z_s} \le c^2. \qquad \square$$

We next examine several special cases and relate our results with earlier results from the literature.

(a) Separable affine functions:

When the variational inequality problem function F is separable, it has components $F_i(x) =$

$g_i x_i + b_i$. In this case the matrix G is diagonal, with diagonal elements $g_i > 0$. In this case $c^2 = 1$ and the bound in Theorem 1 becomes

$$\frac{Z_u}{Z_s} \leq \frac{4}{4-c^2} = \frac{4}{3}$$

originally obtained in Roughgarden and Tardos [36].

(b) Non-separable symmetric affine functions:

When the variational inequality problem function F is non-separable, that is $F(x) = Gx + b$, with G a general symmetric positive definite matrix, then $c^2 = 1$ and thus $Z_u/Z_s \leq 4/3$, thus showing that the bound of $4/3$ holds also for non-separable symmetric affine functions.

(c) Non-separable asymmetric affine functions:

When the matrix G is "not too asymmetric" (in the sense that $c^2 \leq 2$) then the bound becomes $\frac{4}{4-c^2}$. On the other hand, for "rather asymmetric" matrices (in the sense that $c^2 > 2$) then the bound becomes c^2.

The next corollary extends this result to allow for the case where the constant term has negative components $b_i < 0$ corresponding to positive values of $x_i > 0$ in set K. The proof follows similar steps as the one in Theorem 1 and therefore, for the sake of brevity we abbreviate the steps involved.

Corollary 1 *For an affine variational inequality problem with problem function $F(x) = Gx+b$, where G is a general asymmetric, positive definite matrix:*

$$Z_u \leq \begin{cases} \dfrac{4}{4-c^2} Z_s - \dfrac{c^2}{4-c^2} \min\limits_{x \in K} b^t x & \text{if } c^2 \leq 2 \\ c^2 Z_s - (c^2 - 1) \min\limits_{x \in K} b^t x & \text{if } c^2 > 2. \end{cases}$$

The proof can be found in the Appendix.

Remark: Notice that when $b^t x \geq 0, \;\; \forall x \in K$ then this bound coincides with the result in Theorem 1.

The next corollary illustrates how the result in Theorem 1 simplifies when the constant term is zero.

Corollary 2 *When the constant term is zero in the variational inequality problem function, that is, $F(x) = Gx$, where G is a positive definite asymmetric matrix, then*

$$\frac{Z_u}{Z_s} \leq c^2.$$

Proof:

$$\begin{aligned} Z_u &= F(x_u)^t x_u \\ &\le F(x_u)^t x_s \\ &= x_u^t G^t x_s + b^t x_s \\ &\le \|x_u\|_S \, \|S^{-1}G\|_S \, \|x_s\|_S + b^t x_s \quad \text{(from the norm inequality, Cauchy's inequality, and } S \succ 0\text{))} \\ &= c\, \|x_u\|_S \, \|x_s\|_S + b^t x_s. \qquad\qquad \text{(Definition 1)} \end{aligned}$$

For every $a_1,\ a_2 \ge 0$, if we choose $2\sqrt{a_1 a_2} \ge c$, as in Theorem 1, then

$$c\|x_u\|_S\|x_s\|_S \le a_1\|x_u\|_S^2 + a_2\|x_s\|_S^2.$$

We thus obtain that for all $a_1, a_2 \ge 0$ and $a_1 a_2 \ge \frac{c^2}{4}$:

$$\begin{aligned} Z_u &\le a_1\|x_u\|_S^2 + a_2\|x_s\|_S^2 + b^t x_s \\ &= a_1 x_u^t S x_u + a_2 x_s^t S x_s + b^t x_s \\ &= a_1(x_u^t G^t x_u) + a_2(x_s^t G^t x_s). \end{aligned}$$

Since the constant terms are zero it follows that

$$Z_u \le a_1 Z_u + a_2 Z_s.$$

Therefore, if $a_1 \le 1$ then

$$\frac{Z_u}{Z_s} \le \frac{a_2}{1-a_1}.$$

Given that we have freedom to select a_1 and a_2, we find the best upper bound by solving

$$\begin{aligned} \text{minimize} \quad & \frac{a_2}{1-a_1} \\ \text{subject to} \quad & a_1 a_2 \ge \tfrac{c^2}{4} \\ & a_2 \ge 0,\ 0 \le a_1 \le 1. \end{aligned} \tag{6}$$

The optimal solution to Problem (6) is $a_1 = 1/2$, $a_2 = c^2/2$ leading to

$$\frac{Z_u}{Z_s} \le c^2. \square$$

A Key Consequence:

This result has a particular implication in the competitive multi-period pricing application we described in Section 2. The variational inequality formulation we consider in this setting in order to describe the user-optimized problem (i.e., where sellers compete) has a problem function with no constant term, that is, a function of the form $F(x) = -Qx$, where Q is a symmetric matrix. Furthermore, when the Jacobian matrix of the negative demand vector as a function of the price vector (i.e., $-h(p)$), is a positive definite matrix then the bound in Corollary 2 is also one in this application setting. That is, for this class of demand functions $Z_u/Z_s = 1$. This implies that the user-optimized problem (i.e., the setting where sellers compete through pricing) and the system-optimized problem (i.e., where a central authority is forcing sellers to set prices in order to optimize the system's total profit) coincide. The reason behind this result connects with the fact that the variational inequality problem function F has no constant term and a symmetric Jacobian matrix $-Q$. Therefore, the asymmetry constant is $c^2 = 1$. In Perakis and Sood [33] we also provide an independent proof of this result. Nevertheless, it also directly connects with the result of Corollary 2 as in this case we are able to recast the problem as a variational inequality with no constant term and an asymmetry constant $c^2 = 1$.

Remarks:

1. More generally, when the constant term in the affine variational inequality problem is zero and the problem has a symmetric Jacobian matrix then the system-optimized (SO) and the user-optimized (UO) problems coincide. The previous corollary verifies this result since in that case $c^2 = 1$ and therefore, $\frac{Z_u}{Z_s} = 1$. Notice that the bound is tight in this case.

2. We would like to point out that the results in this section allow a feasible region that is a non-convex set. As a result the feasible region can be a subset of integers.

3.3 Tightness of the Bound

In this subsection we discuss some examples that illustrate the tightness of the bounds we established. In particular through the example below we illustrate how the bound becomes tight for some families of problems.

Example:

Consider $F_1(x) = ax^1 + fx^2 + b_1$, $F_2(x) = -fx^1 + gx^2 + b_2$ and feasible region $K = \{x = (x^1, x^2) : \ x^1 + x^2 = d,\ x^1,\ x^2 \geq 0\}$.

Notice that the degree of asymmetry constant $c^2 = \lambda_{max}(S^{-1/2}G^tS^{-1}GS^{-1/2}) = 1 + \frac{f^2}{ag}$, since in this example $G = \begin{pmatrix} a & f \\ -f & g \end{pmatrix}$, $S = \begin{pmatrix} a & 0 \\ 0 & g \end{pmatrix}$.

We will consider the following cases.

- First we consider the case where $a \leq g$. Let $f = a$ and constant terms $b_1 = b = (g-a)d$ and $b_2 = 0$. Notice that the measure of asymmetry becomes $c^2 = 1 + \frac{a}{g} \leq 2$, since $\frac{a}{g} \leq 1$. Furthermore, since $a \leq g$ then $b_1 = b \geq 0$. The user-optimized solution is $x_u^1 = 0$, $x_u^2 = d$ and $Z_u = gd^2$. The system-optimized solution is $x_s^1 = \frac{2gd-b}{2(a+g)} \geq 0$ (since $b = (g-a)d$), $x_s^2 = \frac{2ad+b}{2(a+g)} \geq 0$ and $Z_s = \frac{(3g-a)d^2}{4}$. Notice that then

$$\frac{Z_u}{Z_s} = \frac{4g}{3g-a} = \frac{4}{3-\frac{a}{g}} = \frac{4}{4-c^2}.$$

  Furthermore, when in particular, $a = g$ then $c^2 = 2$ and $\frac{Z_u}{Z_s} = \frac{4}{4-c^2} = 2 = c^2$.

- Consider the case where $a > g$. Choose as $f = -a$ and as constant terms $b_1 = 0$, $b_2 = b = (a-g)d$. Notice that since $a > g$, $b_2 = b > 0$. Moreover, the measure of asymmetry becomes $c^2 = 1 + \frac{a}{g} > 2$. The user-optimized solution is $x_u^1 = d$, $x_u^2 = 0$ and $Z_u = ad^2$. The system-optimized solution is $x_s^1 = \frac{2gd+b}{2(a+g)} \geq 0$, $x_s^2 = \frac{2ad-b}{2(a+g)} \geq 0$ (since $b_2 = b = (a-g)d$) and $Z_s = a(\frac{2gd+b}{2(a+g)})^2 + g(\frac{2ad-b}{2(a+g)})^2 + b\frac{2ad-b}{2(a+g)}$. Suppose that $d \to \infty$. Notice that then as $d \to \infty$,

$$\frac{Z_u}{Z_s} \longrightarrow \frac{a+g}{g} = 1 + \frac{a}{g} = c^2.$$

  These two examples establish that the bound in Theorem 1 is tight. Furthermore, this last example also suggests that the bound is tight even when the demand d is very large.

- Finally, consider the case where the constant terms $b_1 = b_2 = 0$. Choose as $f = -a$. Then $c^2 = 1 + \frac{a}{g}$. The user-optimized solution is $x_u^1 = d$, $x_u^2 = 0$ and $Z_u = ad^2$. The system-optimized solution is $x_s^1 = \frac{gd}{(a+g)} \geq 0$, $x_s^2 = \frac{ad}{(a+g)} \geq 0$ and $Z_s = \frac{agd^2}{a+g}$. Notice that then

$$\frac{Z_u}{Z_s} = \frac{a+g}{g} = 1 + \frac{a}{g} = c^2.$$

  This example establishes that the bound in Corollary 2 is tight (i.e., when the constant term is zero).

In this example $G = \begin{pmatrix} a & f \\ -f & g \end{pmatrix}$ is a positive definite matrix since its symmetrized version is matrix $S = \frac{G+G^t}{2} = \begin{pmatrix} a & 0 \\ 0 & g \end{pmatrix}$. Furthermore, matrix G^2 becomes $\begin{pmatrix} a^2-f^2 & af+fg \\ -af-fg & g^2-f^2 \end{pmatrix}$ with a symmetrized matrix $\frac{G^2+(G^2)^t}{2} = \begin{pmatrix} a^2-f^2 & 0 \\ 0 & g^2-f^2 \end{pmatrix}$. Since in the family of examples above, we chose $f = a$ or $f = -a$, we observe that in both cases the symmetrized matrix of G^2 is

$\begin{pmatrix} 0 & 0 \\ 0 & g^2 - a^2 \end{pmatrix}$. Therefore, G^2 is a positive semidefinite matrix when $a \leq g$. This is equivalent to $w^t G^2 w = (G^t w)^t (Gw) \geq 0$, which suggests that the angle between vectors $(G^t w)$ and (Gw) is less than or equal to 90 degrees. Notice that in this case it is also true that $c^2 = 1 + \frac{a}{g} \leq 2$. Furthermore, G^2 is a negative definite matrix when $a > g$. Since $w^t G^2 w = (G^t w)^t (Gw) < 0$, it follows that the angle between vectors $(G^t w)$ and (Gw) is more than 90 degrees. It is also the case that $c^2 = 1 + \frac{a}{g} > 2$. This latter observation illustrates further the discussion in Subsection 3.1, about how the bound on constant c^2 connects with the degree of asymmetry of matrix G.

4 A Bound for Nonlinear, Asymmetric Functions

In this section, we extend the previous bound from affine to general nonlinear variational inequality problems. Specifically, we assume that the Jacobian matrix is not a constant matrix G but a positive definite, nonlinear and asymmetric matrix $\nabla F(x)$. The positive definiteness assumption of the Jacobian matrix implies that the variational inequality problem has a unique solution (see for example [29] for details).

We introduce the symmetrized matrix, $S(x) = \frac{\nabla F(x) + \nabla F(x)^t}{2}$. We now extend Definition 1 in Section 3, for measuring the **degree of asymmetry** of the problem to a definition that also incorporates the nonlinearity of the cost functions involved. If the problem function F is nonlinear, and as a result the Jacobian matrix $\nabla F(x)$ depends on $x \in K$, in the measure of asymmetry, we also need to account for the feasible region.

Definition 2 *We define a quantity c^2 that measures the degree of asymmetry of the Jacobian matrix $\nabla F(x)$. That is,*

$$c^2 \equiv \sup_{x \in K} \|S(x)^{-1} \nabla F(x)\|^2_{S(x)}.$$

As in Section 3,

$$c^2 = \sup_{x \in K} \lambda_{max}(S(x)^{-1/2} \nabla F(x)^t S(x)^{-1} \nabla F(x) S(x)^{-1/2}).$$

That is, c^2 is the supremum over the feasible region, of the maximum eigenvalue of the positive definite and symmetric matrix $S(x)^{-1/2} \nabla F(x)^t S(x)^{-1} \nabla F(x) S(x)^{-1/2}$. When the Jacobian matrix is positive definite and symmetric, that is, $\nabla F(x) = \nabla F(x)^t$, then $S(x) = \nabla F(x)$ and $c^2 = 1$.

Furthermore, we need to define a measure of the **nonlinearity** of the problem function F. As a result, we consider a property of the Jacobian matrix which always applies to positive definite matrices. This allows us in some cases to provide a tight bound. This bound naturally extends the bound in Theorem 1 from affine to nonlinear problems. The bound involves the constant A that measure the nonlinearity of the problem.

4.1 A Measure of Nonlinearity

The following definition provides a measure of the nonlinearity of the problem function by "comparing" the Jacobian matrix at feasible points.

Definition 3 *(see [38] for more details)*
The variational inequality problem function $F : R^n \to R^n$ satisfies the **Jacobian similarity** *property if it has a positive semidefinite Jacobian matrix ($\nabla F(x) \succeq 0, \ \forall x \in K$) and $\forall w \in R^n, \ \forall x, \ \bar{x} \in K$, there exists $A \geq 1$ satisfying*

$$\frac{1}{A} w^t \nabla F(x) w \leq w^t \nabla F(\bar{x}) w \leq A w^t \nabla F(x) w.$$

Remarks:
1. In the context of nonlinear optimization (i.e., when the variational inequality problem function is the gradient of an objective function f, that is, $F = \nabla f$), Jacobian similarity is referred to as Hessian similarity.
2. The property of Jacobian similarity holds for a positive semidefinite, bounded Jacobian matrix. This property is similar to the property of self-concordance as it applies to barrier functions. In the context of nonlinear optimization, Nemirovskiy and Nesterov [31] have shown that if the self-concordance property holds on a barrier function then the property of Hessian similarity also holds locally.
3. Notice that the left side inequality of the condition always follows from the right side inequality. This is true since the right side inequality of condition (3) holds for every $x, \ \bar{x} \in K$. As a result by interchanging x with $\bar{x}$ implies the left side inequality in condition (3).
The next lemma illustrates two cases where Jacobian similarity holds and establishes constant A.

Lemma 2
- *If the Jacobian matrix is* **strongly positive definite** *(i.e., it has eigenvalues bounded away from zero), then it also satisfies the Jacobian similarity property. In this case, a possible bound for the constant A is*

$$A = \frac{\max_{x \in K} \lambda_{max}(S(x))}{\min_{x \in K} \lambda_{min}(S(x))}.$$

- *If the problem function is* **affine**, *with* **positive semidefinite** *Jacobian matrix G then it also satisfies the Jacobian similarity property. In this case $A = 1$.*

A proof of this Lemma can be found in [38].

4.2 The Bound

The theorem below extends Theorem 1 to the nonlinear case.

Theorem 2 *A variational inequality problem with a continuously differentiable problem function F, a positive definite Jacobian matrix satisfying the Jacobian similarity property and $F(0)^t x \geq 0$, for all $x \in K$, we have:*

$$\frac{Z_u}{Z_s} \leq \begin{cases} \dfrac{4}{4 - c^2 A} & \text{if } c^2 \leq \frac{2}{A} \\ c^2 A^2 - 2(A-1) & \text{if } c^2 > \frac{2}{A}. \end{cases}$$

Proof:

Notice that since x_u solves variational inequality (4) and $x_s \in K$ then the variational inequality formulation implies that

$$Z_u = F(x_u)^t x_u \leq F(x_u)^t x_s.$$

Applying the mean value theorem to function $\Phi(t) = F(x_s + t(x_u - x_s))^t x_s$, $t \in [0,1]$, implies that for some $\bar{t} \in [0,1]$, $\bar{x} = x_s + \bar{t}(x_u - x_s)$ and

$$\begin{aligned} F(x_u)^t x_s &= F(x_s)^t x_s + (x_u - x_s)^t \nabla F(\bar{x})^t x_s \\ &= F(x_s)^t x_s - x_s^t \nabla F(\bar{x})^t x_s + x_u^t \nabla F(\bar{x})^t S(\bar{x})^{-1} S(\bar{x}) x_s && \text{(multiplying with } S(\bar{x}),\ S(\bar{x})^{-1}) \\ &\leq F(x_s)^t x_s - x_s^t \nabla F(\bar{x})^t x_s + \|x_u^t \nabla F(\bar{x})^t S(\bar{x})^{-1}\|_{S(\bar{x})}\ \|x_s\|_{S(\bar{x})} && \text{(from Cauchy's inequality)} \\ &\leq F(x_s)^t x_s - x_s^t \nabla F(\bar{x})^t x_s \\ &\quad + \|x_u\|_{S(\bar{x})}\ \|S(\bar{x})^{-1} \nabla F(\bar{x})\|_{S(\bar{x})} \|x_s\|_{S(\bar{x})} && \text{(from the norm inequality)} \\ &= F(x_s)^t x_s - x_s^t \nabla F(\bar{x})^t x_s + c\ \|x_u\|_{S(\bar{x})}\ \|x_s\|_{S(\bar{x})}. && \text{(Definition 2).} \end{aligned}$$

The previous analysis leads us to conclude that

$$Z_u \ \leq\ F(x_s)^t x_s - x_s^t \nabla F(\bar{x})^t x_s + c \|x_u\|_{S(\bar{x})} \|x_s\|_{S(\bar{x})}.$$

As in Theorem 1, we choose $a_1,\ a_2 \geq 0$ and $a_1 a_2 \geq \frac{c^2}{4}$, so that

$$c\|x_u\|_{S(\bar{x})}\|x_s\|_{S(\bar{x})} \leq 2\sqrt{a_1 a_2}\|x_u\|_{S(\bar{x})}\|x_s\|_{S(\bar{x})} \leq a_1 \|x_u\|^2_{S(\bar{x})} + a_2 \|x_s\|^2_{S(\bar{x})}.$$

This gives rise to

$$\begin{aligned} Z_u &\le F(x_s)^t x_s - x_s^t \nabla F(\bar{x})^t x_s + a_1 \|x_u\|^2_{S(\bar{x})} + a_2 \|x_s\|^2_{S(\bar{x})} \\ &= F(x_s)^t x_s + a_1 \|x_u\|^2_{S(\bar{x})} + (a_2 - 1)\|x_s\|^2_{S(\bar{x})} \\ &= F(x_s)^t x_s + a_1 x_u^t \nabla F(\bar{x}) x_u + (a_2 - 1) x_s^t \nabla F(\bar{x}) x_s \qquad \text{(follows from } S(\bar{x}) = \tfrac{\nabla F(\bar{x}) + \nabla F(\bar{x})}{2}\text{).} \end{aligned} \tag{7}$$

The application of the mean value theorem to functions $\Phi_1(t) = F(x_u - t(x_u - 0))^t x_u$ and $\Phi_2(t) = F(x_s - t(x_s - 0))^t x_s$, $t \in [0, 1]$, imply that

$$(F(x_u) - F(0))^t (x_u - 0) = x_u^t \nabla F(x_1) x_u, \text{ where } x_1 = x_u - t_1 x_u, \ t_1 \in [0, 1], \tag{8}$$

$$(F(x_s) - F(0))^t (x_s - 0) = x_s^t \nabla F(x_2) x_s, \text{ where } x_1 = x_s - t_2 x_s, \ t_2 \in [0, 1]. \tag{9}$$

This gives rise to

$$\begin{aligned} Z_u &\le F(x_s)^t x_s + a_1 x_u^t \nabla F(\bar{x}) x_u + (a_2 - 1) x_s^t (\nabla F(\bar{x}) x_s \\ &\le F(x_s)^t x_s + a_1 A x_u^t \nabla F(x_1) x_u + (a_2 - 1) A x_s^t \nabla F(x_2) x_s \quad \text{(from Definition 3)} \\ &\le F(x_s)^t x_s + a_1 A (F(x_u) - F(0))^t (x_u - 0) \quad \text{(from (8))} \\ &\quad + (a_2 - 1) A (F(x_s) - F(0))^t (x_s - 0) \quad \text{(from (9)).} \end{aligned}$$

In summary,

$$Z_u = F(x_u)^t x_u \le F(x_s)^t x_s + a_1 A (F(x_u) - F(0))^t (x_u - 0) + (a_2 - 1) A (F(x_s) - F(0))^t (x_s - 0).$$

Since $a_2 \ge 1$ then using the fact that $F(0)^t x \ge 0$, it follows, similarly to Theorem 1, that

$$Z_u = F(x_u)^t x_u \le F(x_s)^t x_s + A a_1 F(x_u)^t x_u + A(a_2 - 1) F(x_s)^t x_s.$$

Therefore,

$$[1 - a_1 A] F(x_u)^t x_u \le [1 + (a_2 - 1)A] F(x_s)^t x_s.$$

If $a_1 \le \frac{1}{A}$ then

$$\frac{Z_u}{Z_s} \le \frac{1 + (a_2 - 1)A}{1 - a_1 A}.$$

Given that we have freedom to select a_1 and a_2, we find the best upper bound by solving the following minimization problem,

$$\min \ \frac{1 + (a_2 - 1)A}{1 - a_1 A} \tag{10}$$

$$\text{satisfying } a_1 a_2 \ge \frac{c^2}{4},$$

$$a_2 \geq 1,\ 0 \leq a_1 \leq \frac{1}{A}.$$

Notice that the optimal solution to minimization problem (10) yields

$$\frac{Z_u}{Z_s} \leq \begin{cases} \dfrac{4}{4 - c^2 A} & \text{if } c^2 \leq \frac{2}{A} \\ c^2 A^2 - 2(A-1) & \text{if } c^2 > \frac{2}{A}. \end{cases} \qquad \square$$

Remarks:

1. Theorem 2 directly extends the result of Theorem 1. Indeed when the variational inequality problem function is affine, that is, $F(x) = Gx + b$, then $\nabla F(x) = G$ and as a result Definition 3 holds for $A = 1$. Therefore, $\frac{2}{A} = 2$ and

$$\frac{Z_u}{Z_s} \leq \begin{cases} \dfrac{4}{4 - c^2} & \text{if } c^2 \leq \frac{2}{A} = 2 \\ c^2 & \text{if } c^2 > \frac{2}{A} = 2. \end{cases}$$

2. When the variational inequality problem function has a symmetric Jacobian matrix, that is, $\nabla F(x) = \nabla F(x)^t$, then as we discussed $c^2 = 1$. Therefore,
   i. If $A \leq 2$ it follows that $c^2 = 1 \leq \frac{2}{A}$ and $\frac{Z_u}{Z_s} \leq \frac{4}{4-A}$.
   ii. If $A > 2$ it follows that $c^2 = 1 > \frac{2}{A}$ and $\frac{Z_u}{Z_s} \leq 1 + A^2$.

3. When the term $F(0) = 0$ then using a similar proof to Corollary 2, minimization problem (10) becomes,

$$\min \frac{1 + (a_2 - 1)A}{1 - a_1 A} \tag{11}$$

$$\text{satisfying } a_1 a_2 \geq \frac{c^2}{4},$$

$$a_2 \geq 0,\ 0 \leq a_1 \leq \frac{1}{A}.$$

   Notice that the optimal solution to minimization problem (11) gives rise to the following bound

$$\frac{Z_u}{Z_s} \leq c^2 A^2 - 2(A-1).$$

4.3 Tightness of the Bound

In this subsection, we discuss an example where the bound we established for nonlinear problems is tight.

Example:

Consider the feasible region

$$K = \{x = (x^1, x^2) \in R^2_+ : \quad x^1 + x^2 = 1/2\}.$$

Consider cost functions

$$F_1(x) = (x^1)^2 + x^1 + ax^2, \quad F_2(x) = (x^2)^2 + x^2 - ax^1.$$

These cost functions are nonlinear and non-separable. The Jacobian matrix is

$$\nabla F(x) = \begin{pmatrix} 2x^1 + 1 & a \\ -a & 2x^2 + 1 \end{pmatrix}.$$

Consider the symmetrized matrix $S(x) = \frac{\nabla F(x) + \nabla F(x)^t}{2} = \begin{pmatrix} 2x^1 + 1 & 0 \\ 0 & 2x^2 + 1 \end{pmatrix}$. Notice that this is a strongly positive definite matrix for all $x \in K$, since the minimum eigenvalue is bounded away from zero over the feasible region K, i.e., $\min_{x \in K} \lambda_{min}(x) = 1 > 0$. Furthermore, in this example $A = 2$, since

$$A = \frac{\max_{x \in K} \lambda_{max}(x)}{\min_{x \in K} \lambda_{min}(x)} = \frac{2}{1} = 2.$$

The constant

$$c^2 = \sup_{x \in K} \lambda_{max}(S(x)^{-1/2} \nabla F(x)^t S(x)^{-1} \nabla F(x) S(x)^{-1/2})$$

$$= \sup_{x \in K} \lambda_{max} \left[\begin{pmatrix} 1 - \frac{a^2}{(2x^1+1)(2x^2+1)} & 0 \\ 0 & 1 + \frac{a^2}{(2x^1+1)(2x^2+1)} \end{pmatrix} \right] = 1 + a^2.$$

In this example the system-optimized solution is $x_s = (1/2, 1/2)$. This gives rise to $Z_s = F(x_s)^t x_s = \frac{3}{4}$. Furthermore, the user-optimized solution is $x_u^1 = \frac{2-a}{4}$, $x_u^2 = \frac{2+a}{4}$. Then $Z_u = F(x_u)^t x_u = -\frac{5a^3}{32} + \frac{5a^2}{16} - \frac{3a}{8} + \frac{3}{4}$. As a result,

$$\frac{Z_u}{Z_s} = -\frac{5a^3}{24} + \frac{5a^2}{12} - \frac{a}{2} + 1.$$

Notice that since $A = 2$, and $c^2 = 1 + a^2$, the bound in this case becomes

$$\frac{Z_u}{Z_s} \leq c^2 A^2 - 2(A - 1) = 4(1 + a^2) - 2 = 2 + 4a^2.$$

From this discussion we conclude that the bound is tight for all values of a satisfying

$$-\frac{5a^3}{24} + \frac{5a^2}{12} - \frac{a}{2} + 1 = 2 + 4a^2 \Leftrightarrow 5a^3 + 86a^2 + 12a + 24 = 0.$$

Notice that $a = -17.07588$ solves this third degree polynomial equation. In summary, for all cost functions

$$F_1(x) = (x^1)^2 + x^1 + ax^2, \quad F_2(x) = (x^2)^2 + x^2 - ax^1, \quad \text{with } a: \; 5a^3 + 86a^2 + 12a + 24 = 0,$$

the bound is tight. For example, cost functions

$$F_1(x) = (x^1)^2 + x^1 - 17.07588x^2, \quad F_2(x) = (x^2)^2 + x^2 + 17.07588x^1.$$

5 Bounds via Semidefinite Optimization

In this section, we illustrate how to bound the "price of anarchy" using ideas from semidefinite optimization. We illustrate a method to derive the same bounds as in the previous sections when the Jacobian matrix of the problem function, is not necessarily invertible, i.e., it is a positive semidefinite rather than a positive definite matrix.

5.1 An SDP Approach for Affine Problems

First we consider affine problem functions.

Theorem 3 *For an affine variational inequality problem with problem function $F(x) = Gx + b$, with $G \succeq 0$, $b^t x \geq 0$ for all $x \in K$, the optimal objective function value of the following semidefinite optimization problem,*

$$\min \frac{a_2}{1 - a_1} \tag{12}$$

$$\text{satisfying} \begin{pmatrix} a_1 S & \frac{-G^t}{2} \\ \frac{-G}{2} & a_2 S \end{pmatrix} \succeq 0$$

$$a_2 \geq 1, \; 0 \leq a_1 \leq 1,$$

determines the bound $\frac{Z_u}{Z_s} \leq \frac{a_2^*}{1-a_1^*}$.

Proof:

As in Theorem 1 since x_u solves variational inequality (4) and $x_s \in K$ then

$$F(x_u)^t x_u \leq F(x_u)^t x_s = (x_u)^t G^t x_s + b^t x_s. \tag{13}$$

We wish to find $a_1,\ a_2 \geq 0$, so that

$$\begin{aligned} x_u^t G^t x_s &\leq a_1 x_u^t S x_u + a_2 x_s^t S x_s \quad (\text{since } S = \tfrac{G+G^t}{2}) \\ &= a_1 x_u^t G x_u + a_2 x_s^t G x_s. \end{aligned} \tag{14}$$

Notice that (14) is equivalent to

$$(x_u^t, x_s^t) \begin{pmatrix} a_1 G^t & \frac{-G^t}{2} \\ \frac{-G}{2} & a_2 G^t \end{pmatrix} \begin{pmatrix} x_u \\ x_s \end{pmatrix} \geq 0.$$

This relation follows when the symmetric part of the $2n \times 2n$ matrix $\begin{pmatrix} a_1 G^t & \frac{-G^t}{2} \\ \frac{-G}{2} & a_2 G^t \end{pmatrix} \succeq 0$ (that is, matrix $\begin{pmatrix} a_1 S & \frac{-G^t}{2} \\ \frac{-G}{2} & a_2 S \end{pmatrix} \succeq 0$).
Furthermore, if $a_2 \geq 1$ and $a_1 \geq 0$ then since $b^t x_u,\ b^t x_s \geq 0$, relations (13), (14) imply that

$$Z_u = F(x_u)^t x_u \leq a_1 F(x_u)^t x_u + a_2 F(x_s)^t x_s.$$

Therefore, if $a_1 \leq 1$ then

$$\frac{Z_u}{Z_s} \leq \frac{a_2}{1 - a_1}.$$

Given that we have freedom to select a_1 and a_2, we find the best upper bound by solving the following semidefinite optimization problem,

$$\begin{aligned} &\min \ \frac{a_2}{1-a_1} \\ &\text{satisfying} \begin{pmatrix} a_1 S & \frac{-G^t}{2} \\ \frac{-G}{2} & a_2 S \end{pmatrix} \succeq 0. \\ &a_2 \geq 1,\ 0 \leq a_1 \leq 1. \end{aligned} \tag{15}$$

Notice that the optimal objective function value to the semidefinite optimization problem (15) is the bound determining the "price of anarchy". □

Remark: This proof does not require the matrix G to be positive definite. Matrix G in this case is positive semidefinite. The next corollary connects the bounds in Theorem 3 and 1 illustrating that when the matrix $G \succ 0$, the two bounds coincide.

Corollary 3 *For an affine variational inequality problem $F(x) = Gx + b$, with matrix $G \succ 0$, Theorems 1 and 3 yield the same bound.*

Proof:
Notice that when the matrix $G \succ 0$ (and as a result, its symmetrized matrix $S \succ 0$) then the matrix $\begin{pmatrix} a_1 S & \frac{-G^t}{2} \\ \frac{-G}{2} & a_2 S \end{pmatrix} \succeq 0$ if and only if $a_1 \geq 0$ and the matrix

$$\begin{pmatrix} a_1 \mathcal{I} & \frac{-S^{-\frac{1}{2}} G^t S^{-\frac{1}{2}}}{2} \\ \frac{-S^{-\frac{1}{2}} G S^{-\frac{1}{2}}}{2} & a_2 \mathcal{I} \end{pmatrix} \succeq 0,$$

where matrix $\mathcal{I}$ is the identity matrix. Notice that this is also in turn equivalent to $a_1 \geq 0$ and the matrix $a_1 a_2 \mathcal{I} - \frac{S^{-\frac{1}{2}} G^t S^{-1} G S^{-\frac{1}{2}}}{4}$ being positive semidefinite. But this is equivalent to $a_1 \geq 0$ and $a_1 a_2 \geq \frac{\lambda_{max}(S^{-\frac{1}{2}} G^t S^{-1} G S^{-\frac{1}{2}})}{4} = \frac{c^2}{4}$ (see Definition 1). As a result the SDP involved in Theorem 3 yields in this case the same bound as the optimization problem (5) in Theorem 1. □

Remark: Notice that when the constant term is zero then the SDP problem reduces to

$$\min \frac{a_2}{1 - a_1} \tag{16}$$

$$\text{satisfying} \begin{pmatrix} a_1 S & \frac{-G^t}{2} \\ \frac{-G}{2} & a_2 S \end{pmatrix} \succeq 0$$

$$a_2 \geq 0, \ 0 \leq a_1 \leq 1.$$

Similarly to Corollary 3, one can show that when $G \succ 0$, the bound is c^2. Notice that this is the same bound as in Corollary 2.

5.2 An SDP Approach for Nonlinear Problems

We are now ready to extend the previous results to nonlinear problems.

Theorem 4 *For a variational inequality problem with a positive semidefinite Jacobian matrix (i.e., $\nabla F(x) \succeq 0$) satisfying the Jacobian similarity property, $F(0)^t x \geq 0$, for all $x \in K$, the optimal objective function value of the following semidefinite optimization problem,*

$$\min \frac{a_2}{1 - a_1} \tag{17}$$

$$\text{satisfying} \begin{pmatrix} a_1 S(x) & \frac{-\nabla F(x)^t}{2} \\ \frac{-\nabla F(x)}{2} & a_2 S(x) \end{pmatrix} \succeq 0, \quad \forall x \in K$$

$$a_2 \geq 1, \ 0 \leq a_1 \leq 1,$$

determines the bound $\frac{Z_u}{Z_s} \leq \frac{a_2^*}{1 - a_1^*}$.

Proof:

Notice that since x_u solves variational inequality (4) and $x_s \in K$ then the variational inequality formulation implies that

$$Z_u = F(x_u)^t x_u \leq F(x_u)^t x_s.$$

Applying the mean value theorem to function $\Phi(t) = F(x_s + t(x_u - x_s))^t x_s$, $t \in [0,1]$, implies that for some $\bar{t} \in [0,1]$, $\bar{x} = x_s + \bar{t}(x_u - x_s)$

$$F(x_u)^t x_s \;=\; F(x_s)^t x_s + (x_u - x_s)^t \nabla F(\bar{x})^t x_s.$$

We wish to find $a_1,\ a_2 \geq 0$, so that

$$\begin{aligned} x_u^t \nabla F(\bar{x})^t x_s \;&\leq\; a_1 x_u^t S(\bar{x}) x_u + a_2 x_s^t S(\bar{x}) x_s \qquad \left(\text{since } S(\bar{x}) = \tfrac{\nabla F(\bar{x}) + \nabla F(\bar{x})^t}{2}\right) \\ &=\; a_1 x_u^t \nabla F(\bar{x}) x_u + a_2 x_s^t \nabla F(\bar{x}) x_s. \end{aligned} \tag{18}$$

Notice that (18) is equivalent to

$$(x_u^t, x_s^t) \begin{pmatrix} a_1 \nabla F(\bar{x})^t & \frac{-\nabla F(\bar{x})^t}{2} \\ \frac{-\nabla F(\bar{x})}{2} & a_2 \nabla F(\bar{x})^t \end{pmatrix} \begin{pmatrix} x_u \\ x_s \end{pmatrix} \geq 0.$$

This relation follows when the symmetric part of the $2n \times 2n$ matrix $\begin{pmatrix} a_1 \nabla F(\bar{x})^t & \frac{-\nabla F(\bar{x})^t}{2} \\ \frac{-\nabla F(\bar{x})}{2} & a_2 \nabla F(\bar{x})^t \end{pmatrix} \succeq 0$ (that is, matrix $\begin{pmatrix} a_1 S(\bar{x}) & \frac{-\nabla F(\bar{x})^t}{2} \\ \frac{-\nabla F(\bar{x})}{2} & a_2 S(\bar{x}) \end{pmatrix} \succeq 0$).

$$Z_u \;\leq\; F(x_s)^t x_s + a_1 x_u^t \nabla F(\bar{x}) x_u + (a_2 - 1) x_s^t \nabla F(\bar{x}) x_s \quad \text{(follows from (18))}. \tag{19}$$

The application of the mean value theorem to functions $\Phi_1(t) = F(x_u - t(x_u - 0))^t (x_u)$ and $\Phi_2(t) = F(x_s - t(x_s - 0))^t (x_s)$, $t \in [0,1]$, imply that

$$(F(x_u) - F(0))^t (x_u - 0) = x_u^t \nabla F(x_1) x_u, \text{ where } x_1 = x_u - t_1 x_u,\ t_1 \in [0,1], \tag{20}$$

$$(F(x_s) - F(0))^t (x_s - 0) = x_s^t \nabla F(x_2) x_s, \text{ where } x_1 = x_s - t_2 x_s,\ t_2 \in [0,1]. \tag{21}$$

This gives rise to

$$\begin{aligned} Z_u \;&\leq\; F(x_s)^t x_s + a_1 x_u^t \nabla F(\bar{x}) x_u + (a_2 - 1) x_s^t (\nabla F(\bar{x}) x_s \\ &\leq\; F(x_s)^t x_s + a_1 A x_u^t \nabla F(x_1) x_u + (a_2 - 1) A x_s^t \nabla F(x_2) x_s \quad \text{(from Definition 3)} \\ &\leq\; F(x_s)^t x_s + a_1 A (F(x_u) - F(0))^t (x_u - 0) \qquad \text{(from (20))} \\ &\quad\; + (a_2 - 1) A (F(x_s) - F(0))^t (x_s - 0) \qquad \text{(from (21))}. \end{aligned}$$

In summary,

$$Z_u = F(x_u)^t x_u \leq F(x_s)^t x_s + a_1 A(F(x_u) - F(0))^t (x_u - 0) + (a_2 - 1)A(F(x_s) - F(0))^t (x_s - 0).$$

Since $a_2 \geq 1$ then using the fact that $F(0)^t x \geq 0$ for all $x \in K$, it follows, similarly to Theorem 1, that

$$Z_u = F(x_u)^t x_u \leq F(x_s)^t x_s + Aa_1 F(x_u)^t (x_u) + A(a_2 - 1)F(x_s)^t (x_s).$$

Therefore,

$$[1 - a_1 A]F(x_u)^t x_u \leq [1 + (a_2 - 1)A]F(x_s)^t x_s.$$

If $a_1 \leq \frac{1}{A}$ then

$$\frac{Z_u}{Z_s} \leq \frac{1 + (a_2 - 1)A}{1 - a_1 A}.$$

Given that we have freedom to select a_1 and a_2, we find the best upper bound by solving the following minimization problem,

$$\min \frac{1 + (a_2 - 1)A}{1 - a_1 A} \tag{22}$$

$$\text{satisfying} \begin{pmatrix} a_1 S(x) & \frac{-\nabla F(x)^t}{2} \\ \frac{-\nabla F(x)}{2} & a_2 S(x) \end{pmatrix} \succeq 0, \quad \forall x \in K.$$

$$a_2 \geq 1, \; 0 \leq a_1 \leq \frac{1}{A}.$$

Notice that the optimal objective function value to the semidefinite optimization problem (22) is the bound determining the "price of anarchy". □

Corollary 3 extends to nonlinear problems as well.

Corollary 4 *When the Jacobian matrix is strongly positive definite then the bound in Theorem 4 becomes the same as in Theorem 2. That is,*

$$\frac{Z_u}{Z_s} \leq \begin{cases} \dfrac{4}{4 - c^2 A} & \text{if } c^2 \leq \frac{2}{A} \\ c^2 A^2 - 2(A - 1) & \text{if } c^2 > \frac{2}{A}. \end{cases}$$

Proof: The proof is similar to Corollary 3 and we omit it for the sake of brevity.

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APPENDIX

Corollary 1 *For an affine variational inequality problem with problem function $F(x) = Gx + b$, where G is a general asymmetric, positive definite matrix:*

$$Z_u \leq \begin{cases} \dfrac{4}{4-c^2} Z_s - \dfrac{c^2}{4-c^2} \min\limits_{x \in K} b^t x & \text{if } c^2 \leq 2 \\ c^2 Z_s - (c^2 - 1) \min\limits_{x \in K} b^t x & \text{if } c^2 > 2. \end{cases}$$

Proof:

$$\begin{aligned} Z_u &= F(x_u)^t x_u \\ &\leq F(x_u)^t x_s \\ &= x_u^t G^t x_s + b^t x_s \\ &\leq \|x_u\|_S \; \|S^{-1}G\|_S \; \|x_s\|_S + b^t x_s \quad \text{(from the norm inequality, Cauchy's inequality, and } S \succ 0)) \\ &= c \; \|x_u\|_S \; \|x_s\|_S + b^t x_s. \qquad \text{(Definition 1)} \end{aligned}$$

For every $a_1, \; a_2 \geq 0$, if we choose $2\sqrt{a_1 a_2} \geq c$, as in Theorem 1, then

$$c\|x_u\|_S \|x_s\|_S \leq a_1 \|x_u\|_S^2 + a_2 \|x_s\|_S^2.$$

We thus obtain that for all $a_1, a_2 \geq 0$ and $a_1 a_2 \geq c^2/4$:

$$\begin{aligned} Z_u &\leq a_1 \|x_u\|_S^2 + a_2 \|x_s\|_S^2 + b^t x_s \\ &= a_1 x_u^t S x_u + a_2 x_s^t S x_s + b^t x_s \\ &= a_1 (x_u^t G^t x_u + b^t x_u) + a_2 (x_s^t G^t x_s + b^t x_s) - a_1 b^t x_u - (a_2 - 1) b^t x_s. \end{aligned}$$

If we further select $a_2 \geq 1$, then since $\min_{x \in K} b^t x \leq b^t x_u$ and $\min_{x \in K} b^t x \leq b^t x_s$, we obtain

$$Z_u \leq a_1 Z_u + a_2 Z_s - a_1 \min_{x \in K} b^t x - (1 - a_2) \min_{x \in K} b^t x.$$

If we further impose the condition $a_1 \leq 1$, we obtain

$$Z_u \leq \frac{a_2}{1-a_1} Z_s - \frac{a_1 + a_2 - 1}{1 - a_1} \min_{x \in K} b^t x.$$

Given that we have freedom to select a_1 and a_2, we find the best upper bound by solving

$$\begin{aligned} \text{minimize} \quad & \frac{a_2}{1-a_1} \\ \text{subject to} \quad & a_1 a_2 \geq c^2/4 \\ & a_2 \geq 1, \; 0 \leq a_1 \leq 1. \end{aligned} \tag{23}$$

The optimal solution to Problem (23) is given as follows:

If $c^2 \leq 2$, the optimal solution is $a_1 = c^2/4$, $a_2 = 1$ leading to

$$Z_u \leq \frac{4}{4-c^2} Z_s - \frac{c^2}{4-c^2} \min_{x \in K} b^t x.$$

If $c^2 > 2$, the optimal solution is $a_1 = 1/2$, $a_2 = c^2/2$ leading to

$$Z_u \leq c^2 Z_s - (c^2 - 1) \min_{x \in K} b^t x. \qquad \square$$