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Casimir interaction among heavy fermions in the BCS-BEC crossover

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We investigate a two-species Fermi gas with a large mass ratio interacting by an interspecies short-range interaction. Using the Born-Oppenheimer approximation, we determine the interaction energy of two heavy fermions immersed in the Fermi sea of light fermions as a function of the s -wave scattering length. In the BCS limit, we recover the perturbative calculation of the effective interaction between heavy fermions. The p -wave projection of the effective interaction is attractive in the BCS limit while it turns out to be repulsive near the unitarity limit. We find that the p -wave attraction reaches its maximum between the BCS and unitarity limits, where the maximal p -wave pairing of heavy minority fermions is expected. We also investigate the case where the heavy fermions are confined in two dimensions and the p -wave attraction between them is found to be stronger than that in three dimensions.

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I. INTRODUCTION

Experiments using ultracold atomic gases have achieved great success in realizing a new type of fermionic superfluid. By arbitrarily varying the strength of interaction via the Feshbach resonance, the crossover from the BCS superfluid of fermionic atoms to the Bose-Einstein condensate of tightly bound molecules has been observed and extensively studied [1,2]. Nowadays a portion of interests of the cold atom community is shifting to the two-species Fermi gas with unequal densities [3–7] and with unequal masses [8–10]. Elucidating the phase diagram of such an asymmetric Fermi gas is an important and challenging problem. Such an asymmetric system of fermions will be also interesting as a prototype of high density quark matter in the core of neutron stars, where the density and mass imbalances exist among different quark flavors [11].

So far several quantum phases have been proposed as a ground state of the density-imbalanced Fermi gas. One of such phases is the p -wave superfluid phase by the pairing between the same species of fermions [12]. In Ref. [12], the attraction between the same species of fermions was found to be induced by the interaction with the other species of fermions based on the perturbative calculation in the weak-coupling BCS limit. Although the resulting pairing gap is exponentially suppressed in the BCS limit, it may be possible that the pairing gap becomes large enough near the strong-coupling unitarity limit. Thus an important question is how large the pairing gap can be near the unitarity limit.

However, difficulties for theoretical treatments arise away from the weak-coupling BCS limit because we do not have a controlled tool to analyze the strongly interacting many-body system. Monte Carlo simulations also have limitations in asymmetric systems due to the fermion sign problem. We note that in one dimension the Monte Carlo simulations do not suffer from the sign problem and have been employed to study the Fulde-Ferrel-Larkin-Ovchinnikov phase in the density-imbalanced Fermi gas [13–15].

In order to obtain further insight into the density-imbalanced Fermi gas and the p -wave pairing therein, we investigate the Fermi gas with unequal masses between two

different atomic species. In the limit of large mass ratio, we can determine the effective interaction among heavy fermions induced by the interaction with light fermions using the Born-Oppenheimer approximation. Because we do not need to rely on the weak-coupling approximation, we can go beyond the perturbative BCS regime to the strongly interacting unitary regime in a controlled way.

We will show that the p -wave projection of the effective interaction between two heavy fermions is attractive in the BCS limit being consistent with the perturbative prediction, while it turns out to be repulsive near the unitarity limit. Thus the p -wave attraction has a maximum between the BCS and unitarity limits, where the maximal p -wave pairing of heavy minority fermions is expected. We note that our results have a direct relevance to the recently realized Fermi-Fermi mixture of ^{40}K and ^6Li because of their large mass ratio [8–10].

It is worthwhile to point it out that the effective interaction among heavy fermions immersed in the Fermi sea of light fermions is an analog of the Casimir force among objects placed in a vacuum [16]. The origin of the Casimir force can be traced back to the modification of the spectrum of zero point fluctuations of the electromagnetic field. In our case, the role of vacuum is played by the Fermi sea of light fermions whose spectrum is modified by the presence of heavy fermions [17].

This paper is organized as follows. In Sec. II, we determine the effective interaction between two heavy fermions induced by the interaction with the Fermi sea of light fermions as a function of the s -wave scattering length using the Born-Oppenheimer approximation. (The effective interaction for a general number of heavy fermions is derived using the functional integral method and evaluated in Appendix A.) With having in mind the application to the p -wave pairing of heavy fermions, we compute the p -wave projection of the effective interaction in Sec. III. Here we consider both cases where the heavy fermions are in three dimensions (3D) and where they are confined in two dimensions (2D) while light fermions are always in 3D. The latter case corresponds to the two-species Fermi gas in 2D-3D mixed dimensions studied in Refs. [18,19]. Finally summary and concluding remarks are given in Sec. IV.

II. CASIMIR INTERACTION BETWEEN TWO HEAVY FERMIONS

A. Born-Oppenheimer approximation

Consider a two-species Fermi gas with unequal masses M and m interacting by an interspecies short-range interaction. When the mass ratio is large $M/m \gg 1$, we can employ the Born-Oppenheimer approximation to compute the interaction energy of heavy fermions immersed in the Fermi sea of light fermions. Here we concentrate on the case of two heavy fermions. The formula for a general number of heavy fermions is derived in Appendix A using the functional integral method. The wave function of a light fermion interacting with two heavy fermions fixed at positions \mathbf{x}_1 and \mathbf{x}_2 satisfies the Schrödinger equation (here and below $\hbar=1$):

$$-\frac{\nabla_{\mathbf{y}}^2}{2m}\Psi(\mathbf{y};\mathbf{x}_1,\mathbf{x}_2) = E\Psi(\mathbf{y};\mathbf{x}_1,\mathbf{x}_2). \quad (1)$$

The interspecies short-range interaction between the light and heavy fermions is taken into account by imposing the short-range boundary condition; $\Psi(\mathbf{y} \rightarrow \mathbf{x}_i) \propto \frac{1}{|\mathbf{y}-\mathbf{x}_i|} - \frac{1}{a} + O(|\mathbf{y}-\mathbf{x}_i|)$, where a is the s -wave scattering length. Below we determine the energy spectrum of the light fermion both for bound states and continuum states as a function of a and the separation between the heavy fermions $\mathbf{r} \equiv \mathbf{x}_1 - \mathbf{x}_2$.

For the bound state $E = -\frac{\kappa^2}{2m} < 0$, the solution of Eq. (1) takes the form

$$\Psi_{\pm}(\mathbf{y}) = \frac{e^{-\kappa|\mathbf{y}-\mathbf{x}_1|}}{|\mathbf{y}-\mathbf{x}_1|} \pm \frac{e^{-\kappa|\mathbf{y}-\mathbf{x}_2|}}{|\mathbf{y}-\mathbf{x}_2|}, \quad (2)$$

where \pm is the parity of the wave function. The upper (lower) sign corresponds to the even (odd) parity. From the short-range boundary condition, we obtain the equation

$$-\kappa \pm \frac{e^{-\kappa|\mathbf{r}|}}{|\mathbf{r}|} = -\frac{1}{a}. \quad (3)$$

The solution to this equation exists when $|\mathbf{r}|/a > \mp 1$ and is given by

$$\kappa_{\pm} = \frac{1}{a} + \frac{1}{|\mathbf{r}|} W(\pm e^{-|\mathbf{r}|/a}), \quad (4)$$

where $W(z)$ is the Lambert function satisfying $z = W(z)e^{W(z)}$. Accordingly we find the following binding energy depending on a and $|\mathbf{r}|$:

$$E_{\text{binding}}^{(\pm)} = -\frac{\kappa_{\pm}^2}{2m} \quad (5)$$

for $|\mathbf{r}|/a > \mp 1$.

For the continuum state $E = \frac{k^2}{2m} > 0$, the solution of Eq. (1) takes the form

$$\Psi_{\pm}(\mathbf{y}) = \frac{\sin(k|\mathbf{y}-\mathbf{x}_1| + \delta_{\pm})}{|\mathbf{y}-\mathbf{x}_1|} \pm \frac{\sin(k|\mathbf{y}-\mathbf{x}_2| + \delta_{\pm})}{|\mathbf{y}-\mathbf{x}_2|}, \quad (6)$$

where $0 \leq \delta_{\pm} \leq \pi$ is the phase shift. From the short-range boundary condition, we obtain the equation

$$k \cos \delta_{\pm} \pm \frac{\sin(k|\mathbf{r}| + \delta_{\pm})}{|\mathbf{r}|} = -\frac{\sin \delta_{\pm}}{a}. \quad (7)$$

The solution to this equation is easily found to be

$$\tan \delta_{\pm} = -\frac{k|\mathbf{r}| \pm \sin(k|\mathbf{r}|)}{\frac{|\mathbf{r}|}{a} \pm \cos(k|\mathbf{r}|)}. \quad (8)$$

We then suppose that the system is confined in a large sphere with a radius $R \gg |\mathbf{r}|$ and the two heavy fermions are located near its center; $\mathbf{x}_1 = \frac{\mathbf{r}}{2}$ and $\mathbf{x}_2 = -\frac{\mathbf{r}}{2}$. By imposing $\Psi_{\pm}(|\mathbf{y}| \rightarrow R) \rightarrow 0$ at the boundary of the sphere, the momentum k is discretized as

$$k_n^+ R + \delta_+ = n\pi,$$

$$k_n^- R + \delta_- = \left(n - \frac{1}{2}\right)\pi \quad (9)$$

with $n=1, 2, 3, \dots$. Therefore the energy level becomes

$$E_n^{(\pm)} = \frac{k_n^{\pm 2}}{2m}. \quad (10)$$

Now we fill the above-obtained energy levels with the light fermions. The total energy of such a system is given by

$$E = -\frac{\kappa_+^2 + \kappa_-^2}{2m} + \sum_{n=1}^N \frac{k_n^{+2} + k_n^{-2}}{2m}, \quad (11)$$

where $2N$ is the number of interacting light fermions in the continuum states. We compare the energy of the interacting system with that in the noninteracting limit,

$$E_{\text{free}} = \sum_{n=1}^N \frac{k_n^{+2} + k_n^{-2}}{2m} \Bigg|_{\delta_{\pm}=0}, \quad (12)$$

and define the energy reduction as $\Delta E(|\mathbf{r}|) \equiv E - E_{\text{free}}$. Then we take the thermodynamic limit $N, R \rightarrow \infty$ with the Fermi momentum of the light fermions $k_F \equiv N\pi/R$ fixed. If we notice that the summation over n is dominated by $n \sim N \gg 1$ and neglect small corrections of $O(1/N)$, we obtain the following simple expression for the energy reduction:

$$\Delta E(|\mathbf{r}|) = -\frac{\kappa_+^2 + \kappa_-^2}{2m} - \int_0^{k_F} dk k \frac{\delta_+(k) + \delta_-(k)}{m\pi}. \quad (13)$$

We note that the phase shifts given in Eq. (8) are defined to be in the range $0 \leq \delta_{\pm}(k) \leq \pi$. The same result can be obtained on a more general footing by using the functional integral method (see Appendix A).

B. Effective interaction between two heavy fermions

When the two heavy fermions are infinitely separated, the energy reduction approaches

$$\Delta E(|\mathbf{r}| \rightarrow \infty) \rightarrow 2\mu_{\text{single}}, \quad (14)$$

where $\mu_{\text{single}} < 0$ is the chemical potential of a single heavy fermion immersed in the Fermi sea of light fermions [20]:

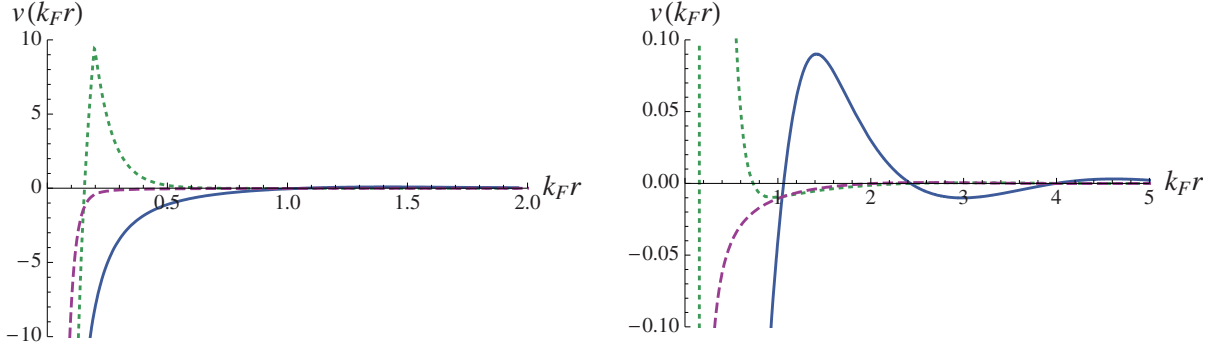


FIG. 1. (Color online) Interaction energy of two heavy fermions $v(k_F r)$ as a function of their separation $k_F r$. Three different curves correspond to the BCS regime $(ak_F)^{-1} = -5$ (dashed curve), unitarity limit $(ak_F)^{-1} = 0$ (solid curve), and BEC regime $(ak_F)^{-1} = 5$ (dotted curve). The right panel is a magnification of the left panel.

$$\mu_{\text{single}} = -\frac{k_F^2 ak_F + [1 + (ak_F)^2][\pi/2 + \arctan(ak_F)^{-1}]}{2m \pi(ak_F)^2}. \quad (15)$$

The energy reduction with $2\mu_{\text{single}}$ subtracted is regarded as the interaction energy of the two heavy fermions:

$$V(|r|) \equiv E(|r|) - E(|r| \rightarrow \infty) = \Delta E(|r|) - 2\mu_{\text{single}}. \quad (16)$$

$V(|r|)$ represents the effective interaction between two heavy fermions induced by the interaction with the Fermi sea of light fermions. We note that the chemical potential of light fermions $\mu_l \equiv \frac{k_F^2}{2m}$ is fixed here instead of their particle number.

It is convenient to measure the interaction energy $V(r)$ in units of the Fermi energy of light fermions and introduce a dimensionless function $v(k_F r)$ as

$$V(r) \equiv \frac{k_F^2}{2m} v(k_F r). \quad (17)$$

$v(k_F r)$ is a function of the separation between the heavy fermions $k_F r$ and also the s -wave scattering length ak_F . $v(k_F r)$ for three typical values of $(ak_F)^{-1} = -5, 0, 5$ are plotted in Fig. 1. One can see the smooth evolution of the effective interaction between the two heavy fermions as a function of $(ak_F)^{-1}$

in Fig. 2. In the BCS regime $(ak_F)^{-1} \lesssim -1$, the effective interaction is attractive at $r \lesssim |a|$ and has a tiny oscillatory behavior at $r \gtrsim |a|$. This oscillatory behavior grows toward the unitarity limit $(ak_F)^{-1} \approx 0$ and makes a small hump at $r \approx k_F^{-1}$ (see the left panel of Fig. 2). This hump further grows in the BEC regime $(ak_F)^{-1} \gtrsim 1$ and eventually develops a repulsive core at $r \approx a$ (see the right panel of Fig. 2).

It is worthwhile to compare our nonperturbative result (16) with the perturbative calculation of the effective interaction in the BCS limit $a \rightarrow -0$. In the BCS limit $|a| \ll r$, k_F^{-1} , we have $\kappa_{\pm} \rightarrow 0$ and the phase shift

$$\delta_{\pm}(k) \rightarrow \frac{|a|}{r} [kr \pm \sin(kr)] \pm \left(\frac{|a|}{r}\right)^2 \cos(kr) [kr \pm \sin(kr)] + O(|a|^3). \quad (18)$$

Thus we can find the interaction energy to be

$$v(k_F r) \rightarrow (|a|k_F)^2 \frac{2k_F r \cos(2k_F r) - \sin(2k_F r)}{2\pi(k_F r)^4} + O(|a|^3), \quad (19)$$

which has the same form as the Ruderman-Kittel-Kasuya-Yosida interaction between magnetic impurities in a Fermi liquid [21]. Accordingly the Fourier transform of the effective interaction $V(|r|)$ in the BCS limit becomes

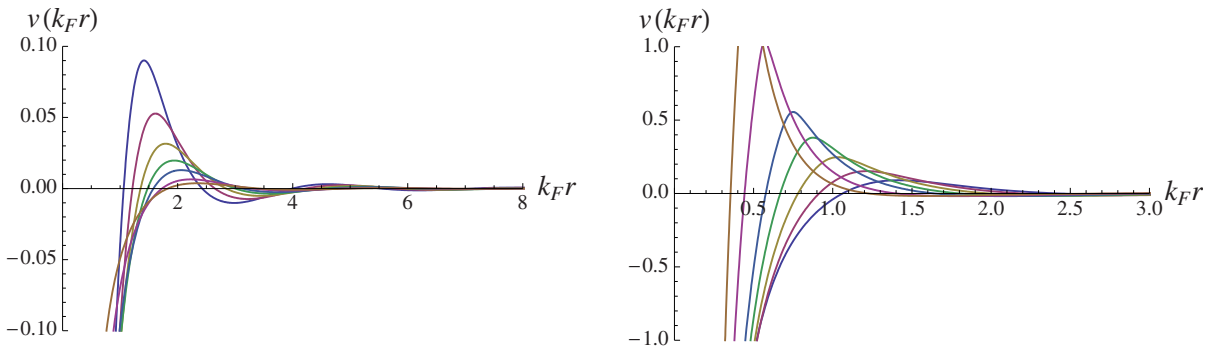


FIG. 2. (Color online) Evolution of the interaction energy $v(k_F r)$ as a function of the inverse s -wave scattering length $(ak_F)^{-1}$. Left panel: $(ak_F)^{-1}$ is varied from -2 (curve with the smallest amplitude) to 0 (largest amplitude). The tiny oscillatory behavior existing in the BCS regime grows toward the unitarity limit and makes a small hump at $r \approx k_F^{-1}$. Right panel: $(ak_F)^{-1}$ is varied from 0 (curve with the smallest height) to 2 (largest height). The hump further grows in the BEC regime and develops a repulsive core at $r \approx a$.

$$\begin{aligned}\tilde{V}(|\mathbf{p}|) &\equiv \int d\mathbf{r} e^{-i\mathbf{p}\cdot\mathbf{r}} V(|\mathbf{r}|) \\ &\rightarrow -\frac{|a|^2 k_F}{2m} \left[2 + \left(\frac{2k_F}{|\mathbf{p}|} - \frac{|\mathbf{p}|}{2k_F} \right) \ln \left| \frac{1 + |\mathbf{p}|/(2k_F)}{1 - |\mathbf{p}|/(2k_F)} \right| \right],\end{aligned}\quad (20)$$

which correctly reproduces the effective interaction to the leading order in a obtained in Ref. [12]¹ as it should be. Our nonperturbative result (16) can go beyond the perturbative BCS regime to the strongly interacting unitary regime in a controlled way by utilizing $M/m \gg 1$.

Analytic expressions of $v(k_F r)$ in various limits are obtained in Appendix B.

III. P-WAVE PROJECTION OF THE EFFECTIVE INTERACTION

A. Lessons from BCS theory

The physics of heavy fermions immersed in the Fermi sea of light fermions is described by the Hamiltonian

$$\begin{aligned}H &= \int d\mathbf{x} \psi_h^\dagger(\mathbf{x}) \left(-\frac{\nabla^2}{2M} - \mu_h \right) \psi_h(\mathbf{x}) \\ &\quad + \frac{1}{2} \int d\mathbf{x} d\mathbf{y} \psi_h^\dagger(\mathbf{x}) \psi_h^\dagger(\mathbf{y}) V(|\mathbf{x} - \mathbf{y}|) \psi_h(\mathbf{y}) \psi_h(\mathbf{x}),\end{aligned}\quad (21)$$

where μ_h is the chemical potential of heavy fermions measured from μ_{single} . Strictly speaking, the pairwise interaction $V(|\mathbf{r}|)$ obtained in the previous section [Eq. (16)] is valid only if there are two heavy fermions in the system. This is because the Casimir interaction energy is not pairwise additive and also because if heavy fermions have a finite density, it will perturb the Fermi sea of light fermions and thus affect the effective interaction $V(|\mathbf{r}|)$. However, in the dilute limit $\sqrt{2M}\mu_h \ll k_F$, we expect that the above Hamiltonian is a good approximation to the system of heavy fermions because the probability to find a third heavy fermion near the two heavy fermions becomes small in the dilute system [22]. Indeed we will confirm in Appendix A that the Casimir interaction among more than two heavy fermions at large separations can be evaluated quite accurately as a sum of pairwise interactions between each of the two heavy fermions. This result also supports the use of our Hamiltonian (21) to describe the physics of dilute heavy fermions immersed in the Fermi sea of light fermions.

Another issue to be addressed is the instability of the system. When the mass ratio M/m exceeds the critical value 13.6 in pure 3D [23] or 6.35 in the 2D-3D mixture [18], the system will not be stable due to the Efimov effect. Therefore we need to assume the mass ratio to be large but smaller than the above critical value. For such a mass ratio, the effective interaction $V(|\mathbf{r}|)$ obtained in the Born-Oppenheimer ap-

proximation is no longer exact but can be considered as a good approximation to the exact one.

One of predictions we can derive from the Hamiltonian (21) is a pairing between the heavy fermions. The pairing gap of heavy fermions Δ_p is defined to be

$$(2\pi)^d \delta(\mathbf{k}) \Delta_p = \int \frac{d\mathbf{q}}{(2\pi)^d} \tilde{V}(|\mathbf{p} - \mathbf{q}|) \left\langle \tilde{\psi}_h\left(\frac{\mathbf{k}}{2} - \mathbf{q}\right) \tilde{\psi}_h\left(\frac{\mathbf{k}}{2} + \mathbf{q}\right) \right\rangle,\quad (22)$$

where $\tilde{\psi}_h(\mathbf{p})$ is the Fourier transform of $\psi_h(\mathbf{x})$ and $\tilde{V}(|\mathbf{p} - \mathbf{q}|)$ is the Fourier transform of $V(|\mathbf{r}|)$ [see Eq. (20)]. Because of the Fermi statistics of heavy fermions, the pairing gap has to have an odd parity; $\Delta_{-\mathbf{p}} = -\Delta_{\mathbf{p}}$. The standard mean-field calculation leads to the self-consistent gap equation

$$\Delta_p = - \int \frac{d\mathbf{q}}{(2\pi)^d} V(|\mathbf{p} - \mathbf{q}|) \frac{\Delta_q}{2E_q} [1 - 2n_F(E_q)].\quad (23)$$

Here $E_p = \sqrt{\left(\frac{p^2}{2M} - \mu_h\right)^2 + |\Delta_p|^2}$ is the quasiparticle energy and $n_F(E_p) = 1/(e^{E_p/T} + 1)$ is the Fermi-Dirac distribution function at temperature T .

When the coupling between different partial waves can be neglected, we can solve the gap equation (23) easily by using the weak-coupling approximation.² For a given orbital angular momentum ℓ (odd integer), the pairing gap and the critical temperature are given by [24]

$$|\Delta_p| \sim T_c \sim \mu_h \exp\left(\frac{1}{N_h \tilde{V}_\ell}\right).\quad (24)$$

Here N_h is the density of states of heavy fermions at the Fermi surface and \tilde{V}_ℓ is the partial-wave projection of the effective interaction $\tilde{V}(|\mathbf{p} - \mathbf{q}|)$ with $|\mathbf{p}| = |\mathbf{q}| = \sqrt{2M}\mu_h$ fixed on the Fermi surface of heavy fermions and assumed to be attractive $\tilde{V}_\ell < 0$. Because the lowest partial wave for identical fermions is $\ell = 1$, the dominant pairing is expected to occur in the p -wave channel. With having in mind the application to the pairing of heavy fermions, we compute $N_h \tilde{V}_\ell$ as a function of the s -wave scattering length ak_F from Eq. (16). Below we consider both cases where the heavy fermions are in three dimensions ($d=3$) and where they are confined in two dimensions ($d=2$).

B. Heavy fermions in $d=3$

When the heavy fermions are in three dimensions $d=3$, the partial-wave projection of the effective interaction $\tilde{V}(|\mathbf{p} - \mathbf{q}|)$ with $|\mathbf{p}| = |\mathbf{q}| \equiv p$ fixed is given by

¹For unequal masses, the effective interaction in the BCS limit becomes $\tilde{V}(|\mathbf{p}|) = -\left[\frac{2\pi(m+M)a}{mM}\right] \frac{\gamma^2 m k_F}{2\pi^2} L\left(\frac{|\mathbf{p}|}{2k_F}\right)$, where $L(z) = \frac{1}{2} + \frac{1-z^2}{4z} \ln\left|\frac{1+z}{1-z}\right|$ is the static Lindhard function. The limit of large mass ratio $M/m \gg 1$ coincides with Eq. (20).

²The weak-coupling approximation even at unitarity is justified in the dilute limit; $\sqrt{2M}\mu_h \ll k_F$. From Eqs. (B2) and (B3), the interaction energy at the mean interparticle distance of heavy fermions $r = (2M\mu_h)^{-1} \gg k_F^{-1}$ becomes $V(r) \sim \frac{k_F^2}{2m} \frac{1}{(k_F r)^3}$ and is parametrically small compared to the kinetic energy $K(r) \sim \frac{1}{2Mr^2}$.

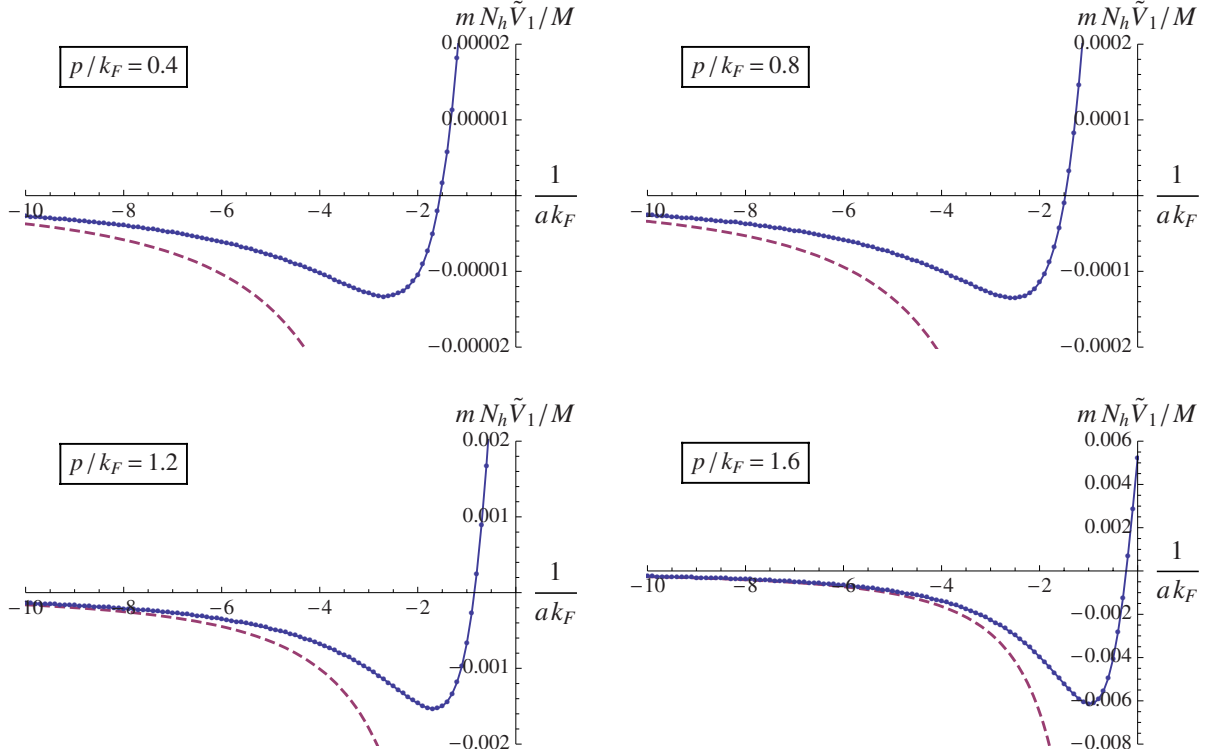


FIG. 3. (Color online) P -wave effective interaction between two heavy fermions in three dimensions as a function of $(ak_F)^{-1}$. $\frac{m}{M}N_h\tilde{V}_{\ell=1}(p)$ in $d=3$ is plotted for $p/k_F=0.4, 0.8, 1.2,$ and 1.6 . The dashed curves are perturbative results at leading order valid in the BCS limit $(ak_F)^{-1} \rightarrow -\infty$.

$$\tilde{V}_\ell(p) = \frac{1}{2} \int_{-1}^1 d \cos \theta P_\ell(\cos \theta) \tilde{V}(|\mathbf{p} - \mathbf{q}|), \quad (25)$$

where $\cos \theta = \hat{\mathbf{p}} \cdot \hat{\mathbf{q}}$. Multiplying it by the density of states of heavy fermions $N_h(p) = \frac{Mp}{2\pi^2}$, we obtain the dimensionless function representing the effective interaction between two heavy fermions for the given partial wave ℓ :

$$N_h \tilde{V}_\ell(p) = \frac{M}{2\pi m} \left(\frac{k_F}{p} \right)^2 \int_{-1}^1 d \cos \theta P_\ell(\cos \theta) \times \int_0^\infty dz \frac{z \sin(z\sqrt{2-2\cos\theta})}{\sqrt{2-2\cos\theta}} v\left(\frac{k_F z}{p}\right). \quad (26)$$

When the projected effective interaction with $p = \sqrt{2M\mu_h}$ being on the Fermi surface of heavy fermions is attractive $N_h \tilde{V}_\ell < 0$, the pairing of the heavy fermions is expected to occur with the pairing gap and the critical temperature given in Eq. (24).

Figure 3 shows $\frac{m}{M}N_h\tilde{V}_\ell(p)$ in the p -wave channel $\ell=1$ as a function of $(ak_F)^{-1}$ for four values of $p/k_F = 0.4, 0.8, 1.2, 1.6$. We note that what is plotted is not the p -wave effective interaction $N_h\tilde{V}_\ell(p) \propto \frac{M}{m}$ itself but $\frac{m}{M}N_h\tilde{V}_\ell(p)$ that is independent of the mass ratio $M/m \gg 1$. We can see that the p -wave effective interaction is attractive in the BCS limit $(ak_F)^{-1} \ll -1$ being consistent with the perturbative prediction [12]. However, it turns out that the perturbative result has only a small range of validity. Our result shows that the

p -wave attraction is generally weaker than the extrapolation of the perturbative calculation and, in particular, the p -wave effective interaction becomes a strong repulsion near the unitarity limit. This may be understood because of the repulsive hump of $v(k_F r)$ developed near the unitarity limit (see Figs. 1 and 2). We also find that the p -wave attraction is stronger for the larger value of p/k_F because the contribution from the attractive part of $v(k_F r) \rightarrow -c^2/(k_F r)^2$ at short distance $r \rightarrow 0$ [see Eq. (B1)] becomes more significant for larger p/k_F . The maximum attraction for each value of $p/k_F = 0.4, 0.8, 1.2, 1.6$ is achieved around $(ak_F)^{-1} \approx -2.7, -2.5, -1.7, -1.0$, respectively. At such a value of the s -wave scattering length, the maximal p -wave pairing of heavy minority fermions immersed in the Fermi sea of light fermions is possible while the pairing gap and the critical temperature would be very small because of the numerically weak attraction $N_h\tilde{V}_{\ell=1}(p) \ll 1$. One can perform the same analysis for the f -wave channel ($\ell=3$) but will find even weaker attraction.

C. Heavy fermions in $d=2$

When the heavy fermions are confined in two dimensions $d=2$, the partial-wave projection of the effective interaction $\tilde{V}(|\mathbf{p} - \mathbf{q}|)$ with $|\mathbf{p}| = |\mathbf{q}| \equiv p$ fixed is given by

$$\tilde{V}_\ell(p) = \frac{1}{\pi} \int_0^\pi d\theta \cos(\ell\theta) \tilde{V}(|\mathbf{p} - \mathbf{q}|), \quad (27)$$

where $\cos \theta = \hat{\mathbf{p}} \cdot \hat{\mathbf{q}}$. Multiplying it by the density of states of heavy fermions $N_h(p) = \frac{M}{2\pi}$, we obtain the dimensionless

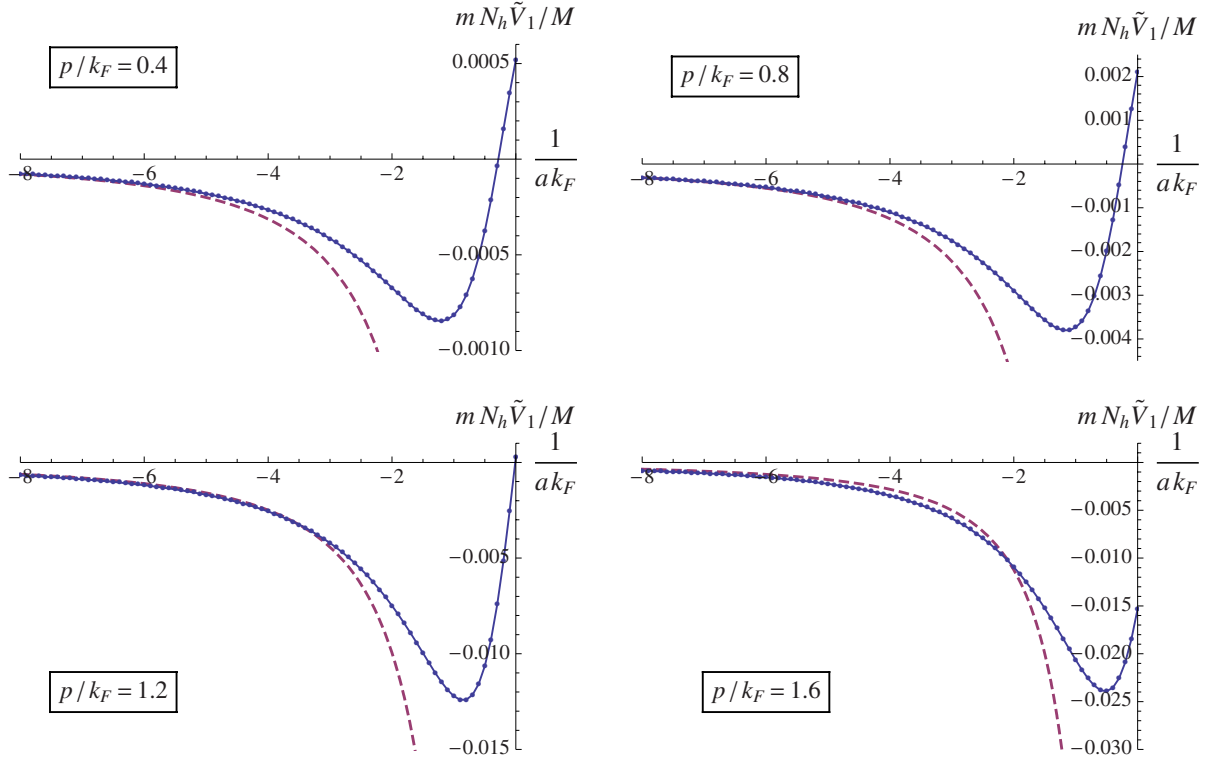


FIG. 4. (Color online) P -wave effective interaction between two heavy fermions confined in two dimensions as a function of $(ak_F)^{-1}$. $\frac{m}{M}N_h\tilde{V}_{\ell=1}(p)$ in $d=2$ is plotted for $p/k_F=0.4, 0.8, 1.2,$ and 1.6 . The dashed curves are perturbative results at leading order valid in the BCS limit $(ak_F)^{-1} \rightarrow -\infty$.

function representing the effective interaction between two heavy fermions for the given partial wave ℓ :

$$N_h\tilde{V}_{\ell}(p) = \frac{M}{2\pi m} \left(\frac{k_F}{p}\right)^2 \int_0^{\pi} d\theta \cos(\ell\theta) \times \int_0^{\infty} dz z J_0(z\sqrt{2-2\cos\theta}) v\left(\frac{k_F z}{p}\right). \quad (28)$$

In the case of $d=2$, the scattering length a should be regarded as the effective scattering length a_{eff} introduced in Ref. [18].

Figure 4 shows $\frac{m}{M}N_h\tilde{V}_{\ell}(p)$ in the p -wave channel $\ell=1$ as a function of $(ak_F)^{-1}$ for four values of $p/k_F=0.4, 0.8, 1.2, 1.6$. One can see the similar behavior to the case of $d=3$. Again the p -wave effective interaction is attractive in the BCS limit $(ak_F)^{-1} \ll -1$ being consistent with the perturbative prediction [19]. Compared to the case of $d=3$, we find that the perturbative result has a wider range of validity. In the case of $d=2$, the p -wave attraction is found to be stronger than that in $d=3$ for the same value of p/k_F and the p -wave effective interaction becomes only a weak repulsion near the unitarity limit. This will be understandable because the contribution from the attractive part of $v(k_F r) \rightarrow -c^2/(k_F r)^2$ at short distance $r \rightarrow 0$ [see Eq. (B1)] is less suppressed by the phase space factor and thus more significant in $d=2$. Our result also shows that the p -wave attraction is stronger for the larger value of p/k_F and the maximum attraction for each value of $p/k_F=0.4, 0.8, 1.2, 1.6$ is achieved

around $(ak_F)^{-1} \approx -1.2, -1.2, -0.85, -0.52$, respectively. At such a value of the s -wave scattering length, the maximal p -wave pairing of heavy minority fermions in two dimensions immersed in the three-dimensional Fermi sea of light fermions is possible and the pairing gap and the critical temperature will be larger than those in the case of $d=3$.

IV. SUMMARY AND CONCLUDING REMARKS

We have investigated a two-species Fermi gas with a large mass ratio interacting by an interspecies short-range interaction. Using the Born-Oppenheimer approximation, we determined the interaction energy of two heavy fermions immersed in the Fermi sea of light fermions, which is an analog of the Casimir force, as a function of the s -wave scattering length. We showed that the p -wave projection of the effective interaction is attractive in the BCS limit being consistent with the perturbative prediction, while it turns out to be repulsive near the unitarity limit. We found that the p -wave attraction reaches its maximum between the BCS and unitarity limits, where the maximal but still weak p -wave pairing of heavy minority fermions is possible.

We also investigated the case where the heavy fermions are confined in two dimensions corresponding to the two-species Fermi gas in the 2D-3D mixed dimensions [18,19]. Because the p -wave attraction between the heavy fermions in two dimensions was found to be stronger than that in three dimensions, we expect the larger p -wave pairing gap of heavy minority fermions. Such a p -wave pairing in two di-

mensions is especially interesting because the resulting system has a potential application to topological quantum computation using vortices with non-Abelian statistics [25,26].

Although our controlled analysis is restricted to the dilute heavy fermions in the limit of large mass ratio, our results indeed shed light on the phase diagram of asymmetric Fermi gases with unequal densities and masses (and spatial dimensions), in particular, in the strongly interacting unitary regime. Our results also have a direct relevance to the recently realized Fermi-Fermi mixture of ^{40}K and ^6Li because of their large mass ratio [8–10]. It will be an important future problem to study corrections to our results when the restrictions of the large mass ratio and the dilute limit of heavy fermions are relaxed.

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APPENDIX A: CASIMIR INTERACTION AMONG HEAVY FERMIONS FROM FUNCTIONAL INTEGRAL METHOD

1. Derivation of Casimir interaction

Here we derive the Casimir interaction among heavy fermions immersed in the Fermi sea of light fermions using the functional integral method [27]. The action that describes the light fermions with the chemical potential μ_l interacting with the heavy fermions fixed at positions \mathbf{x}_i is

$$S = \int d\tau d\mathbf{x} \psi_l^\dagger(\tau, \mathbf{x}) \left(\partial_\tau - \frac{\nabla^2}{2m} - \mu_l \right) \psi_l(\tau, \mathbf{x}) - g_0 \sum_i \int d\tau \psi_l^\dagger(\tau, \mathbf{x}_i) \psi_l(\tau, \mathbf{x}_i), \quad (\text{A1})$$

where τ is an imaginary time. The partition function $Z = e^{-\beta\Omega}$ is given by

$$Z = \int \mathcal{D}\psi_l \mathcal{D}\psi_l^\dagger e^{-S}. \quad (\text{A2})$$

We first insert the following identity into the integrand:

$$1 = \prod_i \int \mathcal{D}\eta_i \mathcal{D}\bar{\eta}_i \delta[\psi_l(\tau, \mathbf{x}_i) - \eta_i(\tau)] \delta[\psi_l^\dagger(\tau, \mathbf{x}_i) - \bar{\eta}_i(\tau)] \quad (\text{A3})$$

and then exponentiate the delta functions by introducing auxiliary fields,

$$1 = \prod_i \int \mathcal{D}\eta_i \mathcal{D}\bar{\eta}_i \mathcal{D}\alpha_i \mathcal{D}\bar{\alpha}_i \times e^{i\int d\tau \bar{\alpha}_i(\tau) [\psi_l(\tau, \mathbf{x}_i) - \eta_i(\tau)] + i\int d\tau [\psi_l^\dagger(\tau, \mathbf{x}_i) - \bar{\eta}_i(\tau)] \alpha_i(\tau)}. \quad (\text{A4})$$

Now the partition function can be written as

$$Z = \int \mathcal{D}\psi_l \mathcal{D}\psi_l^\dagger \prod_i \mathcal{D}\eta_i \mathcal{D}\bar{\eta}_i \mathcal{D}\alpha_i \mathcal{D}\bar{\alpha}_i e^{-S'}, \quad (\text{A5})$$

where the action becomes

$$S' = \int d\tau d\mathbf{x} \psi_l^\dagger(\tau, \mathbf{x}) \left(\partial_\tau - \frac{\nabla^2}{2m} - \mu_l \right) \psi_l(\tau, \mathbf{x}) - g_0 \sum_i \int d\tau \bar{\eta}_i(\tau) \eta_i(\tau) - i \sum_i \int d\tau \bar{\alpha}_i(\tau) [\psi_l(\tau, \mathbf{x}_i) - \eta_i(\tau)] - i \sum_i \int d\tau [\psi_l^\dagger(\tau, \mathbf{x}_i) - \bar{\eta}_i(\tau)] \alpha_i(\tau). \quad (\text{A6})$$

We can easily integrate out ψ_l and ψ_l^\dagger fields and η and $\bar{\eta}$ fields to lead to the partition function

$$Z = Z_0 \int \prod_i \mathcal{D}\alpha_i \mathcal{D}\bar{\alpha}_i e^{-S''}, \quad (\text{A7})$$

where $Z_0 = e^{-\beta\Omega_{\text{free}}}$ is a partition function of noninteracting light fermions and the action S'' in the momentum space becomes

$$S'' = \sum_{i,j} \int \frac{d\omega d\mathbf{p}}{(2\pi)^4} \bar{\alpha}_i(\omega) \frac{e^{i\mathbf{p}\cdot(\mathbf{x}_i - \mathbf{x}_j)}}{-i\omega + \mathbf{p}^2/2m - \mu_l} \alpha_j(\omega) - \frac{1}{g_0} \sum_i \int \frac{d\omega}{2\pi} \bar{\alpha}_i(\omega) \alpha_i(\omega). \quad (\text{A8})$$

Finally, by integrating out α and $\bar{\alpha}$ fields, we obtain the following expression for the partition function:

$$Z = Z_0 \exp \left[\beta \int \frac{d\omega}{2\pi} \ln \det M_{ij}(i\omega) \right], \quad (\text{A9})$$

where M_{ij} is the scattering matrix given by

$$M_{ij} = -\frac{\delta_{ij}}{g_0} + \int \frac{d\mathbf{p}}{(2\pi)^3} \frac{e^{i\mathbf{p}\cdot(\mathbf{x}_i - \mathbf{x}_j)}}{-i\omega + \mathbf{p}^2/2m - \mu_l} = \begin{cases} \frac{m}{2\pi} \left(\frac{1}{a} - \sqrt{-2mi\omega - 2m\mu_l} \right) & \text{for } i=j \\ \frac{m}{2\pi |\mathbf{x}_i - \mathbf{x}_j|} e^{-\sqrt{-2mi\omega - 2m\mu_l} |\mathbf{x}_i - \mathbf{x}_j|} & \text{for } i \neq j. \end{cases} \quad (\text{A10})$$

Here we introduced the s -wave scattering length a through

$$-\frac{1}{g_0} + \int \frac{d\mathbf{p}}{(2\pi)^3} \frac{2m}{\mathbf{p}^2} = \frac{m}{2\pi a}. \quad (\text{A11})$$

Therefore the reduction in the grand potential Ω compared to that in the noninteracting limit Ω_{free} is

$$\Delta\Omega(\{\mathbf{x}_i\}) = - \int \frac{d\omega}{2\pi} \ln \det M_{ij}(i\omega). \quad (\text{A12})$$

By subtracting $\Delta\Omega$ in which all heavy fermions are infinitely separated, the interaction energy of the heavy fermions becomes

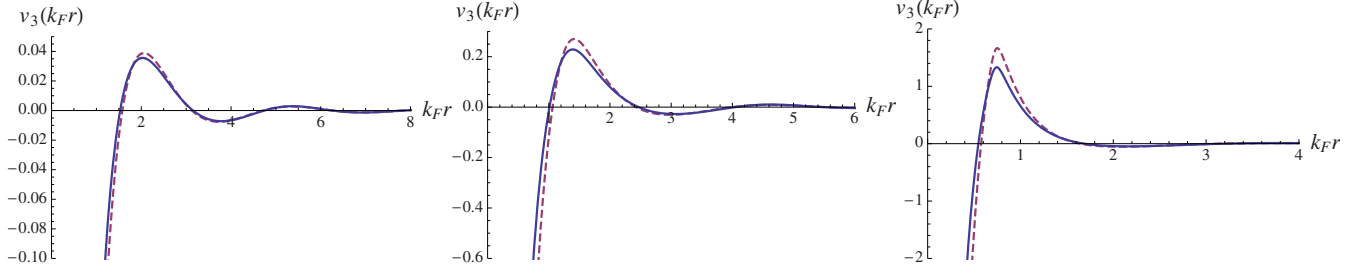


FIG. 5. (Color online) Interaction energy of three heavy fermions fixed with the same separations r . $v_3(k_F r)$ as a function of $k_F r$ is plotted for $(ak_F)^{-1} = -1$ (left), 0 (middle), and 1 (right). The dashed curves represent the sum of pairwise interaction energies $3v_2(k_F r)$.

$$V(\{\mathbf{x}_i\}) = - \int \frac{d\omega}{2\pi} \ln \frac{\det M_{ij}(i\omega)}{\det M_{ij}^\infty(i\omega)}, \quad (\text{A13})$$

where M_{ij}^∞ is a diagonal scattering matrix composed of M_{ii} . This is the generalization of $V(|\mathbf{r}|)$ in Eq. (16) to a general number of heavy fermions fixed at positions \mathbf{x}_i .

2. Casimir interaction between two heavy fermions

It is worthwhile to reproduce the result obtained in Sec. II in the case of two heavy fermions. By deforming the path of the integration over $i\nu \equiv i\omega + \mu_l$ into

$$\int_{-i\infty + \mu_l}^{i\infty + \mu_l} \rightarrow \int_{-\infty - i0^+}^{\mu_l} + \int_{\mu_l}^{-\infty + i0^+}, \quad (\text{A14})$$

we obtain the following expression for the grand potential reduction:

$$\begin{aligned} \Delta\Omega(|\mathbf{r}|) &= i \int \frac{d\nu}{2\pi} \ln \left(\frac{1}{a} - \sqrt{-2m\nu} + \frac{e^{-\sqrt{-2m\nu}|\mathbf{r}|}}{|\mathbf{r}|} \right) \\ &+ i \int \frac{d\nu}{2\pi} \ln \left(\frac{1}{a} - \sqrt{-2m\nu} - \frac{e^{-\sqrt{-2m\nu}|\mathbf{r}|}}{|\mathbf{r}|} \right) \end{aligned} \quad (\text{A15})$$

with $\mathbf{r} = \mathbf{x}_1 - \mathbf{x}_2$. We separate the integral into the contributions from bound states $\text{Re}[\nu] < 0$ and continuum states $\text{Re}[\nu] > 0$. For the bound state contribution, we pick up the singularity in the integrand (A15) as

$$i \int_{\text{Re}[\nu] < 0} \frac{d\nu}{2\pi} \ln \left(\frac{1}{a} - \sqrt{-2m\nu} \pm \frac{e^{-\sqrt{-2m\nu}|\mathbf{r}|}}{|\mathbf{r}|} \right) = \nu_\pm + \text{const}, \quad (\text{A16})$$

where $\nu_\pm < 0$ is the binding energy satisfying

$$\frac{1}{a} - \sqrt{-2m\nu_\pm} \pm \frac{e^{-\sqrt{-2m\nu_\pm}|\mathbf{r}|}}{|\mathbf{r}|} = 0. \quad (\text{A17})$$

For the continuum state contribution, defining the phase shift by

$$\pi - \delta_\pm(\nu) = \arg \left(\frac{1}{a} - \sqrt{-2m\nu - i0^+} \pm \frac{e^{-\sqrt{-2m\nu - i0^+}|\mathbf{r}|}}{|\mathbf{r}|} \right), \quad (\text{A18})$$

the integral in Eq. (A15) can be written as

$$\begin{aligned} & i \int_{\text{Re}[\nu] > 0} \frac{d\nu}{2\pi} \ln \left(\frac{1}{a} - \sqrt{-2m\nu} \pm \frac{e^{-\sqrt{-2m\nu}|\mathbf{r}|}}{|\mathbf{r}|} \right) \\ &= \mu_l - \int_0^{\mu_l} d\nu \frac{\delta_\pm(\nu)}{\pi}. \end{aligned} \quad (\text{A19})$$

Therefore, after dropping the unimportant constants, we find that the grand potential reduction in the case of two heavy fermions is given by

$$\Delta\Omega(|\mathbf{r}|) = \nu_+ + \nu_- - \int_0^{\mu_l} d\nu \frac{\delta_+(\nu) + \delta_-(\nu)}{\pi}. \quad (\text{A20})$$

This result is equivalent to $\Delta E(|\mathbf{r}|)$ in Eq. (13) with $\nu_\pm = -\frac{\kappa_\pm^2}{2m}$, $\nu = \frac{k^2}{2m}$, and $\mu_l = \frac{k_F^2}{2m}$ and thus provides the same interaction energy $V(|\mathbf{r}|)$ as Eq. (16).

3. Casimir interaction among three and four heavy fermions

We now evaluate the Casimir interaction energy (A13) among three and four heavy fermions fixed with the same separations $|\mathbf{x}_i - \mathbf{x}_j| \equiv r$. We measure the interaction energy in units of the Fermi energy of light fermions $V_N(r) \equiv \frac{k_F^2}{2m} v_N(k_F r)$ with the subscript N indicating the number of heavy fermions. The dimensionless functions $v_N(k_F r)$ with three typical values of $(ak_F)^{-1} = -1, 0, 1$ are plotted as functions of $k_F r$ in Fig. 5 for $N=3$ and in Fig. 6 for $N=4$, together with the sum of pairwise interaction energies $\frac{N(N-1)}{2} v_2(k_F r)$ for comparison.

We can see that the Casimir interaction among more than two heavy fermions can be reproduced quite accurately by the sum of pairwise interactions between each of the two heavy fermions, in particular, at long distances, although the behaviors at short distances are slightly overestimated. A similar observation has been made in Ref. [17] in the system of hardcore spheres immersed in a background Fermi sea. These results support the use of the Hamiltonian (21) only with the pairwise interaction to describe the physics of dilute heavy fermions immersed in the Fermi sea of light fermions.

APPENDIX B: INTERACTION ENERGY IN VARIOUS LIMITS

Here we evaluate the effective interaction between two heavy fermions $v(k_F r)$ in Eq. (17) in various limits where analytic expressions are available.

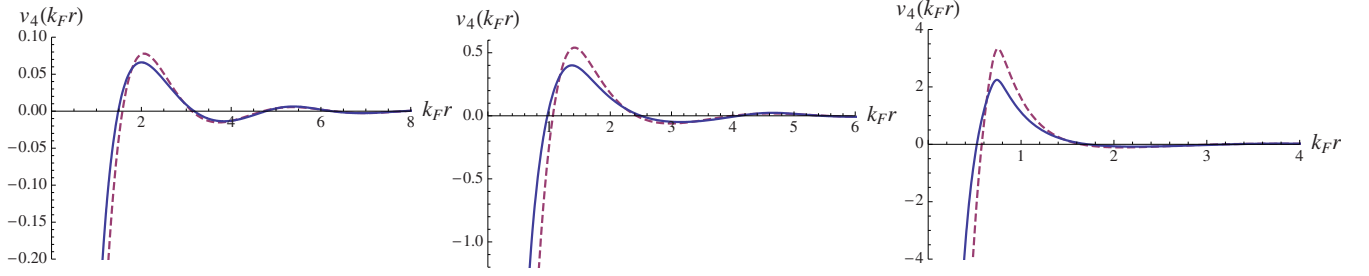


FIG. 6. (Color online) Interaction energy of four heavy fermions fixed with the same separations r . $v_4(k_F r)$ as a function of $k_F r$ is plotted for $(ak_F)^{-1} = -1$ (left), 0 (middle), and 1 (right). The dashed curves represent the sum of pairwise interaction energies $6v_2(k_F r)$.

At short distance $r \ll a$, k_F^{-1} , we obtain

$$v(k_F r) \rightarrow -\frac{c^2}{(k_F r)^2} - \frac{2c}{1+c} \frac{1}{ak_F^2} - \frac{1+c+c^2}{(1+c)^3} \frac{1}{a^2 k_F^2} + \frac{2ak_F + [1 + (ak_F)^2][\pi + 2 \arctan(ak_F)^{-1}]}{\pi(ak_F)^2} - 1 + O(r), \quad (\text{B1})$$

where $c=0.567143$ is a solution to $c=e^{-c}$.

On the other hand, at long distance $r \gg a$, k_F^{-1} , we obtain

$$v(k_F r) \rightarrow \frac{2(ak_F)^3 \sin(2k_F r) - (ak_F)^2 [(ak_F)^2 - 1] \cos(2k_F r)}{\pi [(ak_F)^2 + 1]^2 (k_F r)^3} - \frac{4(ak_F)^3 [(ak_F)^2 - 1] \cos(2k_F r)}{2\pi [(ak_F)^2 + 1]^3 (k_F r)^4} - \frac{(ak_F)^2 [(ak_F)^4 - 6(ak_F)^2 + 1] \sin(2k_F r)}{2\pi [(ak_F)^2 + 1]^3 (k_F r)^4} + O(r^{-5}). \quad (\text{B2})$$

In particular, in the unitarity limit $ak_F \rightarrow \infty$, we find

$$v(k_F r) \rightarrow -\frac{\cos(2k_F r)}{\pi(k_F r)^3} - \frac{\sin(2k_F r)}{2\pi(k_F r)^4} + \frac{2\cos(2k_F r) + \cos(4k_F r)}{4\pi(k_F r)^5} + O[(k_F r)^{-6}]. \quad (\text{B3})$$

In the BCS or BEC limit $|a| \ll r$, k_F^{-1} , we obtain

$$v(k_F r) \rightarrow (ak_F)^2 \frac{2k_F r \cos(2k_F r) - \sin(2k_F r)}{2\pi(k_F r)^4} + (ak_F)^3 \frac{2k_F r \cos(2k_F r) + [2(k_F r)^2 - 1] \sin(2k_F r)}{\pi(k_F r)^5} + O(a^4). \quad (\text{B4})$$

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