

COULOMB SCATTERING IN A MAGNETIC FIELD

by

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## ABSTRACT

Considered was the scattering of a particle of charge  $q$  and mass  $m$  in a uniform magnetic field by the Coulomb potential of a charge  $Q$  fixed at the origin. The scattering was described quantum-mechanically by a formalism in which the presence of the magnetic field was incorporated as the dominant and controlling factor. Also incorporated was the facility for varying the initial position of the gyrocenter with respect to the line on which the scattering charge is located, and for keeping track of the energies perpendicular and parallel to the magnetic field.

The charge  $q$  was represented in a Born approximation cross section by combinations of the energy eigenfunction set obtained as solutions to the Schroedinger equation  $H_0 \psi_{NMk} = E_0 \psi_{NMk}$  in a cylindrical coordinate system. The Hamiltonian  $H_0 = (\vec{p} - q\vec{A})^2/2m$  is that describing the motion of a single particle in the magnetic field generated from the potential  $\vec{A}$ . The parameters  $(NMk)$  are interpreted in terms of energies perpendicular and parallel to the magnetic field and in terms of the radius and radial position of the corresponding classical orbits.

An exact result was obtained for the matrix element of the Coulomb potential energy between these eigenfunctions. The diagonal matrix element is characterized by a logarithmic singularity. The maximum value of the off-diagonal element used in the scattering calculations was about  $10^{-10}$  eV - m. A simple, limiting form was obtained and utilized in a differential cross section.

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PART I  
SUMMARY PAPER

## I. INTRODUCTION

### Content

The problem of interest here is the scattering of a charge  $q$  in a uniform magnetic field by the Coulomb potential of a charge  $Q$  fixed at the origin. The treatment has revealed novel features not present in the zero magnetic field Rutherford problem. The scattering is described quantum-mechanically by a formalism in which the presence of the magnetic field has been incorporated as the dominant and controlling factor. If there is present any magnetic field, no matter how small, this is the only correct approach. The principal physical reason is the very omnipresence of the magnetic field. Even though the Coulomb potential may be considered long range in character, its influence must eventually become inconsequential as the scattered charge moves farther away along the magnetic field. Also incorporated into the formalism is the facility for varying the initial position of the gyrocenter with respect to the line on which the scattering charge is located, and for keeping track of the energies perpendicular and parallel to the magnetic field. At the heart of the scattering calculations presented herein is the matrix element of the Coulomb potential energy between Schroedinger wave functions representing the scattered particle. An exact result has been obtained for this quantum average. The end result is expressed in terms of a Born approximation cross section. Within limits to be discussed later, the general validity of the results are dependent not upon the size of the magnetic field, but upon its existence. Indeed, the results simplify considerably for small magnetic fields. The treatment is spinless and non-relativistic. Other than these, the chief approximations are connected with the fact that we have ignored the coupling of the relative and center of mass motions, and have ignored the possible binding of the charge  $q$  to the Coulomb center  $Q$ .

Specifically, and in more detail, the charge  $q$  was represented by combinations of the energy eigenfunction set obtained as solutions of the Schroedinger equation

$$H_0 \psi_{NMk} = E_0 \psi_{NMk} \quad (1)$$

in the cylindrical coordinate system spanned by the unit vectors  $\hat{e} \times \hat{\phi} = \hat{z}$ . The Hamiltonian  $H_0$  is that describing the motion (see Appendix A) of a single particle of charge  $q$  and mass  $m$  in a magnetic field:

$$H_0 = \frac{(\vec{p} - q\vec{A})^2}{2m} . \quad (2)$$

The magnetic field was generated from the vector potential

$$\vec{A} = \frac{1}{2} \vec{B} \times \vec{r} = \frac{1}{2} B \rho \hat{\phi} \quad (3)$$

through the relation

$$\vec{B} \equiv \text{curl } \vec{A} = B \hat{z} . \quad (4)$$

Since the divergence of this potential is identically zero, we may utilize the commutator  $[\vec{p}, \vec{A}] = 0$  to combine the two cross terms of (2). We note specifically that the term quadratic in  $\vec{A}$  (in  $B\rho$ ) is retained. The radial location of the gyrocenter and the perpendicular and parallel energies are described by the set of eigenparameters (NMk). We shall see that there are actually two orthogonal sets of such eigenfunctions, one corresponding to cyclotron orbits which enclose the origin and a second describing those which do not.

The Coulomb potential energy, with exponential or Debye shielding incorporated, has the functional form (mks rationalized units are employed):

$$\begin{aligned}
 g\Lambda_c &= \frac{gQ}{4\pi\epsilon_0} \frac{e^{-\mu r}}{r} \\
 &= [1.4 \times 10^{-9} \text{ eV-m}] \frac{e^{-\mu r}}{r}
 \end{aligned} \tag{5}$$

where  $r$  is to be replaced by  $\sqrt{\rho^2 + z^2}$ . This is the potential energy of the charge  $q$  (located at  $\vec{r}$ ) due to  $Q$  fixed at the origin. This is the agent or perturbation considered to cause transitions from one quantum representation of the charge  $q$  to another.

Two such representations were employed in the cross section calculations. One was a single eigenfunction  $\Psi_{NMk}$ , leading to a differential cross section. The second representation considered, although not as extensively, was a uniform, flooding beam of sufficient radial extent to encompass as much as desired of the Coulomb potential field. This beam, characterized not only by its radial extent but also by single values of the perpendicular and parallel energies, is of use in the consideration of a total cross section.

One of the novel features of this problem is that we are dealing with transitions from a one dimensional continuum in which are embedded discrete states to a second such continuum-discrete system. The continuum states are associated (through the eigenparameter  $k$ ) with the free or unbound motion of the charge  $q$  along the direction of the magnetic field. The discrete states (belonging to the quantum numbers  $N$  and  $M$ ) are a manifestation of the binding of  $q$  by the magnetic field in the plane perpendicular to the field. The transition probability and cross section expressions must reflect this circumstance. These expressions must contain, loosely speaking, one-dimensional density of states functions for both the initial and the final  $z$  energies and states.

Derived in appendix G is the Born approximation cross section which takes into account this circumstance and which contains these two density of states functions. It is a differential cross section, in a way not connected with  $E_{z_1}$  and to be made clear later. The expression obtained was

$$\sigma_{\alpha\beta} = 2\pi^2 \frac{|\langle \psi_\beta, g\Lambda_c \psi_\alpha \rangle|^2}{\hbar\omega_c E_{z_1}} \frac{(M+2N_1+1)}{\sqrt{1 + \frac{\hbar\omega_c}{E_{z_1}}(N_1-N_2)}} \quad (6)$$

where  $\omega_c \equiv eB/m$ . The index  $\alpha$  refers to the parameters  $(N_1 M_1 k_1)$  characterizing the initial state, and  $\beta$  the final state set  $(N_2 M_2 k_2)$ . Conservation of energy between the initial and final states is implied in this expression, since it has been integrated once on  $dE_{z_2}$  over an energy-conserving delta function. Also incorporated has been a result not yet mentioned, namely, conservation of the quantum number  $M$  in the basic matrix element  $\langle \psi_\beta, g\Lambda_c \psi_\alpha \rangle$ . This result, which has a direct and interesting classical analog, will greatly facilitate formation of a total cross section from the differential expression (6). We return to the results of this investigation after considering other work, and the relation of the scattering and bound state problems.

### Context

Although of fundamental interest, this problem and the closely related bound state problem have been little studied. This is in part due to the formidable mathematical difficulties and in part due to lack of appreciation of the significance of these problems. By the bound state problem we mean the properties associated with the solutions and energy spectrum of the Schroedinger equation

$$(H_0 + g\Lambda_c) \psi_b = E_b \psi_b \quad (7)$$

wherein the terms quadratic in the product of the magnetic field and radial distance are retained. This is commonly known as the problem of the hydrogen atom in a strong magnetic field. However it is obvious from (3) that this is not a complete description. It is also the problem of the hydrogen atom in a (perhaps moderate) magnetic field and with the electron in a highly excited angular momentum state. This is a most interesting region because the electron, though bound (its wavefunction vanishing at infinity in all directions), may have a total positive energy. As the electron occupies states more and more distant from the proton, it may be more strongly bound by the magnetic field than by the Coulomb potential. The always negative and decreasing (as  $1/r$ ) Coulomb binding energy may be overcome by the always positive and increasing (as  $r^2$ ) magnetic binding energy. The electron will always be bound in the direction perpendicular to the magnetic field, whether by the Coulomb potential (negative energy) or by the magnetic field (positive energy). This is not the case along the direction of the magnetic field since the electron does not see the field in this direction. If the electron is bound in this direction, it must have a negative energy, and if not bound, a positive energy. The bound state problem thus approaches the scattering problem as the binding becomes predominantly magnetic in character.

These and other aspects of the bound state problem have been studied by Bitter [1964, 1965, and private communication] and by Praddaude [1964, private communication]. It is probably fair to say that one of the most significant contributions to emerge from their investigations has been the realization that the case of precisely zero magnetic field is singular. That is, the point  $B = 0$  in the treatment of the hydrogen atom as described above is not a limit point as  $B \rightarrow 0$ . They are separate problems having in common only the Coulomb binding. This may be understood from consideration of the energy eigenvalue



spectrum, some aspects of which already have been discussed above. In an approximate solution to the bound state problem (valid for  $\rho \ll |z|$ ), Praddaude obtained an energy spectrum of the form

$$E_b = -\frac{Ry}{n^2} + \hbar\omega_c \left(P + \frac{1}{2}\right). \quad (8)$$

Bitter has obtained a qualitatively similar form by means of semi-classical arguments. The first term is the usual Coulomb binding energy. The second represents the binding by the magnetic field. The quantum number  $P$  is related to the angular momentum, or the energy of azimuthal motion. The feature that we wish to emphasize here is that, as  $B \rightarrow 0$ , these magnetic field states become more and more dense (more and more states per unit energy increment). Then, when  $B = 0$ , these infinitely numerous states discontinuously cease to exist. The existence and behavior of this spectrum is a significant feature of the bound state problem, and has profound implications for the scattering problem. For example, it is conceivable that an electron, initially unbound in the  $z$  direction and incident upon a proton, could be temporarily or permanently delayed in its trip along the field line by occupation of one of the states of this spectrum. Unfortunately, the scattering formalism employed herein is not powerful enough to detect this possibility.

There have been reported sporadic attacks upon the scattering problem, most within the framework of the Born approximation. Each has involved some approximation in the calculation of the matrix elements. Tennenwald [1959] was apparently the first to point out the difficulty of integrating the classical equations of motion and of separating the relative and center of mass motions. Kahn [1960] considered the scattering of Cartesian Landau eigenfunctions against a delta function potential through use of a Greens function in the scattering integral equation. Goldman [1963, 1964] and Goldman and Oster [1963] considered the influence of

Coulomb interactions in the calculation of cyclotron radiation line profiles. There was found one classical approach to the scattering problem, that of Baranankov [1960].

## II. THE CYLINDRICAL LANDAU EIGENFUNCTIONS

### Properties

The cylindrical Landau eigenfunctions  $\Psi_{NMk}$ , solutions of the Schroedinger equation (1), are factorable in each of the coordinates as

$$\Psi_{NMk}(\rho, \phi, z) \equiv R_{NM}(\rho) \Phi_M(\phi) Z_k(z). \quad (9)$$

As derived in appendix C, the factored eigenfunctions have the forms

$$R_{NM} = \sqrt{2 \beta^2 \frac{N!}{(N+M)!}} (\beta^2 \rho^2)^{\frac{M}{2}} e^{-\frac{1}{2} \beta^2 \rho^2} L_N^M(\beta^2 \rho^2) \quad (10)$$

$$\Phi_M = \frac{1}{\sqrt{2\pi}} e^{\pm iM\phi} \quad (11)$$

$$Z_k = \frac{1}{\sqrt{2\pi}} e^{\pm ikz} \quad k = \frac{p_z}{\hbar} = \frac{m v_z}{\hbar} \quad (12)$$

where  $\beta^2 \equiv eB/2\hbar$  (dimensions of  $m^{-2}$ ), and N and M are independent positive integers (including zero) having no formal upper bound.

Johnson and Lippmann [1949] have identified the reciprocal of  $\beta^2$  (or more precisely  $1/2\beta^2$ ) as the minimum area in the x-y plane to which a gyrocenter may be located. It is the minimum area occupied by a single state. We have the numerical relation

$$\beta^2 = [7.6 \times 10^{14} \text{ B}] \text{ in } m^{-2} \text{ for } \text{B in } W/m^2. \quad (13)$$

The Laguerre polynomial, an oscillatory function having N zeroes, has the explicit series representation

$$L_N^M(x) \equiv \sum_{j=0}^N \frac{(N+M)!}{(N-j)!(M+j)!} \frac{(-x)^j}{j!} \quad (14)$$

Other equivalent representations are given in equations (C23).

The factored eigenfunctions are separately normalized to Kronecker and Dirac delta functions such that

$$\langle N'M'k' | NMk \rangle \equiv \int_{\tau} \Psi_{N'M'k'}^* \Psi_{NMk} d\tau = \delta_{N',N} \delta_{M',M} \delta(k'-k) \quad (15)$$

$$\sum_{N'} \sum_{M'} \int_{k'} \langle N'M'k' | NMk \rangle dk' = 1. \quad (16)$$

From these equations there follows the interpretation that the quantity

$$\Psi_{NMk}^*(\vec{r}) \Psi_{NMk}(\vec{r}) d\tau dk \quad (17)$$

represents the probability (a pure number on a scale of unity) of locating the charge q in the volume element  $d\tau$  at  $\vec{r} = (\rho, \phi, z)$  and in the quantum state characterized by the numbers N and M and the continuous wavenumber k in the range dk. The appearance of these eigenfunctions is illustrated in Figures C1 and C2 on pages 53 and 54. These or closely related eigenfunctions have been employed by Dingle [1952], Tannenwald [1959], Goldman [1963, 1964], and Goldman and Oster [1963]. Their relation to the Cartesian Landau eigenfunctions was considered by Johnson and Lippmann [1949].

### Interpretation of the Eigenparameters (NMk)

Interpretation of the parameters (NMk) of the cylindrical Landau eigenfunctions follows from construction of the appropriate quantum operators and eigenvalue equations or from quantum-classical correspondence arguments. The former is of course the fundamentally correct

method; results obtained by the correspondence method should be verified by construction of the operator eigenvalue equations. The details and results of these procedures are to be found in section C3 (page 55). Although of vital importance to the understanding of what follows, a lucid exposition requires more space than is available here. Because of their importance, it is suggested that the interested reader pause in this development and consult the ten or so pages of C3. In particular, one should be aware of the role of the  $\pm$  signs of the  $\phi_M$  eigenfunctions, understand the distinction between the group I and group II states, and have examined the results summarized in Tables C1 and C2 and in Fig. C3.

### III. THE COULOMB MATRIX ELEMENT

The Coulomb matrix element is denoted and defined as

$$\langle q\Lambda_c \rangle \equiv \int_{\tau} \psi_{N_2 M_2 k_2}^* q\Lambda_c \psi_{N_1 M_1 k_1} d\tau. \quad (18)$$

It was also denoted in (6) by  $(\psi_{\beta}, q\Lambda_c \psi_{\alpha})$ . The exact result obtained (section E2) for this integral was

$$\langle q\Lambda_c \rangle = \delta_{M_2, M_1} \cdot \frac{qQ}{4\pi\epsilon_0} \frac{1}{2\pi} \cdot \sqrt{\frac{N_1! N_2!}{(N_1+M)! (N_2+M)!}} \sum_{j=0}^{\min(N_1, N_2)} \binom{N_1+M}{N_1-j} \binom{N_2+M}{N_2-j} \frac{(M+j)! (M+2j)!}{j!} \Psi(1+M+2j, 1-(N_1+N_2-2j), \gamma_{\pm}) \quad (19)$$

where  $\gamma_{\pm}$  stands for the functions

$$\gamma_{\pm} = \frac{\mu^2}{4\beta^2} + N_{z1} \left[ 1 \pm \sqrt{1 + \frac{(N_1 - N_2)}{N_{z1}}} \right]^2 \quad (20)$$

in which conservation of energy has been incorporated and  $\Psi(1+m, 1-n, x)$  denotes a confluent hypergeometric function, the properties of which are discussed in section E3. The parameter  $N_z \equiv E_z / \hbar \omega_c$

is the z energy measured in units of  $h \omega_c$ . The  $\pm$  signs are associated with forward or back scattering transitions in which the direction of the z momentum is either the same before and after the interaction or is reversed. In what follows, the shielding parameter  $\mu$  is set to zero. The effect of a  $\mu > 0$  is to depress the values of both forward and back scattering matrix elements. The properties of the confluent hypergeometric functions are such that the matrix element for forward scattering transitions is always greater in value than that of the back scattering matrix element. The matrix element (19) applies to both group I and group II states even though the signs of  $\Phi_M$  do not explicitly appear. They are contained implicitly in the interpretation which must be supplied to the quantum integers N and M. The steps leading to the appearance of the Kronecker delta (expressing conservation of the angular momentum canonical to the coordinate  $\phi$ , in agreement with the classical equations) indicate that transitions of the type group I  $\rightleftharpoons$  group II are explicitly forbidden. Only intragroup transitions are allowed, and only with M conserved. Energy conservation appears in connection with the cross section. The matrix element is symmetric as regards transitions between any pair of states. It was not possible to determine analytically if the matrix element exhibited a preference for equal upward (increase in N) or downward transitions from a given state. These and other properties of this matrix element are explored in sections E4 and E5. The arguments  $\gamma_{\pm}$  are drawn in Fig. E2. The general appearance of the matrix element is sketched in Fig. E3.

The value of an exact result lies not only with the result itself, but also with the fact that it provides a known reference or base from which to make approximations. Because of the analytical complexity of this general result, we shall utilize in the cross section discussions a simplified form in which is embodied the major contribution of the result (19). The simplification proceeds from the fact that, for  $n \gg 1$  and any  $m \gg 0$ , the confluent hypergeometric function  $\Psi(1+m, 1-n, x)$

approaches a value independent of  $x$  as  $x \rightarrow 0$ . The form is in essence the constant term of a power series expansion in  $B v^2 / E_{z1}$  about the origin of the confluent hypergeometric functions of (19). The procedure is described in section E6. The result is

$$\langle \mathcal{L}_c \rangle_0 / \left( \frac{qQ}{4\pi\epsilon_0} \frac{1}{2\pi} \right) = \frac{1}{v} \sqrt{\frac{(N+M)! (N+v)!}{N! (N+v+M)!}} \quad (21)$$

where  $v \equiv |N_1 - N_2|$  and  $N \equiv \min(N_1, N_2)$ . Since the minimum values of  $v$ ,  $N$ , and  $M$  are respectively 1, 0, and 0, this indicates that the maximum value of the matrix element (19) is about  $10^{-10}$  eV - m. (See equation 5.) The result is valid for  $v^2 / N_{z1} \ll 1$ , the numerical value of which is

$$\frac{v^2}{N_{z1}} = \frac{\hbar \omega_c v^2}{E_{z1}} = \left[ 1.2 \times 10^{-4} \frac{B v^2}{E_{z1}} \right] \quad (22)$$

for  $B$  in  $w/m^2$  and  $E_{z1}$  in eV. This is the principal forward scattering contribution to the matrix element in a region where the back scattering contribution is certainly smaller and may be negligible. As discussed on page 110 in connection with the weaker requirement  $v / N_{z1} \ll 1$ , this inequality places no restriction on the size of  $N_1$  relative to  $N_{z1}$  ( i. e., the partitioning of the total energy into perpendicular and parallel modes), but rather is a restriction on the change of the quantum integer  $N$  compared to  $N_{z1}$ . The milder inequality is equivalent to the requirement that the relative change in  $z$  energy be small, that  $|N_{z2} - N_{z1}| / N_{z1} \ll 1$ . We thus have in (21) a result which describes small angle changes in the momentum vector not only for distant encounters but apparently also for the closest possible encounters (the case  $M=0$ , any  $N$  and  $v$ , is interpreted pictorially or classically as the case where the cyclotron orbit passes through the  $z$  axis, upon which is located the Coulomb center).

#### IV. BORN APPROXIMATION CROSS SECTION

A cross section is a conversion ratio measuring the efficacy of some agent (here, the Coulomb potential) in transferring the particles or states of an incident beam to some other accessible conditions or states. It may be defined operationally as the number per second  $w$  of events (particles, states, or groups of states) arriving at a detector of appropriate configuration, normalized by the product of the incident flux  $\Gamma_z$  and the total number of agents  $N_{sc}$  within the scattering volume:

$$\sigma \equiv \frac{w}{N_{sc} \Gamma_z} . \quad (23)$$

When the scattering agents operate independently of one another, the number  $N_{sc}$  incorporates and corrects for the additive effect of each independent scatterer upon the detector signal (proportional in some way to  $w$ ). In theoretical calculations describing single scattering,  $N_{sc}$  is set to unity, as it is here. By the subscript  $z$  we have implied that the predominant direction of the incident flux is along the  $z$  axis, which is in this problem the direction of the magnetic field.

Considered here is the cross section for Coulomb scattering of an initial cylindrical Landau eigenstate  $\alpha = (N_1, Mk_1)$  to a final eigenstate  $\beta = (N_2, Mk_2)$ . Of interest is the dependence of the cross section upon the initial energies  $E_{\perp 1}$  and  $E_{z1}$ , and upon the initial location of the gyrocenter (in the  $\varrho$ - $\phi$  plane) with respect to the  $z$  axis upon which is located the scattering charge. Also of interest is the most probable change in the perpendicular and parallel energies and the most probable radial gyrocenter displacement. We shall employ the simplified form (21) for the matrix element needed in (6). The numerical values of (22) indicate that the use of the simplified matrix element does not severely restrict the validity of the final results.

Substitution of the simplified form (21) into (6) yields

$$\sigma_{\alpha\beta} = \sigma_0 \frac{M+2N+1}{\nu^2} \frac{(N+M)! (N+\nu)!}{N! (N+\nu+M)!} \quad \begin{array}{c} \uparrow \\ \downarrow \\ N \end{array} \quad (24)$$

where the order-of-magnitude coefficient  $\sigma_0$  is

$$\sigma_0 \equiv 2\pi^2 \frac{\left(\frac{e^2}{4\pi\epsilon_0} \frac{1}{2\pi}\right)^2}{\hbar\omega_c E_{z1}} = \left[ \frac{9.0 \times 10^{-15}}{B E_{z1}} \right] \text{ in } m^2. \quad (25)$$

The numerical expression bears units of (meters)<sup>2</sup> for B in w/m<sup>2</sup> and E<sub>z1</sub> in eV. The explicit B dependence originated in expression for the area-averaged flux of the initial eigenstate, while the factor 1/E<sub>z1</sub> was contributed by the initial and final one-dimensional density of z states functions.

The expression (24) is in fact four cross sections since our notation encompasses upward and downward transitions for group I and group II eigenstates. An upward transition is one in which the quantized variables of the perpendicular motion are increased (by  $\nu$ ). The principal perpendicular variables of interest are the squared gyrocenter distance  $\beta^2 \rho_0^2$  and the squared cyclotron radius  $\beta^2 \rho_c^2$  (equal to the normalized perpendicular energy  $E_{\perp}/\hbar\omega_c$ ). The content of the cross sections (24) is more easily understood when they are rewritten in terms of integers directly representing the perpendicular variables:

$$S \equiv \beta^2 \rho_0^2 - \frac{1}{2} = 0, 1, 2, \dots \quad (26)$$

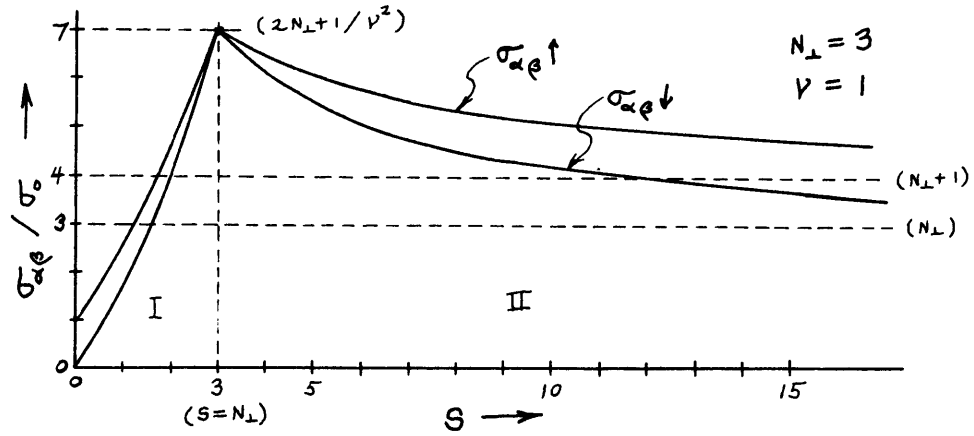
$$N_{\perp} \equiv \beta^2 \rho_c^2 - \frac{1}{2} \equiv \frac{E_{\perp}}{\hbar\omega_c} - \frac{1}{2} = 0, 1, 2, \dots \quad (27)$$

The transformation is accomplished with the aid of the interpretations summarized in Table C1 on page 60. The resulting expressions are listed below. We note that S and N<sub>⊥</sub> refer to initial values, and that  $\nu$  gives the change in these quantities as well as in the energy E<sub>z1</sub>.



Direction of transition	$\sigma_{\alpha e} / \sigma_0$	
	Group I states $e_0^2/e_c^2 \leq 1$ or $S \leq N_{\perp}$	Group II states $e_0^2/e_c^2 \geq 1$ or $S \geq N_{\perp}$
$\begin{array}{c} N_{\perp} + \nu \text{ --- } \uparrow \text{ --- } S + \nu \\   \\ N_{\perp} \text{ --- } \text{ --- } S \end{array}$	$\frac{N_{\perp} + S + 1}{\nu^2} \frac{N_{\perp}! (S + \nu)!}{(N_{\perp} + \nu)! S!}$	$\frac{N_{\perp} + S + 1}{\nu^2} \frac{S! (N_{\perp} + \nu)!}{(S + \nu)! N_{\perp}!}$
$\begin{array}{c} N_{\perp} \text{ --- } \text{ --- } S \\   \\ N_{\perp} - \nu \text{ --- } \downarrow \text{ --- } S - \nu \end{array}$	$\frac{N_{\perp} + S + 1}{\nu^2} \frac{(N_{\perp} - \nu)! S!}{N_{\perp} (S - \nu)!}$	$\frac{N_{\perp} + S + 1}{\nu^2} \frac{(S - \nu)! N_{\perp}!}{S! (N_{\perp} - \nu)!}$

For fixed  $N_{\perp}$  and  $\nu$ , both the upward and downward cross sections exhibit qualitatively similar behavior with respect to  $S$ . The most significant quantitative difference is that  $\sigma_{\alpha e} \uparrow$  is always greater than  $\sigma_{\alpha e} \downarrow$  except at the value  $S = N_{\perp}$ . The tendency is thus toward outward radial motion with a concomitant increase in the cyclotron radius. As  $S$  increases from zero (gyrocircle of squared radius  $N_{\perp}$  centered about the origin, the location of the scattering charge), both cross sections rise from a minimal value (zero for  $\sigma_{\alpha e} \downarrow$  and  $\leq \sigma_0$  for  $\sigma_{\alpha e} \uparrow$ ) at the origin to the same maximum value  $(2N_{\perp} + 1)\sigma_0/\nu^2$  at the group I-group II boundary point  $S = N_{\perp}$ . This is the single point at which  $\sigma_{\alpha e} \uparrow = \sigma_{\alpha e} \downarrow$ . For all other values of  $S$ ,  $\sigma_{\alpha e} \uparrow$  is invariably greater than  $\sigma_{\alpha e} \downarrow$ . The classical picture associated with the point  $S = N_{\perp}$  is that of the set of cyclotron orbits of squared radius  $N_{\perp}$  whose gyrocenters are situated on the circle of squared radius  $S = N_{\perp}$ . That is,  $S = N_{\perp}$  describes the set of orbits which intersect the  $z$  axis upon which is located the scattering charge. The appearance of a maximum at this value of the impact parameter  $S$  is thus physically reasonable. As  $S$  increases beyond the group I-group II boundary point, both  $\sigma_{\alpha e} \uparrow$  and  $\sigma_{\alpha e} \downarrow$  fall to zero as  $S^{-\nu}$  except for the case  $\nu = 1$ . For  $\nu = 1$  the cross sections do not vanish as  $S \rightarrow \infty$ , but instead approach the limiting values  $(N_{\perp} + 1)\sigma_0$  and  $N_{\perp}\sigma_0$ . In illustration of these features, we have sketched on the next page the variation with  $S$  of the  $\nu = 1$  cross sections for  $N_{\perp} = 3$ .



One important result not yet commented upon is that the minimum change  $\nu = 1$  is the most probable, no matter what the values of  $N_{\perp}$  and  $S$ . That is, in classical terms, minimal changes in pitch angle and gyrocenter location are most likely. This is attributed to the strong binding of the scattered charge by the magnetic field. That the  $\nu = 1$  cross sections do not vanish as  $S \rightarrow \infty$  is attributed to the long range character of the Coulomb potential and to the fact that some discrete change must always occur in the quantized perpendicular variables. The quantum integer  $M$  is conserved, and there can be no smooth transition from the minimum change  $\nu = 1$  to  $\nu = 0$ , the case of no change in the quantum integer  $N$  (and through energy conservation, the case of no change in the  $z$  energy).

Although of great conceptual interest, the cross sections  $\sigma_{\alpha\beta\uparrow\downarrow}$  are of little experimental interest (even in the case of massive ions) since the gyrocenter location  $S$  is not under experimental control. As an initial approach to the calculation of a quantity comparable with experiment we should consider the scattering of a beam consisting of a uniform distribution of gyrocenters out to some squared distance  $S = S_{\max}$ . As described in section C4, the beam is further characterized by single values of the perpendicular and parallel energies. The cross section derivation of section G3 must be modified to reflect the different total energy of such a beam. It is only through

the use of such flooding beams that one can arrive at differential and total cross sections which admit of comparison with the familiar  $B=0$ , Rutherford differential and total cross sections. Such a comparison would be made by suppressing the initial perpendicular energy and utilizing the expression which relates the velocity vector pitch angle after the interaction to the change  $\nu$ ,

$$\sin^2 \alpha = \nu \hbar \omega_c / E_{\perp 1} . \quad (28)$$

We are presumably on good grounds for making such comparisons, particularly and most significantly as  $B \rightarrow 0$ , when the simplified matrix element (21) may be used with increasing accuracy. One should also recognize that in summing over groups of states upon the surface of constant total energy (see Fig. C3, page 64), one encounters an additional magnetic field dependence. Although this dependence may in fact be so weak as to be negligible, it originates in the summation limits which define the extent of this surface.

## V. FUTURE WORK

Now that the above results are at hand, and with experimentally more meaningful results near at hand, the single most important question to be answered is, When must the present magnetic field scattering formalism be used in preference to the  $B=0$ , Rutherford scattering formalism? At what magnetic field strength must the  $B=0$ , Rutherford formalism be abandoned? Other questions of theoretical and experimental interest are the relation of these results to the laboratory frame (see the discussion of section G4) and an assessment of the role and effects of bound states. With further clarification of these and other theoretical results, the design of experiments could be undertaken.

PART II  
SUPPORTING APPENDICES

APPENDIX A  
CLASSICAL MECHANICS OF A CHARGED PARTICLE  
IN COULOMB AND MAGNETIC FIELDS

A1 Introduction

Considered in this appendix are the classical mechanics of a non-relativistic particle of charge  $q$  and mass  $m$  in Coulomb and magnetic fields. The equations of motion are constructed by means of the Lagrangian - Hamiltonian formalism in both stationary and rotating cylindrical coordinate systems.

The problem is simplified by considering the seat of the Coulomb potential (the charge  $Q$ ) to be at rest in the reference frame of the charge  $q$ . The transformation from this rest frame to the laboratory frame is considered in connection with the analogous quantum treatment of a later appendix.

The problem is complicated by our interest in the domain where the Larmor theorem cannot legitimately be applied to reduce the problem to the zero magnetic field case. This domain is reached when the Hamiltonian terms quadratic in the product of the magnetic field and the radial distance may not be ignored. Because of this, the effects of the magnetic field cannot in general be removed by rotation of the coordinate system about the direction of the uniform and constant magnetic field.

Since we do retain the terms quadratic in the magnetic field and radial distance, the classical formalism developed should be applicable to the quantum description of highly excited (large angular momentum) bound hydrogenic states in a magnetic field or to unbound states of the charge  $q$  which are perturbed or scattered by the Coulomb potential. This in fact is the main purpose of this appendix -

to serve as an introduction to and foundation for the later quantum treatment of the scattering problem. As we shall see, the interpretation of the numbers and parameters arising in the quantum approach leans heavily upon classical quantities and concepts. Indeed, the starting point of the quantum formalism is the classical Hamiltonian. Further, a coherent and connected treatment displays the often-subtle relationships among the many types of momenta which abound in a system containing a magnetic field.

We proceed from the system Lagrangian which we regard as fundamental and Goldstein-given. From the Lagrangian are derived the various momenta and the system Hamiltonian. The cylindrical coordinate system force equations are found to be non-linear and coupled in at least two dimensions. For the charge  $Q$  located at the origin, rotation of the coordinate system about the magnetic field at a constant, arbitrary velocity leaves the equations of motion invariant.

Generalized coordinates and coordinate systems other than cylindrical were not investigated. Neither were serious attempts made to obtain general solutions of the cylindrical equations. This was due in part to the availability and increased utility of the (guaranteed linear) quantum approach.

## A2 The Lagrangian Formalism

### The Non-relativistic Lagrangian

The non-relativistic Lagrangian (considered to be a function of the generalized coordinates  $x_i$  and their time derivatives  $\dot{x}_i$ ) for a particle of mass  $m$  and charge  $q$  in the magnetic vector potential  $\vec{A}$  and the scalar potential  $\Lambda$  is

$$\mathcal{L} = \frac{1}{2} m v^2 - q\Lambda + q\vec{A} \cdot \vec{v}. \quad (\text{A1})$$

The potentials  $\Lambda$  and  $\vec{A}$  are considered to depend only upon the

coordinates  $x_i$ , and the velocity  $v$  upon both  $x_i$  and  $\dot{x}_i$ . Only in the Cartesian system, in which all coordinate surfaces are planes and all coordinates have the same dimensional footing, are the velocity components given by  $\dot{x}_i$  alone. The expression (A1) follows from expansion (in powers of  $v^2/c^2$ ) of the radical in the relativistic single particle Lagrangian [Goldstein, 1950, p. 207]

$$\mathcal{L} = -mc^2 \sqrt{1 - \frac{v^2}{c^2}} - q\Lambda + q\vec{A} \cdot \vec{v} \quad (\text{A2})$$

with subsequent omission of the rest energy term  $mc^2$ .

#### The Relation of Canonical and Linear Momenta

Suppose now, for the moment only, that the Lagrangian (A1) is expressed in terms of Cartesian coordinates. That is, the generalized coordinates  $x_i$  are chosen to be  $(x, y, z)$ . Then, from the definition of the momentum canonically conjugate to the generalized coordinate  $x_i$ ,

$$p_i \equiv \frac{\partial \mathcal{L}}{\partial \dot{x}_i}, \quad (\text{A3})$$

there follows the oft-quoted vector relation

$$\vec{p} = m\vec{v} + q\vec{A}. \quad (\text{A4})$$

It is important to note that, even though this relation holds for any coordinate system, the momentum components as given by (A4) may be called canonical momenta only for those coordinates satisfying (A3). For coordinates not satisfying (A3), the relation (A4) must be relegated to the lesser role of defining the linear momenta associated with these coordinates.

The relation (A4) is often used in vector proofs and arguments as though it did in fact represent the momentum components canonical

to every possible choice of generalized coordinates. The results of such vector proofs and arguments are valid as long as they remain in vector form or are expressed in terms of Cartesian coordinates. However, when cast in terms of other-than-Cartesian coordinates, the results may appear to be perplexingly inconsistent with the Cartesian expressions. At the root of this inconsistency is the failure to observe the distinction between (A3) and the components of (A4). When casting the vector results in terms of other-than-Cartesian generalized coordinates, this pitfall may be avoided by expressing all non-canonical momentum components in terms of momenta which are canonical to the generalized coordinates.

#### Application to Cylindrical Coordinates

The foregoing distinctions are well illustrated in the familiar cylindrical coordinate system spanned by the unit vectors  $\hat{\rho} \times \hat{\phi} = \hat{z}$ . The generalized coordinates are chosen as the set  $(\rho, \phi, z)$ . We shall employ these coordinates in the majority of our calculations. The components of velocity and acceleration are

$$\vec{v} = \dot{\rho} \hat{\rho} + \rho \dot{\phi} \hat{\phi} + \dot{z} \hat{z} \quad (\text{A5})$$

$$\vec{a} = (\ddot{\rho} - \rho \dot{\phi}^2) \hat{\rho} + (2\dot{\rho} \dot{\phi} + \rho \ddot{\phi}) \hat{\phi} + \ddot{z} \hat{z} . \quad (\text{A6})$$

Here we see that  $v_{\phi}$  is not equal to  $\dot{\phi}$  but to  $\rho \dot{\phi}$ . We also introduce at this time the specific potentials of interest, the magnetic vector potential

$$\vec{A} = \frac{1}{2} \vec{B} \times \vec{r} \quad (\text{A7a})$$

$$= \frac{1}{2} B \rho \hat{\phi} \quad (\text{A7b})$$

and the scalar Coulomb potential,



$$\Lambda = \frac{Q}{4\pi\epsilon_0 |\vec{r} - \vec{r}_e|} \quad (\text{A8a})$$

$$= \frac{Q}{4\pi\epsilon_0} \left[ \rho^2 + \rho_e^2 - 2\rho\rho_e \cos(\phi - \phi_e) + (z - z_e)^2 \right]^{-\frac{1}{2}} \quad (\text{A8b})$$

The expressions for  $\vec{A}$  describe a constant and uniform magnetic field through the relation  $\vec{B} = \text{curl } \vec{A}$ . The particular form (A7b) describes the field  $\vec{B} = B \hat{z}$ . The expressions (A8) describe the potential field at the point  $\vec{r} = (\rho, \phi, z)$  due to the charge  $Q$  located at  $\vec{r}_e = (\rho_e, \phi_e, z_e)$ . With these potentials, the Lagrangian (A1) becomes

$$\mathcal{L} = \frac{m}{2} (\dot{\rho}^2 + \rho^2 \dot{\phi}^2 + \dot{z}^2) - \frac{qQ}{4\pi\epsilon_0} \left[ \rho^2 + \rho_e^2 - 2\rho\rho_e \cos(\phi - \phi_e) + (z - z_e)^2 \right]^{-\frac{1}{2}} \quad (\text{A9})$$

It represents the system of a particle of mass  $m$  and charge  $q$  located at  $\vec{r} = (\rho, \phi, z)$  moving in the magnetic field  $\vec{B} = B \hat{z}$  and in the Coulomb field of a particle of charge  $Q$  fixed at the point  $\vec{r}_e = (\rho_e, \phi_e, z_e)$ . That is, the charge  $Q$  is always at rest in the reference frame of the charge  $q$ .

The canonical momenta are generated from the definition (A3):

$$x_1 = \rho \quad p_1 = m \dot{\rho} \quad (\text{A10a})$$

$$x_2 = \phi \quad p_2 = m \rho^2 \dot{\phi} + \frac{1}{2} q B \rho^2 = L_z \quad (\text{A10b})$$

$$x_3 = z \quad p_3 = m \dot{z} \quad (\text{A10c})$$

Application of the vector relation (A4) yields the triad of linear momentum components:

$$\vec{p} = \begin{Bmatrix} p_\rho \\ p_\phi \\ p_z \end{Bmatrix} = \begin{Bmatrix} m \dot{\rho} \\ m \rho \dot{\phi} + \frac{1}{2} g B \rho \\ m \dot{z} \end{Bmatrix} \quad \begin{array}{l} \text{(A11a)} \\ \text{(A11b)} \\ \text{(A11c)} \end{array}$$

We see that the canonical components  $p_1$  and  $p_3$  are the same as the linear components  $p_\rho$  and  $p_z$ , respectively. The canonical momentum  $p_2$  is an angular momentum which we have identified as the component  $L_z$  of the system angular momentum  $\vec{L}$ . This identification follows from the definition of  $\vec{L}$  in terms of the linear momentum  $\vec{p}$ :

$$\vec{L} \equiv \vec{r} \times \vec{p} \quad \text{(A12)}$$

$$\vec{L} = \begin{Bmatrix} L_\rho \\ L_\phi \\ L_z \end{Bmatrix} = \begin{Bmatrix} -\rho p_\phi z \\ z p_\rho - \rho p_z \\ \rho p_\phi \end{Bmatrix} = \begin{Bmatrix} -(m \rho \dot{\phi} + \frac{1}{2} g B \rho) z \\ m (z \dot{\rho} - \rho \dot{z}) \\ m \rho^2 \dot{\phi} + \frac{1}{2} g B \rho^2 \end{Bmatrix} \quad \begin{array}{l} \text{(A13a)} \\ \text{(A13b)} \\ \text{(A13c)} \end{array}$$

It is to be noted that none of the foregoing momentum relations explicitly reflects the presence of the charge  $Q$ . They would be formally the same for  $Q = 0$ . They do explicitly reflect the presence of the magnetic field.

Equations of Motion

The equations of motion follow upon application of

$$\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{x}_i} \right) - \frac{\partial \mathcal{L}}{\partial x_i} = 0. \quad (\text{A14})$$

For the generalized coordinates ( $\rho$ ,  $\phi$ ,  $z$ ), these equations are

$$m \ddot{\rho} - \frac{qQ}{4\pi\epsilon_0} \frac{\rho - \rho_e \cos \phi}{[\rho^2 + \rho_e^2 - 2\rho\rho_e \cos \phi + z^2]^{3/2}} - qB\rho\dot{\phi} - m\rho\dot{\phi}^2 = 0 \quad (\text{A15})$$

$$\frac{d}{dt} (m\rho^2\dot{\phi} + \frac{1}{2}qB\rho^2) = \frac{qQ}{4\pi\epsilon_0} \frac{\rho\rho_e \sin \phi}{[\rho^2 + \rho_e^2 - 2\rho\rho_e \cos \phi + z^2]^{3/2}} \quad (\text{A16})$$

$$m\ddot{z} - \frac{qQ}{4\pi\epsilon_0} \frac{z}{[\rho^2 + \rho_e^2 - 2\rho\rho_e \cos \phi + z^2]^{3/2}} = 0 \quad (\text{A17})$$

where we have set  $z_e$  and  $\phi_e$  equal to zero. Thus the entire positional dependence of these equations upon the location of the charge  $Q$  is connected with  $\rho_e$ . It is believed that this placement causes no loss of generality which cannot be regained via the initial conditions on the parametric functions ( $\rho$ ,  $\phi$ ,  $z$ ). With the exception of (A16), these equations are identical to those obtained as components of Newton's second law,  $m\vec{a} = q\vec{E} + q\vec{v} \times \vec{B}$ . The equation of motion (A16) is the  $\phi$  component of the torque equation  $\vec{r} \times$  (Newton II). Were this component written out, we would see that the  $qB\rho^2/2$  term in the canonical angular momentum  $L_z$  is the time integral (more correctly, the time primitive) of the torque exerted by the magnetic field upon the charge  $q$ .

From the above equations it follows that the energy is a constant of the motion depending only implicitly upon the magnetic

field:

$$\frac{m}{2} (\dot{\rho}^2 + \rho^2 \dot{\phi}^2 + \dot{z}^2) + \frac{qQ}{4\pi\epsilon_0} [\rho^2 + \rho_e^2 - 2\rho\rho_e \cos\phi + z^2]^{-\frac{1}{2}} = E = \text{const.} \quad (\text{A18})$$

From the  $\phi$  equation of motion (A16) it is apparent that the angular momentum  $L_z$  is a constant of the motion only for  $\rho_e = 0$ . For this location of the charge  $Q$ , the equations may be written in the simpler forms

$$m\ddot{\rho} - \frac{qQ}{4\pi\epsilon_0} \frac{\rho}{(\rho^2 + z^2)^{3/2}} + \left(\frac{qB}{2m}\right)^2 m\rho - \frac{L_z^2}{m\rho^3} = 0 \quad (\text{A19})$$

$$m\ddot{z} - \frac{qQ}{4\pi\epsilon_0} \frac{z}{(\rho^2 + z^2)^{3/2}} = 0 \quad (\text{A20})$$

The condition  $\rho_e = 0$  has permitted the incorporation of (A16) into (A15). The form (A19) is valid for  $\rho_e \neq 0$ , but then  $L_z$  is not a constant of the motion. It appears also from (A16) as if  $L_z$  approaches constant-of-the-motion status as  $Q$  is moved to infinity, i. e., as  $\rho_e \rightarrow \infty$ . In this limit the entire problem approaches that for  $Q = 0$ .

### A3 The System Hamiltonian Construction of the Hamiltonian

The Hamiltonian, a function of the generalized coordinates and the canonical momenta, may be defined in terms of the Lagrangian, the canonical momenta, and the time derivatives of the coordinates:

$$H = \sum p_i \dot{x}_i - \mathcal{L}(x_i, \dot{x}_i) \quad (\text{A21})$$

For the generalized coordinates  $(\rho, \phi, z)$  and the Lagrangian (A9), application of (A21) yields successively the forms

$$H = \frac{m}{2} (\dot{\rho}^2 + \rho^2 \dot{\phi}^2 + \dot{z}^2) + \frac{qQ}{4\pi\epsilon_0} [\rho^2 + \rho_e^2 - 2\rho\rho_e \cos\phi + z^2]^{-\frac{1}{2}} \quad (\text{A22a})$$

$$= \frac{p_\rho^2 + \frac{L_z^2}{\rho^2} + p_z^2}{2m} - \frac{qB}{2m} L_z + \frac{q^2 B^2 \rho^2}{8m} + \frac{qQ}{4\pi\epsilon_0} [\rho^2 + \rho_e^2 - 2\rho\rho_e \cos\phi + z^2]^{-\frac{1}{2}} \quad (\text{A22b})$$

$$= \frac{p_\rho^2 + \left(\frac{L_z}{\rho} - \frac{qB\rho}{2}\right)^2 + p_z^2}{2m} + \frac{qQ}{4\pi\epsilon_0} \frac{1}{\sqrt{\rho^2 + \rho_e^2 - 2\rho\rho_e \cos\phi + z^2}} \quad (\text{A22c})$$

As before, we have set  $z_e$  and  $\phi_e$  equal to zero. We have also replaced in the form c the canonical  $p_1$  and  $p_3$  by the equivalent  $p_\rho$  and  $p_z$  in order to capitalize on their greater mnemonic value. Expressions b and c are the formally correct ones, as they are expressed in terms of the canonical momenta and coordinates. Expression (A22a) states that the Hamiltonian is the sum of the particle kinetic and Coulomb potential energies, which sum we have earlier called E.

Equivalently, one may proceed from the commonly-encountered expression

$$H = \frac{(\vec{p} - q\vec{A})^2}{2m} + q\mathcal{A} \quad (\text{A23})$$

so long as the components of the linear momentum  $p$  are eventually expressed in terms of the canonical momenta  $p_i$ . Equation (A22c) is seen to be of this form since  $p_\phi = (L_z / \rho)$ .

### Equations of Motion

The Hamiltonian equations of motion follow from the pair

$$\dot{x}_i = \frac{\partial H}{\partial p_i} \quad (\text{A24})$$

$$-\dot{p}_i = \frac{\partial H}{\partial x_i} \quad . \quad (\text{A25})$$

The first of these, applied to the Hamiltonian (A22c) for  $x_i = (\rho, \phi, z)$  and  $p_i = (p_\rho, L_z, p_z)$  yields relations identical to (A11a), (A13c), and (A11c). The second leads to the following set of equations:

$$-\dot{p}_\rho = \frac{-1}{m\rho} \left( \frac{L_z}{\rho} - \frac{\partial B\rho}{\partial z} \right) \left( \frac{L_z}{\rho} + \frac{\partial B\rho}{\partial z} \right) + \frac{\partial Q}{4\pi\epsilon_0} \frac{\partial}{\partial \rho} \frac{1}{\sqrt{\rho^2 + \rho_e^2 - 2\rho\rho_e \cos\phi + z^2}} \quad (\text{A26})$$

$$-\dot{L}_z = \frac{\partial Q}{4\pi\epsilon_0} \frac{\partial}{\partial \phi} \frac{1}{\sqrt{\rho^2 + \rho_e^2 - 2\rho\rho_e \cos\phi + z^2}} \quad (\text{A27})$$

$$-\dot{p}_z = \frac{\partial Q}{4\pi\epsilon_0} \frac{\partial}{\partial z} \frac{1}{\sqrt{\rho^2 + \rho_e^2 - 2\rho\rho_e \cos\phi + z^2}} \quad (\text{A28})$$

These equations are equivalent to the set (A15 through (A17).

#### A4 Numerical Estimates of Hamiltonian Energies and Other Quantities of Interest

In Table A1 are collected numerical estimates of quantities pertinent to the motion of an electron ( $q = -e$ ) in magnetic and Coulomb fields. The distances, angular momenta, and energies considered are expressed in units of meters<sup>2</sup>,  $\hbar$ , and electron-volts,

respectively. The estimates are given for two extreme cases.

The first is that of  $q$  so strongly bound to the charge  $Q$  that the magnetic field terms are negligible. Estimates for this case are derived from the Bohr picture of the hydrogen atom. The entries in the H atom column of this table were calculated from the value of  $a_0$  the lowest Bohr orbit radius and the value of the Coulomb potential energy at the distance  $a_0$ . The first values to be calculated were  $\rho^2$  and  $\phi$  from which all others followed. The value of  $\rho^2$  was obtained by equating  $\sqrt{\rho^2 + z^2}$  to  $a_0$  and setting  $z^2 = \rho^2$ . The value of  $\phi$  then followed from the energy  $\frac{1}{2}m\dot{\rho}^2$ . This total average kinetic energy was taken as 13.7 eV on the basis of the virial theorem for the central Coulomb potential.

The opposite extreme is that of a free electron in a magnetic field. For this case  $Q$  is set to zero, and the standard relations for cyclotron motion are utilized. In these relations, the cyclotron radius  $\rho_c$  is normalized by the parameter  $\beta$  defined as

$$\beta^2 \equiv \frac{eB}{2\hbar} = [7.60 \times 10^{14} B] \quad \text{in } \text{m}^{-2} \text{ for } B \text{ in } \text{W/m}^2. \quad (\text{A29})$$

The primary significance of this important quantum parameter is that its reciprocal represents the minimum area in the x-y plane to which a gyrocenter may be located by any measurement [Johnson and Lippmann, 1949]. Thus normalized, we have the relation

$$\beta^2 \rho_c^2 = \left[ \frac{8.63 \times 10^3 E_{\perp}}{B} \right] \quad (\text{A30})$$

for the energy  $E_{\perp}$  in eV and  $B$  in weber/m<sup>2</sup>. Alternatively, if  $E_{\perp}$  is replaced by  $kT$ , this relation becomes

$$\beta^2 \rho_c^2 = \left[ 0.743 \frac{T}{B} \right] \quad (\text{A31})$$

TABLE A1. Numerical estimates of Hamiltonian energies and other quantities of interest, and comparison for the extreme cases of a bound and an unbound electron in a uniform magnetic field.

Quantity or parameter	H atom in a magnetic field	Free electron in a magnetic field
Distances $\sqrt{\rho^2 + z^2}$ $\rho^2$ $1/\beta^2$	$a_0 = 5.3 \times 10^{-11} \text{ m}$ $\frac{1}{2} a_0^2 = 1.4 \times 10^{-21} \text{ m}^2$	any value $\rho_c^2 \equiv \frac{2mE}{e^2 B^2} = \left[ \frac{1.14 \times 10^{-11} E_{\perp}}{B^2} \right] \text{ m}^2$ $1/\beta^2 \equiv \frac{2\hbar}{eB} = \left[ \frac{1.32 \times 10^{-15}}{B} \right] \text{ m}^2$
Angular velocity $\dot{\phi}$	$3.4 \times 10^{16} \frac{\text{rad}}{\text{sec}}$	$\omega_c \equiv \frac{eB}{m} = [1.76 \times 10^{11} \text{ B}] \frac{\text{rad}}{\text{sec}}$
Angular momenta $m\rho_c^2 \dot{\phi}$ $\frac{1}{2} qB\rho_c^2$ $L_z$	$0.41 \hbar$ $- [1.1 \times 10^{-5} \text{ B}] \hbar$ $m\rho_c^2 \dot{\phi} + \frac{1}{2} qB\rho_c^2$	$m\rho_c^2 \omega_c = eB\rho_c^2 = 2\beta^2 \rho_c^2 \hbar$ $-\frac{1}{2} eB\rho_c^2$ $\frac{1}{2} eB\rho_c^2 = \beta^2 \rho_c^2 \hbar = \left[ \frac{8.63 \times 10^3 E_{\perp}}{B} \right] \hbar$
Energies $\frac{qQ}{4\pi\epsilon_0 r}$ $\frac{1}{2} m\rho_c^2$ $\frac{1}{2} m\rho_c^2 \dot{\phi}^2$ $E$ $\frac{L_z^2}{2m\rho_c^2}$ $-\frac{qB}{2m} L_z$ $\frac{q^2 B^2 \rho_c^2}{8m}$ $\hbar\omega_c$	$-27.2 \text{ ev}$ $4.53 \text{ ev}$ $4.53 \text{ ev}$ $\frac{1}{2} m (\dot{\rho}^2 + \rho^2 \dot{\phi}^2)$ $\sim 4.53 \text{ ev}$ $[2.4 \times 10^{-5} \text{ B}] \text{ ev}$ $[3.1 \times 10^{-11} \text{ B}^2] \text{ ev}$	$\frac{e^2}{4\pi\epsilon_0 r} = \left[ \frac{1.44 \times 10^{-9}}{r} \right] \text{ ev (r in m)}$ 0 for a centered orbit $\frac{1}{2} m\rho_c^2 \omega_c^2 = \beta^2 \rho_c^2 \hbar\omega_c$ $\beta^2 \rho_c^2 \hbar\omega_c$ $\frac{e^2 B^2 \rho_c^2}{8m} = \frac{1}{4} E_{\perp}$ $\frac{e^2 B^2 \rho_c^2}{4m} = \frac{1}{2} E_{\perp}$ $\frac{e^2 B^2 \rho_c^2}{8m} = \frac{1}{4} E_{\perp}$ $\hbar \frac{eB}{m} = [1.16 \times 10^{-4} \text{ B}] \text{ ev}$



for T in degrees Kelvin. Note that  $\beta^2 \rho_c^2$  is in general a large number compared to unity. For  $E = 0.1$  eV and  $B = 0.1$  w/m<sup>2</sup>, its value exceeds  $10^3$ . A second important parameter for the case of a free electron in a magnetic field is the cyclotron energy  $\hbar\omega_c \equiv \hbar(eB/m)$ . Its numerical value is

$$\hbar\omega_c = [ 1.16 \times 10^{-4} B ] \quad \text{in eV.} \quad (\text{A32})$$

We shall see that  $\hbar\omega_c$  is the spacing between the levels of the quantized perpendicular energy  $E_{\perp}$ . For  $B = 1$  w/m<sup>2</sup>, a respectable laboratory field, this level spacing is 0.116 milli-eV.

There are several features about this table which are interesting, or will become so in the light of later quantum calculations. We notice first of course that the cyclotron radius is in general much larger than the Bohr orbit radius. Of the two terms which comprise the canonical angular momentum  $L_z$ , the kinetic term for an atom is far larger than the field term  $qB\hbar^2/2$  due to the smallness of  $e^2$ . For a free electron, on the other hand, these terms are of comparable size. A further distinction is that  $L_z$  for the atom is of the order of units of  $\hbar$ , whereas  $L_z$  for the free electron can reasonably be of the order of thousands of  $\hbar$ . Likewise, in the case of the H atom, the energies associated with the magnetic field are negligible compared to the kinetic energies, whereas in the free electron case these energies are comparable. It must be emphasized that the entries are largely estimates and have at best order of magnitude validity. They are intended to encompass the extremes of the dynamical system represented by the Hamiltonian (A22).

## A5 The Effect of Coordinate System Rotation

### Rationale

In the preceding section were considered two limiting cases of the physical system described by the Hamiltonian (A22b). In the

first of these cases we saw that the energy term quadratic in the magnetic field could reasonably be ignored and still leave a term (the energy linear in B) at least partially descriptive of the effects of the magnetic field. This is possible for the charge q in quantum states and magnetic fields such that the energy term proportional to  $B^2 \rho^2$  may be neglected. The Hamiltonian for this limiting case is

$$H = \frac{p_\rho^2 + \frac{L_z^2}{\rho^2} + p_z^2}{2m} - \frac{qB}{2m} L_z + \frac{qQ}{4\pi\epsilon_0} \frac{1}{\sqrt{\rho^2 + \rho_0^2 - 2\rho\rho_0 \cos\phi + z^2}}. \quad (A33)$$

This Hamiltonian has been studied in connection with the Zeeman effect and the Larmor theorem. General solutions have been obtained both classically and quantum-mechanically. The other extreme case for which numerical values were given in Table A1 was for  $Q = 0$ , that is, the case of cyclotron motion of a free charge in a magnetic field. The Hamiltonian is

$$H = \frac{p_\rho^2 + \frac{L_z^2}{\rho^2} + p_z^2}{2m} - \frac{qB}{2m} L_z + \frac{q^2 B^2 \rho^2}{8m}. \quad (A34)$$

General solutions are of course also known for this dynamical system.

In both of these limiting cases, rotation of the coordinate system brings about considerable simplification in the equations of motion and the solutions. Hence it is natural to employ this technique in attempts at simplification of the equations of motion (A15) through (A17), which may be said to result from the more general Hamiltonian (A22).

### Lagrangian-Hamiltonian Formalism in Rotating Cylindrical Coordinates

We again utilize the Lagrangian-Hamiltonian formalism and

begin by constructing the Lagrangian  $\mathcal{L}^*$  from which may be generated the equations of motion referred to a frame rotating with angular velocity  $\vec{\Omega}$ . The prescription is

$$\mathcal{L}^* = \frac{m}{2} \left[ \dot{\vec{r}}(\vec{r}^*, \dot{\vec{r}}^*, \vec{\Omega}) \right]^2 + g \left[ \vec{A}(\vec{r}^*, \vec{\Omega}) \cdot \dot{\vec{r}}(\vec{r}^*, \dot{\vec{r}}^*, \vec{\Omega}) \right] - g \Lambda(\vec{r}^*, \vec{\Omega}). \quad (\text{A35})$$

Quantities referred to the rotating frame are starred; Newtonian reference frame quantities are unstarred. The relations necessary to carry out the prescription are given by Symon [1953, p. 240], among others:

$$\vec{r} = \vec{r}^* \quad (\text{A36})$$

$$\frac{d\vec{r}}{dt} = \frac{d^*\vec{r}^*}{dt} + \vec{\Omega} \times \vec{r}^* \quad (\text{A37})$$

$$\begin{aligned} \frac{d^2\vec{r}}{dt^2} = & \frac{d^{*2}\vec{r}^*}{dt^2} + \vec{\Omega} \times (\vec{\Omega} \times \vec{r}^*) + 2\vec{\Omega} \times \frac{d^*\vec{r}^*}{dt} \\ & + \frac{d\vec{\Omega}}{dt} \times \vec{r}^*. \end{aligned} \quad (\text{A38})$$

The first of these equations says that at any given point in time, the position vector as viewed from either system is fundamentally the same entity. That is, at a given point in time, both observers are considering (from the common origin) the same point in space. The remaining two equations relate the behavior of the position vector over intervals of time. They consequently contain terms describing the effects of frame rotation upon observations of the position vector time behavior. Thus the second of the three vector relations states that the Newtonian frame velocity may be resolved into the velocity

as measured by a rotating observer plus the velocity of the rotating frame itself (the Newtonian frame velocity of a point at rest in the rotating system). The Newtonian frame acceleration similarly may be resolved into components associated with the rotating frame. The first term on the RHS of (A38) is the total acceleration of the position vector as viewed from the rotating frame. The remaining terms give the Newtonian frame components of acceleration due, respectively, to frame rotation (the centripetal acceleration), to motion with respect to the rotating frame (the Coriolis acceleration), and to non-uniform frame rotation.

In what follows, the rotating frame is chosen to be a cylindrical coordinate system rotating about the direction of the magnetic field  $\vec{B} = B \hat{z}$  :

$$\vec{\Omega} = \epsilon \frac{qB}{m} \hat{z} . \quad (\text{A39})$$

The dimensionless parameter  $\epsilon$  measures the rotation angular velocity in units of  $qB/m$ , which for  $q = e$  becomes the cyclotron frequency  $\omega_c$  . We consider that  $\epsilon$  may vary with time, although, as we shall see, conservation of energy requires that  $\epsilon$  be constant. Through use of the foregoing prescription, relations, and choice of  $\vec{\Omega}$  , we write the Lagrangian for a particle of charge  $q$  and mass  $m$  instantaneously located at  $\vec{r} = (r, \phi, z)$  moving in a uniform magnetic field  $\vec{B} = B\hat{z}$  and in the Coulomb field of charge  $Q$  fixed at  $\vec{r}_e = (r_e, \phi_e, z_e)$ :

$$\begin{aligned} \mathcal{L}^* = & \frac{m}{2} (\dot{r}^2 + r^2 \dot{\phi}^2 + \dot{z}^2) + (\epsilon + \frac{1}{2}) q B r^2 \dot{\phi} \\ & + \epsilon(\epsilon+1) \frac{q^2 B^2 r^2}{2m} - \frac{qQ}{4\pi\epsilon_0} \frac{1}{\sqrt{r^2 + r_e^2 - 2rr_e \cos\phi + z^2}} . \end{aligned} \quad (\text{A40})$$

We have omitted the stars from the generalized coordinates ( $r^*$ ,  $\phi^*$ ,

z\*) of the rotating system, and will rely upon the presence of  $\epsilon$  to indicate that these are coordinates in a rotating frame. This form and the Newtonian frame Lagrangian are in essential agreement for  $\epsilon = 0$ . There is apparently no choice of  $\epsilon$  which will completely remove the effects of the magnetic field from this Lagrangian.

We note that if the term quadratic in B can be ignored (due either to the small value of the radial distance or the magnetic field, or both), then the choice  $\epsilon = -1/2$  removes all remaining effects of the magnetic field from this Lagrangian. This is the basis of the Larmor theorem, that the sole effect of the magnetic field upon such a system is a rotation of the system about the field direction at the Larmor frequency  $eB/2m$ . The other obvious choice of rotation speed is  $\epsilon = -1$  describing coordinate system rotation at the cyclotron frequency. For  $Q = 0$  and a cyclotron orbit centered upon the origin, the particle would be at rest for this choice of  $\epsilon$ .

The canonical momenta are

$$\begin{cases} p_1 \\ p_2 \\ p_3 \end{cases} = \begin{cases} m\dot{\rho} \\ m\rho^2\dot{\phi} + (\epsilon + \frac{1}{2})gB\rho^2 \\ m\dot{z} \end{cases} = \begin{cases} p_\rho \\ L_z \\ p_z \end{cases} \quad \begin{matrix} \text{(A41)} \\ \text{(A42)} \\ \text{(A43)} \end{matrix}$$

The Hamiltonian, constructed according to the prescription (A21) is

$$H^* = \frac{p_\rho^2 + \left(\frac{L_z}{\rho} - (\epsilon + \frac{1}{2})gB\rho\right)^2 + p_z^2}{2m} - \epsilon(\epsilon + 1) \frac{g^2 B^2 \rho^2}{2m} + \frac{gQ}{4\pi\epsilon_0} \frac{1}{\sqrt{\rho^2 + \rho_e^2 - 2\rho\rho_e \cos\phi + z^2}} \quad \text{(A44)}$$

The Lagrangian Equations of Motion

The Lagrangian equations of motion for the rotating frame are

$$m\ddot{\rho} - m\rho\dot{\phi}^2 - (\epsilon + \frac{1}{2})2g_B\rho\dot{\phi} - \epsilon(\epsilon + 1)\frac{g^2 B^2 \rho^2}{m} \quad (A45)$$

$$+ \frac{gQ}{4\pi\epsilon_0} \frac{\partial}{\partial \rho} \frac{1}{\sqrt{\rho^2 + \rho_e^2 - 2\rho\rho_e \cos \phi + z^2}}$$

$$\frac{d}{dt}(m\rho^2\dot{\phi} + (\epsilon + \frac{1}{2})g_B\rho^2) = -\frac{gQ}{4\pi\epsilon_0} \frac{\partial}{\partial \phi} \frac{1}{\sqrt{\rho^2 + \rho_e^2 - 2\rho\rho_e \cos \phi + z^2}} \quad (A46)$$

$$m\ddot{z} + \frac{gQ}{4\pi\epsilon_0} \frac{\partial}{\partial z} \frac{1}{\sqrt{\rho^2 + \rho_e^2 - 2\rho\rho_e \cos \phi + z^2}} = 0. \quad (A47)$$

For any value of  $\rho_e$ , the z equation of motion is unchanged by the rotation (compare A47 with A17). For  $\rho_e = 0$  (the charge Q situated at the origin), the remaining equations are also invariant. For this location of Q, the canonical angular momentum  $L_z$  is conserved as before, and (A46) may be incorporated into (A45) with the result

$$m\ddot{\rho} - \frac{L_z^2}{m\rho^3} + \left(\frac{g_B}{2m}\right)^2 m\rho + \frac{gQ}{4\pi\epsilon_0} \frac{\partial}{\partial \rho} \frac{1}{\sqrt{\rho^2 + z^2}} = 0. \quad (A48)$$

The fact that this equation is identical to (A19) does not necessarily imply that the solutions are the same, but only that they are of the same family. It is obvious that for the same physical situation, at least one of the two initial conditions of (A48) would differ from those of (A19). Further,  $L_z$  is not the same constant for (A48) as for (A19). In passing, we note that (A48) is valid also for  $\rho_e \neq 0$  except that  $L_z$

is then no longer a constant of the motion.

By means of the usual techniques for obtaining energy integrals, the equations of motion (A45) through (A47) may be combined to yield

$$\begin{aligned} \frac{d}{dt} \left[ \frac{m}{2} (\dot{\rho}^2 + \rho^2 \dot{\phi}^2 + \dot{z}^2) - \epsilon(\epsilon+1) \frac{g^2 B^2 \rho^2}{2m} + \frac{gQ}{4\pi\epsilon_0} \frac{1}{\sqrt{\rho^2 + \rho_e^2 - 2\rho\rho_e \cos\phi + z^2}} \right] \\ = - \frac{d\epsilon}{dt} \left[ (2\epsilon+1) \frac{g^2 B^2 \rho^2}{2m} + g B \rho^2 \dot{\phi} \right] \end{aligned} \quad (\text{A49})$$

showing explicitly that energy is conserved for constant  $\epsilon$ .

We conclude that, for  $\rho_e = 0$ , there exists no value of  $\epsilon$  which will simplify the equations of motion since they remain invariant under coordinate system rotation about the direction of the magnetic field. For  $\rho_e \neq 0$ , the equations can be somewhat simplified, but apparently cannot be linearized for this coordinate system.

APPENDIX B  
SOME FEATURES OF THE CLASSICAL MOTION OF  
A CHARGED PARTICLE IN A MAGNETIC FIELD

B1 Content

In the following appendix, we shall be faced with the assignment of physical meaning to a quantum representation of a single charged particle moving in a magnetic field. In preparation for this interpretation, we consider here three constants of the classical motion as well as three other dynamical variables whose time dependence has been removed by averaging over one or more gyroperiods. Of particular interest is the dependence of these quantities upon the cyclotron radius and upon the location of the particle gyrocenter with respect to the coordinate system origin. The system considered consists of a particle of mass  $m$  and charge  $q$  moving in the constant and uniform magnetic field  $\vec{B} = B\hat{z}$ .

B2 Coordinates of the Perpendicular Motion

Compared to the motion in the plane normal to the magnetic field, the  $z$  motion is relatively uninteresting and quickly may be eliminated from consideration. In discussing features of the perpendicular motion, we shall utilize the following vectors and coordinates. The particle is located by the vector  $\vec{r}$  extending from the coordinate system origin to the instantaneous particle position. The plane polar (or cylindrical polar) coordinates of this point are labelled  $(r, \theta)$ . The particle gyrocenter is located by the vector  $\vec{r}_0$ , or equivalently, by the pair  $(r_0, \theta_0)$ . The third vector of interest is the cyclotron radius vector  $\vec{r}_c$  extending from the gyrocenter to the particle. These



vectors satisfy the equation

$$\vec{r} = \vec{r}_o + \vec{r}_c . \quad (\text{B1})$$

The associated unit vectors satisfy the relations

$$\hat{r} \times \hat{\theta} = \hat{r}_c \times \hat{\theta}_c = \hat{r}_o \times \hat{\theta}_o = \hat{z} . \quad (\text{B2})$$

This location scheme and the two cases of interest are illustrated in Fig. B1. One is the case  $r_o/r_c < 1$  when the origin is inside the gyrocircle; the other is the case  $r_o/r_c > 1$  when the origin is outside the gyrocircle. There is also the singular, joint case  $r_o/r_c = 1$ .

### B3 Constants of the Motion

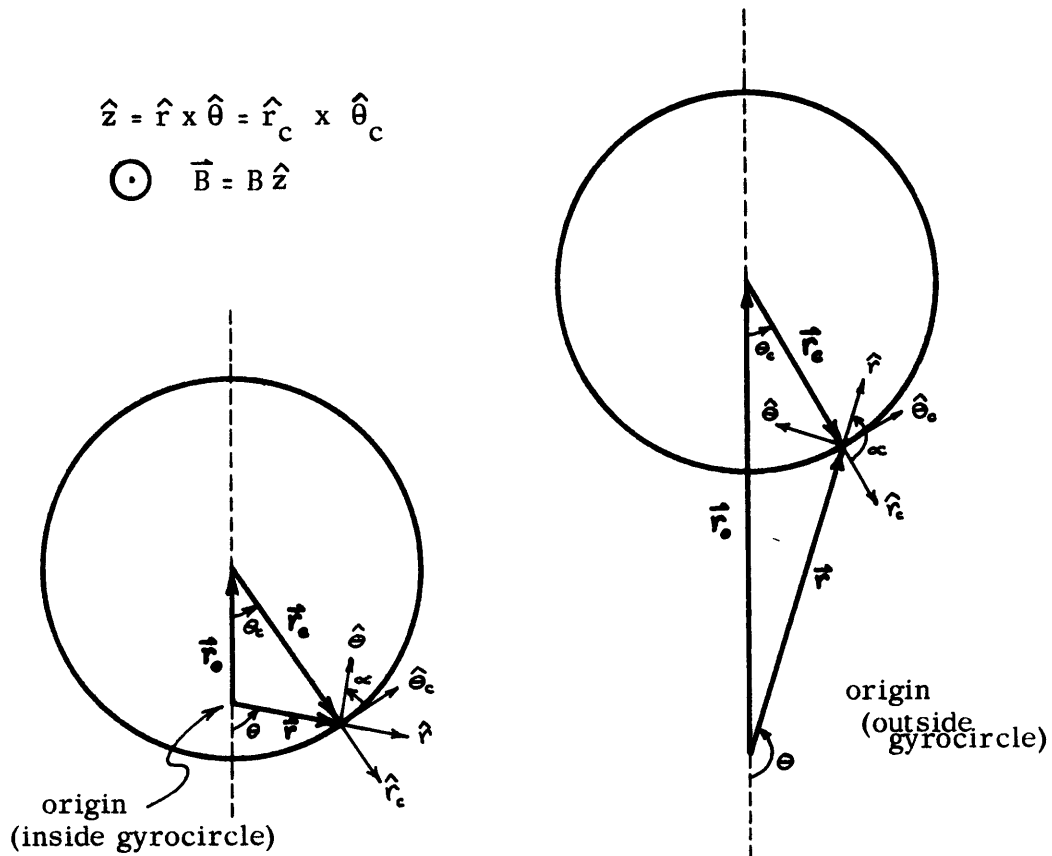
There are three basic constants of the motion. One is associated with the motion along the magnetic field and the others with the motion normal to the field. They are the z energy  $E_z$ , the perpendicular energy  $E_{\perp}$ , and the angular momentum component  $L_z$  (canonical to the coordinate  $\theta$ ). Each may be cast into different though equivalent forms:

$$E_z = \frac{1}{2} m v_z^2 = \frac{p_z^2}{2m} = \text{const.} \quad (\text{B3})$$

$$E_{\perp} = \frac{1}{2} m r_c^2 \omega_c^2 = \frac{1}{2} r_c^2 \hbar \omega_c = \text{const.} \quad (\text{B4})$$

$$L_z = \frac{q B}{2} (r_o^2 - r_c^2) = \text{const.} \quad (\text{B5})$$

That these quantities are constant follows from equations (A16) through (A18) for  $Q = 0$ . The expression of the perpendicular quantities in terms of the distances  $r_o$  and  $r_c$  follows from basic vector definitions. We consider here only the result for  $L_z$ . The z motion is



Particle is located at  $\vec{r}$ , gyrocenter at  $\vec{r}_0$ .

Particle velocity is  $\vec{v}_c = -\frac{q r_c B}{m} \hat{\theta}_c$  for  $\vec{B} = B \hat{z}$ .

Fig. B1. Coordinates used in classical averages-over-orbits.

suppressed.

In general the angular momentum  $\vec{L}$  may be defined as  $\vec{r} \times \vec{p}$  where  $\vec{p}$  is the canonical momentum  $m\vec{v} + q\vec{A}$ . The angular momentum is dependent upon location of the coordinate system origin through the explicit and implicit (in  $\vec{A}$ ) appearance of  $\vec{r}$ . The general vector potential describing (through  $\vec{B} = \text{curl } \vec{A}$ ) a uniform magnetic field is  $(\vec{B} \times \vec{r})/2$ .

Referred to an origin at the particle gyrocenter, the angular momentum is

$$\vec{L}_c = \vec{r}_c \times \left[ m\dot{\vec{r}}_c + \frac{q}{2} \vec{B} \times \vec{r}_c \right]. \quad (\text{B6})$$

The vector expression for  $\dot{\vec{r}}_c \equiv \vec{v}_c$ ,

$$\vec{v}_c = - \frac{qB r_c}{m} \hat{\theta}_c, \quad (\text{B7})$$

satisfies the Lorentz equation  $m\dot{\vec{v}}_c = q(\vec{v}_c \times \vec{B})$  as well as our notions (embodied in the right-hand rule) about the diamagnetism of an unbound charged particle in a magnetic field. The present magnetic field orientation and the suppression of the z motion insures that  $L_c$  and all other angular momenta will have only z components. The first term in (B6) thus has the particular form  $(-qBr_c^2)\hat{z}$ , and the second  $(qBr_c^2/2)\hat{z}$ , with the result

$$\vec{L}_c = - \frac{1}{2} q B r_c^2 \hat{z}. \quad (\text{B8})$$

Recall our convention on charge sign, that  $q = e$  denotes a proton (say), and  $q = -e$  an electron.

Referred to an origin located arbitrarily within or without the gyrocircle, the particle angular momentum is

$$\vec{L}_o = \vec{r} \times \left[ m\dot{\vec{r}} + \frac{q}{2} \vec{B} \times \vec{r} \right]. \quad (\text{B9})$$

Through use of the vector relations cited earlier, and the BAC-CAB triple vector product identity, this expression may be brought to the form

$$\begin{aligned} \vec{L}_o &= \vec{L}_c + \frac{1}{2} q B r_o^2 \hat{z} \\ &+ q B r_o r_c \left[ (\hat{r}_c \cdot \hat{r}_o) \hat{z} - (\hat{r}_o \times \hat{e}_c) \right]. \end{aligned} \quad (\text{B10})$$

However, the third term on the RHS is identically zero since  $\hat{r}_o \times \hat{e}_c = (\hat{r}_c \cdot \hat{r}_o) \hat{z}$ . The final form,

$$\vec{L}_o = \frac{q B}{2} (r_o^2 - r_c^2) \hat{z}, \quad (\text{B11})$$

is manifestly the same for every point of the cyclotron orbit.

#### B4 Three Time-Averaged Dynamical Variables

In these time averages only the two-dimensional motions in the plane normal to the uniform and constant magnetic field  $\vec{B} = B\hat{z}$  are considered; the z motions are suppressed. Again, we are interested in the dependence of these quantities upon the location of the particle gyrocenter with respect to the coordinate system origin. The time averages, denoted by  $\langle \rangle$ , are taken over a single cycle of the cyclotron motion. The quantities considered are:

- $\langle r^2 \rangle$  - squared distance, origin to particle;
- $\langle E_r \rangle$  - radial component of the perpendicular energy;
- $\langle E_\theta \rangle$  - azimuthal component of the perpendicular energy.

#### Squared Distance, Origin to Particle

The first quantity to be considered is the time average of the squared distance ( $\vec{r} \cdot \vec{r}$ ) from the arbitrarily located origin to the

instantaneous particle position. From (B1) we write

$$r^2 = r_o^2 + 2r_o r_c (\hat{r}_o \cdot \hat{r}_c) + r_c^2 \quad (\text{B12a})$$

$$= r_o^2 - 2r_o r_c \cos \theta_c + r_c^2 \quad (\text{B12b})$$

from which the time average follows immediately:

$$\langle r^2 \rangle = r_o^2 + r_c^2. \quad (\text{B13})$$

### Radial and Azimuthal Components of the Perpendicular Energy

The perpendicular energy  $E_{\perp}$  is defined in terms of the velocity as

$$E_{\perp} \equiv \frac{1}{2} m \vec{v} \cdot \vec{v} \quad (\text{B14})$$

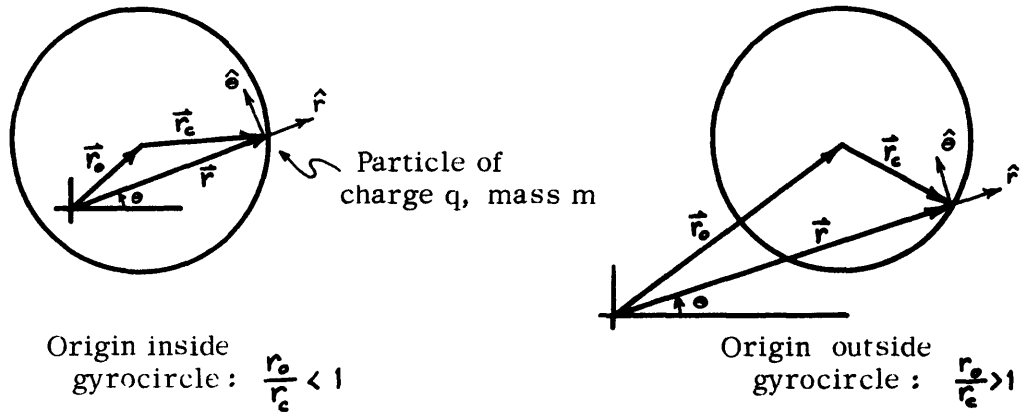
The velocity  $\vec{v} \equiv \dot{\vec{r}}$  is equal to  $\vec{v}_c \equiv \dot{\vec{r}}_c$  since the vector  $\vec{r}_o$  changes in neither magnitude nor direction. In arriving at the expression (B4) for  $E_{\perp}$ , the velocity  $\vec{v}_c$  was resolved along  $\hat{\theta}_c$  as indicated by (B7). In this calculation, however, we resolve  $\vec{v}_c$ , or equivalently  $\hat{\theta}_c$ , into radial and azimuthal components:

$$\hat{\theta}_c = \sin \alpha \hat{r} + \cos \alpha \hat{\theta} \quad (\text{B15})$$

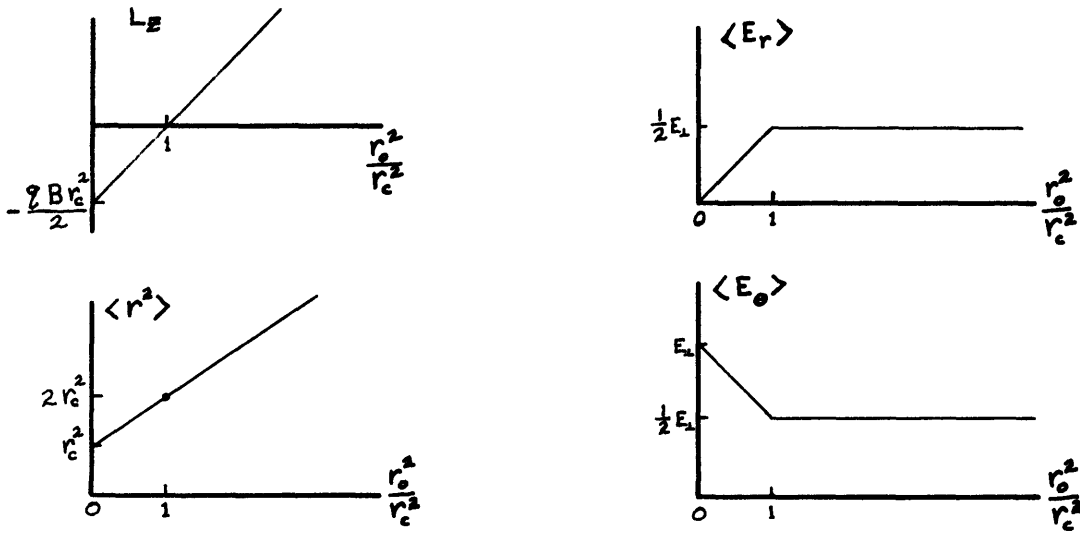
where  $\alpha \equiv \theta - \theta_c$  (see Fig. B1). The radial and azimuthal energies,

$$E_r = E_{\perp} \sin^2 \alpha \quad (\text{B16})$$

$$E_{\theta} = E_{\perp} \cos^2 \alpha, \quad (\text{B17})$$



Locations of coordinate system origin with respect to the orbit of a classical charged particle in a magnetic field (normal to the plane of the paper).



Four quantities of interest for a charged particle in a magnetic field, averaged over 1 gyroperiod, as a function of origin location:

- $L_z$  - z component of canonical angular momentum;
- $\langle r^2 \rangle$  - squared distance, origin to particle;
- $\langle E_r \rangle$  - radial component of particle energy;
- $\langle E_\theta \rangle$  - azimuthal component of particle energy.

$E_\perp$  is the total particle energy in the plane normal to the magnetic field.

Fig. B2. Results of classical averages over a cyclotron orbit.

are not separately constant, although of course their sum is constant. The trigonometric relations,

$$\frac{\sin \alpha}{r_0} = \frac{\sin \theta_c}{r} \quad (\text{B18})$$

$$r^2 = r_0^2 + r_c^2 - 2 r_0 r_c \cos \theta_c, \quad (\text{B19})$$

may be combined to yield an expression for  $\sin^2 \alpha$  in terms of the cyclotron angle  $\theta_c$ . This expression,

$$\sin^2 \alpha = \frac{r_0^2 \sin^2 \theta_c}{r_0^2 + r_c^2 - 2 r_0 r_c \cos \theta_c}, \quad (\text{B20})$$

is averaged with respect to  $\theta_c$ , which is equivalent to a time average. We have to consider the following integral, for the ratio  $r_0/r_c < 1$ , and for  $r_0/r_c > 1$ :

$$\langle \sin^2 \alpha \rangle = \frac{1}{2\pi} \int_0^{2\pi} \frac{r_0^2 \sin^2 \theta_c}{r_0^2 + r_c^2 - 2 r_0 r_c \cos \theta_c} d\theta_c. \quad (\text{B21})$$

This integral was evaluated with the aid of formula (34a), p. 114 in the second volume of the Grobner and Hofreiter Integraltafeln [1961]. The expressions obtained for the time averaged radial and azimuthal components of the perpendicular energy are

$$\langle E_r \rangle = \frac{E_\perp}{2} \frac{r_0^2}{r_c^2} \quad \text{for } r_0/r_c \leq 1 \quad (\text{B22a})$$

$$= \frac{E_\perp}{2} \quad \text{for } r_0/r_c \geq 1 \quad (\text{B22b})$$

$$\langle E_{\bullet} \rangle = E_{\perp} \left( 1 - \frac{1}{2} \frac{r_o^2}{r_c^2} \right) \quad \text{for } r_o / r_c \leq 1 \quad (\text{B23a})$$

$$= \frac{E_{\perp}}{2} \quad \text{for } r_o / r_c \geq 1. \quad (\text{B23b})$$

The time-averaged quantities and the angular momentum are plotted in Fig. B2 as a function of  $r_o^2 / r_c^2$ . It is important to note that these results may be considered also as averages over the angle  $\theta$  about the arbitrarily located origin, due to the lack of any angular dependence.



APPENDIX C  
A QUANTUM REPRESENTATION  
OF A CHARGED PARTICLE IN A MAGNETIC FIELD:  
THE CYLINDRICAL LANDAU EIGENFUNCTIONS

C1 Introduction

For later use in scattering calculations, we shall need a quantum mechanical representation of a single charged particle moving in a uniform and constant magnetic field. The representation and the coordinate system chosen should reflect as nearly as is possible the form and symmetries of the complete Hamiltonian. Hopefully, this will simplify the details of calculation as well as those relating to interpretation of the results.

We have utilized the set of energy eigenfunctions obtained as solutions of the Schroedinger equation  $H_0 \Psi_{NMk} = E_0 \Psi_{NMk}$  in the cylindrical coordinate system spanned by the unit vectors  $\hat{e} \times \hat{\phi} = \hat{z}$ . The Hamiltonian is

$$H_0 = \frac{(\vec{p} - q\vec{A})^2}{2m} . \quad (C1)$$

The eigenfunctions are factorable in each of the coordinates as

$$\Psi_{NMk}(r, \phi, z) \equiv R_{NM}(r) \Phi_M(\phi) Z_k(z) . \quad (C2)$$

After a brief derivation and description of the properties of these eigenfunctions, we consider their interpretation and relation to classical models and physical situations.

C2 Derivation and Properties

Since the Hamiltonian  $H_0$  is independent of time, the time dependence of the solution  $\Psi_0(\vec{r}, t)$  of the Schroedinger equation

$$i\hbar \frac{\partial \Psi_0}{\partial t} = H_0 \Psi_0 \quad (C3)$$

may be split off as a phase factor

$$\Psi_0(\vec{r}, t) = \Psi_0(\vec{r}) e^{-i \frac{H_0}{\hbar} t} \quad (C4)$$

From this is constructed the energy eigenvalue equation

$$H_0 \Psi_0 = E_0 \Psi_0, \quad (C5)$$

the time-independent Schroedinger equation. We henceforth replace  $\Psi_0$  by  $\Psi_{NMk}$  to indicate the specific set (C2).

The cylindrical form of the Hamiltonian (C1) is

$$H_0 = \frac{p_\rho^2 + \frac{L_z^2}{\rho^2} + p_z^2}{2m} - \frac{qB}{2m} L_z + \frac{q^2 B^2 \rho^2}{8m} \quad (C6a)$$

$$= \frac{p_\rho^2 + \left(\frac{L_z}{\rho} - \frac{qB\rho}{2}\right)^2 + p_z^2}{2m} \quad (C6b)$$

As was discussed in appendix A, the momenta canonical to the coordinates  $x_i = (\rho, \phi, z)$  are respectively  $p_\rho$ ,  $L_z$ , and  $p_z$ . Their quantum operator equivalents are  $\frac{\hbar}{i} \frac{\partial}{\partial x_i}$ . Implicit in the steps from (C1) to (C6) is the use of the commutation relation  $[\vec{p}, \vec{A}] = -i\hbar \text{div}\vec{A}$ , and the introduction of the divergenceless vector potential

$$\vec{A} = \frac{1}{2} \vec{B} \times \vec{r} = \frac{1}{2} B \rho \hat{\phi} \quad (C7)$$

describing the uniform and constant magnetic field

$$\vec{B} = \text{curl } \vec{A} = B \hat{z}. \quad (C8)$$

Use in (C6) of the quantum operators mentioned above leads to the coordinate representation of the Schroedinger equation (C5),

$$\left\{ -\frac{\hbar^2}{2m} \nabla^2 - \frac{qB}{2m} \frac{\hbar}{i} \frac{\partial}{\partial \phi} + \frac{q^2 B^2 \rho^2}{8m} - E_0 \right\} \psi_{NMk}(\rho, \phi, z) = 0. \quad (C9)$$

With the solution factored as in (C2), the z dependence is immediately separable as

$$Z_k = \frac{1}{\sqrt{2\pi}} e^{\pm ikz} \quad (C10)$$

where the z wavenumber k is defined by the relation  $p_z = \hbar k$ . The substitution

$$\Phi_M = \frac{1}{\sqrt{2\pi}} e^{\pm iM\phi}, \quad M = 0, 1, 2, \dots \quad (C11)$$

removes the  $\phi$  dependence. We are left with the radial equation

$$\frac{d^2 R}{d\rho^2} + \frac{1}{\rho} \frac{dR}{d\rho} + \left[ -\frac{M^2}{\rho^2} - q^2 \rho^2 + \beta^2 \epsilon \right] R = 0 \quad (C12)$$

where  $\beta^2 \equiv \frac{eB}{2\hbar}$  and  $\beta^2 \epsilon \equiv \left[ \frac{2mE_0}{\hbar^2} - k^2 \pm \frac{qBM}{\hbar} \right]$ .

The parameter  $\beta^2$  has dimensions of inverse length squared ( $m^{-2}$ ). Its reciprocal was shown by Johnson and Lippmann [1949] to represent the minimum area in the x-y plane to which a gyrocenter may be located by any measurement. The z and  $\phi$  eigenfunctions have been normalized to Dirac and Kronecker delta functions, respectively:

$$\int_{-\infty}^{\infty} Z_{k_2}^*(z) Z_{k_1}(z) dz = \delta(k_2 - k_1) \quad (C13)$$

$$\int_0^{2\pi} \Phi_{M_2}^*(\phi) \Phi_{M_1}(\phi) d\phi = \delta_{M_2, M_1} \quad (C14)$$

The connection of this method of z eigenfunction normalization with the density in energy of the states and the interpretation of the probability density  $\Psi^*\Psi$  is discussed later in this appendix. With the z eigenfunction thus normalized and with the yet-to-be-obtained radial eigenfunction normalized to a Kronecker delta, the product  $\Psi_{NMk}^*(\vec{r}) \Psi_{NMk}(\vec{r}) d\tau dk$  is the probability of locating the particle in the volume element  $d\tau$  at  $\vec{r}$  and in the quantum state characterized by the numbers N and M and the continuous wavenumber k in the range dk. It is a pure number whose integral is unity:

$$\int_k \int_{\tau} \Psi_{NMk}^*(\vec{r}) \Psi_{NMk}(\vec{r}) d\tau dk = 1. \quad (C15a)$$

We shall have later use for the more general and informative antecedent of (C15a),

$$\begin{aligned} & \sum_N \sum_M \int_k \int_{\tau} \Psi_{N'M'k'}^* \Psi_{NMk} d\tau dk \\ & \equiv \sum_N \sum_M \int_k \langle \Psi_{N'M'k'} | \Psi_{NMk} \rangle dk \\ & = \sum_N \sum_M \int_k \delta_{N',N} \delta_{M',M} \delta(k'-k) dk = 1. \end{aligned} \quad (C15b)$$

This sequence not only defines the meaning of the matrix element symbol  $\langle \rangle$ , but also expresses the diagonal character of the eigenstate element  $\langle \Psi_{N'M'k'} | \Psi_{NMk} \rangle$ .

The change of variable  $x = \epsilon^2 e^2$  brings the radial differential equation to the form

$$x \frac{d^2 R}{dx^2} + \frac{dR}{dx} + \frac{1}{4} \left( \epsilon - \frac{M^2}{x} - x \right) R = 0 . \quad (C16)$$

It is convenient to determine the asymptotic behavior of the equation and its solution before attempting a complete solution. The foregoing interpretation of the form  $\Psi^* \Psi d\tau dk$  together with the feeling that there is no physical reason to have a pole or other singularity near the origin of  $\Psi$  prompts us to require that the R solution remain finite near the origin (be analytic at the origin). Thus we take as the general form of R the expansion

$$R = \text{const} \cdot x^\lambda \cdot (1 + ax + bx^2 + \dots) .$$

Upon substitution of this form into (C16), it is seen that the coefficient of the lowest power of x (i.e.,  $x^{\lambda-1}$ ) requires satisfaction of the equation

$$\lambda^2 - \frac{M^2}{4} = 0 .$$

We choose  $\lambda = +\frac{M}{2}$ , rejecting the negative solution and its accompanying singularity at the origin. On the other hand, as  $x \rightarrow \infty$ , we require that R and R' remain bounded. In this limit equation (C16) becomes

$$x \frac{d^2 R}{dx^2} - \frac{1}{4} x R = 0$$

With solutions  $\exp(\pm x/2)$ . We reject the plus sign. Guided by the behavior near the origin and for x large, we assume the solution

$$R = e^{-\frac{x}{2}} x^{\frac{M}{2}} F(x) .$$

Substitution into (C16) yields the equation for F(x),

$$x \frac{d^2 F}{dx^2} + (M+1-x) \frac{dF}{dx} + \left(\frac{\epsilon}{4} - \frac{M+1}{2}\right) F = 0. \quad (C17)$$

This is a particular type of confluent hypergeometric equation whose solutions are the Laguerre polynomials  $L_N^M(x)$ , where  $N$  is a non-negative integer

$$N \equiv \frac{\epsilon}{4} - \frac{M+1}{2} = 0, 1, 2, \dots. \quad (C18)$$

This quantity must be an integer in order for the differential equation to have such polynomial solutions and thus satisfy the boundary conditions at infinity. This is easily seen from the general form of the confluent hypergeometric equation, the solution, and its behavior as  $x \rightarrow \infty$ . From HTF I, chapter VI, we write

$$x \frac{d^2 y}{dx^2} + (c-x) \frac{dy}{dx} - ay = 0 \quad (C19)$$

with the solution

$$y = {}_1F_1(a, c, x) \equiv 1 + \frac{a}{c} \frac{x}{1!} + \frac{a(a+1)}{c(c+1)} \frac{x^2}{2!} + \dots. \quad (C20)$$

We see that if  $c$  is to be an integer, it must be positive lest we encounter a zero denominator in one of the terms of  ${}_1F_1$ . For our case this condition is met, since  $c = M + 1$ . Similarly, we see that if the infinite series (C20) is to terminate to a polynomial, then  $a$  must be a negative integer or zero. If  $a$  does not meet this criterion, then

$${}_1F_1(a, c, x) \xrightarrow{x \rightarrow \infty} \frac{\Gamma(c)}{\Gamma(a)} e^{-x} x^{a-c} \quad (C21)$$

(as given in HTF I, p. 278). The appearance here of the  $e^x$  indicates that the confluent hypergeometric function must be forced to terminate to a polynomial if the solution  $R(x)$  is to remain bounded as  $x \rightarrow \infty$ . The forcing is done by adjusting the parameter  $\epsilon$ , which quantizes the perpendicular energy:

$$E_0 - \frac{\hbar^2 k^2}{2m} = \frac{\hbar \omega_c}{2} (2N + M \mp \frac{g}{e} M + 1). \quad (C22)$$

The upper or lower sign in (C22) should be chosen to conform with the choice made in the eigenfunction  $\Phi_M$ . The Laguerre polynomial may be defined in several ways, among which are

$$L_N^M(x) \equiv \frac{(N+M)!}{N! M!} F_1(-N, M+1, x) \quad (C23a)$$

$$\equiv \frac{1}{N!} e^x x^{-M} \frac{d^N}{dx^N} (e^{-x} x^M) \quad (C23b)$$

$$\equiv \sum_{j=0}^N \frac{(N+M)!}{(N-j)! (M+j)!} \frac{(-x)^j}{j!} \quad (C23c)$$

These expressions were taken from chapter X of HTF II. Also given is the normalization integral

$$\int_0^{\infty} e^{-x} x^M L_N^M(x) L_{N'}^M(x) dx = \frac{(N+M)!}{N!} \delta_{N,N'}. \quad (C24)$$

This integral is used to normalize the  $R(\rho)$  eigenfunctions, that is, to insure satisfaction of the condition

$$\int_0^{\infty} R_{NM}^*(\rho) R_{N'M'}(\rho) \rho d\rho = \delta_{N,N'} . \quad (C25)$$

These eigenfunctions are characterized by

- i) the value zero at the origin, unless M is itself zero;
- ii) N zeroes in the range  $0 < \rho^2 < \infty$  (excluding the end points), contributed by the Laguerre polynomial;
- iii) for  $M \gg 1$  and  $N=0$ , a single spike-like function having a power law rise and an exponential fall-off.

We have illustrated these characteristics in terms of the radial probability density RPD

$$\text{RPD} \equiv \frac{N!}{(N+M)!} e^{-x} x^M \left[ L_N^M(x) \right]^2 . \quad (C27)$$

This dimensionless function is just the integrand of the normalization integral (C25) when cast in terms of the variable  $x = \rho^2$ . As such, it satisfies  $\int_0^{\infty} \text{RPD} dx = 1$ . All of the spatial structure of  $\Psi_{NMk}^* \Psi_{NMk}$  is contained in RPD since

$$\Psi_{NMk}^* \Psi_{NMk} = \frac{\beta^2}{2\pi^2} \text{RPD} . \quad (C28)$$

Several RPD functions for small M and N are plotted in Figs. C1 and C2. The values were computed from (C27) using (C23c) to define the Laguerre polynomial. The plots have been normalized to the height of the largest RPD peak.



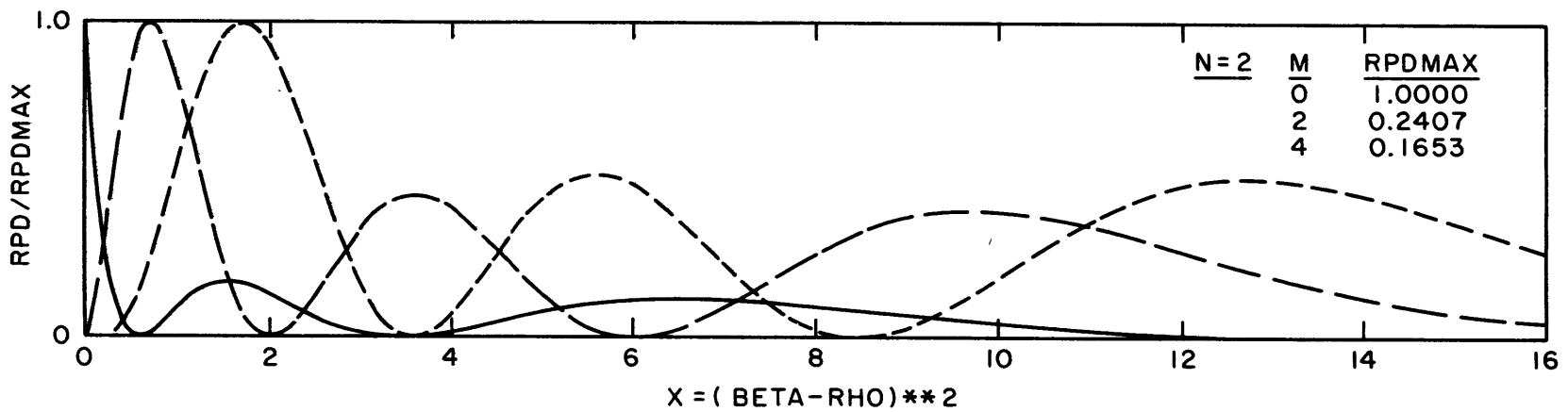
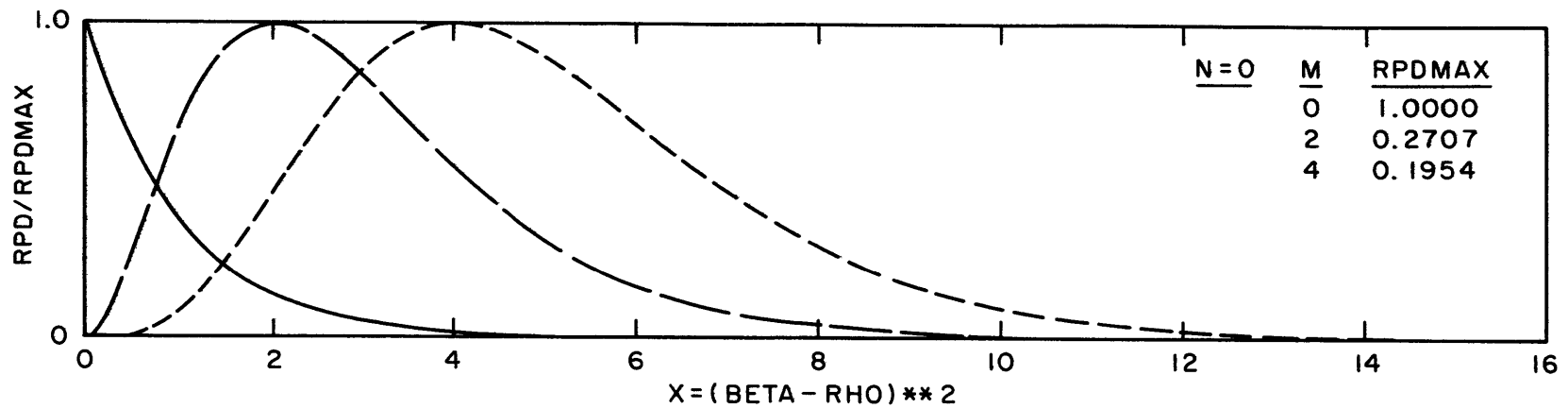


Fig. C1. Radial probability density for several cylindrical Landau eigenfunctions.

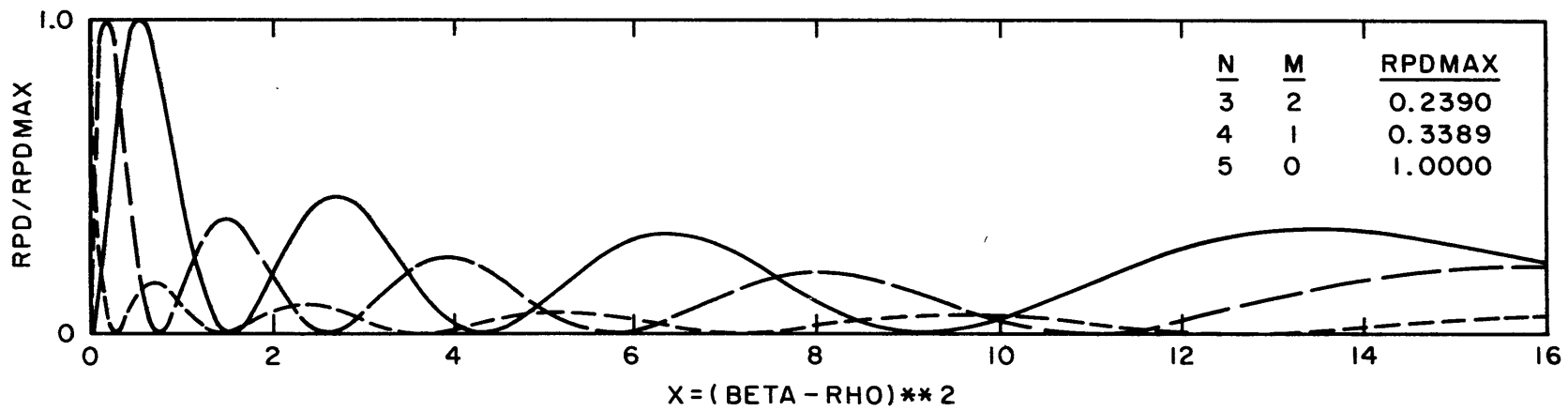
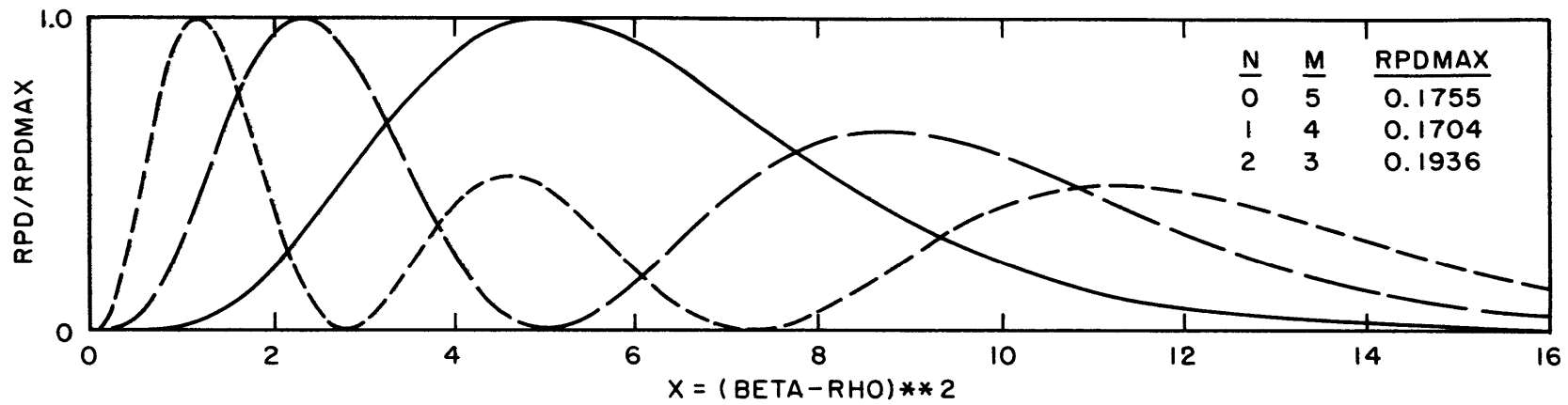


Fig. C2. Radial probability density for several cylindrical Landau eigenfunctions.

C3 Interpretation of the Quantum Numbers and Parameters

The Wavenumber  $k$  of the  $z$  Eigenfunctions

The interpretation of the  $\pm k$  appearing in the  $z$  eigenfunction is straightforward. The continuous wavenumber  $k$  is essentially the  $z$  component of linear momentum. It is also proportional to the  $z$  component of velocity, since the vector potential  $\vec{A}$  lacks such a component. The eigenfunctions  $Z_k$  satisfy the eigenvalue equations

$$p_z Z_k = \pm \hbar k Z_k \quad (C29)$$

$$E_z Z_k = \frac{\hbar^2 k^2}{2m} Z_k \quad (C30)$$

where the operator for  $p_z$  is  $\frac{\hbar}{i} \frac{\partial}{\partial z}$  and is  $p_z^2/2m$  for  $E_z$ . That is, the functions  $Z_k$  are eigen functions of the continuous  $z$  momentum, velocity, and energy.

The  $z$  component of the probability density flux (see a following appendix) reflects the radial structure of  $\psi_{NMk}^* \psi_{NMk}$  :

$$J_z = \frac{\pm \hbar k}{m} \frac{R_{NM}^2}{4\pi^2} . \quad (C31)$$

The area-averaged value for any eigenstate (NMk) is

$$\Gamma_z = \frac{\hbar k}{m} \frac{2 \int_0^{\beta} dk}{(M+2N+1) 2\pi} . \quad (C32)$$

The sign, in conjunction with the time factor  $\exp(-iE_0 t/\hbar)$ , specifies the direction of propagation along the  $z$  axis. The functional form of the  $z$  eigenfunction indicates that there is no reference point or landmark along the  $z$  axis (as there would be if a potential center were introduced), and that motion along the direction of the magnetic field is completely free and unbound.

Density in Energy of the  $z$  Eigenstates

With every set of quantum states characterized by a continuous eigenvalue parameter, there may be associated a function giving the density of such states in the space of that parameter. Depending on the circumstances of use, the function may or may not be normalized to unit length or volume of the configuration space containing the system. For the continuum eigenstates considered immediately above, the eigenvalue parameter was chosen as the energy  $E_z$ . The density of states per unit length along the  $z$  axis is

$$\frac{1}{2\pi} \frac{dk}{dE_z} = \sqrt{\frac{2m'}{\hbar^2}} \frac{1}{2\pi} \frac{1}{2\sqrt{E_z}} \quad (C33)$$

where  $dk/2\pi$  is the number of such states per unit  $z$  length. We shall need this function in the transition probability or cross-section evaluations. The function gives the number of continuum states per unit  $z$  energy and length at a given point in  $z$  energy. It is an energy-averaged limit function in the same sense that, for example, the mass density  $\rho_m \equiv dm/d\tau$  is a point function representing a spatial average over a suitably small volume.

The appearance of the factor  $(1/2\pi)$  with  $dk$  indicates that the periodic or travelling wave boundary conditions

$$Z_k\left(-\frac{L}{2}\right) = Z_k\left(+\frac{L}{2}\right) \quad (C34a)$$

$$Z'_k\left(-\frac{L}{2}\right) = Z'_k\left(+\frac{L}{2}\right) \quad (C34b)$$

were applied for a finite system length  $L$  after which  $L$  was allowed to become arbitrarily large ( $L \rightarrow \infty$ ). The imposition of these conditions insures that, for a given  $k$ , there is as much probability density influx at  $z = -L/2$  as there is outflux at  $z = +L/2$ .

The Quantum Number M of the Azimuthal or  $\Phi_M$  Eigenfunctions

Interpretation of the quantum number M as the component  $L_z$  of the canonical angular momentum is likewise straightforward. With the operator for  $L_z$  as  $\frac{\hbar}{i} \frac{\partial}{\partial \phi}$ , we see that

$$L_z \Phi_M = \pm M \hbar \Phi_M . \quad (C35)$$

That is, the functions  $\Phi_M$  are eigenfunctions of the angular momentum component  $L_z$  canonical to the azimuthal coordinate  $\phi$ . The meaning of the plus and minus signs of  $\Phi_M$  is considered in the next section.

The Signs of  $\Phi_M$  and the Radial Quantum Number N

Recognition of the significance of the  $\pm$  signs of  $\Phi_M$  and the interpretation of the quantum number N proceeds from a comparison of the quantum and classical expressions for the angular momentum component  $L_z$  and averages of the squared origin-to-particle distance  $\rho^2$ . The quantum and classical forms for  $L_z$ , a constant of the motion, are

$$L_z = \pm M \hbar \quad (C36)$$

$$L_z = \frac{qB}{2} (\rho_o^2 - \rho_c^2) \quad (C37)$$

where  $\rho_c$  is the cyclotron radius and  $\rho_o$  is the distance from the origin to the gyrocenter. Notice that there are two sign effects operating in the classical  $L_z$ : the sign of q and the sign associated with origin location. For  $q > 0$ , we see that  $L_z$  is negative for the origin located inside the gyrocircle ( $\rho_o/\rho_c < 1$ ) and is positive for the origin located outside the gyrocircle ( $\rho_o/\rho_c > 1$ ).

The squared distance  $\rho^2$  is not, in general, a constant of the motion. Thus we consider instead the classical time average

(appendix B) and the eigenstate expectation value (appendix D):

$$\langle \hat{e}^2 \hat{e}^2 \rangle = 2N + M + 1 \quad (\text{C38})$$

$$\langle \hat{e}^2 \hat{e}^2 \rangle = \hat{e}^2 (\hat{e}_o^2 + \hat{e}_c^2). \quad (\text{C39})$$

Both the quantum and classical expressions are positive. They do not change sign with origin location or charge sign.

Upon equating the two expressions for  $L_z$  and the two for  $\langle \hat{e}^2 \hat{e}^2 \rangle$ , we obtain the pair

$$\frac{q}{e} \hat{e}^2 (\hat{e}_o^2 - \hat{e}_c^2) = \pm M \quad (\text{C40})$$

$$\hat{e}^2 (\hat{e}_o^2 + \hat{e}_c^2) = 2N + M + 1 \quad (\text{C41})$$

where  $\hat{e}^2 = eB/2\hbar$ . What was implied a few lines earlier now becomes obvious. Consider  $q$  to be positive and equal to  $e$ . Since the classical  $L_z$  is negative, zero, or positive according as to whether the origin is located inside, on, or outside the gyrocircle, we conclude that the + signs in the  $\phi$  eigenfunctions are the quantum descriptors of origin location. That is, for  $q > 0$ ,

-  $M$  denotes  $\frac{r_o}{r_c} < 1$ , origin inside gyrocircle;

+  $M$  denotes  $\frac{r_o}{r_c} > 1$ , origin outside gyrocircle;

$M = 0$  denotes  $\frac{r_o}{r_c} = 1$ , origin on the gyrocircle.

The eigenstates may thus be classified into two groups:

Group I: (sign of  $q$ ) x (sign of  $\Phi_M$ )  $< 0$ ,

Group II: (sign of  $q$ ) x (sign of  $\Phi_M$ )  $> 0$ .

The group I results and eigenfunctions correspond to orbits which enclose the coordinate system origin, whereas the group II orbits do not enclose the origin. For a given charge sign, we must in general use members of both groups to describe an experimental situation unless we can insure either that all gyro-orbits enclose the origin or that no gyro-orbits enclose the origin. It follows also from equations (C40) and (C41) that the quantum number  $N$  is the eigenvalue of the operator representing either  $\hat{p}_o^2 \hat{p}_c^2$  or  $\hat{p}_c^2 \hat{p}_o^2$ , depending upon the various sign possibilities. These are listed in Table C1. That  $N$  is in fact the eigenvalue of one or the other of these quantities may be verified by construction of their operator equivalents. This procedure is guided by the classical equations and is based upon the known results for the eigenvalues of  $L_z$  and the perpendicular energy  $E_{\perp} \equiv E_o - E_z$ . In Table C2, we have listed the eigenstate expectation values of several dynamical variables of the perpendicular motion. Most of these quantities are evaluated in appendix D.

The entries of these tables provide a striking example of the correspondence between classical and quantum constants of the motion, and between classical time averages and eigenstate expectation values. The former correspondence is illustrated by a comparison of the classical equations expressing constancy with the analogous eigenvalue equations. The latter correspondence is illustrated, for instance, by a comparison of equations (B22) and (B23) for the classical time averages of  $E_e$  and  $E_{\phi}$  with their eigenstate expectation value equations. The consistency of such comparisons is comforting, as is the fact that, in each case,  $E_e$  and  $E_{\phi}$  sum to  $E_{\perp}$ . Although not considered classically, it is interesting to observe that the expectation value of  $\dot{\phi}$  vanishes for orbits not enclosing the origin and takes on the value  $\omega_c$  with the correct sign for orbits encircling the origin.

Table C1. Classification and perpendicular eigenvalues of the cylindrical Landau eigenfunctions  $\Psi_{NMk}$ .

Charge Sign		q > 0 (proton)		q < 0 (electron)	
Sign of the set $\{\Phi_M\}$		+M: $e^{+iM\phi}$	-M: $e^{-iM\phi}$	+M: $e^{+iM\phi}$	-M: $e^{-iM\phi}$
Group		II	I	I	II
Significance		$\frac{e_0}{e_c} > 1$ origin outside gyrocircle	$\frac{e_0}{e_c} < 1$ origin inside gyrocircle	$\frac{e_0}{e_c} < 1$ origin inside gyrocircle	$\frac{e_0}{e_c} > 1$ origin outside gyrocircle
Degeneracy		Arbitrarily large	N + M + 1	N + M + 1	Arbitrarily large
Quantity	Operator	Eigenvalues			
$L_z$	$\frac{\hbar}{i} \frac{\partial}{\partial \phi}$	+ M $\hbar$	- M $\hbar$	+ M $\hbar$	- M $\hbar$
$E_o$	$\frac{(\bar{p} - q\bar{A})^2}{2m}$	$E_z + \hbar\omega_c (N + \frac{1}{2})$	$E_z + \hbar\omega_c (N + M + \frac{1}{2})$	$E_z + \hbar\omega_c (N + M + \frac{1}{2})$	$E_z + \hbar\omega_c (N + \frac{1}{2})$
$E_{\perp}$	$E_o - E_z$	$\hbar\omega_c (N + \frac{1}{2})$	$\hbar\omega_c (N + M + \frac{1}{2})$	$\hbar\omega_c (N + M + \frac{1}{2})$	$\hbar\omega_c (N + \frac{1}{2})$
$\beta^2 e_c^2$	$E_{\perp} / \hbar\omega_c$	$N + \frac{1}{2}$	$N + M + \frac{1}{2}$	$N + M + \frac{1}{2}$	$N + \frac{1}{2}$
$\beta^2 e_o^2$	$\frac{e}{q} \frac{L_z}{\hbar} + \beta^2 e_c^2$	$N + M + \frac{1}{2}$	$N + \frac{1}{2}$	$N + \frac{1}{2}$	$N + M + \frac{1}{2}$



Table C2. Eigenstate expectation values of the cylindrical Landau eigenfunctions  $\Psi_{NMk}$ .

Charge Sign		q > 0 (proton)		q < 0 (electron)	
Sign of the set $\{\Phi_M\}$		+M: $e^{+iM\phi}$	-M: $e^{-iM\phi}$	+M: $e^{+iM\phi}$	-M: $e^{-iM\phi}$
Group		II	I	I	II
Significance		$\frac{r_0}{r_c} > 1$ origin outside gyrocircle	$\frac{r_0}{r_c} < 1$ origin inside gyrocircle	$\frac{r_0}{r_c} < 1$ origin inside gyrocircle	$\frac{r_0}{r_c} > 1$ origin outside gyrocircle
Degeneracy		Arbitrarily large	N + M + 1	N + M + 1	Arbitrarily large
Quantity	Operator	Eigenstate expectation values			
$E_e$	$\frac{p_e^2}{2m}$	$2N + M + 1$ $\frac{\hbar\omega_c}{2} (N + \frac{1}{2})$	$2N + M + 1$ $\frac{\hbar\omega_c}{2} (N + \frac{1}{2})$	$2N + M + 1$ $\frac{\hbar\omega_c}{2} (N + \frac{1}{2})$	$2N + M + 1$ $\frac{\hbar\omega_c}{2} (N + \frac{1}{2})$
$E_\phi$	$(\frac{L_z}{e} - \frac{qB\phi^2}{2})/2m$	$\frac{\hbar\omega_c}{2} (N + \frac{1}{2})$	$\frac{\hbar\omega_c}{2} (N+2m+\frac{1}{2})$	$\frac{\hbar\omega_c}{2} (N+2m+\frac{1}{2})$	$\frac{\hbar\omega_c}{2} (N + \frac{1}{2})$
$1/\phi^2$	$1/e^2$	1/M	1/M	1/M	1/M
$\dot{\phi}$	$\frac{L_z}{me^2} - \frac{qB}{2m}$	0	$-\omega_c$	$\omega_c$	0

### Degeneracy of the Perpendicular Energy Levels

Of particular interest is the degeneracy of the energy levels  $E_0 = \hbar^2 k^2 / 2m$  as given by (C22) and in Table C1. We shall for the moment ignore the two-fold degeneracies associated with the sign of  $k$  and with the (fermion) spin. The latter degeneracy would be removed by the explicit introduction of the spin energy and wave functions.

For the Group I (origin inside the gyrocircle) we have listed the degeneracy in energy as  $N + M + 1$ . We are speaking here of degeneracy as it is commonly used in the physical sense of the word. In a purely mathematical sense (that is, divorced from all questions of experiment and considering only the algebraic forms), the degeneracy of this group is 1 (i. e., non-degenerate). The energy eigenvalue expression contains explicitly both  $N$  and  $M$ . For given and separately identifiable  $N$  and  $M$ , there is associated only a single group I eigenfunction. Physical degeneracy, on the other hand, is determined by how many eigenfunctions belong to a given numerical (as opposed to algebraic) value of the energy in this case the sum ( $N + M$ ). At the root of this usage is the recognition that an experimental energy measurement by itself<sup>1</sup> can yield only the sum ( $N + M$ ), and not the separate and individual  $N$  and  $M$  values. In this sense the number of eigenfunctions belonging to a given group I energy level  $\hbar\omega_c (N + M + 1/2)$  is  $N + M + 1$ , as may be inferred from simple numerical examples. There is no infinite degeneracy associated with group I levels. For either  $N$  or  $M$  equal to zero, there is no degeneracy at all.

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<sup>1</sup>This does not preclude a subsequent measurement of another dynamical variable which might select a single eigenstate from the energy degenerate mixture of  $N + M + 1$  states. We are also assuming that the instrumental energy bandwidth is no larger than  $\hbar\omega_c$ , i. e., that it is capable of resolving a single level. This is at present experimentally difficult. The best energy resolution obtained to date is about 20 milli-

The situation is different for the eigenfunctions and energy levels of group II (origin outside the gyrocircle). With each of the group II levels there is associated an arbitrarily large positional degeneracy. Here the energy depends only upon the quantum number  $N$ . The quantum number  $M$ , associated with the location of the gyrocenter with respect to the origin does not appear in the energy eigenvalue expression. Thus an arbitrarily large number of eigenfunctions  $\psi_{NMk}$  belong to a given group II energy level  $\hbar \omega_c (N + 1/2)$ . This degeneracy disappears for  $M = 0$ , but remains for  $N = 0$  and  $M \neq 0$ .

### Surfaces of Constant Energy

The distinction between the degeneracies of the group I and group II discrete energy levels is clearly revealed in the appearance of their surfaces of constant total energy. These surfaces may be portrayed in a Cartesian quarter-space, the axes of which might be labelled  $N\hbar\omega_c$ ,  $\pm M\hbar\omega_c$ , and  $E_z$ . This has been done in Fig. C3 where we have plotted the group I and II surfaces of constant energy for  $q = -e$  (an electron). The surfaces for  $q = e$  follow upon interchange of the axis labels  $+M\hbar\omega_c$  and  $-M\hbar\omega_c$ . In the positive quadrant, labelled I, is plotted the surface

$$E_z + \hbar\omega_c (N + M + \frac{1}{2}) = \text{const.} \equiv E_0 \quad (\text{C42})$$

and in the negative quadrant, labelled II, is plotted

$$E_z + \hbar\omega_c (N + \frac{1}{2}) = \text{const} \equiv E_0 \quad (\text{C43})$$

The particular value of  $E_0$  used for this illustration was slightly less than  $10\hbar\omega_c$ . The same value was used for both surfaces. The allowed positions (states) in this energy space are indicated by large dots. This

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ev [J. F. Waymouth, private communication, 1964] with values ranging up to 60 mev cited in experiments wherein it was desirable to attain the best possible resolution. See for example the abstracts of papers A3 and C3(d) of the Seventeenth Annual Gaseous Electronics Conference [1965], and Kuyatt, Simpson, and Mielczarek [1965]. The value of  $B$  corresponding to  $\hbar\omega_c = 20$  mev is 172 gauss.

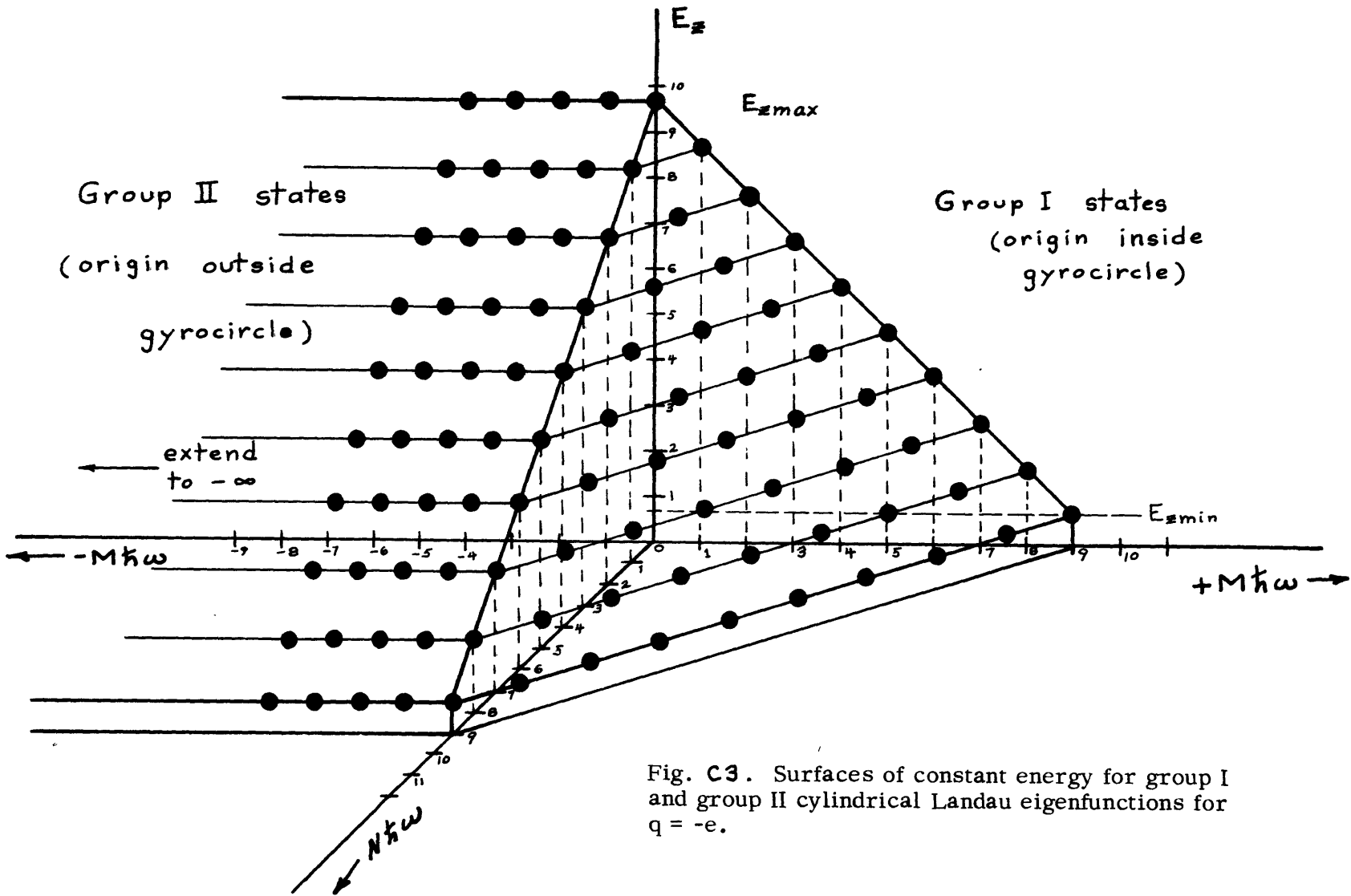


Fig. C3. Surfaces of constant energy for group I and group II cylindrical Landau eigenfunctions for  $q = -e$ .

figure will be helpful in visualizing energy-conserving transitions from one state to another as a result of the Coulomb perturbation. We discuss in detail only the properties of the group I surface since those of the group II surface follow upon obvious and slight modification.

The number of states and the limits of the group I surface are completely determined by specification of the total energy  $E_0$ . We have the relations

$$E_{z \max} = E_0 - \frac{1}{2} \hbar \omega_c \quad (C44)$$

$$N_{\max} = \text{integer part of } \left( \frac{E_0}{\hbar \omega_c} - \frac{1}{2} \right) \quad (C45)$$

$$M_{\max} = \text{integer part of } \left( \frac{E_0}{\hbar \omega_c} - \frac{1}{2} \right) \quad (C46)$$

$$E_{z \min} = \text{fractional part of } \left( \frac{E_0}{\hbar \omega_c} - \frac{1}{2} \right). \quad (C47)$$

Note that  $0 \leq E_{z \min} < \hbar \omega_c$ . Since  $N_{\max} = M_{\max} = (E_{z \max} - E_{z \min}) / \hbar \omega_c$ , the three sides of the surface are of equal length, and the surface normal makes equal angles with the coordinate axes. The surface forms one of the four sides of an equilateral pyramid. The pyramid rests on a base of height  $E_{z \min}$ . Group I energy states having the same value of  $E_z$  and belonging to the same (numerical) value of the perpendicular energy  $\hbar \omega_c (N + M + 1/2)$  are connected by lines parallel to the bottom edge of the pyramid. There are  $N_{\max} + 1$  of these lines having the sequence of  $E_z$  values

$$E_{z_{\max}}, E_{z_{\max}} - \hbar\omega_c, \dots, E_{z_{\max}} - j\hbar\omega_c, \dots, E_{z_{\max}} - N_{\max}\hbar\omega_c = E_{z_{\min}}.$$

These values are generated by the function  $E_z(j)$  where

$$E_z(j) \equiv E_0 - (j + 1/2)\hbar\omega_c \text{ for } j = 0, 1, 2, \dots, N_{\max}. \quad (\text{C48})$$

With these lines (or groups of degenerate states) thus indexed by  $j$ , it is seen that the  $j$ -th line contains  $j + 1$  states. That is, the  $j$ -th perpendicular energy level is  $j + 1$  degenerate ( $j + 1$  is a special case for  $N = j$  and  $M = 0$  of the earlier-discussed degeneracy  $N + M + 1$ ). The total number of states on the surface is obtained by summing over the  $(j + 1)$  - degenerate levels:

$$\text{Number of group I states} = \sum_{j=0}^{N_{\max}} (j + 1) = \frac{(N_{\max} + 1)(N_{\max} + 2)}{2}.$$

If  $N_{\max}$  is increased by unity, the number of states added is  $N_{\max} + 2$ . Conversely, if  $N_{\max}$  is decreased by unity, the number of states is reduced by  $N_{\max} + 1$ . A unit increase in  $N_{\max}$  results from an increase of  $E_0$  by an amount  $\hbar\omega_c - E_{z_{\min}}$ .

Having outlined the properties of the group I surface, we now relax the condition  $E_0 = \text{constant}$  and describe the evolution of the surface as  $E_0$  is increased. We suppose that initially  $E_{z_{\min}} > 0$ . As the total energy  $E_0$  increases and so long as  $E_{z_{\min}} < \hbar\omega_c$ , the number of states remains constant. The surface rises smoothly and continuously in the  $E_z$  direction while maintaining the same area and orientation. The increase in  $E_0$  goes entirely into  $E_z$ , the states riding upwards as if they were beads on rods. However, as  $E_0$  is increased further and as  $E_{z_{\min}}$  passes through the value  $\hbar\omega_c$ , the value of  $N_{\max}$  is increased by one,  $N_{\max} + 2$  states are added, and  $E_{z_{\min}}$  assumes the value zero. An analogous kinematical picture may be developed for a decrease in  $B$  or  $\hbar\omega_c$ .

The properties of the group II energy surface are quite similar. For the same value of  $E_o$ , the values of  $E_{zmax}$ ,  $E_{zmin}$ , and  $N_{max}$  are the same as the group I values. For a given value of  $-M$ , there are  $N_{max} + 1$  states. Unit increase in  $N_{max}$  is matched by unit increase in the number of states. The positional degeneracy is represented by states having the same value of  $E_z$  and belonging to the same  $N$  value. In practice, this degeneracy is very large, but not infinite because of the finite size of the confining experimental apparatus. In stating this, we are ignoring the (probably small) effect of the changed radial boundary condition upon the eigenfunctions and eigenvalue spectrum.

#### C4 Construction of a Uniform Beam

The Born approximation cross section resulting when only a single member of the set  $\Psi_{NMk}$  is used in the perturbation matrix element is differential in nature, with the eigenvalue for  $\beta^2 \rho_o^2$  playing the role of impact parameter. Evaluation of a total cross section requires an incident beam of sufficient radial extent to intercept virtually all of the perturbation potential field (considered here to be seated at the origin). Further, the beam should be of uniform amplitude in the radial direction. In the case of zero magnetic field, such a beam is immediately at hand, and is provided by a single free particle eigenfunction of the form  $\exp(ikz)$ . However, the provision of such a flooding beam is not so simple for a free charge in a magnetic field, due to the radial localization and binding by the magnetic field. The construction of the wave function for this beam from the eigenfunctions  $\{\Psi_{NMk}\}$  is described below, and is guided by the interpretation developed for the eigenvalues of  $\beta^2 \rho_c^2$  and  $\beta^2 \rho_o^2$ . The roles of the group I and group II eigenfunctions will be clearly exhibited, as will the simple relation of this beam wave function to the energy surfaces of Fig. C3.

Definition and Representation

From members of the set  $\{\Psi_{NMk}\}$  we seek to construct the wave function representing a charged particle beam having a uniform distribution of gyrocenters extending from the origin out to the (at present arbitrary) squared radial distance defined by the integer  $S \equiv \beta^2 \rho_{o\max}^2$ . The beam is further to be characterized by single values of the energies  $E_{\perp}$  and  $E_z$ . We measure these energies in units of  $\hbar\omega_c$  by means of the integer  $N_{\perp}$  and the number  $N_z$ :

$$E_{\perp} \equiv \hbar\omega_c (N_{\perp} + \frac{1}{2}) \tag{C49}$$

$$E_z \equiv \hbar\omega_c N_z. \tag{C50}$$

The integer  $N_{\perp}$  will in some cases be the same as the quantum integer  $N$ . From the interpretation of the quantum integers  $N$  and  $M$  in terms of  $\beta^2 \rho_o^2$  and  $\beta^2 \rho_c^2$ , it follows that such a beam is represented by the function

$$U_{N_{\perp} N_z S} = C \left[ \sum_{j=0}^{N_{\perp}-1} \Psi_{j, N_{\perp}-j, k} + \sum_{M=0}^S \Psi_{N_{\perp}, M, k} \right] \tag{C51}$$

where  $\Psi_{r, s, t}$  denotes members of the set  $\{\Psi_{NMk}\}$ . The normalization constant  $C$  is yet to be determined.

The evolution of this form may be understood in a simple graphical manner by drawing, about a definite origin, a series of constant radius circles whose centers begin at the origin and are successively displaced by equal increments along a radial line. Then, referring to Table C1 for the eigenvalues of  $\beta^2 \rho_o^2$  and  $\beta^2 \rho_c^2$ , the circles should be labelled in terms of the numbers  $N$ ,  $M$ , and  $N_{\perp}$ . There will be a finite number of circles which enclose the origin. These are the group I states and are represented by the terms in the first sum (to  $j = N_{\perp} - 1$ ) of  $U_{N_{\perp} N_z S}$ . There will be a single circle passing through the origin.



This is the group I-group II boundary state  $\psi_{N_0 k}$ . It has been included in (C51) as the first term of the second sum, on M. This boundary state could as well have been included instead as the  $j = N_{\perp}$  term of the first sum. The remaining terms in the second sum consists of group II states and have as their analog the circles which do not enclose the origin.

For a beam travelling as nearly as possible along the magnetic field direction, we set  $N_{\perp}$  to zero and omit the group I eigenstates (the first sum in C51). The function  $U_{0 N_z S}$  would then describe a beam composed of particles having all of their energy in the z mode (zero pitch angle).

#### Position Upon the Surfaces of Constant Energy

By locating each eigenstate of the sums in (C51) upon the constant energy surfaces depicted in Fig. C3, it will be seen that the uniform beam is composed of all group I-group II states having the same total and z energy. That is,  $U_{N_{\perp} N_z S}$  is the simple sum of all states situated upon one of the constant  $E_z$  lines (or levels) running axis out to  $-M = -S$ . These  $(N_{\perp} N_z S)$  levels contain  $N_{\perp} + S + 1$  cylindrical Landau eigenstates.

#### Normalization and z Flux

By requiring the wave function  $U_{N_{\perp} N_z S}$  to satisfy the condition

$$\int_k \int_{\tau} U^* U d\tau dk \equiv \int_k \langle U | U \rangle dk = 1, \quad (C52)$$

the normalization constant follows as

$$C = \frac{1}{\sqrt{N_{\perp} + S + 1}}. \quad (C53)$$

This form for C should not be surprising in view of the number and equal weighting of the eigenstates used to construct the wave function U.

More generally, we can show that

$$\langle U_{N'_\perp N'_z S'} | U_{N_\perp N_z S} \rangle = \delta_{N'_\perp, N_\perp} \delta(k'-k) \quad (C54)$$

where

$$C = \frac{1}{\sqrt{N_\perp + \min(S', S) + 1}} \quad (C55)$$

This follows in a straightforward though interesting way through use of (C51) and the eigenstate matrix element contained in (C15b). The calculation of these results is interesting because the group I-group II cross products inherent in (C54) contribute nothing and because the smaller beam controls the result of the group II-group II product.

If we consider the probability density flux component  $J_z$  as an operator, we can show in an almost identical manner that

$$\langle U_{N'_\perp N'_z S'} | J_{zop} | U_{N_\perp N_z S} \rangle = \frac{\pm \hbar k}{m} \delta(k'-k) \delta_{N'_\perp, N_\perp} \quad (C56)$$

The area-averaged flux associated with the  $N_\perp=0$  beam  $U_{0N_z S}$  is

$$\Gamma_z = \frac{\hbar k}{m} \frac{4\beta^2 dk}{(S+2)2\pi} \quad (C57)$$

This flux was calculated in a manner analogous to that of (C32), the details of which are to be found in appendix F.

### C5 Summary

The cylindrical Landau eigenfunctions  $\Psi_{NMk}$  were obtained as solutions to the coordinate representation of the Schroedinger equation

$$H_0 \Psi_{NMk} = E_0 \Psi_{NMk} \quad (C58)$$

The Hamiltonian  $H_0$  is that of a single particle of charge  $q$  and mass  $m$  moving in a uniform magnetic field. As discussed in appendix A, this Hamiltonian is  $(\vec{p} - q\vec{A})^2 / 2m$ . The equation was considered in the cylindrical coordinate system spanned by the triad of unit vectors  $\hat{e} \times \hat{\phi} = \hat{z}$ . The uniform and constant magnetic field was generated from the vector potential

$$\vec{A} = \frac{1}{2} \vec{B} \times \vec{r} = \frac{1}{2} B \rho \hat{\phi} \quad (\text{C59})$$

through the relation

$$\vec{B} = \text{curl } \vec{A} = B \hat{z}. \quad (\text{C60})$$

The solutions were factorable in each of the coordinates as

$$\Psi_{NMk}(\rho, \phi, z) \equiv R_{NM}(\rho) \Phi_M(\phi) Z_k(z). \quad (\text{C61})$$

The factored eigenfunctions have the forms

$$R_{NM} = \sqrt{2 \rho^2 \frac{N!}{(N+M)!}} (\rho^2 e^{\pm 2})^{\frac{M}{2}} e^{-\frac{1}{2} \beta^2 \rho^2} L_N^M(\beta^2 \rho^2) \quad (\text{C62})$$

$$\Phi_M = \frac{1}{\sqrt{2\pi}} e^{\pm i M \phi} \quad (\text{C63})$$

$$Z_k = \frac{1}{\sqrt{2\pi}} e^{\pm i k z} \quad k \equiv \frac{p_z}{\hbar} = \frac{m v_z}{\hbar} \quad (\text{C64})$$

where  $\beta^2 \equiv eB/2\hbar$  (dimensions of  $m^{-2}$ ), and  $N$  and  $M$  are independent positive integers (including zero having no formal upper bound:

$$N = 0, 1, 2, \dots$$

$$M = 0, 1, 2, \dots$$

The Laguerre polynomial, an oscillatory function having N zeroes, is defined in equations (C23). The eigenfunctions are separately normalized to Kronecker and Dirac delta functions such that

$$\langle N'M'k' | NMk \rangle \equiv \int_{\tau} \psi_{N'M'k'}^* \psi_{NMk} d\tau = \delta_{N',N} \delta_{M',M} \delta(k'-k). \quad (C65)$$

By means of quantum-classical correspondence arguments, or by construction of the appropriate operators, the eigenfunctions and states may be divided into two groups. Physically, the group I states correspond to classical orbits which enclose the origin, and group II states to those which do not enclose the origin. Whether a given eigenfunction belongs to group I or group II is determined by the sign of the  $\Phi_M$  eigenfunction and by the charge sign:

$$\text{Group I: } (\text{sign of } q) \times (\text{sign of } \Phi_M) < 0,$$

$$\text{Group II: } (\text{sign of } q) \times (\text{sign of } \Phi_M) > 0.$$

The observables of which M and N are eigenvalues for the group I and group II states are collected in Table C1. Several eigenstate expectation values are listed in Table C2. The energy levels and surfaces of constant energy are depicted in Fig. C3. The construction and properties of a beam characterized by single values of the perpendicular and parallel energies  $E_{\perp}$  and  $E_z$  and by a uniform distribution of gyrocenters are discussed in the preceding section.

APPENDIX D  
MATRIX ELEMENTS OF RADIAL POSITION AND ENERGY  
BETWEEN CYLINDRICAL LANDAU EIGENFUNCTIONS

D1 Content

We give here details of the evaluation of integrals representing quantum averages of dynamical quantities associated with the motion of a particle of charge  $q$  and mass  $m$  in a magnetic field  $\vec{B} = B\hat{z}$ . The system is referred to a cylindrical coordinate system spanned by the unit vectors  $\hat{e}_r \times \hat{e}_\phi = \hat{z}$ . The particle is represented by the cylindrical Landau eigenfunctions

$$\psi_{NMk} = R_{NM}(r) \Phi_M(\phi) \quad (D1)$$

derived in appendix C and rewritten below. If such a dynamical quantity is  $f(\vec{r}, \vec{p})$ , then the quantum average of  $f$ , denoted by  $\langle f \rangle$ , is here defined as

$$\langle f \rangle \equiv \int_{\tau} \psi_{NMk}^* f(\vec{r}, \frac{\hbar}{i} \vec{\nabla}) \psi_{NMk} d\tau, \quad (D2)$$

the integral to be taken over all space. Quantum averages of the type (D2) are more formally known as diagonal matrix elements. The definition may be extended in an obvious manner to include off-diagonal matrix elements here denoted by  $\langle N_2, M_2, k_2 | f | N_1, M_1, k_1 \rangle$ . The matrix element (D2) is interpreted as representing the probabilistic results of repeated measurements of the dynamical quantity  $f(\vec{r}, \vec{p})$  in a system described by the wave function  $\psi_{NMk}$ .

The matrix elements calculated include  $(\hat{e}^2 \hat{e}^2)^0$ ,  $\hat{e}^2 \hat{e}^2$ ,  $(\hat{e}^2 \hat{e}^2)^{-1}$  and  $\mathcal{P}_e^2 / 2m$ .

## D2 The Set of Basis Eigenfunctions

The eigenfunctions used as a basis in these matrix element calculations are the cylindrical Landau eigenfunctions obtained as solutions to the Schroedinger equation  $H_0 \Psi_{NMK} = E_{NMK} \Psi_{NMK}$  for the Hamiltonian  $H_0 = (\vec{p} - q\vec{A})^2 / 2m$ . The cylindrical form of this Hamiltonian was developed in appendix A:

$$H_0 = \frac{p_\rho^2 + \frac{L_z^2}{\rho^2} + p_z^2}{2m} - \frac{qB}{2m} L_z + \frac{q^2 B^2 \rho^2}{8m} \quad (D3a)$$

$$= \frac{p_\rho^2 + \left(\frac{L_z}{\rho} - \frac{qB\rho}{2}\right)^2 + p_z^2}{2m} \quad (D3b)$$

The solutions, factorable as indicated in (D1), and the energy eigenvalues are

$$R_{NM} = \sqrt{2 \beta^2 \frac{N!}{(N+M)!}} (\beta^2 \rho^2)^{\frac{M}{2}} e^{-\frac{1}{2} \beta^2 \rho^2} L_N^M(\beta^2 \rho^2) \quad (D4)$$

$$\Phi_M = \frac{1}{\sqrt{2\pi}} e^{\pm iM\phi} \quad (D5)$$

$$Z_k = \frac{1}{\sqrt{2\pi}} e^{\pm ikz} \quad k = \frac{p_z}{\hbar} = \frac{mv_z}{\hbar} \quad (D6)$$

$$E_{NMK} = \frac{\hbar^2 k^2}{2m} + \frac{\hbar \omega_c}{2} \left(2N + M + \frac{q}{e} M + 1\right) \quad (D7)$$

where  $\omega_c \equiv eB/m$  is the cyclotron frequency,  $\beta^2 \equiv eB/2\hbar$  is a parameter having dimensions of (meters)<sup>-2</sup>, and N and M are independent positive integers having no upper bound:

$$N = 0, 1, 2, \dots$$

$$M = 0, 1, 2, \dots$$

The choice of upper or lower sign in (D7) should conform to the choice made in the eigenfunction  $\bar{\Phi}_M$ . The convention on charge sign is such that, for example,  $q = -e$  denotes an electron. The factored eigenfunctions are separately normalized to unity as either a Kronecker or Dirac delta function. The Laguerre polynomial may be defined as

$$L_N^M(x) \equiv \sum_{j=0}^N \frac{(N+M)!}{(N-j)!(M+j)!} \frac{(-x)^j}{j!} \quad (D8)$$

In evaluating some of the matrix element integrals, we shall employ the generating function

$$\sum_{j=0}^{\infty} t^j L_j^M(x) = (1-t)^{-M-1} e^{-\frac{xt}{1-t}}, \quad (D9)$$

valid for  $|t| < 1$  [HTF 2, p. 189(17)].

### D3 The Matrix Element $\langle e^{2\alpha} \rangle$

The range of the integer  $\alpha$  is  $-M \leq \alpha < \infty$ . Since  $e^{2\alpha}$  is independent of  $\phi$  or  $z$ , we know immediately from the normalization that the general matrix element has the form

$$\langle N_2 M_2 k_2 | e^{2\alpha} | N_1 M_1 k_1 \rangle = \delta_{M_2, M_1} \cdot \delta(k_2 - k_1) \cdot \int_0^{\infty} R_2 e^{2\alpha} R_1 e \, d\rho. \quad (D10)$$

That is, the matrix element  $\langle e^{2\alpha} \rangle$  is diagonal in M and k. Implicit in (D10) is the assumption that either the plus or the minus sign is used in both  $\bar{\Phi}_{M_1}$  and  $\bar{\Phi}_{M_2}$ . If, for example, the minus sign is used

in  $\Phi_{M_1}$ , and the plus sign in  $\Phi_{M_2}$ , then the matrix element survives only for  $M_1 = M_2 = 0$ :

$$\frac{1}{2\pi} \int_0^{2\pi} e^{i(M_1+M_2)\phi} d\phi = \delta_{M_1+M_2, 0} . \quad (D11)$$

This indicates that the only common point or union of the sets  $\{e^{+iM\phi}\}$  and  $\{e^{-iM\phi}\}$  is the point  $M = 0$ . Analogous properties exist in connection with the signs of  $Z_k$ . With these properties in mind, we rewrite (D10) as

$$\langle e^{2\alpha} \rangle = \int_0^\infty R_2(e) e^{2\alpha} R_1(e) e \, de , \quad (D12)$$

omitting the delta functions. The subscript 1 stands for the pair  $(N_1, M_1)$ . This may be cast into the form

$$\langle (ee)^{2\alpha} \rangle = \sqrt{\frac{N_1! N_2!}{(N_1+M)! (N_2+M)!}} I_{N_1, N_2}^{M, \alpha} \quad (D13)$$

where I stands for the integral

$$I_{N_1, N_2}^{M, \alpha} \equiv \int_0^\infty e^{-x} x^{M+\alpha} L_{N_1}^M(x) L_{N_2}^M(x) dx , \quad (D14)$$

taken over the dimensionless variable  $x = \beta^2 e^2$ . By operating upon (D14) with the generating function (D9) and evaluating the resulting RHS integral  $\int_0^\infty \exp[-x(1+\frac{s}{1-s} + \frac{t}{1-t})] \cdot x^{M+\alpha} dx$ , we obtain the identity

$$\sum_{N_1=0}^\infty \sum_{N_2=0}^\infty s^{N_1} t^{N_2} I_{N_1, N_2}^{M, \alpha} \equiv \frac{(M+\alpha)! (1-s)^\alpha (1-t)^\alpha}{(1-st)^{M+\alpha+1}} . \quad (D15)$$

We note the restriction  $M+\alpha \geq 0$ , necessary to insure convergence of the integrals involved (D12, D14). An expansion of the denominator of (D15) in ascending powers of  $st$  ( $< 1$ ) leads to the basic identity



$$\sum_{N_1=0}^{\infty} \sum_{N_2=0}^{\infty} s^{N_1} t^{N_2} I_{N_1, N_2}^{M, \alpha} \equiv (1-s)^{\alpha} (1-t)^{\alpha} \sum_{j=0}^{\infty} \frac{(M+\alpha+j)!}{j!} (st)^j. \quad (D16)$$

Case  $\alpha = 0$ . The Normalization Integral

This case serves as a check upon our procedures thus far.

For  $\alpha = 0$ , (D16) becomes

$$\sum_{N_1=0}^{\infty} \sum_{N_2=0}^{\infty} s^{N_1} t^{N_2} I_{N_1, N_2}^{M, 0} \equiv \sum_{j=0}^{\infty} \frac{(M+j)!}{j!} (st)^j \quad (D17)$$

from which it follows that

$$I_{N_1, N_2}^{M, 0} = \delta_{N_1, N_2} \frac{(M+N)!}{N!} \quad (D18)$$

or

$$\int_0^{\infty} e^{-x} x^M L_{N_1}^M(x) L_{N_2}^M(x) dx = \delta_{N_1, N_2} \frac{(M+N)!}{N!}. \quad (D19)$$

The relation (D19) expresses the orthogonality of the Laguerre polynomials  $L_{N_1}^M(x)$  and  $L_{N_2}^M(x)$  with respect to the weighting function  $e^{-x} x^M$  on the range  $0 \leq x < \infty$ . In its full glory, the general matrix element of  $\hat{\rho}^{\circ} (=1)$  has the form

$$\langle (\hat{\rho}\hat{\rho})^{\circ} \rangle = 1 \cdot \delta_{N_2, N_1} \cdot \delta_{M_2, M_1} \cdot \delta(k_2 - k_1). \quad (D20)$$

Hereafter we shall usually omit the writing of the delta function factors.

Case  $\alpha = 1$ . The Matrix Element  $\langle \beta^2 \rho^2 \rangle$

For  $\alpha = 1$ , the identity (D16) assumes the form

$$\sum_{N_1=0}^{\infty} \sum_{N_2=0}^{\infty} s^{N_1} t^{N_2} I_{N_1, N_2}^{M, 1} \equiv (1-s)(1-t) \sum_{j=0}^{\infty} \frac{(M+1+j)!}{j!} (st)^j \quad (D21)$$

from which we obtain the result

$$I_{N_1, N_2}^{M, 1} = \delta_{N_2, N_1} \frac{(M+N)!}{N!} (M+2N+1) - \left[ \delta_{N_2, N_1+1} \frac{(M+N_2)!}{(N_2-1)!} + \delta_{N_1, N_2+1} \frac{(M+N_1)!}{(N_1-1)!} \right] \quad (D22)$$

It is seen that in general there are only three surviving matrix elements of  $\rho^2$ . The matrix element of  $\rho^2$  which is diagonal in M, N, and k has the value

$$\langle \rho^2 \rho^2 \rangle = M+2N+1, \quad (D23)$$

where we have omitted the delta function factors.

Case  $\alpha = -1$ . The Matrix Element  $(\beta^2 \rho^2)^{-1}$

We utilize the expansion

$$(1-t)^{-1} = \sum_{i=0}^{\infty} t^i \quad (D24)$$

in (D16) to obtain

$$\sum_{N_1=0}^{\infty} \sum_{N_2=0}^{\infty} s^{N_1} t^{N_2} I_{N_1, N_2}^{M, -1} \equiv \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{(M-1+j)!}{j!} s^{j+i} t^{j+k} \quad (D25)$$

The (convergent) sums on the RHS are rearranged according to the transformations  $\eta_1 \equiv j+i$  and  $\eta_2 \equiv j+k$ . We have the option of choosing  $j = \eta_1 - i$  and summing on i from 0 to  $\eta_1$ , or of choosing  $j = \eta_2 - k$  and summing on k from 0 to  $\eta_2$ . Both options are indicated:

$$\sum_{N_1=0}^{\infty} \sum_{N_2=0}^{\infty} s^{N_1} t^{N_2} I_{N_1, N_2}^{M, -1} \equiv \sum_{\gamma_1=0}^{\infty} \sum_{\gamma_2=0}^{\infty} \left\{ \begin{array}{l} \sum_{i=0}^{\gamma_1} \frac{(M-1+\gamma_1-i)!}{(\gamma_1-i)!} \\ \sum_{k=0}^{\gamma_2} \frac{(M-1+\gamma_2-k)!}{(\gamma_2-k)!} \end{array} \right\} s^{\gamma_1} t^{\gamma_2} \quad (D26a)$$

$$(D26b)$$

If we let N stand (momentarily) for either  $N_1$  or  $N_2$  and reverse the order of summation, it follows from (D26) that

$$I_{N_1, N_2}^{M, -1} = \sum_{j=0}^N \frac{(M-1+j)!}{j!} = (M-1)! \sum_{j=0}^N \binom{M-1+j}{j} \quad (D27)$$

With the aid of a formula from Richards [p. 259, 1959], the sum may be performed analytically:

$$I_{N_1, N_2}^{M, -1} = \frac{1}{M} \frac{(M+N)!}{N!} \quad (D28)$$

We must now consider whether N stands for  $N_1$  or  $N_2$ . The answer is that

$$N = \min(N_1, N_2). \quad (D29)$$

In arriving at this identification, we found no comfort in (D26), but were forced to consider the definition (D14). Using the expansion

$$L_{N_1}^M(x) = \sum_{n=0}^{N_1} L_n^{M-1}(x) \quad (D30)$$

( a special case of HTF 2, p. 192 (39)), the definition (D14) may be written

$$I_{N_1, N_2}^{M, -1} = \sum_{n=0}^{N_1} \sum_{m=0}^{N_2} \int_0^{\infty} e^{-x} x^{M-1} L_n^{M-1}(x) L_m^{M-1}(x) dx. \quad (D31)$$

Using (D19), this becomes

$$I_{N_1, N_2}^{M, -1} = \sum_{n=0}^{N_1} \sum_{m=0}^{N_2} \delta_{n,m} \frac{(M-1+n)!}{n!}. \quad (D32)$$

It appears at first sight as if the result of the summations of (D32) depends upon the order of performance. However, this is not so. Supposing that  $N_2 > N_1$ , the basic reason is that, irrespective of the order of summation, the Kronecker delta will select all of the terms of the n-sequence and only the first  $N_1 + 1$  terms of the sequence  $m = 0, 1, 2, \dots, N_1, \dots, N_2$ . Thus the result (D28) is valid for N chosen according to the prescription (D29).

The matrix element of  $(\beta^2 \rho^2)^{-1}$  which is diagonal in N, M, and k is

$$\left\langle \frac{1}{\rho^2 \rho^2} \right\rangle = \frac{1}{M}. \quad (D33)$$

We see that  $M = 0$  is not allowed, in accordance with the restriction  $\alpha + M \geq 0$ . Again, the delta functions have been omitted.

#### D4 Radial Energy. The Matrix Element $\langle p_\rho^2/2m \rangle$

The only term in the Hamiltonian (D3) not yet explicitly evaluated is the radial energy  $p_\rho^2/2m$ . This may be done by calculating the matrix element of the operator  $-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial \rho^2}$  or, knowing as we do all other terms, by subtraction:

$$\left\langle \frac{p_\rho^2}{2m} \right\rangle = \left\langle H_0 - \frac{L_z^2}{2m \rho^2} - \frac{p_z^2}{2m} + \frac{qB}{2m} L_z - \frac{q^2 B^2 \rho^2}{8m} \right\rangle. \quad (D34)$$

Borrowing from appendix C the results  $\langle L_z \rangle = \pm M \hbar$  and  $p_z = \pm \hbar k/m$ , and using (D7), (D23), and (D33), it is seen that

$$\left\langle \frac{p_z^2}{2m} \right\rangle = \frac{\hbar \omega_c}{2} \left( N + \frac{1}{2} \right). \quad (\text{D35})$$

The result is independent of both charge sign and the sign associated with  $\Phi_M$ . It is in agreement with the value listed in Table C2.

## APPENDIX E

### EVALUATION OF THE COULOMB MATRIX ELEMENT BETWEEN CYLINDRICAL LANDAU EIGENSTATES

#### E1 Content

Given in this appendix are the analytical and computational details relevant to the evaluation of the matrix element of the Coulomb potential energy between the cylindrical Landau eigenfunctions  $\Psi_{NMk}$  described in appendix C. The functional dependences and the relation to classical models are discussed. This matrix element is needed for the Born approximation evaluation of the transition probability and the cross section for the scattering by a Coulomb potential of a charge  $q$  in a magnetic field.

Two Coulomb matrix elements were considered, one more directly suited to our needs than the other. They differ primarily in the machinery utilized for mathematically varying the position of the scatterer  $Q$  (the seat of the Coulomb potential) with respect to the initial position of the scattered charge  $q$ . This facility must be present in the formalism since we cannot in experiment control the location of  $q$  with respect to  $Q$ .

#### The Centered Coulomb Matrix Element

The more useful of the two (and the one reported on in detail here) has been called the centered Coulomb matrix element. In this case, the seat of the Coulomb potential was fixed at the origin and one or both of the quantum numbers  $N$  and  $M$  used to vary the radial distance between  $q$  and  $Q$ . The impact parameter is thus a discrete rather than a continuous variable. It is this matrix element which appears in going from the laboratory frame two-body problem to the center of mass frame problem of a single particle moving in a central potential field emanating from the origin.

This matrix element, for which an exact result has been obtained, was denoted by

$$\langle g\mathcal{L}_c \rangle \equiv \langle N_2, M, k_2 | g\mathcal{L}_c | N_1, M, k_1 \rangle. \quad (\text{E1})$$

The subscripts 1 and 2 label the quantum parameters of the initial and final cylindrical Landau eigenstates, respectively. The subscript c indicates that the seat of the Coulomb potential (the charge Q) was centered, i. e., fixed at the origin. The particular form of the Coulomb potential energy used in this matrix element is

$$g\mathcal{L}_c = \frac{gQ}{4\pi\epsilon_0} \frac{e^{-\mu|\vec{r}|}}{|\vec{r}|} \quad (\text{E2a})$$

which in the cylindrical coordinates employed becomes

$$g\mathcal{L}_c = \frac{gQ}{4\pi\epsilon_0} \frac{e^{-\mu\sqrt{\rho^2+z^2}}}{\sqrt{\rho^2+z^2}}. \quad (\text{E2b})$$

The Debye or Yukawa shielding can be incorporated at little additional mathematical cost, may be needed for convergence of the total cross section, and can be removed at any time by letting  $\mu \rightarrow 0$ . We have indicated in (E1) that  $M\hbar$ , the quantized value of the angular momentum canonical to the coordinate  $\phi$ , is conserved in the transition. This reflects the cylindrical symmetry of the system for this location of Q, and is consistent with both the classical and the quantum treatments of the general problem. The conservation of M is expressed by a Kronecker delta  $\delta_{M_1, M_2}$ . This is convenient not only because it leaves N and k as the quantum variables of this matrix element, but also because it will facilitate formation of a total cross section from combinations of the basic matrix element.

When this matrix element is utilized in a cross section, it will be required that energy be conserved in the transition. Thus, as a result of the collision, energy may be transferred into or out of the z mode (change of k) at the expense or gain of the perpendicular energy.

There will in general also be an accompanying change in the gyrocenter location. The cross section derived from this basic matrix element is differential in that it describes the effect of the Coulomb center  $Q$  upon a cylindrical shell beam characterized by a single value of the perpendicular energy and located at a certain radial distance from  $Q$ . Cross sections derivable from the centered Coulomb matrix elements are considered in appendix H.

After a cursory examination in the following paragraphs of the physical situation represented by the other matrix element, we present in section E2 the details of execution of the centered Coulomb matrix element. As will be seen, the matrix element is reduced to a finite series of confluent hypergeometric psi functions. The remarkably simple properties of this formidable-appearing function are described in the third section. In section E4 is discussed the variation of the continuous argument of this function, with conservation of energy incorporated. The section following combines the results of the previous two into a picture of the behavior and properties of the centered Coulomb matrix element. The sixth and final section contains details of a physically interesting limiting case of the general expression.

### The Off-Center Coulomb Matrix Element

In the second Coulomb matrix element considered, but not reported on in detail here, the problem of  $q$ - $Q$  location was approached in a different manner. In this case the scatterer  $Q$  was located arbitrarily with respect to the origin. For a Coulomb center  $Q$  fixed at the point  $\vec{r}_e = (\rho_e, \phi_e, z_e)$ , the potential energy of a charge  $q$  located at the point  $\vec{r} = (\rho, \phi, z)$  is given by



$$q\Lambda_{oc} = \frac{qQ}{4\pi\epsilon_0} \frac{e^{-\mu|\vec{r}-\vec{r}_e|}}{|\vec{r}-\vec{r}_e|} . \quad (\text{E3a})$$

This location of the seat of the potential away from the origin was accomplished at the expense of an infinite sum and integral expansion:

$$q\Lambda_{oc} = \frac{qQ}{4\pi\epsilon_0} \sum_{\nu=0}^{\infty} (2-\delta_{0,\nu}) \cos \nu(\phi-\phi_e) \cdot \int_0^{\infty} J_{\nu}(\lambda\rho) J_{\nu}(\lambda\rho_e) \frac{e^{-\sqrt{\mu^2+\lambda^2}|z-z_e|}}{\sqrt{\mu^2+\lambda^2}} \lambda d\lambda . \quad (\text{E3b})$$

This form was obtained from the expansion of the Green's function  $(\exp ikr)/R$  for the scalar Helmholtz equation by letting this  $k$  be purely imaginary. The expansion is given by Morse and Feshbach [1963, p. 888] and by Magnus and Oberhettinger [1949, p. 155]. We will see that  $\phi_e$  and  $z_e$  may be set to zero without loss of generality.

Since there is incorporated into this matrix element the facility for arbitrary location of the scatterer  $Q$  with respect to the origin, we are at liberty to dispense with the positioning feature contained in the incident state representation of the scattered charge  $q$ . This is accomplished by setting  $N$  to zero in the eigenfunctions  $\psi_{NMk}$ . Thus the group I members of this  $N = 0$  subset represent a cylindrical shell beam centered about the  $z$  axis and characterized by a squared cyclotron radius proportional to  $M$ , the minimal origin-to-gyrocenter value, and a  $z$  energy proportional to  $k^2$ . The members correspond physically to a non-degenerate beam composed of particles having the same  $z$  and perpendicular energies, and a common gyrocenter. In the same sense, the group II members of the  $N = 0$  subset represent a cylindrical shell beam whose squared radius is proportional to  $M$  and which is composed

of particles travelling as nearly as possible along the direction of the magnetic field with a z energy proportional to  $k^2$ . The cross section derivable from this matrix element is differential in that it describes the effect of a differential ring of charge at the radial distance  $\rho_e$  upon the group I or II cylindrical shell beam described above.

The matrix element is denoted by

$$\langle q\Lambda_{oc} \rangle \equiv \langle N_2 M_2 k_2 | q\Lambda_{oc} | N_1 M_1 k_1 \rangle \quad (E4)$$

where  $N_1$  is to be set to zero. We have indicated that the final value  $M_2\hbar$  of the canonical angular momentum  $L_z$  may differ from the initial value  $M_1\hbar$ . This is not surprising since all of the foregoing classical and quantum considerations indicated that  $L_z$  is not conserved for Q located away from the z axis. Indeed, if the value of the matrix element is found to have a maximum as a function of  $(M_2 - M_1)$ , then this preferred M change may be taken as an estimate of the classical integral-over-an-encounter

$$\Delta L_z = -\frac{qQ}{4\pi\epsilon_0} \int_{-\infty}^{\infty} \frac{\partial}{\partial \phi} \frac{1}{\sqrt{\rho^2 + \rho_e^2 - 2\rho\rho_e \cos \phi}} dt \quad (E5)$$

obtained from equation (A16).

Even with the simplification provided by setting  $N_1 = 0$ , the mathematical difficulties introduced by the expansion (E3b) could not be completely overcome. This matrix element could be reduced only to a single integral. Because of this and because of the more general character of the centered matrix element, the off-center matrix element was not utilized. However it is mentioned here because it describes a conceptually simple situation, and because it was attempted.

## E2 Execution of the Centered Coulomb Matrix Element

The centered Coulomb matrix element is denoted and defined by the expressions

$$\begin{aligned} \langle g\Lambda_c \rangle &\equiv \langle N_2, \pm M_2, \pm k_2 | g\Lambda_c | N_1, \pm M_1, \pm k_1 \rangle \\ &\equiv \int_{\text{all space}} \Psi_{N_2 M_2 k_2}^* g\Lambda_c \Psi_{N_1 M_1 k_1} d\tau. \end{aligned} \quad (\text{E6})$$

The cylindrical Landau eigenfunctions  $\Psi_{NMk}$  were given in appendix C as

$$\Psi_{NMk}(e, \phi, z) \equiv R_{NM}(e) \Phi_M(\phi) Z_k(z) \quad (\text{E7})$$

$$R_{NM} = \sqrt{2\beta^2 \frac{N!}{(N+M)!}} (\beta^2 \rho^2)^{\frac{M}{2}} e^{-\frac{1}{2}\beta^2 \rho^2} L_N^M(\beta^2 \rho^2) \quad (\text{E8})$$

$$\Phi_M = \frac{1}{\sqrt{2\pi}} e^{\pm iM\phi} \quad (\text{E9})$$

$$Z_k = \frac{1}{\sqrt{2\pi}} e^{\pm ikz} \quad k = \frac{p_z}{\hbar} = \frac{mv_z}{\hbar} \quad (\text{E10})$$

where  $\beta^2 = eB/2\hbar$  (dimensions of  $m^{-2}$ ) and N and M are independent positive integers (including zero) having no upper bound:

$$N = 0, 1, 2, \dots,$$

$$M = 0, 1, 2, \dots$$

This shorthand notation for the  $\Psi_{NMk}$  allows us to write the matrix element as

$$\langle g\Lambda_c \rangle = \int_0^{2\pi} \Phi_{M_2}^* \Phi_{M_1} d\phi \cdot \int_0^\infty d\rho \cdot \rho \cdot R_{N_2 M_2}^* R_{N_1 M_1} \cdot \int_{-\infty}^\infty Z_{K_2}^* \frac{gQ}{4\pi\epsilon_0} \frac{e^{-\mu\sqrt{\rho^2+z^2}}}{\sqrt{\rho^2+z^2}} Z_{K_1} dz . \quad (\text{E11})$$

The matrix element (E11) has been analytically evaluated in three ways, giving in each case the same final result in terms of the confluent hypergeometric function  $\Psi(\alpha, \gamma, z)$ . Definitions and properties of this function are given in HTF I [1953, chapter VI], in the book by Lebedev [1965, chapter 9], and in NBS [1964, chapter 13]. In NBS, this function is denoted by  $U(\alpha, \gamma, z)$ . We defer a discussion of the properties of this function until it is seen precisely which forms must be considered.

### Method I

Let us consider the  $\phi$  integral in (E11) for the transitions

$$\left\{ \begin{array}{l} +M_1 \longrightarrow +M_2 \\ -M_1 \longrightarrow -M_2 \end{array} \right\} . \quad (\text{E12})$$

The result is

$$\int_0^{2\pi} \Phi_{M_2}^* \Phi_{M_1} d\phi = \frac{1}{2\pi} \int_0^{2\pi} e^{\pm i(M_1 - M_2)\phi} d\phi = \delta_{M_1, M_2} . \quad (\text{E13})$$

Had we instead considered the transitions

$$\left\{ \begin{array}{l} +M_1 \longrightarrow -M_2 \\ -M_1 \longrightarrow +M_2 \end{array} \right\} , \quad (\text{E14})$$

the result would have been

$$\int_0^{2\pi} \Phi_{M_2}^* \Phi_{M_1} d\phi = \delta_{M_1,0} \cdot \delta_{M_2,0} \quad , \quad (\text{E15})$$

a result already contained in (E13).

We consider next the z integral for the transitions

$$\left\{ \begin{array}{l} + k_1 \rightarrow + k_2 \\ - k_1 \rightarrow - k_2 \end{array} \right\} \quad (\text{E16a})$$

$$\left\{ \begin{array}{l} - k_1 \rightarrow - k_2 \\ + k_1 \rightarrow + k_2 \end{array} \right\} \quad (\text{E16b})$$

We have the sequence

$$\begin{aligned} & \int_{-\infty}^{\infty} Z_{k_2}^* \frac{qQ}{4\pi\epsilon_0} \frac{e^{-\mu\sqrt{\rho^2+z^2}}}{\sqrt{\rho^2+z^2}} Z_{k_1} dz \\ &= \frac{qQ}{4\pi\epsilon_0} \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{\pm i(k_1-k_2)z - \mu\sqrt{\rho^2+z^2}}}{\sqrt{\rho^2+z^2}} dz \end{aligned} \quad (\text{E17a})$$

$$= \frac{qQ}{4\pi\epsilon_0} \frac{1}{\pi} \int_0^{\infty} \frac{e^{-\mu\sqrt{\rho^2+z^2}}}{\sqrt{\rho^2+z^2}} \cos|k_1-k_2|z dz \quad (\text{E17b})$$

$$= \frac{qQ}{4\pi\epsilon_0} \frac{1}{\pi} K_0(\rho\sqrt{\mu^2+|k_1-k_2|^2}) \quad (\text{E17c})$$

Only the real part of the integral(E17a) survives, the imaginary part being an odd function of z. The fact that the cosine is an even function of  $(k_1 - k_2)$  as well as of z allows the insertion of the absolute value signs. The step from(E17b) to(E17c) was made with the aid of TIT I, p. 17(27). Had we considered the transitions

$$\left\{ \begin{array}{l} + k_1 \rightarrow - k_2 \\ - k_1 \rightarrow + k_2 \end{array} \right\} \quad (\text{E18a})$$

$$\left\{ \begin{array}{l} - k_1 \rightarrow + k_2 \\ + k_1 \rightarrow - k_2 \end{array} \right\} \quad (\text{E18b})$$

in similar detail, the end result would have been

$$\int_{-\infty}^{\infty} Z_{k_2}^* \frac{qQ}{4\pi\epsilon_0} \frac{e^{-\mu\sqrt{\rho^2+z^2}}}{\sqrt{\rho^2+z^2}} Z_{k_1} dz$$

$$= \frac{qQ}{4\pi\epsilon_0} \frac{1}{\pi} K_0(\rho\sqrt{\mu^2+|k_1+k_2|^2}). \quad (\text{E19})$$

The symbol  $K_0(x)$  stands for the modified Bessel function of the third kind, of order zero. It is also known as MacDonald's function. As a function of the real variable  $x$ , it is everywhere positive and monotonically decreasing. The dependences for small and large  $x$  are

$$K_0(x) \xrightarrow{x \rightarrow 0^+} -\ln\left(\frac{x}{2}\right) \quad (\text{E20})$$

$$K_0(x) \xrightarrow{x \rightarrow \infty} \sqrt{\frac{\pi}{2x}} e^{-x}. \quad (\text{E21})$$

Definitions and properties are given in NBS [1964, chapter 9], in HTF II [1953, chapter VII], and in Lebedev [1965, chapter 5].

We are left with an integral on the radial coordinate  $\rho$ , which we rewrite in terms of the dimensionless variable  $x \equiv \beta^2 \rho^2$ :

$$\langle q\Lambda_c \rangle = \frac{qQ}{4\pi\epsilon_0} \frac{1}{\pi} \sqrt{\frac{N! (N+\nu)!}{(N+M)! (N+\nu+M)!}}$$

$$\cdot \int_0^{\infty} K_0(2\sqrt{\gamma x}) e^{-x} x^M L_N^M(x) L_{N+\nu}^M(x) dx. \quad (\text{E22})$$

The integers  $N$  and  $\nu$  are defined in terms of  $N_1$  and  $N_2$  by the relations

$$N \equiv \min(N_1, N_2)$$

$$\nu \equiv |N_1 - N_2|.$$

The parameter  $\gamma$  is proportional to the squared  $z$  momentum transfer and inversely proportional to the magnetic field  $B$ :

$$\gamma \equiv \frac{\mu^2 + |k_1 + k_2|^2}{4\beta^2}. \quad (\text{E23})$$

The minus sign in this definition refers to the forward scattering transitions (E16) and the plus sign to the back scattering transitions (E18). We shall evaluate (E22) by alternating between two integral representations of the aforementioned confluent hypergeometric psi function. The forms we shall use have been taken from Lebedev [1965, p. 268 and p. 278]:

$$\Psi(\alpha, 1, z) = \frac{2}{[\Gamma(\alpha)]^2} \int_0^{\infty} K_0(2\sqrt{zt}) e^{-t} t^{\alpha-1} dt \quad (\text{E24})$$

$$\Psi(\alpha, \gamma, z) = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} e^{-zt} t^{\alpha-1} (1+t)^{\gamma-\alpha-1} dt \quad (\text{E25})$$

The restrictions cited are that  $\text{Re } \alpha > 0$  and  $\text{Re } z > 0$ .

In preparation, we evaluate the simpler integral A defined as

$$A \equiv \int_0^{\infty} K_0(2\sqrt{zt}) e^{-t} t^m L_n^m(t) dt. \quad (\text{E26})$$

Use of the explicit expression for the Laguerre polynomial  $L_n^m(t)$  together with the representations (E24) and (E25) allows us to proceed through the sequence

$$A = \sum_{j=0}^n \frac{(n+m)! (-1)^j}{(n-j)! (m+j)! j!} \int_0^{\infty} K_0(2\sqrt{zt}) e^{-t} t^{m+j} dt \quad (\text{E27a})$$

$$= \sum_{j=0}^n \frac{(n+m)! (-1)^j}{(n-j)! (m+j)! j!} \frac{[(m+j)!]^2}{2} \Psi(m+j+1, 1, z) \quad (\text{E27b})$$

$$= \frac{1}{2} \int_0^{\infty} e^{-zt} \frac{t^m}{(1+t)^{m+1}} \left\{ \sum_{j=0}^n \frac{(n+m)!}{(n-j)! j!} \left(\frac{-t}{1+t}\right)^j \right\} dt \quad (\text{E27c})$$

$$= \frac{(n+m)!}{2 n!} \int_0^{\infty} e^{-zt} t^m (1+t)^{-(m+n+1)} dt \quad (\text{E27d})$$

$$= \frac{1}{2} \frac{(n+m)! m!}{n!} \Psi(1+m, 1-n, z). \quad (\text{E27e})$$

Returning now to (E22), we employ an expansion cited by Erdélyi [1938] in order to transform the product of two Laguerre polynomials into a finite sum over a single such polynomial:

$$L_N^M(t) L_{N+P}^M(t) = \sum_{j=0}^N \frac{(2N+P-2j)! (N+M)! (N+P+M)!}{j! (N-j)! (N+P-j)! (M+j)! (2N+P+M)!} \cdot t^{2j} L_{2N+P-2j}^{M+2j}(t) \quad (E28)$$

When this expansion is inserted into (E22), the resultant integral has the form of A, and we may immediately write the end result:

$$\langle g\Lambda_c \rangle = \frac{gQ}{4\pi\epsilon_0} \frac{1}{2\pi} \sqrt{\frac{N! (N+P)!}{(N+M)! (N+P+M)!}} \sum_{j=0}^N \frac{(N+M)! (N+P+M)! (M+2j)!}{(N-j)! (N+P-j)! (M+j)! j!} \Psi(1+M+2j, 1-(2N+P-2j), \gamma) \quad (E29)$$

### Two Alternate Methods

Before proceeding with a discussion of this result, we indicate how it may be obtained in two other ways. This will provide reassuring checks on the expansion (E28) and the representations (E24) and (E25).

We utilize the identity

$$\frac{1}{\sqrt{\rho^2 + z^2}} \equiv \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-(\rho^2 + z^2)t^2} dt \quad (E30)$$

to recast the original form (I11) into

$$\langle g\Lambda_c \rangle = \frac{gQ}{4\pi\epsilon_0} \delta_{M_1, M_2} \int_0^\infty d\rho \cdot \rho \cdot R_{N_2, M_2}^* R_{N_1, M_1} \cdot \int_0^\infty e^{-\rho^2 t^2} \left\{ \frac{2}{\sqrt{\pi}} \int_{-\infty}^\infty z_{k_2}^* e^{-\mu\sqrt{\rho^2 + z^2} - t^2 z^2} z_{k_1} dz \right\} dt \quad (E31)$$

After executing steps similar to (E17a) and (E17b), the integral in braces (denoted by  $I_z$ ) may be written

$$\{I_z\} = \frac{2}{\sqrt{\pi}} \frac{1}{\pi} \int_0^\infty \cos|k_1 + k_2|z e^{-\mu\sqrt{\rho^2 + z^2} - t^2 z^2} dz \quad (E32a)$$

$$= \frac{1}{\pi t} e^{-\frac{\gamma \rho^2}{t}} \quad (E32b)$$



The transition from a to b was made by means of a slight generalization ( $\mu > 0$ ) of TIT I, p. 15 (11). We are left with an integral on t and one on  $\rho$ . In order to check the generalization and to proceed further with this second evaluation of  $\langle q \Lambda_c \rangle$ , we rewrite the t integral

$$\int_0^{\infty} e^{-\rho^2 t^2 - \frac{\gamma \rho^2}{t}} \frac{dt}{t} = \frac{1}{2} \int_0^{\infty} e^{-\rho^2 t - \frac{\gamma \rho^2}{t}} \frac{dt}{t} \quad (\text{E33a})$$

$$= \frac{1}{2} \int_0^{\infty} e^{-\rho^2 \rho'^2 t - \frac{\gamma}{t}} dt. \quad (\text{E33b})$$

The check is provided by (E33a). Upon integration by means of TIT I, p. 146 (29), the radial integral remaining is just the previously encountered (E22). Choosing now, however, to integrate first on  $\rho$  and then on t, we utilize (E33b) and write the matrix element in the form

$$\langle q \Lambda_c \rangle = \frac{q Q}{4\pi \epsilon_0} \frac{1}{2\pi} \sqrt{\frac{N_1! N_2!}{(N_1+M)! (N_2+M)!}} \int_0^{\infty} \frac{dt}{t} e^{-\frac{\gamma}{t}} \int_0^{\infty} e^{-(t+1)\rho'^2} (\rho'^2)^M L_{N_1}^M(\rho'^2) L_{N_2}^M(\rho'^2) d(\rho'^2). \quad (\text{E34})$$

If we again let  $x \equiv \beta^2 \rho'^2$ , we have to consider the integral

$$I_x \equiv \int_0^{\infty} e^{-(t+1)x} x^M L_{N_1}^M(x) L_{N_2}^M(x) dx. \quad (\text{E35})$$

The integral  $I_x$  is quite close to the integral which expresses the N orthogonality of  $L_{N_1}^M(x)$  and  $L_{N_2}^M(x)$  with respect to the weight function  $x^M e^{-x}$ ,

$$\int_0^{\infty} e^{-x} x^M L_{N_1}^M(x) L_{N_2}^M(x) dx = \delta_{N_1, N_2} \frac{(N+M)!}{N!}. \quad (\text{E36})$$

See Lebedev [1965, p. 84], HTF II [1953, chapter 10], or section 3 of appendix D. What spoils the picture is the lack of agreement

between the coefficients of  $x$  in the exponential and in the Laguerre polynomials. This suggests that we force these arguments to be the same, in order to utilize (E36). This is accomplished by means of a generalization of HTF II, p. 192 (40),

$$L_n^m(\lambda x) = \gamma^{-n} \sum_{j=0}^n \binom{n+m}{j} \lambda^{n-j} (\gamma - \lambda)^j L_{n-j}^m(\gamma x). \quad (\text{E37})$$

Note that the superscript  $m$  remains unchanged in this scale-changing expansion. This is the point at which the finite sum of (E29) again makes its entry. If we now choose  $\lambda = 1$  and  $\gamma = 1 + t$ , the expansion (E37) becomes

$$L_n^m(x) = (1+t)^{-n} \sum_{j=0}^n \binom{n+m}{j} t^j L_{n-j}^m[(1+t)x]. \quad (\text{E38})$$

Use of (E38) and (E36) allows us to take  $I_x$  through the following steps:

$$I_x = (1+t)^{-N_1-N_2-M-1} \sum_{j=0}^{N_1} \sum_{k=0}^{N_2} \binom{N_1+M}{N_1-j} \binom{N_2+M}{N_2-k} t^{N_1-j+N_2-k} \cdot \int_0^\infty e^{-s} s^M L_j^M(s) L_k^M(s) ds \quad (\text{E39a})$$

$$= \frac{t^{N_1+N_2}}{(1+t)^{N_1+N_2+M+1}} \sum_{j=0}^{\min(N_1, N_2)} \binom{N_1+M}{N_1-j} \binom{N_2+M}{N_2-j} \frac{(M+j)!}{j!} \frac{1}{t^{2j}}. \quad (\text{E39b})$$

At this stage, the matrix element has the form

$$\langle g \Lambda_c \rangle = \frac{g Q}{4\pi \epsilon_0} \frac{1}{2\pi} \sqrt{\frac{N_1! N_2!}{(N_1+M)! (N_2+M)!}} \sum_{j=0}^{\min(N_1, N_2)} \binom{N_1+M}{N_1-j} \binom{N_2+M}{N_2-j} \frac{(M+j)!}{j!} \cdot \int_0^\infty e^{-\frac{\gamma}{t}} \frac{t^{N_1+N_2-2j-1}}{(1+t)^{N_1+N_2+M+1}} dt. \quad (\text{E40})$$

However, the  $t$  integral is just the representation (E25) for the confluent hypergeometric psi function:

$$I_t \equiv \int_0^{\infty} e^{-\frac{\gamma}{t}} \frac{t^{N_1+N_2-2j-1}}{(1+t)^{N_1+N_2+M+1}} dt \quad (\text{E41a})$$

$$= \int_0^{\infty} e^{-\gamma t} t^{M+2j} (1+t)^{-N_1-N_2-M-1} dt \quad (\text{E41b})$$

$$= (M+2j)! \Psi(M+2j+1, 1-(N_1+N_2-2j), \gamma) \quad (\text{E41c})$$

The end result,

$$\langle \mathcal{L}_c \rangle = \frac{qQ}{4\pi\epsilon_0} \frac{1}{2\pi} \sqrt{\frac{N_1!}{(N_1+M)!} \frac{N_2!}{(N_2+M)!}}$$

$$\cdot \sum_{j=0}^{\min(N_1, N_2)} \binom{N_1+M}{N_1-j} \binom{N_2+M}{N_2-j} \frac{(M+j)! (M+2j)!}{j!} \Psi(M+2j+1, 1-(N_1+N_2-2j), \gamma) \quad (\text{E42})$$

is the same as (E29).

Finally, the results (E29) and (E42) may also be obtained in a much more roundabout and tenuous manner through use of (E28), TIT II, p. 152 (82), and the relations connecting the Whittaker functions  $M_{k,m}$  and  $W_{k,m}$  with the Laguerre polynomial and the confluent hypergeometric psi function, respectively.

### E3 Properties of the Confluent Hypergeometric Psi Function

#### General Relations

The functional properties we shall need may be derived from three general relations. The first of these is the expansion in ascending powers of the argument

$$\Psi(a, 1+n, z)$$

$$= \frac{(-1)^{n+1}}{\Gamma(a-n)} \sum_{k=0}^{\infty} \frac{(a)_k z^k}{(n+k)! k!} \left[ \psi(a+k) - \psi(1+k) - \psi(1+n+k) + \ln z \right]$$

$$+ \frac{1}{\Gamma(a)} \sum_{k=0}^{n-1} \frac{(-1)^k (n-k-1)! (a-n)_k}{k!} z^{k-n} \quad (\text{E43})$$

where  $\text{Re } a > 0$  and  $n = 0, 1, 2, \dots$ . The finite series is to be omitted for  $n = 0$ . This form was taken from Lebedev [1965, p. 264]. It is equivalent to the forms given in HTF I, p. 261 (13) and in NBS, p. 504 (13.1.6). The series is convergent almost everywhere in the finite complex plane:  $0 \leq |z| < \infty$ ,  $|\text{phase } z| < \pi$ . Pochhammer's symbol  $(a)_k$  is defined as

$$(a)_k \equiv a(a+1)(a+2)\cdots(a+k-1) = \frac{\Gamma(a+k)}{\Gamma(a)} \quad (\text{E44})$$

The (lower case) symbol  $\Psi(z)$  stands for the logarithmic derivative of the gamma function,  $\Gamma'(z)/\Gamma(z)$ . The machine evaluation of this function employed the starting and recursion relations

$$\Psi(1) = -\gamma = -0.5772157 \quad (\text{E45})$$

$$\Psi(1+z) = \Psi(z) + \frac{1}{z} \quad (\text{E46})$$

where  $\gamma$  stands for the Euler-Mascheroni constant. Note that  $\Psi(1)$  is negative, whereas  $\Psi(2), \Psi(3), \dots$  are all positive. We list also the dependence

$$\Psi(z) \xrightarrow{z \rightarrow \infty} \ln z \quad (\text{E47})$$

This function is discussed in HTF I [1953, section 1.7].

Secondly, we shall need the Kummer transformation relation

$$\Psi(a, 1-n, z) = z^n \Psi(a+n, 1+n, z) \quad (\text{E48})$$

in order to extend the definition (E43) to psi functions having negative second indices.

Finally, we cite the asymptotic expansion

$$\Psi(a, b, z) = z^{-a} \left[ \sum_{k=0}^{\infty} \frac{(-1)^k (a)_k (1+a-b)_k}{k!} z^{-k} + \mathcal{O}(|z|^{-t-1}) \right] \quad (\text{E49})$$

useful for  $z$  sufficiently large. The meaning of sufficiently will emerge in subsequent discussion.

The relations (E48) and (E49) may be found in any of the three references cited earlier in connection with the expansion (E43).

### The Particular Form of Interest

The confluent hypergeometric function appearing in the centered Coulomb matrix element (E29) has the form  $\Psi(1+m, 1-n, x)$  where in general  $m$  and  $n$  are positive integers including zero. For our application  $m$  and  $n$  have the values

$$\begin{aligned} m &= M + 2j = 0, 1, 2, \dots \\ n &= 2N + \nu - 2j = \nu, \nu + 1, \dots \end{aligned} \quad (\text{E50})$$

The general relations become

$$\Psi(1+m, 1-n, x) = x^n \Psi(1+m+n, 1+n, x) \quad (\text{E51})$$

$$\Psi(1+m, 1-n, x)$$

$$\begin{aligned} &= \frac{(-1)^{n+1}}{m!} \sum_{k=0}^{\infty} \frac{(m+n+k)! x^{k+n}}{(m+n)! (n+k)! k!} \left[ \Psi(1+m+n+k) - \Psi(1+k) - \Psi(1+n+k) + \ln x \right] \\ &+ \frac{1}{(m+n)!} \sum_{k=0}^{n-1} \frac{(-1)^k (n-k-1)! (m+k)!}{m! k!} x^k \end{aligned} \quad (\text{E52})$$

$$\Psi(1+m, 1-n, x) = x^{-1-m} \left[ \sum_{k=0}^{\infty} \frac{(-1)^k (m+n+k)! (m+k)!}{(m+n)! m! k!} x^{-k} + \mathcal{O}(|x|^{-t-1}) \right]. \quad (\text{E53})$$

In the following paragraphs, we investigate the behavior of this function for the two extremes  $x \rightarrow 0$  and  $x \rightarrow \infty$ , and for  $x$  finite.

### Leading Terms for Small Values of the Argument

The behavior near the origin of the function  $\Psi(1+m, 1-n, x)$  depends upon the value of  $n$ . From the series definition(E52), we must select the leading terms as  $x \rightarrow 0$  for the cases  $n = 0$ ,  $n = 1$ , and  $n \geq 2$ . For  $n = 0$ , it is fairly easy to ascertain that the function goes as

$$[-\ln x - \Psi(1+m) + 2\Psi(1)]/m! \quad (\text{E54})$$

for small  $x$ . Here one is tempted to ignore the lower case psi functions in deference to the  $\ln x$  term and its limit

$$\ln x \xrightarrow{x \rightarrow 0^+} -\infty. \quad (\text{E55})$$

However the logarithm exhibits such a mild singularity for  $x$  small that, for a large range of  $x$  small, the lower case psi functions cannot be neglected in comparison with  $\ln x$ . For example, the numerical values corresponding to  $x = 10^{-4}$  and  $m = 0$  are

$$[1.74 - 0.577].$$

Our point here is that the psi functions should be retained unless one is dealing with an  $x$  small enough that  $\ln x$  is clearly much larger. We consider next a case where they must be retained.

For  $n = 1$ , the finite series contains and contributes the single constant term  $1/(m + 1)!$ . The second term, for  $x \rightarrow 0$ , must come from the  $k = 0$  term of the infinite series. Here again one is tempted to ignore the lower case psi functions in deference to the logarithm. However, in this case this is not legitimate for any small  $x$  since it overlooks the important role played by the single power of  $x$ . In view of the limit

$$x^p \ln x \xrightarrow{x \rightarrow 0^+} \frac{-x^p}{p} \quad (\text{E56})$$

( $p > 0$ , not necessarily integer), it is clear that  $x$  times a lower case psi function is of the same order as  $x \ln x$ . Hence all four terms must be retained. We conclude that  $\Psi(1 + m, 1 - n, x)$  goes as

$$\frac{1}{(m+1)!} - \frac{x}{m!} \left[ -\ln x - \psi(2+m) + 2\psi(1) \right] \quad (\text{E57})$$

for  $n = 1$  and  $x$  approaching zero. For small  $x$ , the quantities in square brackets in (E57) and (E54) are always positive. For  $n \geq 2$ , the finite series always contains a constant term and a term linear in  $x$ , whereas the lowest term in the finite series is at least quadratic in  $x$ . Thus the finite series contributes both the constant and the linear terms of  $\Psi(1 + m, 1 - n, x)$  for  $n \geq 2$  as  $x \rightarrow 0$ :

$$\Psi(1+m, 1-n, x) \approx \frac{(n-1)!}{(m+n)!} - \frac{n!}{(m+n)!} (m+1)x. \quad (\text{E58})$$

Note that the quadratic term would contain contributions from both the finite and the infinite series. The results are collected in Table E1, as well as being transcribed according to the prescriptions (E50). In the transcribed entries, we have indicated in parentheses the only values of  $j$  and  $\nu$  which will meet the conditions  $n = 0$  and  $n = 1$ .

Table E1. Behavior of the confluent hypergeometric function  $\Psi(1+m, 1-n, x)$  as  $x \rightarrow 0$ .

Behavior of $\Psi(1+m, 1-n, x)$ as $x \rightarrow 0$	
$n$	$\Psi(1+m, 1-n, x)$
0	$[-\ln x - \Psi(1+m) + 2\Psi(1)] / m!$
1	$\frac{1}{(m+1)!} \left\{ 1 - (m+1)x[-\ln x - \Psi(2+m) + 2\Psi(1)] \right\}$
$\geq 2$	$\frac{(n-1)!}{(m+n)!} \left\{ 1 - (m+1)nx \right\}$
Transcribed: $m = M + 2j \quad n = 2N + \nu - 2j$	
$2N + \nu - 2j$	$\Psi(1+M+2j, 1-(2N+\nu-2j), x)$
0 $\left( \begin{matrix} j=N \\ \nu=0 \end{matrix} \right)$	$[-\ln x - \Psi(M+2N+1) + 2\Psi(1)] / (M+2N)!$
1 $\left( \begin{matrix} j=N \\ \nu=1 \end{matrix} \right)$	$\frac{1}{(M+2N+1)!} \left\{ 1 - (M+2N+1)x[-\ln x - \Psi(M+2N+2) + 2\Psi(1)] \right\}$
$\geq 2$	$\frac{(2N+\nu-2j-1)!}{(M+2N+\nu)!} \left\{ 1 - (M+2j+1)(2N+\nu-2j)x \right\}$



Behavior for Large Values of the Argument

As was indicated above, for  $x \gg (m + 1) (m + n + 1)$

$$\Psi(1 + m, 1 - n, x) \approx x^{-1-m} \tag{E59}$$

which when transcribed becomes

$$\Psi(1 + M + 2j, 1 - (2N + \nu - 2j), x) \approx x^{-1-M-2j} \tag{E60}$$

for  $x \gg (1 + M + 2j) (1 + M + 2N + \nu)$ . As a function of  $x$ , the  $j = 0$  function is the largest, the  $j = 1$  function being smaller by a factor of  $x^2$ . As a function of  $M$ , the  $M = 0$  function is the largest. For  $x$  large, this function exhibits no  $N$  or  $\nu$  dependence.

General Behavior

With the behavior of  $\Psi(1 + m, 1 - n, x)$  for the extremes of  $x$  in mind, we consider now the general behavior of this function for finite  $x$ . The remarkably simple features of this behavior may be inferred from the integral representation

$$\Psi(1+m, 1-n, x) \equiv \frac{1}{m!} \int_0^{\infty} e^{-xt} \frac{t^m}{(1+t)^{1+m+n}} dt, \tag{E61}$$

a transcription of (E25). We see immediately that the function is always positive and is non-oscillatory. Its first and second derivatives are likewise featureless. Since the first derivative is always negative, and the function approaches zero as  $x \rightarrow \infty$ , the function  $\Psi(1+m, 1-n, x)$  must decrease monotonically from a maximum located at the origin. This behavior is supported by the second derivative which is always positive.

The behavior of the function  $\Psi(1 + m, 1 - n, x)$  with the indices  $m$  and  $n$  is similar to its behavior with  $x$ . For fixed  $x$  and  $n$ , it follows from (E61) that the  $m = 0$  function is the largest, and that the functional

values decrease monotonically as  $m$  increases from zero. This behavior stems not only from the factorial coefficient, but also from the integrand factor  $[t/(1+t)]^m$ .

The variation with  $n$  is qualitatively similar to that of  $m$ . Here again, for fixed  $x$  and  $m$ , the  $n = 0$  function is the largest, and the functional values decrease monotonically with  $n$ . Because of the factorial coefficients multiplying the confluent hypergeometric function  $\Psi(1 + M + 2j, 1 - (2N + \nu - 2j), \gamma)$  in the finite series (E29) for the centered Coulomb matrix element, we postpone a discussion of these results in terms of these particular values of  $n$  and  $m$ .

### Computation

In Fig. E1 is displayed the variation with  $x$  of the confluent hypergeometric function  $\Psi(1 + m, 1 - n, x)$  for several values of  $m$  and  $n$ . These numerical values illustrate and confirm the limiting values and general behavior described in the preceding sections.

The functional values plotted in Fig. E1 were computed through use of subroutine CHGF3, a Fortran II listing of which is available. This subroutine evaluates the series expansion (E52) in a direct and straightforward manner. The expansion was regarded as the sum of a four-term infinite series plus a finite series:

$$\begin{aligned} & \Psi(1 + m, 1 - n, x) \\ &= \frac{(-1)^{n+1}}{m!} \sum_{k=0}^{\infty} [T_k^{(1)} + T_k^{(2)} + T_k^{(3)} + T_k^{(4)}] + \frac{1}{(m+n)!} \sum_{k=0}^{n-1} T_k^{(5)} \end{aligned} \quad (E62)$$

The general terms  $T_k^{(i)}$  were evaluated recursively by means of the relations

$$T_k^{(i)} = \left[ \left( \frac{(m+n+k)x}{(n+k)k} \right) \left( 1 + \frac{1}{(m+n+k) \cdot \Psi(m+n+k)} \right) \right] T_{k-1}^{(i)} \quad (E63)$$

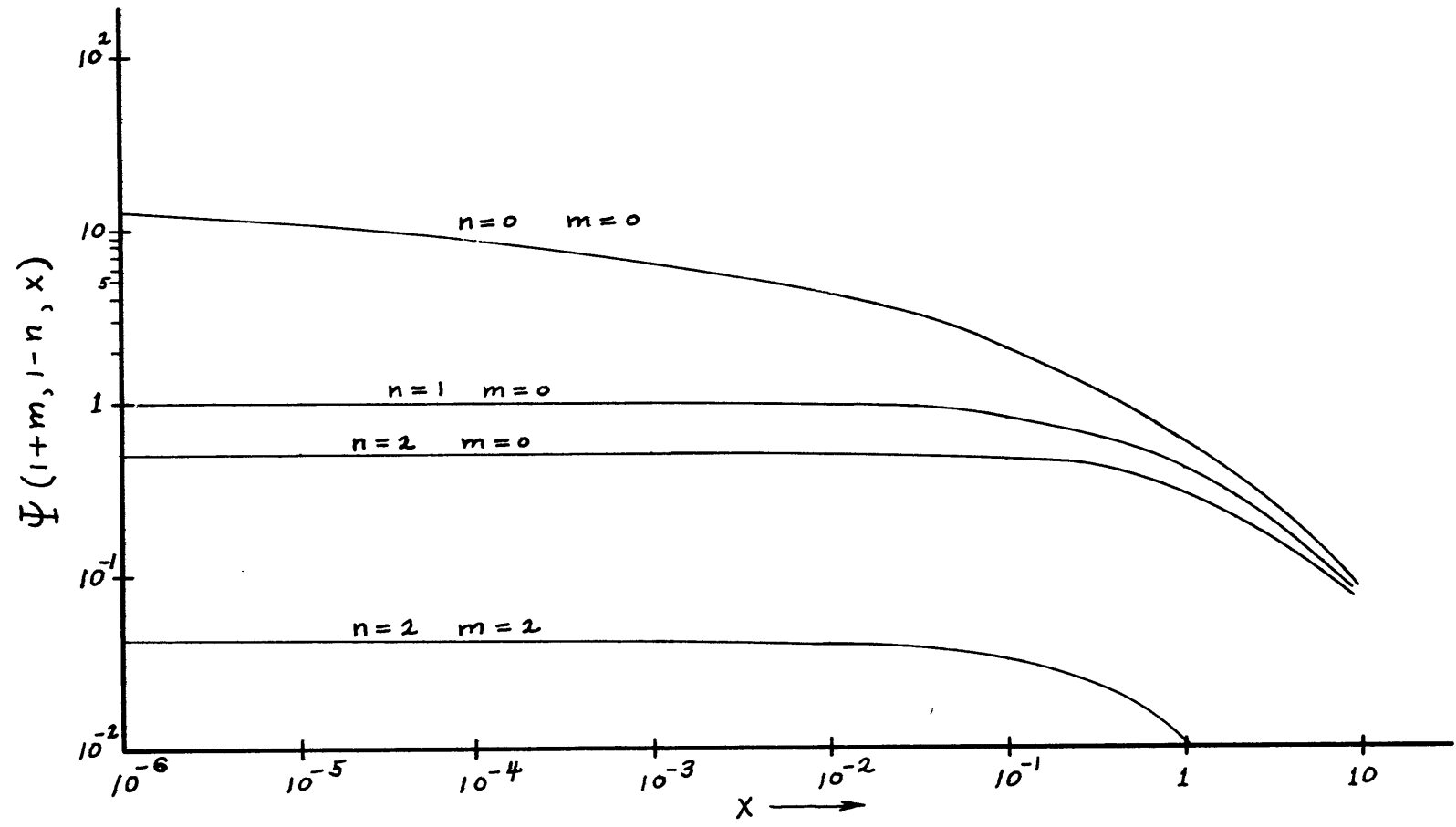


Fig. E1. Variation with  $x$  of the confluent hypergeometric function  $\Psi(1+m, 1-n, x)$  for several values of  $m$  and  $n$ .

$$T_k^{(2)} = \left[ \left( \frac{(m+n+k)x}{(n+k)k} \right) \left( 1 + \frac{1}{k \cdot \Psi(k)} \right) \right] T_{k-1}^{(2)} \quad (\text{E64})$$

$$T_k^{(3)} = \left[ \left( \frac{(m+n+k)}{(n+k)k} \right) \left( 1 + \frac{1}{(n+k) \cdot \Psi(n+k)} \right) \right] T_{k-1}^{(3)} \quad (\text{E65})$$

$$T_k^{(4)} = \left[ \frac{(m+n+k)}{(n+k)k} \right] T_{k-1}^{(4)} \quad (\text{E66})$$

$$T_k^{(5)} = \left[ \frac{(-1)(m+k)x}{k(n-k)} \right] T_{k-1}^{(5)} \quad (\text{E67})$$

where  $k = 1, 2, 3, \dots$  (or up to  $k = n - 1$  for  $T_k^{(5)}$ ). The starting ( $k = 0$ ) values are

$$T_0^{(1)} = \frac{x^n}{n!} \Psi(1+m+n) \quad (\text{E68})$$

$$T_0^{(2)} = \frac{x^n}{n!} [-\Psi(1)] \quad (\text{E69})$$

$$T_0^{(3)} = \frac{x^n}{n!} [-\Psi(1+n)] \quad (\text{E70})$$

$$T_0^{(4)} = \frac{x^n}{n!} \ln x \quad (\text{E71})$$

$$T_0^{(5)} = (n-1)! \quad (\text{E72})$$

The infinite series was evaluated until the successive values of the running sum  $S_{14}(p)$  of terms satisfied the inequality

$$\frac{S_{14}(p) - S_{14}(p-1)}{S_{14}(p-1)} < \text{Error} . \quad (\text{E73})$$

Error values of  $10^{-5}$  were used. Decreasing error to  $10^{-8}$  did not materially change the 8 digit results.

Also written was subroutine CHGF4 for the evaluation of the asymptotic expansion (E53) of the function  $\Psi(1+m, 1-n, x)$ . The recursion and starting relations are

$$T_k^{(a)} = \left[ \frac{(-1)(m+n+k)(m+k)}{kx} \right] T_{k-1}^{(a)} \quad (\text{E74})$$

$$T_0^{(a)} = 1 \quad (\text{E75})$$

where (E53) was considered in the form

$$\Psi(1+m, 1-n, x) = x^{-1-m} \sum_{k=0}^t T_k^{(a)}. \quad (\text{E76})$$

We see that the asymptotic expansion cannot be usefully employed unless

$$x \gg (m+1)(m+n+1). \quad (\text{E77})$$

The routine is coded to evaluate the sum in (E76) up to but not including the smallest term. This smallest term  $T_{t+1}$  is considered to be the uncertainty in the value of the sum. See, for example, Morse and Feshbach [1953, p. 434] or the monograph by Copson [1965].

#### E4 Variation of the Argument $\gamma$

The argument of the confluent hypergeometric function appearing in the centered Coulomb matrix element (E29) is the quantity

$$\gamma \equiv \frac{|k_1 \pm k_2|^2}{4\beta^2} = \frac{|P_{z1} \pm P_{z2}|^2}{2eB\hbar} = \frac{E_{z1}}{\hbar\omega_c} \left| 1 \pm \frac{P_{z2}}{P_{z1}} \right|^2 \quad (\text{E78})$$

where  $p_z^2 = \hbar^2 k^2$  and  $\phi^2 = eB/2\hbar$ . The shielding parameter  $\mu$  has been set to zero. The plus sign refers to transitions in which the direction of the z momentum is reversed (back scattering) while the minus sign denotes forward scattering transitions in which the direction of the z momentum is the same before and after the encounter. We are interested in the size and variation of  $\gamma$  with parameters such as the magnetic field B and the initial z energy  $E_{z1}$ .

The matrix element will be employed in a transition probability or cross section which takes on non-zero values only when energy is conserved. Thus for a given  $N_1$ ,  $p_{z1}$ , and B, the only values of  $p_{z2}$  which hold any interest are those given by

$$p_{z2} = \pm p_{z1} \sqrt{1 + \frac{\hbar\omega_c}{E_{z1}} (N_1 - N_2)} \quad (E79)$$

When this relation is substituted into (E78), we obtain the expression for  $\gamma$  in which conservation of energy has been incorporated

$$\gamma = \frac{E_{z1}}{\hbar\omega_c} \left[ 1 \pm \sqrt{1 + \frac{\hbar\omega_c}{E_{z1}} (N_1 - N_2)} \right]^2 \quad (E80)$$

Recall from appendix C that the group I (origin inside the gyrocircle) and group II (origin outside the gyrocircle) energy eigenvalue equations are

$$E_I = \hbar^2 k^2 / 2m + \hbar\omega_c (N + M + \frac{1}{2}) \quad (E81)$$

$$E_{II} = \hbar^2 k^2 / 2m + \hbar\omega_c (N + \frac{1}{2}) \quad (E82)$$

Because M is conserved in the transition, the forms (E79) and (E80) apply to both  $I \rightarrow I$  and  $II \rightarrow II$  transitions. In addition, the conservation of M precludes the need for consideration of  $I \rightleftharpoons II$  transitions.

Table E2. Properties of the function  $\gamma$  near the origin and at the end points of the independent variable  $(N_1 - N_2)/N_{z1}$ , with energy conservation incorporated.

Value of $(N_1 - N_2)/N_{z1}$	Value of $\gamma$		Slope of $\gamma$	
	$\gamma_+$ (back)	$\gamma_-$ (forward)	$\gamma_+$ (back)	$\gamma_-$ (forward)
$\left  \frac{N_1 - N_2}{N_{z1}} \right  \ll 1$	$4N_{z1} \left[ 1 + \frac{(N_1 - N_2)}{2N_{z1}} \right]$	$\frac{(N_1 - N_2)^2}{4N_{z1}} \left[ 1 - \frac{(N_1 - N_2)}{2N_{z1}} \right]$	$2 \left[ 1 - \frac{(N_1 - N_2)}{4N_{z1}} \right]$	$\frac{(N_1 - N_2)}{2N_{z1}}$
0	$4N_{z1}$	0	2	0
-1	$N_{z1}$	$N_{z1}$	$+\infty$	$-\infty$
$N_1/N_{z1}$ ( $\ll 1$ )	$4N_{z1} \left[ 1 + \frac{N_1}{2N_{z1}} \right]$	$\frac{N_1^2}{4N_{z1}} \left[ 1 - \frac{N_1}{2N_{z1}} \right]$	$2 \left[ 1 - \frac{N_1}{4N_{z1}} \right]$	$\frac{N_1}{2N_{z1}}$
$N_1/N_{z1}$ ( $\gg 1$ )	$N_1 \left[ 1 + 2 \sqrt{\frac{N_{z1}}{N_1}} \right]$	$N_1 \left[ 1 - 2 \sqrt{\frac{N_{z1}}{N_1}} \right]$	$1 + \sqrt{\frac{N_{z1}}{N_1}}$	$1 - \sqrt{\frac{N_{z1}}{N_1}}$

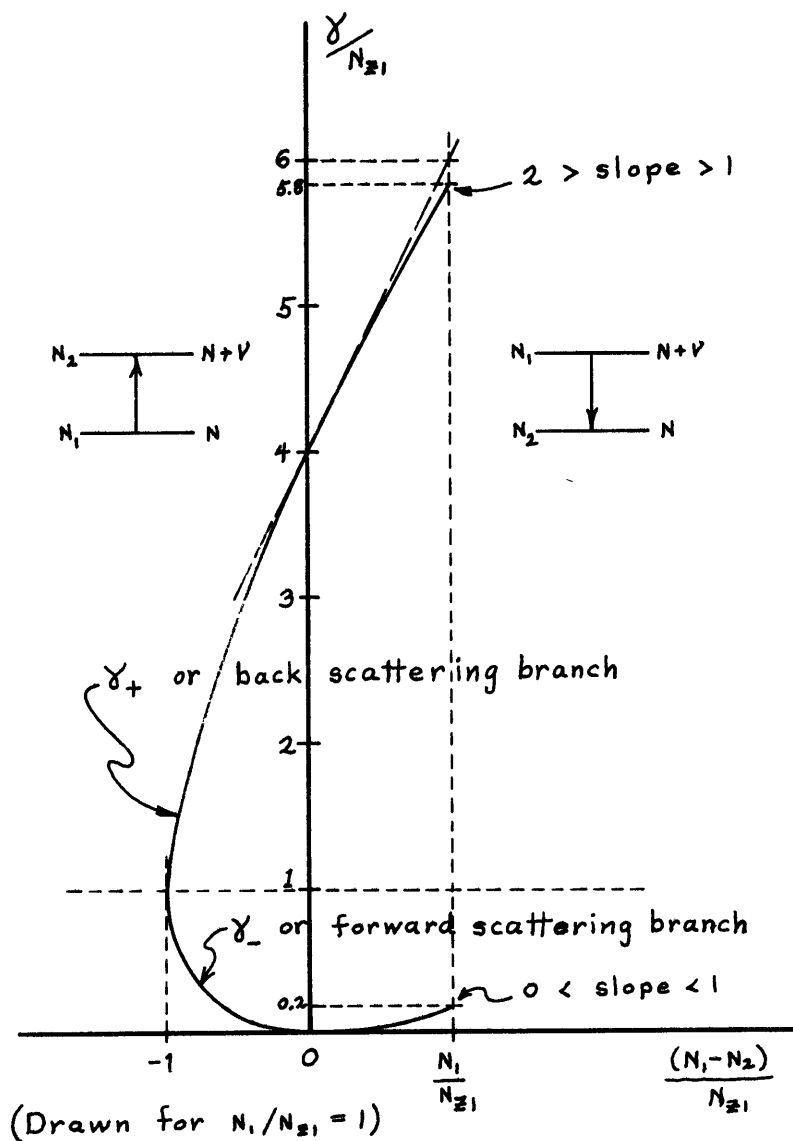


Fig. E2. Variation of the argument  $\gamma$  of the centered Coulomb matrix element, with conservation of energy incorporated. Approximately to scale.



The extent of the branches on the positive side of the origin (the region of downward transitions and increasing  $z$  energy) is controlled by the ratio  $N_1/N_{z1}$ . The curves of Fig.E2 were drawn for  $N_1/N_{z1} = 1$ . For this value, there are about as many states available for upward transitions as for downward transitions. As this boundary point moves in toward the origin (that is, for  $N_1/N_{z1} < 1$ ), the number of energetically accessible downward states decreases in relation to the number of accessible upward states. For  $N_1/N_{z1} > 1$ , the curves extend further to the right with the consequence that there are relatively more states available for downward transitions than for upward transitions.

As  $N_1/N_{z1} \rightarrow \infty$ , the branches coalesce and become asymptotic (from above and below) to the line extending from the origin at a forty-five degree angle. The figure presented by the branches thus has the appearance of an airfoil, rather than a skew-truncated hyperbola. If we view this limit as resulting from the limit  $N_{z1} \rightarrow 0$ , then (E83) becomes, approximately,

$$\gamma_{\pm} = |N_1 - N_2| \pm 2 \sqrt{N_{z1} |N_1 - N_2|} . \quad (\text{E86})$$

It follows from the definition of  $N_{z1}$  that this is the limiting form of  $\gamma$  for the extremes  $B \rightarrow \infty$  or  $E_{z1} \rightarrow 0$ .

In the regions near enough the origin that the inequality

$$\frac{|N_1 - N_2|}{N_{z1}} \ll 1 \quad (\text{E87})$$

is satisfied, the leading terms in the expansion of (I83) are

$$\gamma_+ \approx 4N_{z1} \left[ 1 + \frac{(N_1 - N_2)}{2N_{z1}} \right] \quad (\text{E88})$$

$$\gamma_- \approx \frac{(N_1 - N_2)^2}{4 N_{z1}} \left[ 1 - \frac{(N_1 - N_2)}{2 N_{z1}} \right]. \quad (\text{E89})$$

The behavior of the energy conserving  $\gamma$  as  $B \rightarrow 0$  or as  $E_{z1} \rightarrow \infty$  is contained in these expressions. The forward scattering values  $\gamma_-$  can in this case be very small, and the back scattering values  $\gamma_+$  very large. Note that the inequality (E87) places no restriction upon the size of  $N_1$  relative to  $N_{z1}$ , but rather upon the change of the quantum integer  $N$  compared to the initial  $N_z$ . That is, we can adjust  $N_{z1}$  over a fairly wide range of values and still satisfy (E87). This inequality is equivalent to the requirement that the relative change in  $z$  energy be small,

$$\frac{|N_{z2} - N_{z1}|}{N_{z1}} \ll 1. \quad (\text{E90})$$

Thus the forward scattering expression (E89) for  $\gamma_-$  may be said to describe the small angle values of  $\gamma$ .

#### E5. Behavior and Properties of the Centered Coulomb Matrix Element

In the preceding sections we have discussed the behavior of the confluent hypergeometric psi functions and the variation of their argument  $\gamma$ . Many of the results of these investigations appear in the figures and tables numbered E1 and E2. Our task now is to combine these results so as to gain some insight into the behavior and meaning of the rather formidable-appearing expressions (E29) and (E42) for the centered Coulomb matrix element.

Our interest centers primarily upon the off-diagonal or  $\nu \geq 1$  matrix elements since these correspond to some change in the incident eigenstate  $\psi_{NMk}$ . We shall consider its symmetries and general behavior, and obtain a relatively simple expression embodying the principal contribution of the matrix element. The diagonal or  $\nu = 0$  matrix elements

correspond to no change at all in the incident eigenstate since M is conserved and a change of N implies a change of k (if energy is to be conserved between initial and final states). The shielding parameter  $\mu$  holds little interest presently, and is set to zero. The general effect of a  $\mu > 0$  is to increase  $\gamma$  and consequently to decrease the value of the matrix element, as we shall see.

In general, as a function of  $\gamma$ , the matrix element has a maximum at the origin, or if  $\gamma$  does not go to zero, at the minimum value of  $\gamma$ . The matrix element is always positive definite, and decreases monotonically as  $\gamma$  increases from its minimum value. This behavior reflects that of the component confluent hypergeometric psi functions plus the fact that the terms of the finite series do not alternate in sign. Whether the shielding is present or not, the forward scattering small angle (small change in z energy) transitions contribute most to the values of the matrix element.

The Diagonal Matrix Element

The  $\nu = 0$  diagonal matrix element contains at least a logarithmic singularity for any N and M. As may be seen from the following representation of (E29), this singularity is contributed by the  $j = N$  term of the finite series, since  $\nu = 0$  implies  $\gamma_- = 0$ :

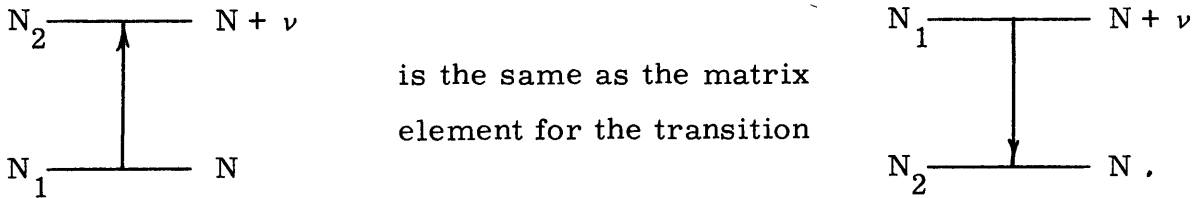
$$\begin{aligned} \langle g_{\mathcal{L}_c} \rangle = & \frac{Z Q}{4\pi\epsilon_0} \frac{1}{2\pi} \sqrt{\frac{(N+M)! (N+\nu+M)!}{N! (N+\nu)!}} \\ & \cdot \left\{ \left[ \right]_{j=N} \Psi(1+M+2N, 1-(\nu); \gamma) \right. \\ & + \left[ \right]_{j=N-1} \Psi(1+M+2N-2, 1-(\nu+2); \gamma) \\ & + \dots + \left. \left[ \right]_{j=0} \Psi(1+M, 1-(\nu+2N); \gamma) \right\}. \end{aligned} \tag{E91}$$

This is a manifestation of the long range character of the Coulomb potential. The singularity would, of course, not appear for  $\mu > 0$ .

The Off-Diagonal Matrix Elements

It is the off-diagonal or  $\nu \geq 1$  matrix elements which will be used in the Born approximation cross section calculations of the following appendix.

The matrix elements are symmetric as regards transitions between any pair of states. The matrix element for the transition



Recall our useage of N as the minimum of ( $N_1, N_2$ ) and of  $\nu$  as the absolute value of the difference. This symmetry property is manifestly exhibited by the forms (E29) and (E42), and is expressed in an abbreviated manner by the equation

$$\langle N | g \Lambda_c | N + \nu \rangle = \langle N + \nu | g \Lambda_c | N \rangle, \tag{E92}$$

The general appearance of the forward and back-scattering matrix elements is sketched in Fig. E3. These curves represent a crude mapping of the energy-conserving  $\gamma$  vs  $(N_1 - N_2)/N_{z1}$  curves onto the corresponding  $\langle q \Lambda_c \rangle$  plane, such a mapping being guided by the functional behavior of the confluent hypergeometric functions and by the fact that the terms of the  $\langle q \Lambda_c \rangle$  finite series do not alternate in sign. The points at the origin should be excluded as they are associated with the  $\nu = 0$  matrix elements.

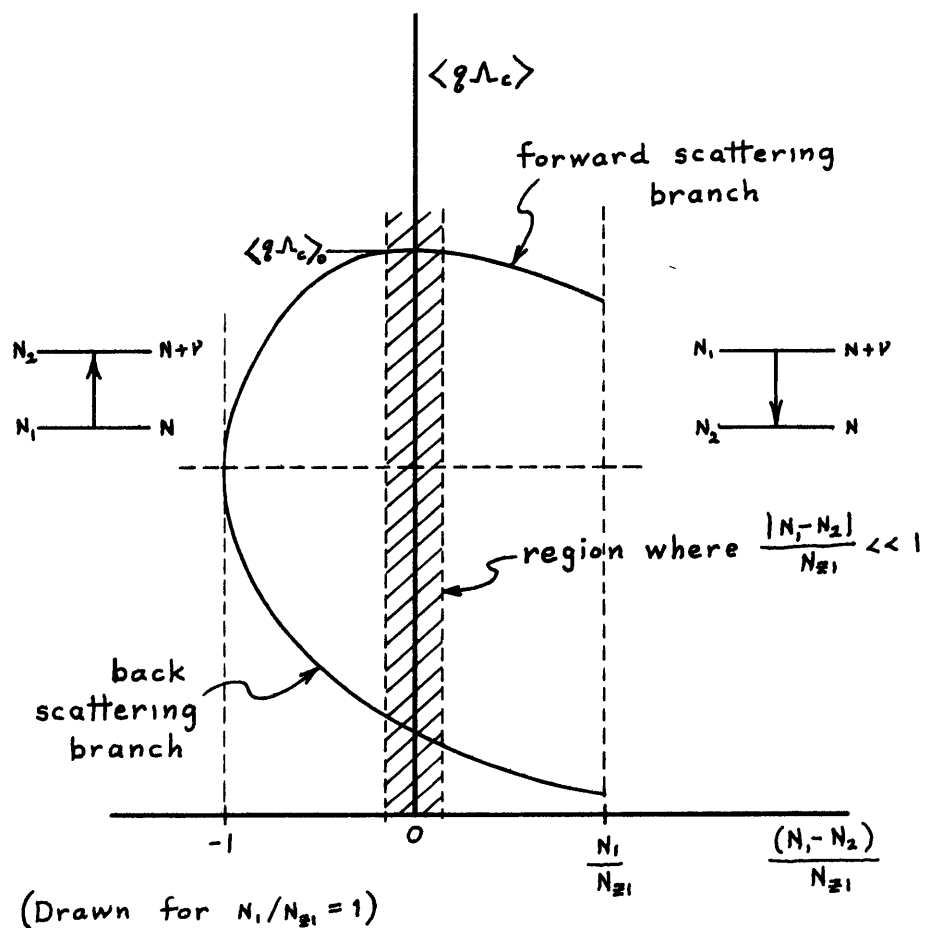
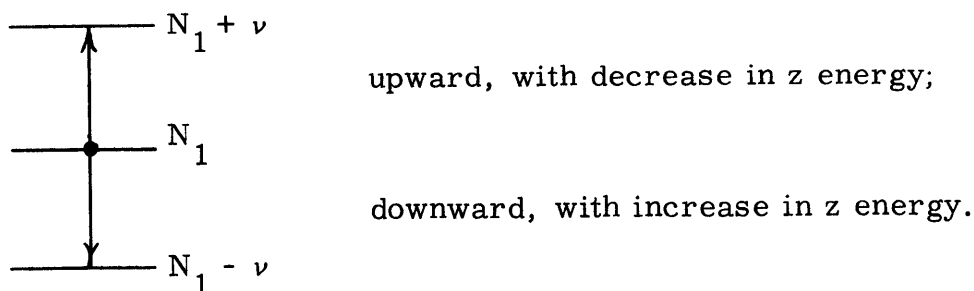


Fig. E3. Sketch indicating the variation of the off-diagonal centered Coulomb matrix element, with conservation of energy incorporated. Ordinate not to scale.

In these curves are, of course, several flaws or points of ambiguity which are attributable to the qualitative procedure used in generating them. The dependence upon  $M$  is not indicated, although we shall see that the matrix element decreases as  $M$  is increased. The ordinate of this graph is compressed and is not linear as was that of its antecedent, Fig. E2. Another significant feature which is not indicated with fidelity in Fig. E3 is the ordinate value corresponding to  $\gamma = N_{z1}$ . This ordinate value separates the forward and back scattering branches and may be moved up or down by adjusting the value of  $N_{z1}$ . For example, by choosing  $N_{z1} \gg 1$ , this ordinate value may be quite small compared to the maximum value. Since we have seen that  $N_{z1}$  can reasonably be quite large, this is a case of some importance in that it may justify neglect of the back scattering contribution to a cross section.

Even with these faults, the figure does display two significant properties of the centered Coulomb matrix element. The first of these, as already pointed out, is that the matrix element for forward scattering is always greater in value than that for back scattering. The second property displayed in Fig. E3 is the preference for equal upward or downward transitions from a given state. This property should not be confused with the detailed balancing symmetry discussed at the beginning of this section. We are here interested in which of the following two transitions is favored:



Note that this question involves three states, whereas the earlier involved upward and downward transitions between any two states.

The shape of the forward scattering curve in Fig. E3 indicates that downward transitions are favored over equal upward transitions. That is, provided states are available (and this is controlled by the ratio  $N_1/N_{z1}$ ), the forward scattering matrix element apparently exhibits a tendency for the z energy to increase at the expense of the perpendicular energy, and for the radial extent or size of the emergent eigenstate to be smaller than that of the original state. These statements lose validity for small angle transitions. As we shall be able to analytically demonstrate, upward transitions are favored in the forward scattering region quite close to the ordinate axis, where  $(\nu/N_{z1}) \ll 1$ . This discrepancy may be attributed to the fact that, in our mapping of  $\gamma$  into  $\langle q \Lambda_c \rangle$ , the  $\nu$  dependence of the factorial coefficient in the finite series was ignored. In the only region where the confluent hypergeometric functions exhibit no dependence upon  $N_{z1}$  (the small angle region  $\nu/N_{z1} \ll 1$  of the forward scattering branch), the  $\nu$  dependence of the factorial coefficient becomes dominant, and indicates that upward transitions are favored. In all other regions, it is assumed that these coefficients (which have no  $N_{z1}$  dependence) do not dominate the behavior of the hypergeometric functions sufficiently to modify the appearance of Fig. E3.

The back scattering matrix element exhibits no such subtleties. In this case, any upward transition is favored over any downward transition (it is unnecessary to restrict ourselves to equal upward or downward transitions). Thus in back scattering, the dominant tendency is for a decrease to occur in the z energy with accompanying increases in the perpendicular energy and eigenstate size.

At this point there arises an interesting question, whether the sum of transitions from a given state to all accessible states is predominantly upward or downward. Although this question is more properly discussed in terms of a transition probability or a cross section, we may at this point begin thinking about it in terms of the matrix elements. Considering these only (ignoring the role of the density of states function),

it appears conceivable that for smaller values of  $N_{z1}$  (or for some optimum value if the sum of  $N_1$  and  $N_{z1}$  is to remain constant), the total area under the two LHS curves of Fig. E3 could be greater than the area under the two curves on the RHS. The existence of a critical value of  $N_{z1}$  (or of the ratio  $N_1/N_{z1}$ ) separating regions of predominantly upward and downward transitions has interesting implications since this number is to some extent under experimental control. For example, as this boundary is crossed, there should be enhanced radial diffusion. An electron cyclotron maser might become self-pumping, or even oscillatory as the induced radiation acts to decrease the perpendicular energy. One would expect these effects to appear as the magnetic field is slowly increased. An increasing magnetic field corresponds to a decreasing  $N_{z1}$  which in turn raises the value of the matrix element boundary point separating the forward and back scattering branches.

E6 The Centered Coulomb Matrix Element in the Limit of Small Angle Scattering

We have seen that in the region near the origin, that is, when  $|N_1 - N_2| \ll N_{z1}$ , the arguments  $\gamma$  may be approximated by the expressions (E88) and (E89). We give here a simplified form of the forward scattering matrix element valid when this inequality is satisfied. This form is in essence the constant and linear terms of a power series expansion in  $B v^2/E_{z1}$  about the origin. It is obtained by collecting the zeroth and first order terms of (E29) and then substituting  $\gamma_- = v^2/4N_{z1}$ .

The zeroth order term is

$$\begin{aligned} \langle \rho \Lambda_c \rangle_0 & / \left( \frac{q Q}{4\pi \epsilon_0} \frac{1}{2\pi} \right) & (E93) \\ & = \frac{\sqrt{(N+M)! (N+v+M)! N! (N+v)!}}{(M+2N+v)!} \cdot S_0 \end{aligned}$$



where  $S_0$  is the finite sum

$$S_0 \equiv \sum_{j=0}^N \frac{(M+2j)!}{(M+j)! j!} \frac{(2N+\nu-2j-1)!}{(N+\nu-j)! (N-j)!} \quad (\text{E94a})$$

$$= \sum_{j=0}^N \binom{M+2j}{j} \binom{2N+\nu-2j}{N-j} \frac{1}{2N+\nu-2j} \quad (\text{E94b})$$

$$= \frac{1}{\nu} \binom{M+2N+\nu}{N} = \frac{1}{\nu} \frac{(M+2N+\nu)!}{(M+N+\nu)! N!} \quad (\text{E94c})$$

The analytic value of this sum was obtained by guessing. The zeroth order term assumes the simple form

$$\langle \mathcal{L}_c \rangle_0 / \left( \frac{qQ}{4\pi\epsilon_0} \frac{1}{2\pi} \right) = \frac{1}{\nu} \sqrt{\frac{(N+M)! (N+\nu)!}{N! (N+\nu+M)!}} \quad (\text{E95})$$

This is the exact value of the forward scattering matrix element near the origin (see Fig.E3). Note that we are explicitly forced to exclude the value  $\nu = 0$ .

The first order term is at once more complicated because of the necessity to provide separately for the cases  $\nu = 1$  and  $\nu \geq 2$ . For  $\nu \geq 2$ , the linear term of the confluent hypergeometric psi function is

$$\begin{aligned} & \Psi(1+M+2j, 1-(2N+\nu-2j); \gamma) \\ &= (-\gamma) \frac{(2N+\nu-2j)!}{(M+2N+\nu)!} (M+2j+1) \end{aligned} \quad (\text{E96})$$

This form applies for all  $j$  from 0 to  $N$ . If, however,  $\nu = 1$ , the  $j = N$  term must be modified to

$$\Psi_1(1+M+2N, 1-(1); \gamma) = \frac{(-\gamma)}{(M+2N)!} \left[ -\ln \gamma - \psi(M+2N+2) + 2\psi(1) \right]. \quad (\text{E97})$$

The necessity for these two forms may be understood by examining equation (E91) in conjunction with Table E1. Upon combining these linear terms with the factorial coefficients of the finite sum, the linear term of the matrix element may be written in the form

$$\begin{aligned} \langle \rho \Lambda_c \rangle_1 / \left( \frac{q^2}{4\pi\epsilon_0} \frac{1}{2\pi} \right) &= (-\gamma) \frac{\sqrt{(N+M)! (N+\nu+M)! N! (N+\nu)!}}{(M+2N+\nu)!} S_1 \\ &+ \delta_{\nu,1} (-\gamma) \sqrt{(N+1)(N+M+1)} \left[ -1 + (-\ln \gamma - \psi(M+2N+2) + 2\psi(1)) \right] \end{aligned} \quad (\text{E98})$$

where  $\gamma$  is to have the value  $\nu^2/4N_{z1}$  and  $S_1$  is the finite sum

$$S_1 \equiv \sum_{j=0}^N \frac{(M+2j+1)!}{(M+j)! j!} \frac{(2N+\nu-2j)!}{(N+\nu-j)! (N-j)!} \quad (\text{E99a})$$

$$= \sum_{j=0}^N \binom{M+2j}{j} \binom{2N+\nu-2j}{N-j} (M+2j+1). \quad (\text{E99b})$$

The Kronecker delta in (E98) simply subtracts off the  $j = N$  term of the finite series when  $\nu = 1$ , and adds on the logarithm-lower case psi

function combination. Since this combination is always positive for the present regime of  $\nu$ , the value of (E98) is always negative. Unfortunately the simplicity and utility of this result is diminished by our present inability to perform this sum. This in spite of its similarity to the  $S_0$  of (E94). Nevertheless, what we have in this term is a parabolic fit about the origin of the forward scattering branch depicted in Fig. E3. It is particularly frustrating to be unable to simplify  $S_1$  since we now have at hand the constant and linear dependence in the magnetic field of a matrix element in which this field has been incorporated as a major effect. A simple form for  $S_1$  would thus encourage further calculations of an averaging or statistical nature.

Before moving on to consider the matrix element for back-scattering when  $|N_1 - N_2| \ll N_{z1}$ , let us see what the constant (in  $N_{z1}$ ) term has to say about the preference for upward or downward transitions. For  $M \neq 0$ , there is a slight tendency for upward transitions to be favored over downward transitions. This result follows from the observation that the ratio

$$\begin{array}{c}
 \begin{array}{|c}
 \hline
 N+\nu \\
 \hline
 N \\
 \hline
 N-\nu \\
 \hline
 \end{array}
 \end{array}
 \begin{array}{l}
 \frac{\langle g_{\Lambda c} \rangle_{N \rightarrow N+\nu}}{\langle g_{\Lambda c} \rangle_{N \rightarrow N-\nu}}
 \end{array}$$

$$= \sqrt{\frac{\left(1 + \frac{\nu}{N}\right) \left(1 + \frac{\nu}{N-1}\right) \cdots \left(1 + \frac{\nu}{N-\nu+1}\right)}{\left(1 + \frac{\nu}{N+M}\right) \left(1 + \frac{\nu}{N-1+M}\right) \cdots \left(1 + \frac{\nu}{N-\nu+1+M}\right)}} \quad (\text{E100})$$

is greater than unity for  $M > 0$ . As discussed earlier, this result serves to point up the significance of the  $\nu$  dependence of the factorial coefficients

in the region near the ordinate axis on the forward scattering curve of Fig. E3.

We consider next the value of the back scattering matrix element when  $\nu \ll N_{z1}$ . In this region, the argument  $\gamma_+ = 4N_{z1}$ , the numerical value of which may reasonably exceed  $10^4$  (equation (E84)). We assume that the  $\gamma$  value is large enough that we may employ the asymptotic form (E60) of the confluent hypergeometric functions. In this limit the matrix element assumes the form

$$\langle \varrho \Lambda_c \rangle_\infty / \left( \frac{\varrho_0}{4\pi \epsilon_0} \frac{1}{2\pi} \right) = \sqrt{\frac{(N+M)! (N+\nu+M)!}{N! (N+\nu)!}} \gamma^{-1-M} \sum_{j=0}^N \frac{N! (N+\nu)! (M+2j)!}{(N-j)! (N+\nu-j)! (M+j)! j!} \gamma^{-2j} \quad (\text{E101a})$$

$$= \sqrt{\frac{(N+M)! (N+\nu+M)!}{N! (N+\nu)!}} \gamma^{-1-M} \left[ 1 + \frac{N(N+\nu)(M+2)}{\gamma^2} + \dots \right]. \quad (\text{E101b})$$

It is evident that all terms after the first may be ignored if

$$\gamma^2 \gg N(N+\nu)(M+2). \quad (\text{E102})$$

Using only the first term of (E101b), the ratio  $\langle \varrho \Lambda_c \rangle_\infty / \langle \varrho \Lambda_c \rangle_0$  for the value  $\gamma_+ = 4N_{z1}$  is

$$\frac{\langle \varrho \Lambda_c \rangle_\infty}{\langle \varrho \Lambda_c \rangle_0} = \frac{(N+\nu+M)!}{(N+\nu)!} \frac{\nu}{(4N_{z1})^{1+M}} \leq \frac{\nu}{4N_{z1}} \left( \frac{N+\nu+M}{4N_{z1}} \right)^M \ll 1. \quad (\text{E103})$$

This ratio is much less than unity, as indicated, if the original criterion for the use of the asymptotic form (E60) is satisfied.

APPENDIX F  
PROBABILITY DENSITY FLUX ASSOCIATED WITH  
CYLINDRICAL LANDAU EIGENFUNCTIONS

F1 Content

Presented in this appendix are expressions for the probability density flux  $\vec{J}$  associated with any prescribed wave function  $\Psi(\vec{r}, t)$  and in particular with the cylindrical Landau eigenfunctions

$$\Psi_{NMK} = e^{-i \frac{E_{NMK} t}{\hbar}} R_{NM}(\rho) \Phi_M(\phi) Z_K(z). \quad (F1)$$

Discussed in addition is the derivation from  $J_z$  of the flux component  $\Gamma_z$  utilized in the expression connecting the cross section  $\sigma$  and the transition probability or rate  $w$ ,

$$w = N_{sc} \Gamma_z \sigma, \quad (F2)$$

where  $N_{sc}$  is the total number of scatterers effective in the volume under consideration. The flux  $\Gamma_z$  is an areal average of  $J_z$  in the  $\rho$ - $\phi$  plane. It is necessary because in  $J_z$  is preserved the radial structure of the eigenfunctions (F1).

F2 Origin of the Flux Concept, and General Expressions

The concept of probability density flux  $\vec{J}$  arises because of the presence of the expression for  $\vec{J}$  in the sourceless conservation equation for the probability density  $\Psi^* \Psi$  :

$$\frac{\partial}{\partial t} \Psi^* \Psi + \nabla \cdot \vec{J} = 0 \quad (F3)$$

where  $\Psi = \Psi(\vec{r}, t)$  and

$$\vec{J} = \frac{\hbar}{2im} \left( \Psi^* \nabla \Psi - (\nabla \Psi)^* \Psi \right) - \frac{\partial}{\partial t} \Psi^* \Psi \vec{A}. \quad (F4)$$

The conservation equation follows from the time-dependent Schrodinger equation (of which  $\Psi(\vec{r}, t)$  is a solution) together with the requirement of time invariance of the probability density norm  $\int \Psi^* \Psi d\tau$  :

$$\frac{\partial}{\partial t} \int \Psi^* \Psi d\tau = 0 . \quad (\text{F5})$$

From this also follows the condition that the Hamiltonian H be Hermitian, i. e. , that it satisfy

$$\int \Psi^* H \Psi d\tau = \int \Psi (H \Psi)^* d\tau , \quad (\text{F6})$$

$d\tau$  denoting the usual three-dimensional volume element.

In the cylindrical coordinate system spanned by the unit vectors  $\hat{e} \times \hat{\phi} = \hat{z}$ , the components of  $\vec{J}$  are

$$J_{\rho} = \frac{\hbar}{2im} \left[ \Psi^* \frac{\partial \Psi}{\partial \rho} - \left( \frac{\partial \Psi}{\partial \rho} \right)^* \Psi \right] - \frac{q}{m} \Psi^* \Psi A_{\rho} \quad (\text{F7})$$

$$J_{\phi} = \frac{\hbar}{2im} \left[ \Psi^* \frac{1}{\rho} \frac{\partial \Psi}{\partial \phi} - \left( \frac{1}{\rho} \frac{\partial \Psi}{\partial \phi} \right)^* \Psi \right] - \frac{q}{m} \Psi^* \Psi A_{\phi} \quad (\text{F8})$$

$$J_{z} = \frac{\hbar}{2im} \left[ \Psi^* \frac{\partial \Psi}{\partial z} - \left( \frac{\partial \Psi}{\partial z} \right)^* \Psi \right] - \frac{q}{m} \Psi^* \Psi A_z . \quad (\text{F9})$$

### F3 Flux Associated with Cylindrical Landau Eigenfunctions

The set of eigenfunctions indicated in (F1) were obtained as solutions to the Schrodinger equation for the Hamiltonian  $H_0 = (\vec{p} - q\vec{A})^2 / 2m$ . The factored spatial wave functions, normalized to unity as Kronecker and Dirac delta functions, are:

$$R_{NM} = \sqrt{2 \beta^2 \frac{N!}{(N+M)!}} (\beta^2 \rho^2)^{\frac{M}{2}} e^{-\frac{1}{2} \beta^2 \rho^2} L_N^M(\beta^2 \rho^2) \quad (\text{F10})$$

$$\Phi_M = \frac{1}{\sqrt{2\pi}} e^{\pm i M \phi} \quad (\text{F11})$$

$$\chi_k = \frac{1}{\sqrt{2\pi}} e^{\pm i k z} \quad k = \frac{p_z}{\hbar} = \frac{m v_z}{\hbar} \quad (\text{F12})$$

where  $\beta^2 \equiv eB/2\hbar$  (dimensions of  $m^{-2}$ ), and N and M are independent positive integers (including zero) having no upper bound:

$$N = 0, 1, 2, 3, \dots$$

$$M = 0, 1, 2, 3, \dots$$

The derivation of these eigenfunctions and the interpretation of the quantum integers N and M were considered in appendix C. The magnetic field  $\vec{B} = B\hat{z}$  was generated from the vector potential

$$\vec{A} = \frac{1}{2} B \rho \hat{\phi} \quad (\text{F13})$$

through the relation  $\vec{B} = \text{curl } \vec{A}$ . This form may be recognized as a particular case (for the present coordinate system and orientation of  $\vec{B}$ ) of the general vector potential describing a constant and uniform magnetic field,

$$\vec{A} = \frac{1}{2} \vec{B} \times \vec{r} \quad (\text{F14})$$

The cylindrical components of  $\vec{J}$  generated from the Landau eigenfunctions indicated in (F1) and given explicitly in (F10) through (F12) are:



$$J_{\rho} = 0 \quad (F15)$$

$$J_{\phi} = \frac{R_{NM}^2}{4\pi^2} \left( \frac{\pm M\hbar}{m\rho} - \frac{qB\rho}{2m} \right) \quad (F16)$$

$$J_z = \frac{\pm \hbar k}{m} \frac{R_{NM}^2}{4\pi^2} \quad (F17)$$

The component  $J_{\rho}$  is zero because the  $\rho$  eigenfunctions are real and because the potential  $\vec{A}$  has no radial component. The factors  $1/4\pi^2$  in the  $\phi$  and  $z$  components originate in the normalization constants of the corresponding eigenfunctions. The choice of upper or lower sign should conform to the choice made in the eigenfunctions.

That these flux components have dimensions of  $(m \text{ sec})^{-1}$  instead of the expected  $(m^2 \text{ sec})^{-1}$  is due to the method of normalization of the  $z$  eigenfunctions. They are presently normalized to the Dirac delta function  $\delta(k_1 - k_2)$  over the range  $-\infty < z < \infty$ . In view of this normalization (reflecting the presence of a continuous eigenparameter), the probability density is given by  $\Psi_{NMk}^* \Psi_{NMk} dk$  instead of the customarily encountered  $\psi^* \psi$ . Similarly, the flux of probability density associated with the eigenstate  $(N, M, k)$  is correctly specified by  $\vec{J} dk$ .

#### F4 The Averaged z Flux

The expression (F2) may be viewed as an experimental relation giving the number per second of events arriving at a detector due to scattering of the incident flux  $\Gamma_z$  by agents  $(N_{sc})$  of them

in the scattering volume, acting independently) of cross section  $\sigma$ . The nature of both the incident flux and the events to which the detector is sensitive must be carefully specified and must be consistent with the conversion process which the cross section represents. Further, the flux  $\Gamma_{\underline{z}}$  must in principle be an experimentally measurable quantity. This in turn implies that  $\Gamma_{\underline{z}}$  must be expressible in terms of expectation values or diagonal matrix elements. We calculate here the  $\Gamma_{\underline{z}}$  associated with a single eigenfunction  $\psi_{NMk}$ .

The radially-varying flux component  $J_z dk$  of equation (F17) represents the particle current associated with the state (NMk) which flows across a unit area erected normally to the z axis at the radial position  $\rho$ . Thus the quantity  $J_z(\rho) dk \rho d\rho d\phi$  represents the particle current (dimensions of  $\text{sec}^{-1}$ ) flowing through the area  $\rho d\rho d\phi$ . The total current associated with the state (NMk) follows as

$$I_z \equiv \int_0^{2\pi} \int_0^{\infty} J_z(\rho) dk \rho d\rho d\phi \quad (\text{F18a})$$

$$= \pm \frac{\hbar k}{m} \frac{dk}{2\pi} \int_0^{\infty} R_{NM}^2(\rho) \rho d\rho \quad (\text{F18b})$$

$$= \pm \frac{\hbar k}{m} \frac{dk}{2\pi} . \quad (\text{F18c})$$

We next calculate the area through which this spatially-localized particle current flows. This average radial area is defined as

$$a \equiv \int_0^{2\pi} \int_0^{\infty} \frac{R_{NM}(\rho)}{2\pi} \pi \rho^2 \frac{R_{NM}(\rho)}{2\pi} \rho d\rho d\phi \quad (\text{F19a})$$

$$= \frac{M+2N+1}{2\beta^2} . \quad (\text{F19b})$$

The radial integral in (F19a) was evaluated in section D3.

The flux  $\Gamma_z$  is then defined as the total current divided by the average area through which the current flows:

$$\Gamma_z \equiv \frac{I_z}{a} = \frac{\hbar k}{m} \frac{2\beta^2 dk}{(M+2N+1)2\pi} \quad (F20)$$

This result is interesting in several respects. It says that the area-averaged flux associated with the state (NMk) increases with the magnetic field B (through  $\beta^2$ ) and decreases as the integers N and M become larger. The magnetic field dependence stems from the fact that the argument of the radial eigenfunctions  $R_{NM}$  is  $\beta^2 \rho^2$ . Thus as B is increased, the eigenfunctions occupy smaller and smaller areas. The same total current must flow through a smaller area with the result that the flux (referred to a constant area) increases. The integer dependence reflects the fact that the radial area occupied by the eigenfunctions  $\psi_{NMk}$  increases as these integers increase. Both of these effects are evident from Figures C1 and C2, pages 53 and 54. Finally, if we may be allowed to compare the  $\Gamma_z$  of (F20) with the product  $n v$ , then the quantity  $2\beta^2(dk/2\pi)/(M+2N+1)$  may be thought of as the number density associated with the state (NMk).

The flux  $\Gamma_z$  may be defined equivalently as the ratio of expectation values,

$$\Gamma_z \equiv \frac{\int_{-\infty}^{\infty} \int_0^{2\pi} \int_0^{\infty} J_z(\rho) dk \rho d\rho d\phi dz}{\int_{-\infty}^{\infty} \int_0^{2\pi} \int_0^{\infty} \psi_{NMk}^* \pi \rho^2 \psi_{NMk} \rho d\rho d\phi dz} \equiv \frac{\langle J_{zop} \rangle}{\langle \pi \rho^2 \rangle} \quad (F21)$$

It is this more general definition (with  $\psi_{NMk}$  replaced by  $U_{N_{\perp}N_z S}$ ) which may be used to calculate the area-averaged flux associated with the uniform beam  $U_{N_{\perp}N_z S}$  of section C4.

APPENDIX G  
DERIVATION OF THE DIFFERENTIAL CROSS  
SECTION FOR SCATTERING IN A  
MAGNETIC FIELD

G1 Introduction

When the cylindrical Landau eigenfunctions are employed as the basis set for Born approximation calculations of scattering in a magnetic field, the transition probabilities and the related cross sections involve transitions from a one dimensional continuum in which are embedded discrete states to a second such continuum-discrete system. The transition probability and cross section expressions must reflect this circumstance. Most expressions found in texts do not. Although they may be generalized on an intuitive and dimensional basis so as to encompass the present situation, it was thought prudent to derive the expressions by some standard method. Accordingly, we present here, in abbreviated form, the Dirac time-dependent perturbation theory derivation of the differential transition probability for scattering (by a time independent potential) from one Landau eigenstate to another. No attempt is made in this discussion to connect the formalism with experiment, other than the interpretation of the expansion coefficients. We begin with a review of the method, following closely the development given in Roman [1965].

G2 General Formalism

The aim of the method is a solution of the complete Schrodinger equation

$$i\hbar \frac{\partial \Psi}{\partial t} = (H_0 + H_1) \Psi \quad (G1)$$

where  $\Psi = \Psi(\vec{r}, t)$ , in terms of the eigenfunction set  $\{\psi_\alpha\}$  of the unper-

turbed problem, described by the time - independent Hamiltonian  $H_0$ . However the solution, once obtained, is not used directly. Rather, the expansion coefficients are interpreted in terms of a probability of transition from one eigenstate of  $H_0$  to another, such transitions caused by the agent or perturbation  $H_1$ .

The time independent Hamiltonian  $H_0$  is presumed to satisfy the energy eigenvalue equation

$$H_0 \Psi_\alpha = E_\alpha \Psi_\alpha \quad (G2)$$

where the subscript  $\alpha$  stands for the set of eigenparameters (simultaneous observables) which characterizes the state represented by the eigenfunction  $\Psi_\alpha$ . These eigenfunctions are also presumed to be orthogonal,

$$\int \Psi_\rho^* \Psi_\alpha \, d\tau \equiv (\Psi_\rho, \Psi_\alpha) = \delta_{\rho, \alpha} \quad (G3)$$

and complete in the sense that any arbitrary (though well-behaved) function may be described in terms of the  $\Psi_\alpha$ :

$$\Phi_\rho = \sum_\alpha c_{\rho, \alpha} \Psi_\alpha(\vec{r}) e^{-i \frac{E_\alpha}{\hbar} t} \quad (G4)$$

Here  $\beta$  stands for the parameters characterizing the state  $\Phi_\beta$ .

The expansion coefficients follow upon application of (G3),

$$c_{\rho, \gamma} = (\Psi_\gamma, \Phi_\rho) e^{i \frac{E_\gamma}{\hbar} t} \quad (G5)$$

The meaning of the coefficients follows from the requirement that  $\Phi_\beta$  be normalized to unity. Consider first the sequence

$$\begin{aligned}
 (\Phi_\epsilon, \Phi_\eta) &= \sum_\alpha \sum_\rho c_{\epsilon,\rho}^* c_{\eta,\alpha} (\psi_\rho, \psi_\alpha) e^{i\omega_{\rho\alpha}t} \\
 &= \sum_\alpha c_{\epsilon,\alpha}^* c_{\eta,\alpha} .
 \end{aligned} \tag{G6}$$

Now set  $\epsilon = \eta$  and sum on  $\eta$  to obtain

$$\sum_\eta (\Phi_\eta, \Phi_\eta) = \sum_\eta \sum_\alpha |c_{\eta,\alpha}|^2 \equiv 1 . \tag{G7}$$

Thus  $|c_{\eta,\alpha}|^2$  is the probability (a pure number) of finding the system described by the state function  $\Phi_\eta$  in the basis eigenstate  $\alpha$ .

With these preliminaries in mind, we define an expansion solution of the complete problem (G1) which is analogous to the expansion solution of the unperturbed problem:

$$\Psi_\eta = \sum_\alpha g_{\eta,\alpha}(t) \psi_\alpha(\vec{r}) e^{-i\frac{E_\alpha}{\hbar}t} . \tag{G8}$$

The coefficient  $g$  takes into account the effect of the perturbation  $H_1$  (whether time dependent or not). Substitution of this solution into (G1) followed by use of the orthogonality relation (G3) yields the set

$$\frac{dg_{\eta,\rho}}{dt} = \frac{-i}{\hbar} \sum_\gamma g_{\eta,\gamma} (\psi_\rho, H_1 \psi_\gamma) e^{i\omega_{\rho\gamma}t} \tag{G9}$$

where

$$\omega_{\rho\gamma} \equiv (E_\rho - E_\gamma) / \hbar . \tag{G10}$$

This set of coupled (by  $H_1$ ) equations is exact and is equivalent to (G1). The problem of solving (G1) has been transformed to one of determining the coefficients  $g$ . Once the  $g$  coefficients are in hand, they are interpreted in precisely the same way as were the coefficients  $c$  in the general solution (G4) of the unperturbed problem.

The solution of the set (G9) proceeds by iteration. As a first guess, it is assumed that  $\Psi(\vec{r}, t)$  may be approximated by  $\Phi(\vec{r}, t)$ ,

$$\Psi_{\alpha}^{(0)}(\vec{r}, t) = \Phi_{\alpha}(\vec{r}, t) , \quad (G11)$$

which implies that

$$g_{\alpha, \gamma}^{(0)}(t) = c_{\alpha, \gamma} . \quad (G12)$$

This approximation, when substituted into the RHS of (G9), permits an integration yielding a once-refined expression for  $g$ ,

$$g_{\alpha, \beta}^{(1)}(t) = \frac{-i}{\hbar} \sum_{\gamma} c_{\alpha, \gamma} \int_0^t (\psi_{\alpha}, H_1 \psi_{\gamma}) e^{i\omega_{\alpha\gamma} t'} dt' + g_{\alpha, \beta}^{(0)}(0) . \quad (G13)$$

To proceed further, we must know more about  $H_1$ . When the actual  $H_1$  is time independent (as is the Coulomb potential energy), the  $H_1$  in (G1) and in (G13) is assigned the time dependence of a step function. It is considered to have been turned on at  $t' = 0$  and remain constant in time until  $t' = t$  at which time it is turned off. The transients thus introduced are later removed by a limiting and averaging procedure. Adoption of this time dependence for  $H_1$  enables the integration of (G13) to the form

$$g_{\alpha, \beta}^{(1)}(t) = \frac{-i}{\hbar} \sum_{\gamma} c_{\alpha, \gamma} (\psi_{\alpha}, H_1 \psi_{\gamma}) \frac{e^{i\omega_{\alpha\gamma} t} - 1}{\omega_{\alpha\gamma}} + g_{\alpha, \beta}^{(0)}(0) . \quad (G14)$$

The next step is the application of the initial conditions. Information about the state of the system at  $t=0$  must be supplied. Since the system was described by the state function  $\Phi$  at  $t = 0$ , this means a specification of the coefficients  $c$  on the RHS of (G14). Since our avowed interest is in transitions from one basis eigenstate to another, we specify that initially the system was in the single eigenstate  $\alpha$ .

That is, we subject (G14) to the initial condition

$$c_{\eta,\nu} = \delta_{\alpha,\nu} = c_{\alpha,\nu} . \quad (\text{G15})$$

To reflect the fact that the initial system is entirely in a single eigenstate, we have set the index  $\eta$  to  $\alpha$ . The expression becomes

$$g_{\alpha,\beta}^{(1)}(t) = \frac{-1}{\hbar} (\psi_\beta, H, \psi_\alpha) \frac{e^{i\omega_{\beta\alpha}t} - 1}{\omega_{\beta\alpha}} + \delta_{\beta,\alpha} . \quad (\text{G16})$$

Thus for a system initially in the basis eigenstate  $\alpha$ , the probability of a transfer (caused by  $H_1$ ) to the eigenstate  $\beta$  is

$$\begin{aligned} W_{\alpha\beta}^t &\equiv |g_{\alpha\beta}(t)|^2 \approx |g_{\alpha\beta}^{(1)}(t)|^2 \\ &= \frac{4 |(\psi_\beta, H, \psi_\alpha)|^2}{\hbar^2} \frac{\sin^2 \omega_{\beta\alpha} \frac{t}{2}}{\omega_{\beta\alpha}^2} . \end{aligned} \quad (\text{G17})$$

The transition probability per unit time is then defined by the transient smoothing limit

$$w_{\alpha\beta} \equiv \lim_{t \rightarrow \infty} \frac{W_{\alpha\beta}^t}{t} . \quad (\text{G18})$$

If we apply this definition to (G17) and utilize the relation

$$\lim_{t \rightarrow \infty} \frac{\sin^2 \omega t}{\omega^2 t} = \pi \delta(\omega) \quad (\text{G19})$$

we obtain the general expression

$$w_{\alpha\beta} = \frac{2\pi}{\hbar} |(\psi_\beta, H, \psi_\alpha)|^2 \delta(E_\beta - E_\alpha) \quad (\text{G20})$$

for the transition probability per unit time.



### G3 Application to Cylindrical Landau Eigenfunctions

The formalism developed above is now applied to describe transitions from one cylindrical Landau eigenstate to another. The relations (G2) through (G4) become, respectively,

$$H_0 \Psi_{NMk} = E_{NMk} \Psi_{NMk} \quad (G21)$$

$$(\Psi_{N'M'k'}, \Psi_{NMk}) = \delta_{N',N} \delta_{M',M} \delta(k'-k) \quad (G22)$$

$$\Phi_{TUV} = \sum_N \sum_M \int_k c_{TUV, NMk} \Psi_{NMk}(\vec{r}) e^{-i \frac{E_{NMk}}{\hbar} t} dk, \quad (G23)$$

where T, U, N, and M are quantum integers and  $\nu$  and k are continuous wavenumbers. Because of these wavenumbers and because the general symbol  $\Phi$  now contains an integration over a wavenumber, the interpretation of the coefficients c, developed on the basis of equation (G7), must be modified. In the present case, speaking with reference to (G23), the quantity

$$|c_{TUV, NMk}|^2 d\nu dk$$

is the probability (a pure number) of finding the system described by the state function  $\Phi_{TUV}$  occupying the basis eigenstate NMk. The continuous wavenumbers  $\nu$  and k are understood to lie in the respective intervals  $d\nu$  and  $dk$ . This interpretation is the root of all modifications of the general expressions and permeates all that follows.

The expression (G15) which selects a single initial eigenstate becomes

$$C_{N_1, M_1, k_1, N_2, M_2, k_2} = \delta_{N_1, N_2} \delta_{M_1, M_2} \delta(k_1 - k_2). \quad (G24)$$

When substituted into (G14), one obtains the analog of (G16),

$$g_{N_1, M_1, k_1, N_2, M_2, k_2}^{(1)} = \frac{-1}{\hbar} (\psi_{N_2, M_2, k_2}, H, \psi_{N_1, M_1, k_1}) \frac{e^{i\omega_{21}t} - 1}{\omega_{21}}. \quad (G25)$$

The transition probability per unit time becomes

$$W_{\alpha\beta} = \frac{2\pi}{\hbar} |(\psi_\beta, H, \psi_\alpha)|^2 \delta(E_\beta - E_\alpha) dk_1 dk_2 \quad (G26)$$

where  $\alpha$  refers to the set  $(N_1, M_1, k_1)$  and  $\beta$  to the set  $(N_2, M_2, k_2)$ . Since  $E_z = \hbar^2 k^2 / 2m$ , differentials of wavenumber are related to differentials of z energy as

$$dk = \sqrt{\frac{2m}{\hbar^2}} \frac{1}{2\sqrt{E_z}} dE_z. \quad (G27)$$

In terms of z energy differentials, the transition probability rate has the form

$$W_{\alpha\beta} = \frac{2\pi}{\hbar} |(\psi_\beta, H, \psi_\alpha)|^2 \frac{2m}{\hbar^2} \frac{1}{4\sqrt{E_{z1} E_{z2}}} \delta(E_\beta - E_\alpha) dE_{z1} dE_{z2}. \quad (G28)$$

Because the quantum number M is conserved in the centered Coulomb matrix element, the argument of the delta function has the form

$$E_\beta - E_\alpha = E_{z2} - E_{z1} - \hbar\omega_c (N_1 - N_2). \quad (G29)$$

If we insert this into (G28) and integrate over the final z energy, we obtain

$$W_{\alpha\beta} = \frac{2\pi}{\hbar} \left| (\psi_\beta, H_1 \psi_\alpha) \right|^2 \frac{2m}{\hbar^2} \frac{1}{4 E_{z1} \sqrt{1 + \frac{\hbar\omega_c}{E_{z1}} (N_1 - N_2)}} dE_{z1} \quad (G30)$$

where energy conservation is implied in the matrix element of the perturbation  $H_1$ . The expression (G30) is the differential probability of transition from Landau state  $\alpha = (N_1, M, k_1)$  to Landau state  $\beta = (N_2, M, k_2)$  per unit time for  $E_{z1}$  in the range  $dE_{z1}$ . The adjective differential is not connected in any way with  $E_{z1}$ , but with the value of  $M$  (and in some cases  $N_1 + M$ ). The cross section follows upon division by the z flux

$$\begin{aligned} \Gamma_z &= \frac{\hbar k_1}{m} \frac{2\beta^2 dk_1}{(M+2N_1+1) 2\pi} \\ &= \frac{\beta^2 dE_{z1}}{\pi \hbar (M+2N_1+1)}. \end{aligned} \quad (G31)$$

The expression obtained is

$$\sigma_{\alpha\beta} \equiv \frac{W_{\alpha\beta}}{\Gamma_z} = 2\pi^2 \frac{\left| (\psi_\beta, H_1 \psi_\alpha) \right|^2}{\hbar\omega_c E_{z1}} \frac{(M+2N_1+1)}{\sqrt{1 + \frac{\hbar\omega_c}{E_{z1}} (N_1 - N_2)}}. \quad (G32)$$

#### G4 Separation of the Two-Particle Hamiltonian

The problem as stated above and as treated in this work has been simplified in at least one important respect. The problem has been considered from the beginning in a frame in which the seat of the perturbation potential energy is at rest. That is, the original, laboratory frame, two body problem has been treated in terms of an equivalent single particle moving in a fixed potential field. We consider here the implications of this simplification in the context of the present problem of interest.

For two particles of charge and mass  $(q, m_1)$  and  $(Q, m_2)$  moving in a uniform magnetic field, the Hamiltonian describing their motion is approximately,

$$H = \frac{(\vec{p}_1 - q\vec{A}_1)^2}{2m_1} + \frac{(\vec{p}_2 - Q\vec{A}_2)^2}{2m_2} + q\Lambda_c(|\vec{r}_1 - \vec{r}_2|) \quad (\text{G33})$$

where

$$\vec{A}_1 = \frac{1}{2} \vec{B} \times \vec{r}_1 \quad \vec{A}_2 = \frac{1}{2} \vec{B} \times \vec{r}_2 \quad (\text{G34})$$

The charge  $q$  is instantaneously located by the position vector  $\vec{r}_1$  and the charge  $Q$  by  $\vec{r}_2$ . Included in this Hamiltonian has been the kinetic energy of each particle and the Coulomb interaction potential energy, depending only upon the distance between the charges. Their magnetic interaction, for example, has been ignored. The vector potentials (G34) satisfy the gauge condition  $\vec{\nabla} \cdot \vec{A}_i = 0$  so that the commutator  $[\vec{p}_i, \vec{A}_i] = 0$ .

As is customary in analogous zero magnetic field treatments, we transform the system to the relative ( $\vec{r}$ ) and center of mass ( $\vec{R}$ ) coordinates by means of the prescriptions

$$\vec{R} \equiv \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2} \quad \vec{r} \equiv \vec{r}_1 - \vec{r}_2 \quad (\text{G35})$$

$$\vec{r}_1 = \vec{R} + \frac{m_2}{m_1+m_2} \vec{r} \quad \vec{r}_2 = \vec{R} - \frac{m_1}{m_1+m_2} \vec{r} . \quad (\text{G36})$$

Note the effect of making  $m_2$  (the charge  $Q$ ) so massive that it may be considered fixed. As  $m_2 \rightarrow \infty$ , the vector  $\vec{R}$  goes into  $\vec{r}_2$ , in this limit a constant vector which may be set to zero. This corresponds to location of  $Q$  at the origin. The motion of the system is described, in the limit  $m_2 \rightarrow \infty$ , by the vector  $\vec{r} = \vec{r}_1$ .

When the Hamiltonian  $H(\vec{r}_1, \vec{r}_2)$  of (G33) is subjected to these transformations, the resulting form  $H(\vec{r}, \vec{R})$  may be written as the sum of three parts,

$$H(\vec{r}, \vec{R}) \equiv H_R(\vec{R}) + H_r(\vec{r}) + H_c(\vec{r}, \vec{R}) \quad (\text{G37})$$

where, as indicated,  $H_R$  describes the center of mass motion and  $H_r$  the relative motion. The component  $H_c$  consists of terms representing the coupling of the relative and center of mass motions.

The forms of these components are as follows:

$$H_R = \frac{\mathcal{P}_R^2}{2M} - \left( \frac{g}{m_1} + \frac{Q}{m_2} \right) \vec{A}_R \cdot \vec{\mathcal{P}}_R + \left( \frac{g^2}{m_1} + \frac{Q^2}{m_2} \right) \frac{A_R^2}{2} \quad (\text{G38})$$

$$\begin{aligned} H_r = & \frac{\mathcal{P}_r^2}{2M^2} \left( \frac{m_2^2}{m_1} + \frac{m_1^2}{m_2} \right) - \frac{\vec{A}_r \cdot \vec{\mathcal{P}}_r}{M^2} \left( \frac{g m_2^2}{m_1} + \frac{Q m_1^2}{m_2} \right) \\ & + \frac{A_r^2}{2M^2} \left( \frac{g^2 m_2^2}{m_1} + \frac{Q^2 m_1^2}{m_2} \right) + g \mathcal{L}_c(|\vec{r}|) \end{aligned} \quad (\text{G39})$$

$$H_c = \frac{\vec{P}_R \cdot \vec{P}_r}{M} \left( \frac{m_2}{m_1} - \frac{m_1}{m_2} \right) - \frac{(\vec{A}_r \cdot \vec{P}_R + \vec{A}_R \cdot \vec{P}_r)}{M} \left( \frac{q m_2}{m_1} - \frac{Q m_1}{m_2} \right) + \frac{\vec{A}_R \cdot \vec{A}_r}{M} \left( \frac{q^2 m_2}{m_1} - \frac{Q^2 m_1}{m_2} \right) \quad (G40)$$

where  $M \equiv (m_1 + m_2)$  and  $\mu \equiv m_1 m_2 / (m_1 + m_2)$ .

The object of these manipulations is to force the coupling terms (of this or any other transformation) to vanish under conditions which do not severely restrict the generality of the problem at hand. Then the remaining two components of the motion may be studied separately, there being no interaction between them. It is seen from (G40) that we cannot in general achieve this objective by means of this transformation alone. Only if the charges and the masses of the two particles are the same does  $H_c$  vanish. For this case, when  $Q = q$  and  $m_1 = m_2 = m$ , the Hamiltonian becomes

$$H = \frac{(\vec{P}_R - q \vec{A}_R)^2}{2\mu} + \frac{(\vec{P}_r - q \vec{A}_r)^2}{2M} + q \Lambda(|\vec{r}|) \quad (G41)$$

where  $\mu = m/2$  and  $M = 2m$ . The center of mass moves as a free particle in the magnetic field. The relative motion takes place in a Coulomb field fixed at the origin, as well as in a magnetic field.

In the limit  $m_2 \rightarrow \infty$ , the Hamiltonian becomes

$$H = \frac{(\vec{P}_1 - q \vec{A}_1)^2}{2m_1} + q \Lambda(|\vec{r}_1|) \quad (G42)$$

where we have set  $R = r_2 = \text{constant} = 0$  as described earlier.

Even though the usual center of mass and relative coordinate transformation does not by itself separate the motion of two charged particles moving in an external magnetic field, Knox [1963] and

Dexter and Knox [1965] have cited a further transformation upon the Schroedinger equation constructed with the transformed Hamiltonian (G37). This apparently goes further but does not completely separate the motion. The possibility remains, however, that the coupling terms could be considered as small perturbations. This further transformation was not investigated.

REFERENCES

- Barananenkov, Yu. N., 1960, "Coulomb Scattering of Charges in a Strong Magnetic Field", translation of Soviet Physics JETP, 11, pp. 1091-1092 (Russian original 38, pp.1512-1514).
- Bitter, F., 1964, "Transition from Predominantly Electric to Predominantly Magnetic Binding in the Hydrogen Atom in a Magnetic Field", Quarterly Progress Report No. 72, Research Laboratory of Electronics, MIT, Cambridge, Mass., pp. 45-48.
- Bitter, F., 1965, "Ultrastrong Magnetic Fields", Scientific American, July, 1965.
- Copson, E. T., 1965, Asymptotic Expansions, Cambridge University Press, London.
- Dexter, D. L., and Robert S. Knox, 1965, Excitons, Interscience Publishers, pp. 54-56.
- Dingle, R. B., 1952, "Some Magnetic Properties of Metals", Proc. Roy. Soc. A, 211, 500.
- Erdélyi, A., 1938, J. London Math. Soc., 13, 154-156.
- Goldman, Robert, 1963, "Coulomb Interactions in a Strong Magnetic Field", Ph.D. thesis, Yale University. Also available as AD 412 506.
- Goldman, Robert, and Ludwig Oster, 1963, "Coulomb Interactions in a Strong Magnetic Field. I. Adiabatic Case", Phys. Rev. 129, 1469-1478.
- Goldman, Robert, 1964, "Coulomb Interactions in a Strong Magnetic Field. II. Nonadiabatic Case. Line Profile of Cyclotron Radiation", Phys. Rev. 133, A647-A661.
- Goldstein, Herbert, 1950, Classical Mechanics, Addison-Wesley, Reading, Massachusetts.
- Gröbner, W. and N. Hofreiter, 1961, Integraltafeln, two volumes, Springer-Verlag, Wien.



- HTF I, II, 1953, Higher Trancendental Functions, Volumes I and II, A. Erdélyi, et. al., McGraw-Hill, New York.
- Johnson, M. H., and B. A. Lippmann, 1949, "Motion in a Strong Magnetic Field", Phys, Rev. 76, 828-832.
- Kahn, Arnold H., 1960, "Electron Scattering in High Magnetic Field", Phys, Rev. 119, 1189-1192.
- Knox, Robert S., 1963, Theory of Excitons , Supplement 5 of the series, Solid State Physics, Advances in Research and Applications, Academic Press, New York, pp. 80-81.
- Kuyatt C., J. Simpson, and S. Mielczarek, 1965, Phys. Rev. 138, A385.
- Lebedev, N. N., 1965, Special Functions and Their Applications, trans. by R. A. Silverman, Prentice-Hall, Englewood Cliffs, N. J.
- Magnus, Wilhelm, and Fritz Oberhettinger, 1949, Formulas and Theorems for the Functions of Mathematical Physics, Chelsea Publishing Co., New York.
- Morse, P. M., and H. Feshbach, 1953, Methods of Theoretical Physics, McGraw-Hill, New York,
- NBS, 1964, Handbook of Mathematical Functions, Applied Mathematics Series 55, National Bureau of Standards, U. S. Government Printing Office, Washington.
- Richards, Paul I., 1959, Manual of Mathematical Physics, Pergamon Press, New York.
- Roman, Paul, 1965, Advanced Quantum Theory, Addison-Wesley, Reading, Massachusetts.
- Seventeenth Annual Gaseous Electronics Conference, abstracts thereof appearing in Bull. Am. Phys. Soc., Series II, 10, 178-193, 1965.
- Symon, Keith R., 1953, Mechanics, Addison-Wesley, Reading, Massachusetts.

Tannenwald, L. M., 1959, "Coulomb Scattering in a Very Strong Magnetic Field", Phys. Rev. 113, 1396-1405.

TIT I, II, 1953, Tables of Integral Transforms, Volumes I and II, A. Erdélyi, et. al., McGraw-Hill, New York.

## BIOGRAPHICAL NOTE

The author spent his childhood on farms in the midwestern and southern United States. He was graduated in 1958 from the University of California, Santa Barbara with a bachelor of arts degree in physics. While in attendance there, he held a variety of part-time and summer jobs, including installation and maintenance of marine radiotelephones and fathometers, tutoring at a school for mentally retarded children, and announcer-engineer for two local radio broadcasting stations. It was in Santa Barbara that he had his first contact with geophysics, as a powder monkey and oscillograph operator on a United Geophysical Corporation seismic water crew off the coast of California.

The following two years were spent in research and study for a master of science degree in physics at the Case Institute of Technology in Cleveland. The research constituted the basis of an experimental thesis, "The Pressure Derivatives of the Elastic Constants of Single Crystal Silicon, to 10,000 Bars".

The year beginning June, 1960 was spent in Northwest Greenland operating a remote 3-man geophysical observation station for the American Geographical Society. The site was located within two degrees of the geomagnetic pole.

The summer of 1961 was spent at Los Alamos Scientific Laboratory in New Mexico, studying air fluorescence.

The author enrolled at MIT in the Fall of 1961. During his stay in Cambridge, he spent two summers and other periods in the employ of Edgerton, Germeshausen, and Grier, Inc. This association was most beneficial. Activities there included an absolute measurement of the transient 3914 Å air fluorescence pulse from the nuclear detonation Starfish, study and interpretation of the EM and optical signals from such detonations, feasibility of a Lyot birefringent optical filter operating simultaneously at four wavelengths, and photometric measurements of the outer corona during the solar eclipse of 30 May 1965.