A NONLINEAR INVESTIGATION OF SINGULAR LEVELS FOR INTERNAL ATMOSPHERIC GRAVITY WAVES

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ABSTRACT

For internal gravity waves in a stratified fluid, the height at which the horizontal phase speed is equal to the mean motion or wind is a singular level. The inviscid, adiabatic, linearized equations are singular at this height and predict infinite values for the wave density and the wave horizontal motion.

In this work the behavior of internal gravity waves near a singular level is investigated by means of a transient, two-space-dimensional, finite difference model which includes all the important nonlinear terms as well as viscosity and thermal conduction. It is assumed that the medium is incompressible, but this has a negligible effect on events near the singular level.

It is concluded that the nonlinear terms are quite important near a singular level, but that the viscous and heat conduction terms are not. Some of the qualitative wave behavior near a singular level can be predicted from simple linear theory, but the actual interaction of the wave and wind is nonlinear. For a horizontal wavelength of five kilometers the interaction region is found to be several hundred meters thick.

The nonlinear terms generate changes in the wind which absorb most of the incident wave's momentum and energy when the Richardson number is greater than 0.25. If the incident wave has a horizontal phase speed greater than the wind speed, the wave carries positive horizontal momentum and energy. This wave is absorbed symmetrically around the singular level increasing the wind speed there. The higher harmonics are generated on the side of the singular level away from the source. When the horizontal phase speed is less than the wind speed, the incident wave carries negative horizontal momentum and energy. This wave is absorbed before it reaches the singular level, where it decreases the wind speed. The higher harmonics are generated on the side of the singular level near the source. When the Richardson number is less than 0.25. the incident wave is largely transmitted through the singular level and over-reflection occurs. The excess momentum and energy is supplied by the wind.

Near the singular level the horizontal phase speed is observed to differ from that of the source and to be a function of height. The associated shearing of the wave pattern accompanies the decrease of the vertical wave-length. The change in the horizontal phase speed results in the actual singular level being further from the source than the linear theory predicts.

Thesis Supervisor: Theodore R. Madden
Title: Professor of Geophysics

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List of Symbols

B_T = $\omega_B \bar{\rho}/g$

c = speed of sound = 320 m/s

$\frac{D}{Dt}$ = $\partial/\partial t + U\partial/\partial x + v\partial/\partial y + w\partial/\partial z$ = convective or intrinsic derivative

e = internal energy per unit mass

g = 9.8 m/s = acceleration due to gravity

H = scale height = 8 km

I = integer denoting x position in finite difference array

J = integer denoting z position in finite difference array

k = k_x = horizontal wave number = $2\pi/\lambda_x$

K = coefficient of thermometric conductivity

i = $\sqrt{-1}$

L = integer denoting the time step

m = k_z = vertical wavenumber = $2\pi/\lambda_z$

$\vec{n}$ = unit vector

N = number of finite difference points per wavelength

p = p(x, y, z, t) = perturbation pressure

$\bar{p}$ = $\bar{p}(z)$ = mean pressure

P = $\bar{p} + p$

q = thermal energy per unit mass

Q = $u\tau/\lambda_x$

R = 287.04 $m^2 / (s^{2\,o}K)$ = gas constant for air

Ri = Richardson Number = $\omega_B^2/(du/dz)^2$

s = entropy per unit mass

t = time

T = temperature

u = perturbation motion in x direction

$\bar{u}$ = $\bar{u}(z)$ = mean motion in x direction

U = $\bar{u} + u$

v = $v(x, y, z, t)$ = perturbation motion in y direction

$\vec{v}$ = $\vec{a}_x u + \vec{a}_y v + \vec{a}_x w$

$\vec{V}$ = $\vec{a}_x \bar{u} + \vec{v}$

w = $w(x, y, z, t)$ = perturbation motion in z direction

x = eastward direction in Cartesian coordinate system

y = north direction in Cartesian coordinate system

z = upward direction in Cartesian coordinate system

α = relaxation parameter

β = $\bar{\rho} u$

γ = $\bar{\rho} w$

δ = proximity to critical level parameter (ch. 3)

$\vec{\nabla}$ = $a_x \partial/\partial x + a_y \partial/\partial y + a_z \partial/\partial z$

∇^2 = $\partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$ = Laplacian operator

ϵ = small number

χ = displacement in x direction from original position

ζ = displacement in z direction from original position

λ = wavelength

κ = thermal conductivity

μ = dynamic viscosity

η = kinematic viscosity = $\mu/\bar{\rho}$

ξ = vorticity

ρ = perturbation density

$\bar{\rho}$ = mean density

$\rho_T = \bar{\rho} + \rho$

τ = period = $2\pi/\omega$

τ = Brunt-Vaisala period = $2\pi/\omega_B$

ψ = stream function

ω = radian frequency = $2\pi f$

ω_B = Brunt-Vaisala radian frequency

ω_E = radian frequency of the earth's rotation = 2π/day

Ω = Doppler or intrinsic frequency = $\omega - k\bar{u}$

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Chapter One

Introduction

Internal gravity waves form a portion of the spectrum of internal waves in fluids. This spectrum is divided into three frequency ranges on the basis of which of the Coriolis, gravity, or compressibility terms is the most important in the complete equations. The acoustic branch is comprised of waves with periods of less than a minute or so for which the gravity and Coriolis terms in Newton's law may be neglected. The study of rotational waves considers waves which have a period which is a sizable fraction of one day, and for which the Coriolis term dominates the gravity term in Newton's law and the fluid can be considered incompressible. The gravity branch consists of those waves of intermediate frequency and period, for which the gravity term in Newton's law is the most important. The periods of internal gravity waves range from roughly five minutes to an hour or two. When both the gravity term and the compressibility are kept in the equations, the phrase acoustic-gravity is often used, even though the periods of the waves being considered may not lie in the transition region where the retention of both terms is mandatory.

In the early theoretical work only simple models were used. For waves with periods much shorter than one day this means an isothermal, inviscid, adiabatic, irrotational model with constant wind. With these assumptions the equations are quite tractable and analytic solutions may be readily obtained (Hines, 1960). Relaxation of one or more of these assumptions renders the problem considerably more complex and simple analytic solutions are no longer available.

In the case where the wind is a function of height, if there is a height at which the horizontal component of the phase velocity is equal to the wind velocity, this height is known as a critical level or a singular level. The two terms will be used interchangeably. At this height the intrinsic frequency or Doppler frequency is zero. This frequency is that which would be observed by someone at rest with respect to the fluid.

There are two problems associated with a critical level. The first and simpler of the two, with which this work is primarily concerned, is the nature of wave behavior near a critical level. The second, and related problem, is the role, if any, which critical levels play in the source mechanism which is responsible for the gravity waves observed on the ground and in the ionosphere.

At a singular level the simple linear equations predict that the wave density and the wave horizontal motion will be infinite and that the wave pressure and the wave vertical motion will be zero. This can certainly not be the case because it contradicts what we know about the real, physical world. The vertical wavelength is also predicted to be zero at the singular level. Due to this increase in the magnitude of the wave horizontal motion and decrease in the vertical wavelength as a singular level is approached, any analysis which neglects the nonlinear effects cannot correctly describe events near a critical level. In this study appropriate nonlinear terms as well as the viscous and heat conduction terms are retained in the equations, so that more realistic conclusions about what happens near a singular level are obtained. With all these terms included, the resulting equations are analytically intractable, so they are handled by finite difference methods. Insofar as known, this is the first study of singular levels to include the nonlinear terms.

Chapter two contains a brief review of previous work which has a direct bearing on the critical level problems. In chapter three the complete basic equations are analysed in order to discover which terms in these equations are likely to be of significant size near a critical level. The details of the finite difference scheme are presented in chapter four, and chapter five consists of the results of the finite difference calculations.

Chapter Two

Review of Previous Work

2.1 Theory

This section contains a brief review of those papers which theoretically treat the critical level. The reader who is unfamiliar with internal gravity waves is advised to first consult Hines (1960), Eckart (1960a) or Tolstoy (1963) where the basic linear equations for simple atmospheric models are presented.

Before the geophysicists and wave propagation theorists took up internal gravity waves, a few meteorologists had done some work with mountain lee waves. These large scale disturbances which are formed when a steady wind blows over a large mountain range are a special case of internal gravity waves. They are stationary waves because the obstacle causing them is fixed and a nonzero wind is necessary for their existence. Thus one of the earlier works which has a direct bearing on critical levels is Eliassen and Palm (1960). (The review of the earlier lee wave studies in Eliassen and Palm is adequate except for the omission of Scorer (1949).) Although Eliassen and Palm did not consider a critical level as such, their linear analysis showed how the energy and momentum fluxes depended on the wind speed. They also considered layer boundaries and derived the quantities which must be continuous there. Much of the later work has been the extension and application of ideas in their paper.

The behavior of a gravity wave near a critical level has been investigated by Bretherton (1966) and extended by Garrett (1968) using the W.K.B. approximation. To do this, it must be assumed that the problem is linear, adiabatic, inviscid, and non-rotational. In addition, it is required that the Richardson number is large and that the vertical wavelength is small with respect to distances over which such ambient quantities as the Brunt frequency and wind speed change by a significant amount. These conditions are unlikely to be satisfied for the actual atmosphere. The conclusion reached in this unrealistic model was that the wave packet approaching a critical level is neither reflected nor absorbed, but that the packet never reaches the critical level due to the vertical group velocity becoming increasingly smaller as the distance from the critical level decreases. Thus the energy remains in the vicinity of the critical level. An extension and a more general consideration of conserved quantities has been made by Bretherton and Garrett (1968).

A more realistic treatment of the same simple equations is that of Booker and Bretherton (1967). They treated the singular level by taking the frequency to be complex and applying contour integration. They found that the Reynolds stress of a wave was attenuated by a factor of $\exp(-2\pi(R_c-0.25)^{1/2}$ upon passage through a critical singular level, where R_c is the Richardson number at the critical level.

This basic analysis was extended by Jones (1967) to include rotation and by Hazel (1967) to include viscosity and heat flow. Inclusion of the Earth's rotation does not remove the singularity at $\Omega = 0$ but introduces two additional singularities at $\Omega = \pm \omega_R$. Jones showed that the basic conclusions of Booker and Bretherton concerning attenuation held but that the nature of the solutions very near the singular level was quite different. The Richardson number of unity was used and the reflected wave was found to be about 0.026 of the incident wave.

By including the viscosity and thermal conductivity Hazel found that the singularity in the equations was removed. Like Jones he used the linearized equations but Hazel considered the problem in the mountain lee wave form rather than the propagating wave form. The four additional solutions resulting from the abandonment of the adiabatic and inviscid requirements are naturally small away from the critical level. Hazel calculated all six solutions numerically from assymptotic expansions matched at appropriate heights for Ri = 3. He found that the transmitted wave was indeed attenuated by the factor found by Booker and Bretherton and that there was no reflected wave. The wave energy and momentum were absorbed by the wind in the region around, but mostly below, the singular level for an upward traveling wave. For the critical layer, defined to be that region where the viscosity is important, Hazel found a height of five to ten meters for a horizontal wavelength of ten kilometers.

Hazel used the molecular (laminar flow) values for the viscosity and thermal conductivity, but the evidence from meso-scale meteorology (Sutton, 1953, 1955) is that turbulence is almost always present on scales smaller than the wavelengths of internal gravity waves. Therefore the eddy (turbulent flow) values for the viscosity and thermal conductivity would be more appropriate. Multiplying Hazel's values by 10^4 gives a critical layer one hundred meters thick, which seems more reasonable for the atmosphere. Since the value of the Prandtl number is unaffected by this change, Hazel's analysis is unchanged except for the value of z_o, the normalization length.

Jones (1968) has calculated reflection coefficients and complex normal mode frequencies for a simple model atmosphere using the linearized incompressible equations. His model consisted of a region of constant shear below a region of no wind, so that reflections occurred both at the critical level and at the boundary between the two regions. The reflected wave was considerably enhanced when the wave was evanescent in the upper region, in which case reflections might also be expected at the point in the sheared region where the wave becomes evanescent. The analytic solutions used in the lower region were Whitaker's functions, which imply that w is zero and u is infinite at the critical level. Jones found that the downgoing wave below both the interface and the critical level could be larger than the upgoing wave if the Richardson number was small enough and the other parameters fell within certain limits. The upper limit on Ri for this over-reflectivity was 0.25 in the case where the wave was evanescent in the upper region, and 0.115 in the case where the wave was propagating in the upper region. If the reflected wave is smaller than the incident wave, the excess energy and momentum go into the wind; and if the reflected wave is larger, then the extra energy and momentum is supplied by the wind. In these studies the wave approached the critical level from the low wind speed side, and these energy remarks refer only to this case. In the stability analysis portion of the work, Jones found that instabilities occurred for low Richardson numbers only if the wind had an inflection point.

Bretherton (1969) has also considered the interaction between gravity waves and the wind. Unfortunately his linear perturbation analysis applies only when the Froude number ($|u|^2/\omega_B^2 H^2$) and the ratio $|u|/v_g$ are much less than unity, where v_g is the group velocity. This last requirement restricts the validity of the conclusions to regions well away from a critical level.

Utilizing the Lagrange equations and a variational principle, Drazin (1969) has considered one facet of the nonlinear behavior of an internal gravity wave, that of propagation to great heights where the perturbation density is a significant fraction of the ambient density. While there was no mean flow and the analysis is not applicable to the critical level problem as it is, it does indicate a new approach through which future progress might be made. In the Lagrange equations, the equations do not become singular at a critical level.

The importance of an inflection point in the wind profile has been emphasized by those investigations which are of a more mathematical nature. With the usual inviscid, irrotational, adiabatic and linear assumptions, the basic equations can be combined into one tractable equation, and for many years mathematicians have been examining the roots to this differential equations for various assumptions and various wind profiles. Drazin and Howard (1966) present an exhaustive review of this approach to the problem of the stability of parallel fluid flow. The basic Kelvin-Helmholtz instability is given in Lamb (1945, p. 373, 458) and the Richardson number as a stability parameter is discussed by Taylor (1931).

Another area of inquiry which may have implications regarding the critical level problem is that of resonant interactions among different internal gravity waves. A short review of this field is given by Kelly (1968), and this paper together with Davis and Acrivos (1967) and Phillips (1968) reference most of the important earlier work. In general only cases with no wind shear are considered. Exceptions include Craik (1968) and Kelly (1967, 1968). Craik's work is the most interesting because he finds that most of the energy transfer takes place in the vicinity of a critical level. He states, however, that the resonance condition is rather severe because it requires larger velocity gradients than may be expected to occur in nature.

In wave propagation studies where the ambient quantities vary slowly the layered media approach has proven extremely valuable (Pierce, 1966, Hines and Reddy, 1967). However this technique cannot validly be applied to the study of singular levels because the ambient quantities vary too quickly and because the quantities otherwise continuous at layer boundaries are not continuous across a singular level.

Turbulence as a source for acoustic and gravity waves has been considered by Stein (1967). Hs approach, based on some earlier ideas of Lighthill, appears to be more successful in analyzing the acoustic wave generation than it is for the gravity wave generation.

2.2 Finite Difference Studies

There appear to be only two finite difference studies which have any bearing on the critical level problem. Foldvik and Wurtele (1967) set up a model to study the development of nonlinear effects near the mountain for the lee wave problem. Houghton and Jones (1968, 1969) were concerned with the behavior of propagating waves at a singular level.

Foldvik and Wurtele were primarily interested in the details of the fluid flow near the mountain, and they evidently made no attempt to investigate the critical level problem. They note that the main perturbation cells were set up very rapidly, which is to be expected since their model is incompressible in the perturbations, and these main cells are a direct result of the fluid flow over the obstacle. They found this no drawback since their interest was in the steady state form of the lee waves rather than in their development. A number of their techniques have been used in this study such as the staggered grid system and the use of the stream function - vorticity equations instead of the basic equations. While they report no stability or error analysis, they do mention that an occasional forward time step was found to be helpful for stability.

Houghton and Jones, on the other hand, were primarily interested in the critical level problem and worked in the propagating wave mode. To obtain sufficient vertical resolution they used only a one dimensional matrix of points, having eliminated the x dimension by linearizing and taking exp (ikx) dependence. Viscosity, heat flow, and Coriolis terms were also omitted, although the pressure and a finite value for the compressibility were kept. A moving lower boundary was the source and a region with Rayleigh damping at the top served to absorb the energy.

The results of these calculations showed good agreement with the predictions of Booker and Bretherton (1967) regarding attenuation upon passage through a critical level. Also, by making the wind vary with respect to time, they demonstrated that the momentum flux varied as the intrinsic frequency, which had been suggested by Bretherton and Garrett (1968) and by Claerbout (1967).

2.3 Experiments

As far as known to the author there have been no experiments involving internal gravity waves in a gas. On the other hand, experiments on stratified liquids have been going on for over half a century. In two recent papers Thorpe (1968a, 1968b) reviews the previous work and presents his own experimental and theoretical work. One series of experiments involved internal waves in a stratified fluid at rest. The largest waves were obtained when the forcing frequency differed slightly from one of the tank's natural frequencies. Irregularities in the wave motion and overturning were ascribed to distortion of internal wave rays by standing waves rather than to instabilities because the local Richardson numbers generated by the wave motion were always very large. Both two-layer and multi-layer experiments were conducted.

Thorpe's other paper concerns the instability of shear flow. No internal gravity waves were present. The formation of regular spiral structure and later decay to irregular turbulence was observed for cases when the Richardson number was less than 0.25.

A fairly simple experiment that demonstrates the failure of a gravity wave to propagate through a critical level has been reported by Bretherton, Hazel, Thorpe and Wood (1967). Density stratification was obtained by means of solutions of salt in water with various concentrations. Their rectangular tank was briefly tilted to initiate a shear flow. A train of lee waves behind an obstacle of triangular cross-section had apparently reached a steady state before the effects of the ends of the tank changed the flow significantly. For Richardson numbers of greater than 0.5 they found no detectable transmission of the lee wave through the critical level. They also note that the amplitudes are large enough so that the linearized theory is not applicable near the critical level.

Chapter Three

The Basic Equations near a Critical Level

3.1 The Relative Importance of the Terms in the Basic Equations

The purpose of this section is to analyse the complete basic equations in order to determine which terms are important as a critical level is approached. First the notation used is described and then the complete basic equations are presented. Next two of the commonly used linear approximations are written down and discussed. The predictions of how the wave parameters will vary as a critical level is neared are obtained from the linearized equations. These predictions are used to determine at what distances and for which cases each of the terms in the complete equations is important. This analysis is lengthy and the details are contained in appendix D. Here only the results and their implications are presented.

Let the unit vectors in a Cartesian coordinate system be $\vec{a}_x$, $\vec{a}_y$, and $\vec{a}_z$. The positive x direction is eastward, the positive y direction is northward, and the positive z direction is upward. The ambient or time-independent pressure, density, and fluid flow are represented by $\bar{p}$, $\bar{\rho}$, and $\bar{u}$, where it is assumed that the ambient flow or mean wind is in the x direction only. The perturbation or wave pressure, density and velocity are represented by p, ρ , and $\vec{v} = \vec{a}_x u + \vec{a}_y v + \vec{a}_z w$. The total pressure density and velocity are given by

$$p = \bar{p} + p$$
$$\rho_T = \bar{\rho} + \rho$$
$$\vec{V} = \vec{a}_x \bar{u} + \vec{v}$$

MKS units are used throughout. The total or convective derivative is defined by

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{V} \cdot \vec{\nabla}$$

where

$$\vec{\nabla} = \vec{a}_x \frac{\partial}{\partial x} + \vec{a}_y \frac{\partial}{\partial y} + \vec{a}_z \frac{\partial}{\partial z}$$

Assuming that the curvature of the Earth can be neglected, that there are no sources or sinks of heat, and that the mean wind and ambient density are functions of height only: the complete basic equations are:

$$\rho_T \left[\frac{D\vec{V}}{D\tau} + 2 \vec{\omega}_R \times \vec{V} \right] + \vec{\nabla} P - \vec{g} \rho_T - \mu \left[\frac{1}{3} \vec{\nabla} (\vec{\nabla} \cdot \vec{V}) + \nabla^2 \vec{V} \right] = 0 \qquad 3.1\text{-}1A,B,C$$

$$\frac{D\rho_T}{D\tau} + \rho_T \vec{\nabla} \cdot \vec{V} = 0 \qquad 3.1\text{-}1D$$

$$\frac{D\rho_T}{Dt} - \frac{1}{C^2}\frac{DP}{Dt} - K\,\nabla^2\rho = 0 \qquad 3.1\text{-}1E$$

The first equation is Newton's law for the change of momentum. $\vec{\omega}_R$ is the radian frequency of the Earth's rotation and has the direction of the axis of rotation. $\vec{g}$ is the acceleration of gravity and is downward, and μ is the dynamic viscosity. The second equation expresses the conservation of matter and the last equation expresses the conservation of heat. K is the coefficient of thermometric conductivity, and c represents the speed of sound.

The equation of heat transfer, 3.1-1E, has already been simplified somewhat. The effect of the vertical temperature and density gradients on the heat conduction is shown to be negligible in appendix D, so they have been omitted from the conduction term, thereby eliminating the temperature from the equation. The effect of the pressure on the conduction has also been neglected as mentioned in appendix A.

The viscous and heat conduction parameters are discussed at some length in appendix D. Besides defining these and other parameters, the appropriate values to use for them are considered. While the above equations do not take turbulence and convection which have the same scale as the internal gravity waves into account, it is shown that these random processes on scales smaller than the scale of the gravity waves can be treated by adopting the eddy values for μ and K.

The momentum equation has been written in vector form for compactness. When written out the x, y, and z components will be the A, B, and C equations respectively. The mass conservation and heat transfer equations will be the D and E equations whenever this set is written. The coupling between the three momentum equations is contained in the Coriolis term through the cross product and in the viscous term through the divergence of the velocity. If these terms are not included, the x and z momentum equations will not contain the north-south component of motion v. In this case it is common to consider the problem to have only two spatial dimensions and to drop the y momentum equation from consideration. If this is done there will be no B equation in the set. Since ignoring the motion in the y direction results in a considerable simplification of the equations, this is often done when the viscosity is kept by simply requiring that v = 0 everywhere.

The momentum equation contains several nonlinear terms. Some of these come from multiplying the first term by the total density rather than by just the ambient density and some are implicitly contained in the convective derivative. The convective derivative appears because the Eulerian form of the equations is being used. These nonlinear terms may be seen by expanding the convective derivative:

$$\frac{D}{Dt}(\bar{u}+u) = \frac{\partial u}{\partial t} + \bar{u}\frac{\partial u}{\partial x} + w\frac{d\bar{u}}{dz} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$

The momentum equation also includes some terms, ambient terms, which contain only ambient variables and no perturbation variables. By definition these terms must sum to zero, and the resulting equations are written out in appendix D. It is common to remove the ambient terms by subtracting these equations from the complete component momentum equations.

When the Coriolis, nonlinear, viscous, and heat conduction terms are also removed, and it is assumed that there is no motion in the y direction, the following equations are obtained:

$$\bar{\rho}\left[\left(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\right)\vec{V} + \vec{a}_x w\frac{d\bar{u}}{dz}\right] + \vec{\nabla}p + \vec{a}_z g\rho = 0 \qquad 3.1\text{-}2A,C$$

$$\left(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\right)\rho + w\frac{d\bar{\rho}}{dz} + \bar{\rho}\,\vec{\nabla}\cdot\vec{V} = 0 \qquad 3.1\text{-}2D$$

$$\left(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\right)p + w\frac{d\bar{p}}{dz} = \frac{1}{c^2}\left[\left(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\right)\rho + w\frac{d\bar{\rho}}{dz}\right] \qquad 3.1\text{-}2E$$

This set of equations will be referred to as the simple linear equations, and these equations are those upon which most internal acoustic-gravity wave studies are based. The ambient quantities are invariably assumed to be independent of x and t so that when exp $(ikx - i\omega t)$ is assumed for these variables the dependence of these equations on x and t may be eliminated.

Equations 3.1-1-2 can be simplified further by assuming that variations of the density with time and position are unimportant in the equation of mass conservation. This leads to

$$\vec{\nabla} \cdot \vec{v} = 0 \qquad 3.1\text{-}3$$

which is the expression used when the density is constant. This approximation is valid when the period of the wave is several times the Brunt period.

An extended form of the equation above,

$$\vec{\nabla} \cdot (\bar{\rho}\, \vec{v}) = 0 \qquad 3.1\text{-}4$$

is often used when dealing with stratified fluids. This extension takes the change of ambient density with height into account by including the $w(d\bar{\rho}/dz)$ term. Equation 3.1-4 is a valid approximation to the complete equation of mass conservation over a much wider range of a frequency than is 3.1-3. This is shown in detail in appendix D.

Equation 3.1-3 implies that the density, and thus the volume, of a fluid parcel does not change with respect to time or position. Equation 3.1-4 implies that the parcel's density and volume are functions of the parcel's height because the ambient density is a function of height.

Although the writing of the equation of mass conservation in one of the simpler forms just discussed always comes to mind when incompressibility is mentioned, incompressibility strictly means only that the density is not affected by the pressure. When the density is unaffected by the pressure, the speed of sound is infinite, but it is always invalid in this problem to let c approach infinity indiscriminantly.

The two vertical derivatives of ambient variables in 3.1-2E are generally combined to give

$$\omega_B^2 = \frac{g}{\bar{\rho}}\left[\frac{1}{c^2}\frac{d\bar{p}}{dz} - \frac{d\bar{\rho}}{dz}\right] \qquad 3.1\text{-}5$$

This equation defines the Brunt frequency, which can also be written in terms of the vertical ambient temperature derivative:

$$\omega_B^2 = \frac{g}{T}\left[\frac{\partial T}{\partial z} - \left(\frac{\partial T}{\partial z}\right)_{\text{adiabatic}}\right] \qquad 3.1\text{-}6$$

(The Brunt frequency, also called the Väisälä frequency, is a measure of the static stability of the atmosphere and is denoted by N by most meteorologists.) It turns out that it is a very bad approximation to let the speed of sound become infinite in the Brunt frequency, so that when incompressibility is assumed for gravity waves a realistic value of c is used for the Brunt frequency and an infinite value elsewhere. It is very convenient to treat the Brunt frequency as a constant. For an isothermal atmosphere this is the case, but a more realistic constant may be obtained by the slightly inconsistant procedure of using a more normal value for dT/dz (such as the standard lapse rate of 6.5°C/km) and an average value for T in 3.1-6.

Making the approximations just discussed, equations 3.1-2 become:

$$\bar{\rho}\left[\left(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\right)\vec{V} + \vec{a}_x W \frac{d\bar{u}}{dz}\right] + \vec{\nabla}\rho + \vec{a}_z g\rho = 0 \qquad 3.1\text{-}7A,C$$

$$\vec{\nabla}\cdot\vec{v} = 0 \quad \text{or} \quad \vec{\nabla}\cdot(\bar{\rho}\,\vec{v}) = 0 \qquad 3.1\text{-}7D$$

$$\left(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\right)\rho - W\left(\frac{\omega_B^2\bar{\rho}}{g}\right) = 0 \qquad 3.1\text{-}7E$$

If the left equation in 3.1-7D is used this set will be referred to as the simple or regular linear incompressible equations, while extended linear incompressible equations will apply if the right equation is used.

In the case that the mean wind u and the Brunt Frequency ω_B are constant, either 3.1-2 or 3.1-7 can be solved analytically. These solutions and their applications form the bulk of the literature on atmospheric internal gravity waves. The basics are given by Hines (1960). The ambient quantities are almost always taken to be independent of x and t so that exponential dependence of the perturbation variables on x and t may be assumed. With this assumption

$$\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x} = -i\Omega$$

where the intrinsic or Doppler frequency Ω is defined by

$$\Omega = \omega - k\bar{u}$$

ω is the radian frequency, k is the horizontal wavenumber, and m is the vertical wavenumber.

Note that this frequency is equal to the radian frequency if the mean wind is zero, that it will be a function of height if the mean wind is, and that it will be negative if the wind speed is greater than the horizontal phase speed. The critical level is often defined as that level where $\Omega = 0$. Ω is the frequency which would be seen by an observer moving with the wind, and it is the relevant frequency for almost all discussions. For example, for an isothermal atmosphere with scale height H, the dispersion relation is

$$m^2 = k^2\left(\frac{\omega_B^2}{\Omega^2} - 1\right) + \frac{\Omega^2}{c^2} - \frac{1}{4H^2} \qquad 3.1\text{-}8$$

where $\bar{\rho}u$, $\bar{\rho}w$, $\bar{\rho}$, and p were assumed to have $\exp(i(-\omega t + kx + (m+i/2H)z))$ dependence. It can be shown that m is real for all values of Ω except those within a small range bounded approximately by ω_B and $\omega_z = c/2H$. The higher range of frequencies are acoustic waves which will not be considered. Only those waves which have Ω less than ω_B will be dealt with, and it is obvious that the smaller Ω is with respect to ω_B the less effect the compressibility will have. Thus the validity of the incompressibility assumption depends on the relationship of Ω and ω_B.

As a critical level is approached, Ω approaches zero. This is also true if waves with a very long period are being considered, and there is no fundamental difference between the two cases. Either way the wave is nearly a zero frequency wave with respect to the fluid. As discussed in the appendix, the linear analysis predicts that other wave parameters will change in certain ways as Ω approaches zero. Specifically, the vertical wavelength λ_z goes at $1/\Omega$, the magnitudes of ρ and u go as $1/\sqrt{\Omega}$, and the magnitudes of p and w go as $\sqrt{\Omega}$. This behavior causes some of the Coriolis, viscous, thermal conduction, and nonlinear terms which are validly neglected otherwise to become important near a critical level.

Each equation in 3.1-1 is treated in detail in the appendix and here only the results will be presented. Because u and ρ become large with proximity to a critical level and the vertical wavelength becomes very short, it is expected that terms involving these variables and/or the vertical derivative will take precedence over the others. Because they change in very different ways, it is necessary to treat the components of the vector momentum equation separately. The terms containing only ambient variables sum to zero independently of the other terms so these terms have been removed leaving perturbation equations.

In equation 3.1-1A the perturbation density occurs only in the sum $\rho_T = \bar{\rho} + \rho$ and it never becomes large enough to be significant with respect to the ambient density, so ρ may be neglected entirely in this equation. Thus the x momentum equation may be written:

$$\begin{aligned}
&\bar{\rho}\left[\left(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\right)u + w\frac{d\bar{u}}{dz}\right] + \frac{\partial p}{\partial x} + \\
&\bar{\rho}\,(\vec{v}\cdot\vec{\nabla})u + \bar{\rho}\,2\,\omega_R\,(w\cos\phi - v\sin\phi) \qquad 3.1\text{-}9A \\
&-\mu\left[\frac{1}{3}\frac{\partial}{\partial x}(\vec{\nabla}\cdot\vec{v}) + \nabla^2 u\right] = 0
\end{aligned}$$

where ϕ is the latitude north of the equator. The first line of this equation comprises the terms kept in the simple linear approximation, the second line contains the nonlinear and Coriolis terms, and the viscous term is on the last line. (Additional nonlinear terms, called the nonlinear density terms, which are entirely negligible, would appear if the wave density had been kept.) For the waves in which we are interested all three of the additional terms may be validly neglected far from a critical level, but as a critical level is neared each term eventually becomes larger than the linear term. (The first line of the equation is referred to as the linear term.) The Coriolis term becomes large because v, like u, approaches infinity at a singular level. It is not valid to include the Coriolis term and require that $v = 0$ because whenever u is not zero there is a force in the y direction.

Which of these three additional terms is the largest unfortunately depends on the wave parameters so that no completely general conclusion can be drawn. The nonlinear term is the most important for most of the waves in which we are interested, but if the period is more than a few Brunt periods and the wave amplitude is small the Coriolis term will be the dominant one. Or, if the quantity k^2/ω is large enough and the amplitude is small the viscous term will be the largest. For the x momentum equation it is concluded that the wave density may be dropped entirely, and, of the nonlinear, Coriolis and viscous terms, the nonlinear term will be the most important if the amplitude is large or if the period is not too much greater than the Brunt period.

The y momentum equation behaves in exactly the same manner as a critical level is approached, so the above conclusions apply here as well.

The z component of the vector equation of Newton's law is:

$$(\bar{\rho}+\rho)\left[\left(\frac{\partial}{\partial t}+\bar{u}\frac{\partial}{\partial x}\right)w\right]+g\rho+\frac{\partial p}{\partial z}+$$
$$(\bar{\rho}+\rho)(\vec{V}\cdot\vec{\nabla})w-2\omega_R\cos\phi(\bar{\rho}u+\bar{u}\rho+\rho u) \quad 3.1\text{-}9C$$
$$-\mu\left[\frac{1}{3}\frac{\partial}{\partial z}(\vec{\nabla}\cdot\vec{V})+\nabla^2 w\right]=0$$

Unlike the equation for the horizontal momentum, none of the additional terms ever become important in this equation. Not only are the nonlinear, Coriolis and viscous terms completely negligible, but some of the terms retained in the simple linear approximation become small enough to neglect also. Within a kilometer or so of a singular level

$$g\rho+\frac{\partial p}{\partial z}=0 \qquad 3.1\text{-}10$$

is a valid approximation to the vertical momentum equation.

In 3.1-9C it is seen that the Coriolis term becomes large as a critical level is approached. However, the terms retained in 3.1-10 also become large. Two of the three quantities in the Coriolis term increase at the same rate as the terms in 3.1-10 and thus are always much smaller than these terms. The other quantity in the Coriolis term increases faster than the terms in 3.1-10, but it is so much smaller to begin with that, for reasonable scales, it never becomes large enough to be significant. The viscous term also becomes large as a singular level is neared, but like the Coriolis term it becomes equal in magnitude to the gravity term only for distances which are much smaller than reasonable scales for this problem. Thus both the Coriolis and viscous terms may be neglected in this equation.

The equation of mass conservation has already been discussed and it need not be written again. The nonlinear terms are completely negligible. Within a kilometer or two of a critical level the extended incompressible equation 3.1-4 is a valid approximation, and within several hundred meters the regular incompressible equation 3.1-3 is a valid approximation.

When written out using the definition of the Brunt frequency the heat transfer equation 3.1-1E is:

$$\left(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\right)\left(\rho + \frac{1}{c^2}p\right) - \left(\frac{\omega_B^2 \bar{\rho}}{g}\right)w + (\vec{v}\cdot\vec{\nabla})\rho - \frac{1}{c^2}(\vec{v}\cdot\vec{\nabla})p - K\nabla^2\rho = 0 \qquad 3.1\text{-}9E$$

The first line contains the terms kept in the simple linear approximation and the second line contains the non-linear density term, the nonlinear pressure term, and the heat conduction term. The nonlinear pressure term is completely negligible. The linear pressure term decreases in importance as a critical level is approached and may be neglected completely within about half a kilometer of the critical level. The nonlinear density term and the heat conduction term are of about equal importance for many of the cases of interest. For large amplitude waves the importance of the nonlinear term is increased and the conduction term is negligible while if the value of k^2/ω is large the nonlinear term may not be significant. In general both terms should be included for unquestioned validity. As with the viscosity term in the horizontal momentum equation, only the z derivative in the heat conduction term is important due to the predicted vertical wavelength shortening. Use of the molecular (laminar flow) value rather than the eddy value for the conductivity would make this damping term completely negligible also.

An additional term is one of those neglected in the simple linear approximation. For frequencies in which we are interested all such terms are validly excluded if the region being considered is far from a critical level. In the foregoing an additional term has been said to be important if its magnitude becomes equal to that of the largest linear term and if it is not always dominated by another additional term. Of course the influence of an additional term will extend some distance beyond the point at which it is equal in magnitude to the linear term.

The intrinsic frequency Ω is linearly related to

z_d, the distance from the critical level, and from equations 3.1-9 it may be seen that in both the horizontal momentum equation and the heat transfer equation the largest nonlinear term varies as $z_d^{-3/2}$ and the damping term varies as z_d^{-3} with respect to the largest linear term. In the horizontal momentum equation the Coriolis term varies as z_d^{-1}. No additional terms ever become important in the vertical momentum equation or the mass conservation equation so this discussion does not apply to these equations.

Due to this dependence on the distance from the singular level the nonlinear term will affect the wave's behavior over a larger region than will the damping term. Since the nonlinear term removes the singularity in the equations, by the time the wave has moved close enough to the critical level for the damping term to be of significant size according to the linear prediction, the effect of the nonlinear term may have altered the wave so that the damping term has little or no effect.

Although the Corilis term will be of significant size over a larger range than the nonlinear term, the inclusion of the Coriolis force does not remove the singularity in the equations or alter the basic nature of the wave behavior near a singular level. Thus it is unlikely that the Coriolis term will alter the wave in such a manner that the nonlinear term would be ineffective.

In this section the complete basic equations and two commonly used linear approximations have been discussed.

The predictions about how the perturbation quantities will vary with proximity to a critical level have been used to determine which approximations are valid for different distances from the critical level. The predictions were found from the linear, inviscid, adiabatic, irrotational equations and so cannot be expected to be accurate when one of the excluded terms becomes large with respect to the terms included.

None of the additional terms ever become important in the vertical momentum equation or in the mass conservation equation so that the simple linear approximations remain valid as a critical level is neared. In fact, some of the linear terms become negligibly small, thus simplifying the equations even further.

The horizontal momentum and heat transfer equations, however, become increasingly complicated as a critical level is approached, and the simple linear approximation is invalid. In the horizontal momentum equation the Coriolis force, the viscous damping, and those nonlinear terms which do not involve the wave density must all be kept for general validity. In the heat transfer equation, the pressure terms are negligible near a critical level but the nonlinear density term and the heat conduction terms are too large to be excluded generally. While the Coriolis, nonlinear, or damping term may be the dominant term if the wave parameters are appropriately chosen, the nonlinear term in both equations will have the greatest effect for most of the cases of interest.

The damping terms are completely negligible unless the eddy values are used for the coefficients of viscosity and conduction.

3.2 Wave behavior close to a singular level

In this section two examinations of how an internal gravity wave may be expected to behave in the region close to a critical level will be undertaken. Each examination is based on a different approximation to the complete set of equations. While not strictly valid, these approximations are necessary for an analytic treatment due to the complexity of the complete equations. The insights gained from these analyses are considered to be helpful, even though the approximations on which they are based allow them to be considered only as tentative indications.

Let the height range near the critical level where the nonlinear terms dominate the horizontal momentum equation and the heat transfer equations be called the strongly nonlinear region. In this region equations 3.1-1 become:

$$u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} = 0 \qquad 3.1\text{-}1A$$

$$g\rho + \frac{\partial p}{\partial z} = 0 \qquad 3.2\text{-}1C$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \qquad 3.2\text{-}1D$$

$$u\frac{\partial \rho}{\partial x} + w\frac{\partial \rho}{\partial z} = 0 \qquad 3.2\text{-}1E$$

In the preceding section it was demonstrated that the middle two equations are valid approximations to the vertical momentum and continuity equations for a region extending several hundred meters from the critical level. If the horizontal wavelengths are not too long, the Coriolis force may be neglected, and this allows simplification of the problem to two space dimensions, x and z, and the dropping of the y momentum, or "B", equation.

The first and the last of the four equations above, however, are valid approximations to the horizontal momentum and heat transfer equations only for fifty meters or so around a critical level, and only for large amplitude perturbations. The other terms which would extend the region of validity of these equations are not included because they make the set of equations analytically intractable.

Combining 3.2-1A and 3.2-1D by eliminating $\partial u/\partial x$ one obtains

$$\frac{\partial}{\partial z}\left(\frac{u}{w}\right) = 0 \qquad 3.2\text{-}2$$

This implies that the ratio u/w is independent of height. This is in direct contradiction to the linear prediction in which w approaches zero at a critical level while u approaches infinity.

Since w would not be zero and u would not be infinity at that distance from the critical level where this set of equations becomes valid, this linear prediction can be ruled out. It is possible, as far as the above equation is concerned, that u and w could both go to infinity or zero together, however.

Very large values of u (and ρ) are not compatible with purely physical considerations, and because the equations which contain $\partial p/\partial z$ and $\partial w/\partial z$ remain linear near a critical level, the linear predictions concerning the behavior of p and w near a critical level are much more likely to be correct than those for u and ρ . Thus it appears more reasonable that u would become small with w at a critical level than that w would approach infinity with u. All that can be definitely concluded though is that the ratio u/w remains constant with height for a region in which these equations are valid.

By equation 3.2-2 u and w may be related by u = wf where f = f(x,t) is independent of z. Then 3.2-1D is

$$\frac{\partial u}{\partial x} + \frac{1}{f}\frac{\partial u}{\partial z} = 0 \qquad 3.2\text{-}3$$

If we let s = x + fz, then this equation becomes

$$\frac{\partial u}{\partial s} = 0 \qquad 3.2\text{-}4$$

From this it is seen that u is constant along lines in the xz plane which are perpendicular to the lines s = constant. Now the ratio u/w = f is likely to be a reasonably large number. This is because u will have increased and w will have decreased in the linear region as the wave approached the critical level. If f is large, then a line given by s = constant will be nearly horizontal, and it follows that u will be constant along lines which are nearly vertical.

In the previous section it was seen that the linear theory predicts that the vertical wavelengths will become increasingly shorter as a critical level is neared. This implies that u would change rapidly along a vertical line. But here we see that these nonlinear equations show that u is constant along a line which is nearly vertical. This rules out the rapid fluctuation of u with height and the very short vertical wavelengths. Since the ratio u/w is constant with respect to z, it follows that w also is constant along the same nearly vertical line.

From 3.2-1D one can obtain

$$u\left[\frac{\partial \rho}{\partial x} + \frac{1}{f}\frac{\partial \rho}{\partial z}\right] = 0$$

u = 0 identically over the entire region is an uninteresting solution, and the other solution gives an equation analogous to 3.2-3 for ρ . Thus all the conclusions regarding u above hold for ρ also unless u = 0 everywhere. And from 3.2-1C it can be seen that the conclusions for the density hold for the vertical derivative of the pressure as well.

For the region in which equations 3.2-1 are a valid approximation to the complete equations it has been shown that the ratio u/w is independent of z and that u, w, ρ , and $\partial p/\partial z$ are constant along lines of slope $f = u/w$. These conclusions do not agree with the behavior predicted from the simple linear approximation. If f is large, which appears likely, then u, w, ρ, and $\partial p/\partial z$ are constant along a line which is nearly vertical.

For the values of the wave parameters in appendix D which include horizontal and vertical wavelengths of about twenty km, these conclusions hold rigorously only for a region extending about fifty meters from a critical level and only for a large amplitude wave. From the linear and nonlinear predictions of what happens at a critical level it is difficult to draw a general conclusion. The attenuation in the most reasonable linear study, that of Booker and Bretherton (1967), occurs all at once right at the critical level, and the linear theory is certain to give less reliable results at that point than the nonlinear theory. On the other hand the nonlinear theory is applicable over such a narrow region that about all that can be definitely stated is that infinite values of u and ρ will not occur. Also, if the linear results are extended very close to but not through the critical level on each side, and joined by the nonlinear results, no attenuation at all occurs, which does not agree with the experimental results. It would appear that wave behavior near a critical level depends on the region in which both the linear and the nonlinear terms are too large to be omitted, and for which the equations are analytically intractable.

These conclusions do not depend on the use of 3.1-3 instead of 3.1-4. If the extended expression for the conservation of mass is used in place of 3.2-1D, the results obtained are essentially equivalent to those above.

Chapter Four

The Numerical Computation Scheme

4.1 Possible approaches to the critical level problem

It was shown in section 3.1 that the Coriolis, nonlinear and damping (viscous and thermal conduction) terms must be kept if the set of equations used for an investigation is to have unquestioned validity for all cases in the vicinity of a critical level. The complexity and the nonlinearity makes this complete set of equations very formidible. Even with just the Coriolis force (Jones, 1967) or just the damping terms (Hazel, 1968) the equations though linear are far from simple. The next step would seem to be either keeping both the Coriolis and damping terms in which case the equations remain linear, or inclusion of only the nonlinear terms.

If the nonlinear terms are included most of the familiar techniques are no longer available. There is no point in assuming $\exp(-i\omega t + ikx)$ dependence because this factor can no longer be factored out to leave the equations dependent on z only. A numerical approach is practically dictated. A few analytic considerations such as those in section 3.2 may be made, but the approximations necessary to make the equations amenable to analytic treatment are strictly valid in a very restricted region if at all.

Before a numerical scheme can be designed, one must decide which set of basic equations to use, and whether to attack the problem as an initial value problem in time or in height.

In deciding on the set of basic equations to use, the first question would seem to be, are there any drawbacks to inclusion of the Coriolis and damping terms as well as the nonlinear terms. For the damping terms the answer is no, but keeping the Coriolis terms means that three space dimensions must be used. If the Coriolis term is neglected, it is reasonable to assume that there is no motion in the y direction and to reduce the problem to two space dimensions. If we wished to have 10^2 points in each dimension, considering only two space dimensions instead of three means that only 10^4 points need be considered instead of 10^6. The storage capacity and speed of the present computers make it infeasible to consider three space dimensions, so the Coriolis term will not be included. Since the Coriolis force does not remove the singularity or change the basic singular behavior, its exclusion should not alter the wave behavior close to a critical level. It does make the equations an invalid approximation in long period and small amplitude cases, but should not alter the basic conclusions about the singular level, even in these cases.

The next question is whether or not to assume that the equations are incompressible. It has been shown that the

3.1-4 equation is a good approximation to the mass conservation equation for quite a wide region around a critical level, and that the pressure terms in the heat transfer equation are of diminishing importance near a critical level, so by making the incompressible assumption there is no danger of an invalid approximation near the critical level. The incompressible approximation is not valid far from a critical level for frequencies near the Brunt period, but since the main interest is in behavior near the critical level this is not important.

If the initial value in time approach is to be used, there is an important argument in favor of using the incompressible equations. It was seen that the linear prediction is that the pressure terms in the heat transfer equation become small and that the equation of mass conservation approaches $\vec{\nabla}\cdot\vec{v} = 0$ as a critical level is neared. For the progress in time scheme equations 3.1-1D and 3.1-1E would be solved for the time derivative in p:

$$\frac{D\rho}{Dt} = -c^2\left[\left(\frac{\omega_\beta^2\bar{\rho}}{g}\right)w + \vec{\nabla}\cdot\bar{\rho}\,\vec{v}\right]$$

or

$$\frac{\partial p}{\partial t} = -\vec{V}\cdot\vec{\nabla}p + g\bar{\rho}w - c^2\bar{\rho}\,\vec{\nabla}\cdot\vec{v}$$

Now c^2 is large number , and $\partial u/\partial x$ and $\partial w/\partial z$ which together form $\vec{\nabla}\cdot\vec{v}$ are large quantities, so in solving for $\partial p/\partial t$ we are adding two large numbers which very nearly sum to zero, and then multiplying this small quantity by a large number. Because of the small word size of the IBM 360 computers a preliminary study showed that double precision would be necessary in order to make the above calculations with sufficient accuracy. If incompressibility is assumed, the stream function - vorticity formulation can be used and the entire problem avoided because the perturbation pressure does not appear.

Thus it appears that the incompressible equations without the Coriolis terms are the most suitable for a numerical study of internal gravity wave behavior near a critical level. This set of equations is valid near a critical level except for the exclusion of the Coriolis term which has already been discussed. This set of equations is most easily worked with in the stream function - vorticity formulation which is presented in the next section. This formulation has the advantage of reducing the working variables to three, and the important nonlinear terms and the damping terms may be kept throughout.

The other main question was whether to use a progress in z approach or a progress in time approach. While the latter is the more familiar form of the initial value problem, since gravity waves are continuously monitored at the ground for a number of locations, the wave's dependence on x and t can be considered known there. By solving the set of equations for the z derivatives the values of the wave parameters at successive heights could be found. Since there are no measurements of gravity waves in the atmosphere over a considerable height range at nearly the same time, the initial values for the progress in z approach are much better known.

At first this progress in z approach was tried, with simple $\exp(-i\omega t + ikx)$ dependence at the ground and cyclical or repetitive boundary conditions in the x and t directions. After some investigation this approach was abandoned. The reasons for this failure will be briefly discussed.

It is clear from equations 3.1-1 that one may solve explicity for $\frac{\partial w}{\partial z}$ and $\frac{\partial P}{\partial z}$ so advancing those two variables will pose no problem. While one may also solve for $\frac{\partial u}{\partial z}$ and $\frac{\partial \rho}{\partial z}$, one is constrained to divide by w when so doing, and for those points where w is near or equal to zero this is incorrect. Further the terms containing $\frac{\partial u}{\partial z}$ and $\frac{\partial \rho}{\partial z}$ are nonlinear terms, and solving the equations this way is certainly going to be very inaccurate far from a critical level.

An implicit scheme in which a relaxation procedure was used to find u and ρ at the new z step was tried but was unsuccessful because the relaxation procedure did not work correctly. Apparently the equations used in the relaxation were ones for which relaxation does not converge.

Another scheme involving matrix inversion was also tried, again without success. Let the two dimensional matrix F^J contain the N^2 values for each of the four variables u, w, ρ, ρ for any z-step J. The basic equations may be written as a balanced difference scheme of the form

$$AF^{J+1} = BF^J$$

where A and B are 4N x 4N matrices. F^{J+1} is obtained by inversion of A and matrix multiplication. Unfortunately when only the linear terms are used A is a singular matrix and inversion is impossible. When the nonlinear terms are included A becomes invertable but since the determinant involves only the nonlinear terms the results are very inaccurate unless the equations are extremely nonlinear.

The reason for the singularity of A in the linear case is that the equation for some point (I,L) is the exact negative of the equation for the point (I+N/2,L) half a wavelength away. When the nonlinear terms are added the two equations are no longer exact negatives of each other. By using only the first N/2 points a nonsingular A is obtained and the resulting scheme works very well in the linear region. In the region where the nonlinear terms are of significant size, this scheme amounts to imposing a very artificial reflection condition. No satisfactory method was found to treat the region in which the nonlinear terms were large enough to be significant but not large enough to allow using all N points in A.

Upon discovery that the progress in z approach appeared to be intractable, work was begun on the transient or progress in T approach. In this approach the cyclical boundary conditions are imposed in the x direction. The lower z direction boundary is the ground, a w = 0 surface. It is very difficult to find a feasible upper boundary condition which successfully simulates the real infinite atmosphere for all times and for all conditions. The exact method of treating this boundary will be considered at some length in the next chapter where the details of the finite difference scheme are presented. While artificial when compared to physical reality, the condition imposed on the upper boundary has as minimal an effect as possible on the events near the critical level.

4.2 The stream function - vorticity form of the equations

In this section the basic equations on which the finite difference scheme is based will be manipulated into the more appropriate stream function - vorticity form. Because of the use of 3.1-4 instead of 3.1-3 for the equation of the conservation of mass, the resulting equations will appear somewhat different from the usual ones. In addition to the stream function and the vorticity, two new momentum variables are introduced.

As discussed in the preceeding chapter it is a good approximation near a critical level to use equation 3.1-4 instead of the complete equation for mass conservation and to let the speed of sound be infinite everywhere but in the Brunt frequency. Because of computational limitations only two space dimensions can be included, so the Coriolis force is neglected and it is assumed that there is no motion in the y direction. The basic equations to be used for the numerical model are then obtained:

$$\bar{\rho}\frac{Du}{Dt} + \bar{\rho} w \frac{d\bar{u}}{dz} + \frac{\partial p}{\partial x} = \mu \nabla^2 u \qquad 4.2\text{-}1A$$

$$\bar{\rho}\frac{Dw}{Dt} + g\rho + \frac{\partial p}{\partial z} = \mu \nabla^2 w \qquad 4.2\text{-}1C$$

$$\vec{\nabla} \cdot (\bar{\rho}\vec{V}) = 0 \qquad 4.2\text{-}1D$$

$$\frac{D\rho}{Dt}+\bar{\rho}w\left(\frac{\omega_\beta^2}{g}\right) = K\nabla^2\rho \qquad 4.2\text{-}1E$$

These four equations can not be used for a finite difference scheme as is because they do not determine the pressure at the new time step. In order to eliminate the pressure from these equations, the stream function - vorticity form for these equations is adopted.

Define new momentum variables β, γ by

$$\beta = \bar{\rho}u \qquad \gamma = \bar{\rho}w \qquad 4.2\text{-}2A$$

and define the stream function ψ and the vorticity ξ by

$$\beta = -\frac{\partial\psi}{\partial z} \qquad \gamma = \frac{\partial\psi}{\partial x} \qquad 4.2\text{-}2B$$

$$\xi = \frac{\partial\gamma}{\partial x} - \frac{\partial\beta}{\partial z} = \nabla^2\psi \qquad 4.2\text{-}2C$$

Note that by the definition of the stream function equation 4.2-1D is automatically satisfied. The wind could have been included in the definitions of ψ and ξ but it has been omitted for two reasons. Inclusion of the wind adds a large term which is a function of z only to ψ and reduces the accuracy with which the perturbation motions may be calculated. Secondly, it increases the complexity of the calculations. Changes in the average horizontal motion for any given height may occur by means of the perturbation variable u acquiring a nonzero average, so keeping $\bar{u}$ constant does not rule out interaction between the wind and the wave.

With these definitions equations 4.2-1 become:

$$\frac{D\beta}{Dt} - \beta\gamma\frac{1}{\bar{\rho}^2}\frac{d\bar{\rho}}{dz} + \gamma\frac{d\bar{u}}{dz} + \frac{\partial p}{\partial x} = \mu\nabla^2(\beta/\bar{\rho}) \quad 4.2\text{-}3A$$

$$\frac{D\gamma}{Dt} - \gamma^2\frac{1}{\bar{\rho}^2}\frac{d\bar{\rho}}{dz} + g\rho + \frac{\partial p}{\partial z} = \mu\nabla^2(\gamma/\bar{\rho}) \quad 4.2\text{-}3C$$

$$\frac{\partial\beta}{\partial x} + \frac{\partial\gamma}{\partial z} = 0 \quad 4.2\text{-}3D$$

$$\frac{D\rho}{Dt} - \gamma\left(\frac{\omega_B^2}{g}\right) = K\nabla^2\rho \quad 4.2\text{-}3E$$

To get the vorticity equation, the result of operating on 4.2-3A with $\partial/\partial z$ is subtracted from the result of operating on 4.2-3C with $\partial/\partial x$. After some algebra one obtains:

$$\frac{\partial \xi}{\partial t} = -\bar\rho \frac{\partial}{\partial x}\left[\frac{(\bar\rho \bar u + \beta)\xi}{\bar\rho^2}\right] - \bar\rho \frac{\partial}{\partial z}\left[\frac{\gamma \xi}{\bar\rho^2}\right] - \beta\gamma \frac{d}{dz}\left[\frac{1}{\bar\rho^2}\frac{d\bar\rho}{dz}\right]$$

$$-\gamma \frac{d^2 \bar u}{dz^2} - \bar\rho \frac{\partial}{\partial x}\left[\frac{g\rho}{\bar\rho}\right] + \mu \nabla^2\left[\frac{\xi}{\bar\rho} + \beta \frac{1}{\bar\rho^2}\frac{d\bar\rho}{dz}\right]$$

the terms involving the second or third derivative of the ambient density or the square of the first derivative of the ambient density are small with respect to other terms (see appendix C). Their neglect corresponds to the Boussinesq approximation in the more common form of the vorticity equation based on 3.1-3 instead of 3.1-4. In general, wind profiles with constant shear will be used, so that the second derivative of the wind will not be carried further. Because of the importance of inflection points in the mathematical studies, the case where the second derivative of the wind is nonzero is not entirely excluded.

For the few cases where the shear is not constant, the nature of the additional term in the following equations is evident.

Thus the equations from which the finite difference equations will be obtained are:

$$\frac{\partial \xi}{\partial t} = -\bar{\rho}\left[\frac{\partial}{\partial x}\left\{\frac{(\bar{\rho}\bar{u}+\beta)\xi}{\bar{\rho}^2} + \frac{g\rho}{\bar{\rho}}\right\} + \frac{\partial}{\partial z}\left\{\frac{\gamma\xi}{\bar{\rho}^2}\right\}\right] + \mu\nabla^2\left\{\frac{\xi}{\bar{\rho}}\right\} \qquad 4.2\text{-}4A$$

$$\frac{\partial \rho}{\partial t} = -\frac{1}{\bar{\rho}}\left[\frac{\partial}{\partial x}\left\{(\bar{\rho}\bar{u}+\beta)\rho - \psi\left(\frac{\omega_B^2\bar{\rho}}{g}\right)\right\} + \frac{\partial}{\partial z}\left\{\gamma\rho\right\}\right] + K\nabla^2\rho \qquad 4.2\text{-}4B$$

$$\nabla^2\psi = \xi \qquad 4.2\text{-}4C$$

$$\beta = -\frac{\partial\psi}{\partial z} \qquad \gamma = \frac{\partial\psi}{\partial x} \qquad 4.2\text{-}4D$$

In order to see how these equations are used for the transient calculation assume that all the variables are known for time step L and all preceeding time steps. First ξ and ρ for the next time step are calculated from 4.2-4A and 4.2-4B. Using this ξ, ψ is found from 4.2-4C, and then the momentum variables are found from

4.2-4D. All the variables are now known at the new time step and the sequence is complete.

4.3 The grid system and the finite difference equations

Before equations 4.2-4 can be written in finite difference form the grid system to be used must be described. Let the positive integers I,J,L refer to values in the x,z,t dimensions respectively. Δx, Δz, and Δt are the step sizes. The grid system to be used is the staggered one used by Foldvik and Wurtele (1967). ξ and ρ are defined at points given by $x = \Delta x(I-1)$, $z = \Delta z(J-1.5)$. ψ is defined at points given by $x = \Delta x(I-0.5)$, $z = \Delta z(J-1)$. β is defined at points given by $x = \Delta x(I-0.5)$, $z = \Delta z(J-1.5)$. γ is defined at points given by $x = \Delta x(I-1)$, $z = \Delta z(J-1)$.

It can be seen that β is defined at points halfway between the points on the same row where ξ and ρ are defined, and that γ is defined at points halfway between the points in the same column where ξ and ρ are defined. The top and bottom boundaries of the region being considered are taken to coincide with rows of ψ and γ, and to be halfway between rows of ξ, ρ and β .

There is no staggering in the time dimension, and $t = \Delta t(L-1)$ always. The value chosen for Δt will depend on the results of the stability and error analysis and will be considered further in later sections. While Δz can generally be chosen at one's discretion, the value of Δx must depend on the choice of a basic horizontal wavelength. This will be discussed in more detail when boundaries are considered.

The notation $\rho(I,J)$ is used to represent the value of the perturbation density at the values of x and z given above for this variable. It is assumed that time step L is meant unless a different value is indicated by a superscript. Note that, for example, $\rho(I,J)$ and $\psi(I,J)$ do not refer to values at the same point in space.

Equations 4.2-4D are simply expressed due to the staggered grid system:

$$\beta(I,J) = (\ \psi(I,J-1) = \psi(I,J)\ )\ /\ \Delta z \qquad 4.3\text{-}1A$$

$$\gamma(I,J) = (\ \psi(I,J) - \psi(I-1,J)\ )\ /\Delta x \qquad 4.3\text{-}1B$$

The method of handling the Poisson's equation 4.2-4C will be covered in the next section.

The time step is treated in the balanced or leapfrog manner:

$$\xi^{L+1}(I,J) = \xi^{L-1}(I,J) - 2\,\Delta t\ F_{\xi}(I,J) \qquad 4.3\text{-}2A$$

$$\rho^{L+1}(I,J) = \rho^{L-1}(I,J) - 2\,\Delta t\ F_{\rho}(I,J) \qquad 4.3\text{-}2B$$

where $-F_{\xi}$ and $-F_{\rho}$ represent the finite difference analogs of the right sides of equations 4.2-4A and 4.2-4B. These quantities can be written:

$$F_{\xi}(I,J) = \bar{\rho}(J)\Bigg[\Big\{\Big(T_x(\xi,I,J) - T_x(\xi,I-1,J)\Big)\Big/\Big(\bar{\rho}(J)\Big)^2$$

$$+ \Big(g/\bar{\rho}(J)\Big)\Big(T_x(\rho,I,J) - T_x(\rho,I-1,J)\Big)\Big\} \Big/ \Delta x$$

$$+ \Big\{\Big(T_x(\xi,I,J)/r_s(J)\Big) - \Big(T_z(\xi,I,J-1)/r_x(J-1)\Big)\Big\}\Big/\Delta z\Bigg]$$

$$-\mu D_L(\xi/\bar{\rho},I,J)$$

4.3-3A

$$F_{\rho}(I,J) = \Big(\bar{\rho}(J)\Big)^{-1}\Bigg[\Big\{\Big(T_x(\rho,I,J) - T_x(\rho,I-1,J)\Big)$$

$$- B(J)\Big(T_p(I,J) - T_p(I-1,J)\Big)\Big\} \Big/ \Delta x$$

$$+ \Big\{T_z(\rho,I,J) - T_z(\rho,I,J-1)\Big\} \Big/ \Delta z\Bigg] \qquad 4.3\text{-}3B$$

$$-K\, D_L(\rho,I,J)$$

where the following quantities have been used:

$$B(J) = \omega_B^2(J)\,\bar{\rho}(J)/g \qquad 4.3\text{-}4A$$

$$T_p(I,J) = \Big(\psi(I,J) + \psi(I,J-1)\Big) \;/2 \qquad 4.3\text{-}4B$$

$$r_s(J) = \Big[\Big(\bar{\rho}(J) + \bar{\rho}(J+1)\Big) \;/2\Big]^2 \qquad 4.3\text{-}4C$$

and letting f represent $\xi/\bar{\rho}$ or ρ:

$$D_L(f,I,J) = \left[f(I+1,J) + f(I-1,J) - 2f(I,J)\right]/(\Delta x)^2$$

$$+ \left[f(I,J+1) + f(I,J-1) - 2f(I,J)\right]/(\Delta z)^2$$

4.3-4D

For T_x and T_z let f represent ξ or ρ :

$$T_x(f,I,J) = \left[\bar{\rho}(J)u(J) + \beta(I,J)\right] S_x(f,I,J) \quad 4.3\text{-}5A$$

$$T_z(f,I,J) = \gamma(I,J)\ S_z(f,I,J) \quad 4.3\text{-}5B$$

The exact form of S_x and S_z will depend on which method of averaging is used. In the following definitions the simple two-point average is obtained by setting $M_a = 0$ and a six-point average is obtained by setting $M_a = 1$:

$$S_x(f,I,J) = \left[f(I,J) + f(I+1,J)\right] (4-M_a)/8$$

$$+ \left[f(I,J+1) + f(I+1,J+1) + f(I,J-1)\right.$$

$$\left. + f(I+1,J-1)\right] M_a/16 \quad 4.3\text{-}5C$$

$$S_z(f,I,J) = \left[f(I,J) + f(I,J+1)\right](4-M_a)/8$$

$$+ \left[f(I+1,J) + f(I+1,J+1) + f(I-1,J)\right.$$

$$\left. + f(I-1,J+1)\right] M_a/16 \qquad 4.3\text{-}5D$$

The choice of which method of averaging to use will be discussed in conjunction with the error analysis.

The complete finite difference analogs to equations 4.2-4A and 4.2-4B may be obtained by the use of 4.3-4, 4.3-4 and 4.3-5 in 4.3-2 but there seems to be little point in writing out the whole equations.

Of course any numerical model can consist of only a finite number of points, and for a transient calculation two boundaries are required for each spatial dimension. Let the maximum values of the horizontal and vertical indices be I_{max} and J_{max} respectively. This means, for example, that I_{max} different values of x are being considered, and that there are I_{max} columns of points, for the different variables.

In the horizontal, cyclical boundary conditions are imposed. This is equivalent to considering an infinite repetitive model, and each wave considered is infinite in the x direction. If a region with a repetition length of $N \Delta x$ is desired, then $I_{max} = N + 2$ is used. Values on all sides of each point at which ξ and ρ are defined are needed to advance these variables one time step.

Assume that all the variables are known for all I_{max} points for step L and preceding time steps. The values for step L+1 are first found for values of I from I = 2 to I = I_{max}-1 using the equations of section 4.2. Then the cyclical boundary conditions are invoked and the values at the new time step for I = 1 are defined to be those found for I = I_{max}-1 and the values for I = I_{max} are defined to be those found for I = 2.

4.4 Solving Poisson's equation

Foldvik and Wurtele solved Poisson's equation 4.2-4C by a relaxation procedure, and this method was used initially in this work also. However, when the variable vertical spacing was introduced the relaxation procedure failed to work satisfactorily and the Fourier transform method described below was developed.

The fact that the linear theory predicts that the vertical wavelength will become increasingly shorter with proximity to a critical level creates an undesirable situation. Although the nonlinear consideration of section 3.2 indicates that this will not continue right up to the critical level, it is not known how short the wavelength might become and some shortening must be expected.

Use of a value of Δz which provides reasonable resolution elsewhere will probably give insufficient resolution near the critical level. Use of a much smaller value of Δz which might be expected to provide sufficient resolution near the critical level would mean the calculation of many thousands of unnecessary values in the region away from the critical level. The obvious solution is to use some Δz_s, much smaller than Δz, only in an expanded region around the critical level, and Δz elsewhere. Using these two different spacings has its drawbacks, however. It has been found in practice that the constants and the harmonics of the wave variables, which depend upon the nonlinear terms for their generation, often come to exhibit erratic behavior near the boundaries of the expanded region. This is not unexpected. Certainly disturbances which have a vertical wavelength less than z will not be able to propagate outside of the expanded region, so that these waves will be reflected at the boundaries of the expanded region.

Implementation of this idea was hindered by a few minor complications due to the staggered grid system, but the real difficulty was that the relaxation procedure either failed to converge at all or converged only for very inefficient values of the relaxation parameter.

Since Poisson's equation is linear many techniques are available for its solution. Each variable is represented by values at a finite number of points so that Fourier analysis and synthesis should provide very accurate results at these points. The existence of fast transform routines means that the time required for this method will be comparable with that for the relaxation method.

Let ψ_J be the solution to Poisson's equation when ξ is nonzero for only the J^{th} row of points. Because of the linearity superposition is valid and the total solution is

$$\psi(x,z) = \sum_J \psi_J(x,z) \qquad 4.4\text{-}1$$

where the summation is over all the rows of ξ in the region. ψ_J satisfies the equation

$$\nabla^2 \psi_J = \xi_J(x)\,\delta(z-z_J)$$

where

$$\xi_J(x) = \Delta z\ \xi(x,z_J)$$

and z_J is the height of the J^{th} row. (Note that due to the use of a finer vertical spacing near the critical level it is no longer true that $z_J = \Delta z(J-1.5)$.

The transform functions H_J and F_J are defined by

$$H_J(k,z) = \int_{-\infty}^{\infty} \psi_J(x,z)\exp(-ikx)dx$$

$$F_J(k) = \int_{-\infty}^{\infty} \xi_J(x)\exp(-ikx)dx$$

with the appropriate inverse transforms. H_J is composed of a homogeneous and an inhomogeneous part:

$$H_J(k,z) = G_J(k,z) + Ae^{kz} + Be^{-kz}. \qquad 4.4\text{-}2A$$

The equation for the inhomogeneous part is

$$(-k^2 + \frac{\partial^2}{\partial z^2})\, G_J(k,z) = F_J(k)\,\delta(z-z_J)$$

for which the solution is

$$G_J(k,z) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[-F_J(k)/(k^2+m^2)\right] \exp(im(z-z_J))dm$$

This integral is done by contour integration with the result that

$$G_J(k,z) = -F_J(k)\exp(-k|z-z_J|)/2k \qquad 4.4\text{-}2B$$

With the boundary conditions that $\psi_J = 0$ at $z = 0$ and at $z = h$ the values of A and B may be found:

$$A = \exp(-kh)\,G_J(k,0) - G_J(k,h) \quad /2\sinh(kh) \qquad 4.4\text{-}3A$$

$$B = -G_J(k,0) - A \qquad 4.4\text{-}3B$$

More general top boundary conditions will be considered shortly.

Thus the final form for H_J is

$$H_J(k,z) = (F_J(k)/2k)\left[-\exp(-k|z-z_J|) + C_A e^{kz} + C_B e^{-kz}\right]$$

4.4-4

where

$$C_A = e^{-kh}\sinh(kz_J)/\mathrm{sink}(kh)$$

$$C_B = \exp(-kz_J) - C_A.$$

The solution for the stream function can now be written

$$\psi(x,z) = \sum_J \frac{1}{2\pi}\int_{-\infty}^{\infty} H_J(k,z)e^{ikx}dk = \frac{1}{2\pi}\int_{-\infty}^{\infty}\sum_J H_J(k,z)e^{ikx}dk$$

4.4-5

Interchanging the order of the summation and the inverse transform means that a transform must be done for each source row and an inverse transform only for each value of z at which values of ψ are desired.

For computation the sums

$$S_A(I) = \sum_{JX} T_a(I,JX)\ F(I,JX) \qquad 4.4\text{-}6A$$

$$S_B(I) + \sum_{JX} T_b(I,JX)\ F(I,JX) \qquad 4.4\text{-}6B$$

$$S_C(I,JP) = \sum_{JX} T_c(I,JP,JX)\ F(I,JX) \qquad 4.4\text{-}6C$$

are formed, where

$$T_a(I,JX) = C_A(I,JX)\ /\ 2k(I)$$

$$T_b(I,JX) = C_B(I,JX) / 2k(I)$$

$$T_c(I,JP,JX) = -\exp(-k(I)\ |z_p(JP)-z_x(JX)|) / 2k(I)$$

are quantities which can be calculated once in the beginning of the program and stored for future use. JX refers to a row of ξ values at height $z_x(JX)$ and JP refers to a row of ψ values at height $z_p(JP)$. F(I,JX) is the finite difference analog of $F_J(k)$. In these transformed quantities, I refers to the value of the wavenumber k(I) where

$$k(I) = 2\pi(I-1)/N\Delta x \qquad 4.4\text{-}7$$

N is the number of points in a row and should be an integer power of two for most of the fast Fourier transform routines. $N\Delta x$ is the basic horizontal wavelength or the repetition length.

One drawback to this method of solving Poisson's equation is the need to store the matrix T_c. Because T_c is three dimensional, only moderate values of the three indices imply a huge array. If T_c is too large to be stored, the factors of T_c, which are two dimensional, may be stored instead and T_c recalculated at each step.

Since ξ and ψ are defined at values of x separated by $\Delta x/2$, after the $\sum H_J$ is complete it is multiplied by $\exp(ik\Delta x/2)$ prior to the inverse transformation so that ψ will be evaluated at the appropriate values of x.

The above method will not account for any average in the horizontal wave motion. Because $k(1) = 0$, equation 4.4-4 cannot be used for the first (constant) term in the transform of each row of ξ. The two poles of the contour integral merge on the contour, so that the integral cannot be evaluated. These constant values are saved as the other values in the transform are treated, and together these values are represented by $\xi_a(J)$. For this case there is no x dependence in Poisson's equation, so

$$\frac{d^2\psi_a}{dz^2} = \xi_a$$

is solved for ψ_a which is added to the result above.

ψ is undetermined to within an additive constant, so $\psi_a(z=0) = 0$ can always be required. A value of $\psi_a(z=h)$ cannot also be specified, however, because so doing places an unjustifiable restriction on $\int_0^h u_a dz$, where $\bar{\rho} u_a = - d\psi_a/dz$ and $\xi_a = -d(\bar{\rho} u_a)/dz$. Only one boundary value of u_a may be given if the problem is not to be overspecified, and as long as the viscosity is nonzero this is provided by the requirement that $u_a(z=0) = 0$. Using the boundary values just discussed, and with u_a defined at the same values of z at which ξ is defined, ψ_a is found by using

$$u_a(J) = u_a(J-1) - \Delta z(\xi_a(J) + \xi_a(J-1))/2$$

$$\psi_a(J) = \psi_a(J-1) - \Delta z u_a(J)$$

The case in which the top boundary acts as a source is easily treated by adding to the above solution a function ψ_s which is a solution of Laplace's equation and which satisfies the top boundary condition. The source is taken to have a trigonometric dependence on a single wavenumber k in the horizontal, and the lower boundary condition is still $\psi = 0$ so that

$$\psi_s(x,z) = \text{Real}\left[C_s e^{ikx}\right] \frac{\sinh(kz)}{\sinh(kh)}$$

where C_s, containing the magnitude and phase, will be constant for each solution of Poisson's equation at a given time step but will in general be a function of time.

A line vorticity source has been found to be a more satisfactory wave source than motion of the top boundary. Before beginning the solution of Poisson's equation for ψ, row JX_s of ξ values is replaced by $\xi_s \cos(kx-\omega t)$, where ξ_s, and the wavenumber k and the radian frequency ω may be specified at the experimenter's discretion. Because each ξ row enters the solution of Poisson's equation as $\xi \Delta z$, if the source strength is to be independent of the spacing, $\xi_s \Delta z$, not ξ_s, is the source strength. This product conveniently has the units of m/s and is roughly equal to the magnitudes of the wave motions it generates. Because ξ_s may be much larger than the other values of ξ, especially for small Δz, the nonlinear and damping terms in 4.3-3A must be omitted when applying 4.3-2A to the rows adjacent to JX_S if huge erroneous source terms are to be avoided. This also has the advantage of making the source completely free from the tendency to generate nonlinear terms close to the source.

The alternative to placing a rigid top surface at z = h is to place another region above z = h, because a free surface is very hard to incorporate in the stream function - vorticity formulation. Let the subscript u refer to the region above z = h and the subscript r to the region below z = h. The existence of this upper region makes it necessary to redefine A and B appropriately, and to consider what values will be used for ξ in region u. Solving the boundary condition equations properly for A and B does not eliminate the need for values of ξ above z = h, although it is true that values very far above the interface will have a negligible influence on region r.

The obvious choice for region u is an infinitely high region with constant wind. The boundary conditions are easily solved for A and B (see appendix B), but the proper values of ξ in region u are not readily found. For a steady state, ξ could be found from the analytic solutions. However, in the early stages of these calculations the transient wave has spread only a very short distance into the upper region and using the steady state ξ values for the entire region leads to large errors. It may be possible to treat the propagation of the wave into the upper region correctly, by Laplace transforms perhaps, but this must be a future development. It is much simpler to ignore ξ above z = h entirely or to generate it from the steady state solutions, but both of these options are unsatisfactory because the wave does not propagate out through the upper boundary properly and large values of the wave variables occur there which dominate the development elsewhere. Professor T. R. Madden suggested that the values of ξ below the source could be used above the source as well. This mirror technique for generating the values of ξ in region u has worked well in practice. Although this amounts to placing another critical level and a rigid boundary above the source, reflections from them have caused no problem because they take so long to propagate to the lower critical level.

For an infinitely high region r, A = 0 in 4.4-2A, and since the mirrored upper boundary at z = 2h is far from the lower critical level, it is a good approximation with the mirrored upper region. ψ in region u is never calculated, and the contributions to ψ in region r from ξ in region u have the form $(F_J/k)\sinh(kz)\exp(-kz_J)$. The source is at height $z_x(JX_s)$ and JP_h is defined by $z_p(JP_h) = h$; then equations 4.4-6A and 4.4-6B become

$$S_A(I) = \sum_{J_a=1}^{J_r-2} M_a(I,J_a)\ F(I,J_r-J_a)$$

$$S_B(I) = \sum_{JX=2}^{JP_h} T_b(I,JX)\ F(I,JX) - \sum_{J_a=1}^{J_r-2} M_a(I,J_a)\ F(I,J_r-J_a)$$

where

$$J_r = JX_s + (JX_s - JP_h),$$

$$T_b(I,JX) = \exp(-k(I)\ z_x(JX))/2k(I),$$

and

$$M_a(I,J_a) = -\exp(-k(I)\ (z_x(JP_h) + J_a \Delta z))/2k(I)$$

(Note that $z_x(JP_h) = h - \Delta z/2$, so that JP_h denotes the topmost row of ξ values in region r.) In practice it has been found sufficient to carry the summations over J_a to ten or twenty instead of $J_r - 2$. Because the effect of the ξ rows in region u decreases as $\exp(-J_a \Delta z)$, exactly how many rows should be included depends on the value of Δz.

4.5 Stability analysis

In this section the methods of Richtmyer (1957) are applied to the finite difference scheme just described in order to determine whether or not the scheme is stable and if so what restrictions on t are necessary to achieve this stability. The available methods for assessing stability are applicable only to equations which are linear, so the complete equations will be linearized for this analysis. The nonlinear terms are expected to be small except in a small region right around the critical level, and the results of this analysis should prove adequate.

Linearizing the finite difference equations described in section 4.3 one obtains:

$$\xi^{L+1}(I,J) = \xi^{L-1}(I,J) - 2\Delta t \Big[\bar{u}(J) \left\{ S_x(\xi, I, J) - S_x(\xi, I-1, J) \right\} / \Delta x + g \left\{ S_x(\rho, I, J) - S_x(\rho, I-1, J) \right\} / \Delta x - \mu D_L(\xi, I, J) \Big] \qquad 4.5\text{-}1A$$

$$\rho^{L+1}(I,J) = \rho^{L-1}(I,J) - 2\Delta t \Big[\bar{u}(J) \Big\{ S_x(\rho,I,J) - S_x(\rho,I-1,J) \Big\} / \Delta x - \omega_B^2/g \Big\{ T_p(I,J) - T_p(I-1,J) \Big\} / \Delta x - KD_L(\rho,I,J) \Big] \qquad 4.5\text{-}1B$$

where for simplicity it has been assumed that the ambient density may be taken outside the Laplacian operator in the viscous term and $B(J)/\bar{\rho}(J) = \omega_B^2/g$ has been used. S_x, T_p, and D_L retain the meanings given them in section 4.3. Letting f represent ξ or ρ and assuming exp(ikx+imz) depedence:

$$S_x(f,I,J) - S_x(f,I-1,J) = f(I,J)\,(i\sin(k\Delta x))\,(1 + M_a(\cos(m\Delta z))/4)$$

$$D_L(f,I,J) = f(I,J)\Big[(2\cos(k\Delta x) - 2)/(\Delta x)^2 + (2\cos(m\Delta z) - 2)/(\Delta z)^2\Big]$$

In theory the Fourier transform method solves Poisson's equation without any error due to the finite differencing, so using

$$\psi(I,J) = \xi(I,J)\exp(ik(\Delta x/2) + im(\Delta z/2))/(-k^2-m^2)$$

where the exponential factors are introduced to account for the fact that ψ and ξ are defined at different points:

$$T_p(I,J) - T_p(I-1,J) = \left[-\xi(I,J)/(k^2 + m^2)\right]$$

$$\left(2i\sin k(\Delta x/2)\right) \cos(m\Delta z/2)$$

Now let

$$T_u = -2\bar{u}(\Delta t/\Delta x)\ i\sin(k\Delta x)\left[1 + M_a(\cos(m\Delta z) - 1)/4\right]$$

$$T_a = T_u g/\bar{u}$$

$$T_b = -2(\omega_B^2/(g(k^2 + m^2)))(\Delta t/\Delta x)2i\sin(k\Delta x/2)\cos(m\Delta z/2)$$

$$T_v = 2\mu\Delta t\left[(2\cos(k\Delta x) - 2)/(\Delta x)^2 + (2\cos(m\Delta z) - 2)/(\Delta z)^2\right]$$

$$T_h = T_v K/\mu$$

and defining the new variables r and q by the equations:

$$r^{L+1}(I,J) = \rho(I,J)$$

$$q^{L+1}(I,J) = \xi(I,J)$$

where a time step value of L is to be assumed when no superscript is present. Suppressing the (I,J) dependence, the equations for the advance of one time step can now be written in matrix form:

$$\begin{bmatrix} r^{L+1} \\ q^{L+1} \\ \rho^{L+1} \\ \xi^{L+1} \end{bmatrix} = A \begin{bmatrix} r \\ q \\ \rho \\ \xi \end{bmatrix}$$

where the matrix A, called the amplification matrix, is

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & T_u+T_h & T_b \\ 0 & 1 & T_a & T_u+T_v \end{bmatrix}$$

A difference scheme is said to be stable for a system in which exponential growth in time is not allowed only if

$$|\lambda|_{max} \leq 1$$

where the λ are the eigenvalues of the amplification matrix for the difference scheme.

Unfortunately the expression for λ is exceedingly cumbersome when all the terms in A are retained. However, the damping terms T_v and T_h are small with respect to T_u for moderate wind speeds, so taking $T_v = 0 = T_h$:

$$\lambda = -iB \pm (1 - B^2)^{1/2} \qquad 4.5\text{-}2A$$

with

$$B = (-T_u \pm (T_a T_b)^{1/2})/2i \qquad 4.5\text{-}2B$$

It may be seen that B is a real quantity and the stability condition can be shown to be

$$|B| < 1.$$

Using the above definitions, this condition may be written

$$(\Delta t/\Delta x) \Big| \bar{u} \sin(k\Delta x)\ (1 + M_a(\cos(m\Delta z)/4) \pm (\omega_B/(k^2+m^2)^{1/2}) \sin(k\Delta x) \Big[1 + M_a(\cos(m\Delta z) - 1)/4) \cos(m\Delta z/2)\Big]^{1/2} \Big| \leq 1$$

The value of Δt used should be chosen such that this inequality holds for all values of k and m which have meaning for the difference scheme. For example, all wavelengths between $2\Delta x$ and the width of the region must be considered in the horizontal. In the simplest case where $M_a = 0$ and only the first order and zero order terms in the trignometric expansions are kept, the stability condition is

$$\Delta t \leq \left[(\bar{u}_{max} / \Delta x) + \omega_B \right]^{-1} = \Delta t_s \qquad 4.5\text{-}3$$

where $\bar{u}_{max}$ is the maximum wind speed in the region. This equation defines the stability limit Δt_s.

The effect of neglecting the second and higher order terms in the trigonometric expansions has been to make Δt_s smaller than it would be otherwise. Thus this stability condition was found to be perfectly adequate when the relaxation method was used for solving Poisson's equation. However, when the Fourier transform method was adopted it was found that even though Δt was well below Δt_s very small horizontal wavelengths tended to increase rapidly when the viscosity and thermal conductivity were zero. This effect was magnified by increasing the number of horizontal points which decreased the size of the smallest wavelength considered. It was discovered that this problem could be eliminated by using small but non-zero values for the damping constants.

Since the shortest wavelengths are accounted for in the above stability analysis, this effect is not understood. Even though truncation errors give rise to finite values for the shortest wavelengths, there is no known reason why these wavelengths should increase in magnitude. Because values of the damping constants which are so much smaller than the eddy values that the damping terms remain insignificant very close to a critical level are sufficient to eliminate this problem, it will not be pursued further.

Foldvik and Wurtele state that their scheme is stable provided that the centered time step is replaced by a forward time step every t_f steps, where t_f was determined by experiment.

The equations in matrix form for the forward time step are

$$\begin{bmatrix} \rho^{L+1} \\ \xi^{L+1} \end{bmatrix} = A \begin{bmatrix} \rho \\ \xi \end{bmatrix}$$

with

$$A = \begin{bmatrix} 1+(T_u+T_h)/2 & T_b/2 \\ T_a/2 & 1+(T_u+T_v)/2 \end{bmatrix}$$

For the case where $T_h = 0 = T_v$, the eigenvalues are

$$\lambda = 1 \pm iB/2$$

where B is defined in equation 4.5-1B. Since B is a real quantity it is seen that this method is unstable for all values of Δt. It is not surprising, then, that an occasional forward time step was found to be of no help in this work.

4.6 Error analysis

In this section the error which results from using finite difference equations instead of partial differential equations is evaluated. Since this requires a known analytic solution, only the linear equations for the case with no shear can be treated, but this provides an indicative result for the complete equations.

Assume that the wave variables have $\exp(-i\omega t+ikx+imz)$ dependence where ω, k, and m are related by the dispersion relation. The notation of the preceding section is used, but note that k and m are here wavenumbers which must satisfy the dispersion relation while in the preceding section they were any meaningful wavenumbers.

Let the error factor E_u be defined by

$$E_u = (\Delta f/\Delta x)/(\partial f/\partial x)$$

where $\Delta f/\Delta x$ represents the finite difference operator and f represents ρ or ξ. If there is no error at all, $E_u = 1$. From the preceding section, it is seen that E_u is given by:

$$E_u = \frac{-T_u f(I,J)}{2\Delta t\bar{u}(\partial f/\partial x)} = \sin(k\Delta x)\left[1 + M_a(\cos(m\Delta z) - 1)/4\right]/(k\Delta x)$$

If Δx and Δz approach zero, E_u approaches one as would be expected. In an analogous manner other error factors are formed from T_a, T_b, T_v, and T_h which are all defined in the preceding section:

$$E_a = E_u$$

$$E_b = 2\sin(k\Delta x/2)\cos(m\Delta z/2) / (k\Delta x)$$

$$E_v = \left[\frac{-2}{k^2+m^2}\right]\left[\frac{\cos(k\Delta x) - 1}{(k\Delta x)^2} + \frac{\cos(m\Delta z) - 1}{(m\Delta z)^2}\right]$$

$$E_h = E_v$$

A time error factor E_t is also needed:

$$E_t = \frac{f^{L+1} - f^{L-1}}{2\Delta t(\partial f/\partial t)} = \sin(\omega\Delta t)/(\omega\Delta t)$$

where a balanced time step has been used.

By means of these error factors the actual finite difference equations being used can be written using the partial differential terms. Thus one obtains

$$E_t \frac{\partial \xi}{\partial t} = - E_u\left[\bar{\rho}\,\bar{u}\frac{\partial \xi}{\partial x} + g\frac{\partial \rho}{\partial x}\right] + E_v \mu \nabla^2 \xi \qquad 4.6\text{-}1A$$

$$E_t \frac{\partial \rho}{\partial t} = - E_u \bar{u}\frac{\partial \rho}{\partial x} + E_b \omega_B^2/g(-1/(k^2+m^2))\frac{\partial \xi}{\partial x}$$
$$+ E_h K \nabla^2 \rho \qquad 4.6\text{-}1B$$

where it has been assumed that Poisson's equation is solved without any error due to the finite differences.

The case where there is no wind and $\mu = 0 = K$ is easily treated. The dispersion relation obtained from equations 4.6-1 is

$$\omega^2(1 + m^2/k^2) = \omega_B^2 E_s^2 \qquad 4.6\text{-}2A$$

where

$$E_s = (E_u E_b)^{1/2}/E_t \qquad 4.6\text{-}2B$$

The error factor in the impedance is $(E_u/E_b)^{1/2}$. Note that changing the vertical spacing will change the impedance and thereby cause spurious reflections.

From 4.6-2 it is seen that the best time dependence that can be expected from the finite difference system is $\exp(-i\omega E_s t)$. The array values for any variable will be then obtained from an expression of the form Real $(F \exp(-i\,\phi_e))$ where F is the correct complex value, and $F = f\exp(-i\omega t)$ with f being a complex function of x and z. The phase error $\phi_e = \omega t(E_s - 1)$. The theoretical error A_t due to this phase shift is computed in the same manner as the actual error:

$$A_t = \frac{1}{\lambda_x \lambda_z}\int_0^{\lambda_z}\int_0^{\lambda_x}\left[\frac{|\text{Real }(F\exp(-i\ \ _e)) - \text{Real}(F)|}{|\text{Real}(F)|_{max}}\right]dxdz \qquad 4.6\text{-}3A$$

This expression can be shown to be equivalent to

$$A_t = \frac{1}{2\pi} \int_0^{2\pi} \left| \text{Real}(\exp(-i\phi - i\phi_e)) \right| d\phi = \frac{2}{\pi}(\sin \phi_e + \cos \phi_e - 1) \qquad 4.6\text{-}3B$$

It is clear that a value of Δt which makes $E_s = 1$ will eliminate this error. This value of Δt, denoted Δt_e, is called the minimum error value because there will be other sources of error which will keep the total error from being zero.

For the case where the wind is a nonzero constant it can be shown that it is necessary that $E_t = E_b = E_u$ in order to eliminate this source of error. In general it is not possible to choose Δx and Δz so that $E_b = E_u$ because there are other requirements which these quantities must meet, but in practice E_b and E_u are about the same size and it is adequate to use $E_s = 1$ as the minimum error condition.

If terms above second order are neglected in the series representations for the sine and cosine, expanding the error factors gives

$$E_s = 1 - \frac{5}{48}(k\,\Delta x)^2 - \frac{1+M_a}{16}(m\,\Delta z)^2 + \frac{1}{6}(\omega\,\Delta t)^2$$

$$\Delta t_e = \tau \left[\frac{5}{8} \left(\frac{\Delta x}{\lambda x}\right)^2 + \frac{3}{8}(1 + M_a) \left(\frac{\Delta z}{\lambda z}\right)^2 \right]^{1/2}$$

If $\lambda_x/\Delta x = N = \lambda_z/\Delta z$ where N is some integer, then

$$\Delta t_e = \tau/N = \tau\Delta x/\lambda_x \qquad 4.6\text{-}4$$

For any analytic solution of interest the viscous and thermal conduction terms are quite small, so their neglect throughout is justified.

The results of three computer runs in which an analytic solution appropriate to the region was given as the initial condition are presented in table 4.1.

Table 4.1

Theoretical and actual error after L steps for three values of Δt: $\bar{u} = 0$, $N = 20$, $\tau = 900s$, $\tau_B = 345s$, $\Delta t_s = 54.0s$, $\Delta t_e = 55.1s$.

	$\Delta t = 25.0s$		$\Delta t = 50.0s$		$\Delta t = 55.0s$	
L	Th.	Actual	Th.	Actual	Th.	Actual
4	0.017	0.004	0.008	0.031	0.0002	0.031
8	0.035	0.011	0.016	0.028	0.0004	0.029
16	0.070	0.023	0.031	0.043	0.0008	0.066
32	0.14	0.051	0.062	0.030	0.0016	0.066
64	0.28	0.105	0.12	0.050	0.0032	0.17

These runs were made when the relaxation procedure was still being used, and with the finite difference error in the relaxation taken into account

$$E_s = 1 + (2\pi)^2 \left[\frac{1}{3}\left(\frac{\Delta t}{\tau}\right)^2 - \frac{1}{2}\left(\frac{1}{N}\right)^2 \right].$$

The theoretical error is calculated from equation 4.6-3B to first order. The actual error is calculated using 4.6-3A and is the average over the three working variables ρ, ξ, and ψ. In one case the error is large because the value of Δt used is far from Δt_e and in the other case it is large because the stability condition is violated.

The error is large in the beginning for $\Delta t = 50s$ and $\Delta t = 55s$ because the first step was a forward time step which is inherently inaccurate. The time derivatives are calculated for $t = 0$ and used as if they were the values for $t = \Delta t/2$. For the $\Delta t = 50s$ case the program was terminated after 180 steps when the error was 0.08.

Unfortunately, for all interesting cases the stability and minimum error requirements on Δt are incompatible. Of course it is desirable that

$$\Delta t_s \geq \Delta t_e$$

so that $\Delta t = \Delta t_e$ may be used. From 4.5-3 and 4.6-4 it is seen that this inequality is

$$1 > \bar{u}_{max}/v_{px} + \omega B$$

where $v_{px} = \lambda_x/\tau$ is the horizontal component of the phase velocity. For a critical level to exist the maximum wind speed in the region must be greater than v_{px} and this condition cannot be satisfied.

The error can be reduced by using as small a value as possible for τ and by using as large a value as possible for N. However, the Brunt period places a lower limit on τ and computation time varies as N^3, so not much can be done in this regard. It is seen to be advantageous to choose the two-point averaging method ($M_a = 0$) in order to make Δt_e as small as possible.

Chapter Five

Results of Calculations

5.1 Specification of parameters and description of output.

The numerical model based on the finite difference scheme described in the preceding chapter has been run for different combinations of the many parameters describing the ambient atmosphere and the wave source. These results and the inferences drawn from them are presented in the following sections. In this section the restrictions on the various parameters and the reasons for choosing certain values for them are discussed. Limitations of the model and the type of output produced by the program are also described.

Wave behavior near a critical level is largely independent of the temperature gradient, so an isothermal atmosphere, Brunt period $\tau_B = 345s$, has been used throughout. The important parameter to vary is the Richardson number, Ri, and since the Brunt frequency is constant, it will be a function of the wind shear only. The maximum and minimum wind speeds are related to the stability of the finite difference scheme and the propagation or nonpropagation of the wave fundamental and its higher harmonics, however, so they cannot be chosen indiscriminantly.

While the 'harmonics' can refer to multiples of either the frequency or the wavenumber, it is the horizontal wavenumber which is meant in this case. The reason for choosing wavenumber rather than frequency will be discussed shortly. In the case that all the harmonics travel with the same horizontal phase speed, the two are equivalent. For a linear system only the fundamental would be of concern since the source contains only the fundamental (to the extent that this is possible in a finite difference system). In this model the nonlinear terms are capable of generating constants and higher harmonics, so that these must be considered.

From the simple dispersion relation it can be shown that for the vertical wavenumber m to be real, the wind must satisfy the following relation:

$$v_{px} - \omega_B/k(I) < \bar{u} < v_{px} + \omega_B/k(I) \qquad 5.1\text{-}1$$

where the horizontal wavenumber k is given by equation 4.4-7. $v_{px} = \omega/k$ is the horizontal phase speed of the source. The maximum range of the wind for real m is seen to be

$$\Delta\bar{u}_p = 2\omega_B/k(I) = 2v_{px}\,\tau/\tau_B(I\text{-}1).$$

Note that if the entire range of $\bar{u}$ is to be used, the values of $\bar{u}$ must be centered on v_{px} for 5.1-1 to be satisfied.

It is desirable that the stability limit be as large as possible and that the wave period be as small as possible so that a given number of time steps will equal as many wave periods as possible. Using equation 4.5-2:

$$\frac{\tau}{\Delta t_s} = \frac{N\,\tilde{u}_{max}}{v_{px}} + 2\,\tau/\tau_B$$

There is a definite lower limit on this ratio because $\tilde{u}_{max}$ must be greater than v_{px} for a critical level to exist and the wave period must be greater than the Brunt period for propagation with no wind. Keeping N small will make this ratio small, and it also makes the number of points to be calculated small. While both of these factors decrease the amount of computer time required, N cannot be made so small that the results are entirely inaccurate. In practice, N = 8 has generally been used. This provides somewhat less accuracy than might be desired, but as long as the third harmonic (which has a wavelength of $2.7\Delta x$) is not too large, it appears to be adequate.

The ((N/2) + 1)th value of the transform must be real to produce a real inverse transform, and since this term is usually largely imaginary it is arbitrarily attenuated by either requiring this term to be real or by taking only the real portion of the inverse transform. Thus the Fourier transform method of solving Poisson's equation makes this term, the fourth harmonic for N = 8, unreliable.

It is generally true that the second harmonic is larger than the third, and the third larger than the fourth since they are all generated by the interactions of lower harmonics. If the third harmonic is small with respect to the fundamental, the attenuation of the fourth harmonic is insignificant. If the third harmonic is an appreciable fraction of the fundamental, however, the opposite must be assumed and the model is unreliable. In practice it has been found that soon after the third harmonic becomes of significant size the wave quantities become locally larger than is physically reasonable and the program terminates. This may be because the energy which would normally go down-scale to higher harmonics and eventual viscous dissipation is blocked and accumulates in the third harmonic where it causes instability.

While τ should be reasonably close to τ_B, there are no restrictions on $\lambda_x = N \Delta x$ and thus on v_{px}. However, from section 4.4 it is seen that for a source row at height z_s ψ will have $\exp(-k|z - z_s|)$ dependence. This implies that unless the height of the region, h, is on the order of or greater than $\lambda_x/2$, the disturbance from the source will fill the entire region after only one time step. If the critical level is in the near field of the source, then it will not be possible to observe the wave's arrival at the critical level, and the effect of the critical level on energy transmission will be difficult to evaluate since the wave will be about the same size on both sides of the critical level in the beginning. So, while λ_x is arbitrary, ratios like h/λ_x and

$\overline{u}_{max}/v_{px}$ are important.

A horizontal wavelength of 5000m has been used throughout, but the results apply to any wavelength as long as various quantities are scaled appropriately. If the wavelengths, wave motions and wind speed are all multiplied by a factor f while the period is unchanged, it can be seen from the basic equations that they are unchanged if ρ is multiplied by f, and if p, ψ, μ, and K are all multiplied by f^2. The magnitude of ξ is unaffected.

With τ, λ_x, h, and Ri chosen, $\overline{u}_{max}$ and $\Delta \overline{u}$ are chosen taking into account the propagation or nonpropagation of the wave harmonics, and then Δz_w is chosen to give the specified value of Ri. The shear layer, containing the critical level is placed as far as possible from the source so that the wave parameters will be small there in the beginning. On the other hand, the shear layer must be far enough from the ground so that the events near the critical level are not obscured by the effect of the rigid surface at z = 0.

In presenting the results of this model the emphasis is more on horizontal wavelength as opposed to frequency because it is convenient to Fourier analyse the arrays for ρ, w, and u by rows and to present magnitude and phase angle of the variable in tabular form as a function of wavelength and height. The repetition length is a fixed constraint in this model, and cannot change with time.

While the source has a constant frequency as well as a constant wavelength, in the transient early stages the frequency exhibited by the wave is often quite different from that of the source. To Fourier analyse in the time dimension, the arrays of the variables would have to be stored for many time steps which is not feasible.

A fairly good idea of the frequency as a function of time, height and wavelength can be obtained by comparing the phase angles at successive print steps. Unfortunately this was not done for some of the earlier runs. It has been found easier to think in terms of speed than frequencies, so the relation used is

$$HPS^{L}(I,J) = (\phi^{L}(I,J) - \phi^{L-L_o}(I,J))/L_o \Delta t k(I)$$

$\phi^{L}(I,J)$ is the phase angle of the I^{th} term in the Fourier transform for the J^{th} row at time step L. L_o is the number of time steps between successive print steps. A value of $L_o \Delta t$ roughly equal to $\tau_B/2$ has been found best. HPS is seen to be the average horizontal phase speed over the preceding period $L_o \Delta t$.

The other portion of the model's output which will be shown here are contour plots of the variables ρ, w, and u. When an expanded region is present, in addition to a plot of the entire region, a plot of the same size for the expanded region is also produced, thereby showing the region of interest in greater detail. Because of similarities between the plots for different variables, and the troubles with averages in the horizontal motion, only the plots of the wave density are usually shown.

As mentioned in section 4.4 some local effects develop at the boundaries of the expanded region where the vertical spacing changes. Large values of $\Delta z/\Delta z_s$ such as eight and sixteen must be used in order to have h as large as λ_x and yet have adequately small spacing near the critical level. Disturbances with vertical wavelengths less than $2\Delta z$ will not be able to propagate outside the expanded region and so their reflection at these boundaries is expected. It is found that the second and third harmonics which have shorter vertical wavelengths than the fundamental are occaionally quite large at these heights. In a few cases large spurious values of ξ_a, the constant term in the transform of ξ, are generated at one of the edges of the expanded region. Since u_a, the change in the wind, is obtained from integrating ξ_a upward from the ground, the values of u_a above the spurious value for ξ_a will be offset. There is no reason to doubt the relative changes of u_a within the expanded region, though.

5.2 Results of calculations

The finite difference model described in the preceding chapter has been run for a number of different cases. The reference, case A, will be presented in some detail. The other cases, most of which differ from case A by having different values for only a few parameters, are treated more briefly. Then some figures which contain the results of most of the cases are discussed, and finally some observations about energy and momentum flow are made.

Some new parameters will be needed to describe the model completely. Let z_{wb} and z_{wt} be the heights of the bottom and top of the layer in which the wind shear is nonzero. $\bar{u}_b$ and $\bar{u}_t$ denote the wind speed below z_{wb} and above z_{wt} respectively. The wind is a linear function of height between z_{wb} and z_{wt} and is continuous everywhere. z_{eb} and z_{et} are the heights of the bottom and top of the expanded region. h is the height of the region. v_{px} is the constant horizontal phase speed of the source. The source strength $s_s = \xi_s \Delta z$ is approximately equal to the magnitude of the horizontal wave motion it generates. z_s is the height of the source. z_c is the height of the critical level assuming that the wave moves with v_{px} and that the wind remains unchanged.

Those parameters which are the same for all cases are: $\Delta t = 15s$, $\Delta x = 625m$, $\tau_B = 345s$, $\lambda_x = 5000m$, and

$\Delta z_s = 25m$.

Now it is not expected that the actual critical level will remain at the theoretical value of z_c. In the first place the wind will change if the incident wave brings momentum which is absorbed near the critical level. The original wind $\bar{u}$ is independent of time, so the total wind is $\bar{u} + u_a$, where $u_a(z,t)$, the change in the wind since time zero, is the average of u over a row of points at height z and at time t. u_a is also the constant term in the Fourier transform of a u row. In addition, the actual critical level may change because the wave at a given height may be moving with a speed different from v_{px}. Further, the different wave variables may move at different speeds, so that the critical level may be different for each variable.

Case A (Reference)

The parameters for this case are: h = 6400m, $\Delta z = 400m$, $z_{wb} = z_{eb} = 2000m$, $z_{wt} = z_{et} = 2800m$, $z_s = 6200m$, $u_b = 0$, $\bar{u}_t = 20m/s$, $\tau = 450s$, $s_s = 1.12m/s$, Ri = 0.53, $\mu = 0.02kg/ms$, and $K = 0.02m^2/s$. This gives $v_{px} = 11.11m/s$ and $z_c = 2444m$. The upper boundary is treated by the mirror technique.

Tables 5.1 and Figures 5.1 contain the row transforms and contour plots for three times this case which ran for 4500s. Changes after 3465s were not too great, and the tables and contour plots are presented at this time in order to faciliate comparison with other cases. The vertical phase speed above z_{wt} is 8.6m/s so it takes about 400s for the wave to reach the shear layer.

With these tables and figures there is no need for a detailed description of this case, but attention must be drawn to a number of features. From figures 5.1H,L it may be seen that the angle which the pattern makes with the vertical just above the critical level is much greater than it is near the top of the expanded region. The wave pattern is being sheared and in order to do this it is necessary that the horizontal phase speed be different at different heights. From tables 5.1D,F it may be seen that just above z_cHPS increases with height as the figures suggest. This shearing of the wave decreases its vertical wavelength. Also, $\bar{u}$(2437.5m) = 10.94m/s so that by 2025s the actual critical level for ρ and u is about 25m below z_c.

The linear theory predicted that ρ and u would increase without limit as a critical level is approached, and that w would decrease to zero. From tables 5.1 it is seen that these predictions are partially true. w does decrease but does not go to zero, and the increase of ρ and u stops about 100m from the critical level, below which height these variables decrease in size.

In the 1000s following 3465s the fundamentals remain about the same size but the second harmonics roughly double. u_a in the expanded region increases in magnitude by approximately one third, but u_a above z_{et} changes sign. The importance of the sign of u_a will be discussed later when energy and momentum are considered.

TABLE 5.1A   FOURIER TRANSFORM OF RHO BY ROWS FOR CASE A   TIME STEP 64   TIME = 945.0
MAGNITUDE AND HPS IN MKS UNITS   ANGLE IN RADIANS
HPS IS THE HORIZONTAL PHASE SPEED CALCULATED FROM THE CHANGE IN THE PHASE ANGLE IN THE LAST 180.00 SECONDS

		CONSTANT	FUNDAMENTAL			SECOND HARMONIC			THIRD HARMONIC		
J	Z		MAGNITUDE	ANGLE	H P S	MAGNITUDE	ANGLE	H P S	MAGNITUDE	ANGLE	H P S
2	200.0	-0.000000	0.000020	0.896	10.00	0.000000	0.504	9.15	0.000000	0.0	0.0
3	600.0	-0.000000	0.000055	1.113	9.60	0.000000	0.643	9.45	0.000000	0.0	0.0
4	1000.0	-0.000000	0.000076	1.551	8.89	0.000000	0.732	10.00	0.000000	0.0	0.0
5	1400.0	-0.000000	0.000084	2.127	8.53	0.000000	1.116	9.23	0.000000	1.787	6.63
6	1800.0	-0.000000	0.000099	2.842	7.45	0.000000	1.589	8.19	0.000000	1.464	5.84
7	2012.5	-0.000000	0.000124	3.127	7.08	0.000001	1.495	7.99	0.000000	0.736	-0.32
8	2037.5	-0.000000	0.000129	3.137	7.30	0.000001	1.617	9.33	0.000000	0.0	7.35
9	2062.5	-0.000001	0.000134	-3.110	7.45	0.000001	1.713	9.56	0.000000	2.122	5.17
10	2087.5	-0.000001	0.000142	-3.043	7.44	0.000001	2.270	8.50	0.000000	2.310	5.84
11	2112.5	-0.000002	0.000158	-2.961	7.27	0.000001	2.581	6.90	0.000000	2.317	6.35
12	2137.5	-0.000004	0.000185	-2.893	7.03	0.000002	3.137	6.42	0.000000	2.260	6.56
13	2162.5	-0.000006	0.000225	-2.863	6.83	0.000003	3.076	6.35	0.000000	2.172	6.67
14	2187.5	-0.000009	0.000277	-2.877	6.71	0.000004	2.945	6.38	0.000000	2.076	6.71
15	2212.5	-0.000012	0.000338	-2.930	6.66	0.000006	2.787	6.44	0.000000	1.985	6.71
16	2237.5	-0.000016	0.000406	-3.010	6.67	0.000008	2.615	6.49	0.000000	1.907	6.68
17	2262.5	-0.000019	0.000475	-3.109	6.71	0.000010	2.431	6.55	0.000000	1.842	6.59
18	2287.5	-0.000021	0.000543	3.060	6.78	0.000013	2.233	6.62	0.000000	1.774	6.49
19	2312.5	-0.000022	0.000605	2.937	6.89	0.000016	2.016	6.71	0.000000	1.678	6.41
20	2337.5	-0.000022	0.000659	2.805	7.02	0.000019	1.779	6.84	0.000001	1.530	6.37
21	2362.5	-0.000021	0.000704	2.667	7.19	0.000022	1.523	7.01	0.000001	1.323	6.39
22	2387.5	-0.000018	0.000738	2.525	7.38	0.000026	1.250	7.22	0.000001	1.064	6.47
23	2412.5	-0.000014	0.000762	2.381	7.61	0.000029	0.966	7.45	0.000001	0.760	6.60
24	2437.5	-0.000010	0.000777	2.237	7.86	0.000032	0.677	7.69	0.000001	0.422	6.76
25	2462.5	-0.000005	0.000785	2.096	8.11	0.000035	0.388	7.94	0.000001	0.061	6.94
26	2487.5	-0.000001	0.000786	1.958	8.38	0.000037	0.103	8.18	0.000002	-0.317	7.13
27	2512.5	0.000003	0.000781	1.827	8.64	0.000039	-0.175	8.41	0.000002	-0.706	7.32
28	2537.5	0.000007	0.000771	1.703	8.90	0.000040	-0.444	8.64	0.000002	-1.102	7.50
29	2562.5	0.000009	0.000757	1.587	9.15	0.000040	-0.704	8.85	0.000002	-1.502	7.68
30	2587.5	0.000011	0.000739	1.479	9.39	0.000039	-0.951	9.04	0.000001	-1.913	-1.40
31	2612.5	0.000012	0.000719	1.378	9.63	0.000038	-1.183	9.22	0.000001	-2.348	8.06
32	2637.5	0.000012	0.000696	1.284	9.87	0.000037	-1.397	9.36	0.000001	-2.843	8.35
33	2662.5	0.000013	0.000672	1.196	10.11	0.000036	-1.588	9.46	0.000001	2.832	-0.44
34	2637.5	0.000012	0.000647	1.112	10.35	0.000035	-1.747	9.46	0.000001	2.173	0.11
35	2712.5	0.000012	0.000622	1.032	10.61	0.000034	-1.870	9.26	0.000001	1.571	0.50
36	2737.5	0.000011	0.000596	0.953	10.88	0.000034	-1.943	8.21	0.000001	0.975	0.64
37	2762.5	0.000012	0.000574	0.885	11.14	0.000038	-1.983	3.86	0.000008	1.627	7.34
38	2787.5	-0.000001	0.000544	0.788	11.55	0.000006	2.631	5.31	0.000007	-1.992	5.90
39	3000.0	-0.000007	0.000773	0.529	11.66	0.000051	-2.056	10.58	0.000001	-1.414	7.33
40	3400.0	-0.000009	0.001174	0.243	10.85	0.000084	-2.431	7.15	0.000000	0.284	3.09
41	3800.0	0.000007	0.001242	-0.131	9.78	0.000074	-2.849	7.13	0.000003	-0.156	3.65
42	4200.0	0.000006	0.001061	-0.842	10.26	0.000029	1.256	11.37	0.000003	0.084	5.32
43	4600.0	-0.000002	0.001035	-1.838	11.73	0.000070	1.761	10.77	0.000001	1.599	3.95
44	5000.0	-0.000003	0.001050	-2.822	12.02	0.000035	1.618	7.17	0.000002	1.985	4.66
45	5400.0	-0.000008	0.001130	2.559	12.13	0.000018	-0.740	-1.02	0.000001	1.894	7.06
46	5800.0	-0.000000	0.001212	1.871	11.41	0.000014	1.012	8.32	0.000001	-0.507	-0.53
47	6200.0	0.0	0.001340	1.515	11.23	0.000080	2.765	0.26	0.000001	-0.885	-1.28

TABLE 5.1B FOURIER TRANSFORM OF W BY ROWS FOR CASE A TIME STEP 64 TIME = 945.0
MAGNITUDE AND HPS IN MKS UNITS ANGLE IN RADIANS
HPS IS THE HORIZONTAL PHASE SPEED CALCULATED FROM THE CHANGE IN THE PHASE ANGLE IN THE LAST 180.00 SECONDS

J	Z	CONSTANT	FUNDAMENTAL MAGNITUDE	FUNDAMENTAL ANGLE	FUNDAMENTAL H P S	SECOND HARMONIC MAGNITUDE	SECOND HARMONIC ANGLE	SECOND HARMONIC H P S	THIRD HARMONIC MAGNITUDE	THIRD HARMONIC ANGLE	THIRD HARMONIC H P S
2	400.0	-0.000000	0.009629	-0.046	9.98	0.000007	-0.534	9.11	0.000000	-0.472	6.87
3	800.0	-0.000000	0.017143	0.209	9.46	0.000019	0.198	7.76	0.000000	-0.082	6.60
4	1200.0	0.000000	0.023563	0.809	7.89	0.000062	0.717	6.87	0.000001	1.388	5.01
5	1600.0	-0.000000	0.035981	1.469	6.21	0.000199	1.099	6.15	0.000003	1.632	5.10
6	2000.0	0.0	0.057427	1.985	5.02	0.000587	1.394	5.56	0.000006	1.114	5.92
7	2025.0	0.000000	0.059925	2.013	4.95	0.000635	1.407	5.53	0.000006	0.942	6.17
8	2050.0	-0.000000	0.061858	2.040	4.88	0.000679	1.414	5.51	0.000007	0.678	6.54
9	2075.0	0.0	0.063827	2.065	4.82	0.000721	1.409	5.50	0.000008	0.419	6.87
10	2100.0	0.000000	0.065744	2.089	4.75	0.000756	1.396	5.51	0.000009	0.177	7.18
11	2125.0	0.000000	0.067490	2.112	4.69	0.000777	1.380	5.52	0.000009	-0.073	7.51
12	2150.0	-0.000000	0.068950	2.137	4.62	0.000777	1.372	5.53	0.000010	-0.306	-1.42
13	2175.0	0.000000	0.070064	2.166	4.55	0.000754	1.389	5.49	0.000010	-0.549	-1.02
14	2200.0	-0.000000	0.070874	2.203	4.47	0.000711	1.457	5.37	0.000010	-0.812	-0.50
15	2225.0	0.000000	0.071574	2.251	4.36	0.000667	1.617	5.09	0.000009	-1.135	0.27
16	2250.0	0.000000	0.072543	2.314	4.24	0.000675	1.892	4.63	0.000008	-1.610	1.44
17	2275.0	0.000000	0.074320	2.389	4.12	0.000811	2.195	4.18	0.000008	-2.343	2.98
18	2300.0	0.000000	0.077518	2.473	4.00	0.001114	2.391	4.04	0.000012	-3.096	4.37
19	2325.0	0.000000	0.082672	2.558	3.94	0.001567	2.459	4.20	0.000019	2.672	5.24
20	2350.0	-0.000000	0.090086	2.633	3.96	0.002149	2.438	4.53	0.000032	2.310	5.77
21	2375.0	0.000000	0.099771	2.693	4.06	0.002839	2.363	4.90	0.000049	2.010	6.15
22	2400.0	0.000000	0.111491	2.733	4.23	0.003614	2.255	5.26	0.000071	1.735	6.45
23	2425.0	-0.000000	0.124879	2.755	4.45	0.004448	2.128	5.57	0.000099	1.472	6.69
24	2450.0	0.000000	0.139522	2.760	4.70	0.005307	1.989	5.85	0.000132	1.214	6.92
25	2475.0	-0.000000	0.155026	2.754	4.94	0.006152	1.845	6.10	0.000167	0.961	7.12
26	2500.0	-0.000000	0.171052	2.738	5.17	0.006944	1.701	6.32	0.000202	0.714	7.31
27	2525.0	0.000000	0.187316	2.715	5.38	0.007647	1.561	6.52	0.000234	0.474	7.48
28	2550.0	-0.000000	0.203605	2.689	5.57	0.008236	1.426	6.69	0.000259	0.242	7.65
29	2575.0	-0.000000	0.219759	2.660	5.74	0.008696	1.300	6.85	0.000273	0.020	-1.45
30	2600.0	0.000000	0.235673	2.629	5.89	0.009028	1.184	6.98	0.000275	-0.189	-1.30
31	2625.0	0.000000	0.251284	2.598	6.03	0.009247	1.079	7.10	0.000263	-0.383	-1.15
32	2650.0	-0.000000	0.266564	2.568	6.14	0.009379	0.987	7.19	0.000236	-0.563	-0.99
33	2675.0	-0.000000	0.281508	2.537	6.25	0.009457	0.907	7.28	0.000196	-0.723	-0.83
34	2700.0	-0.000000	0.296128	2.508	6.34	0.009518	0.838	7.34	0.000146	-0.858	-0.68
35	2725.0	0.000000	0.310444	2.480	6.42	0.009598	0.778	7.40	0.000091	-0.938	-0.56
36	2750.0	-0.000000	0.324478	2.452	6.50	0.009730	0.727	7.46	0.000037	-0.732	-0.83
37	2775.0	0.000000	0.338242	2.426	6.57	0.009930	0.678	7.51	0.000035	0.745	6.34
38	2800.0	-0.000000	0.355642	2.401	6.64	0.010332	0.630	7.58	0.000048	1.021	6.09
39	3200.0	0.000000	0.517606	2.061	7.45	0.008580	-0.368	9.29	0.000441	-2.551	3.73
40	3600.0	-0.000000	0.589395	1.687	8.19	0.022736	-2.100	-1.13	0.000950	-2.716	6.48
41	4000.0	0.000000	0.571148	1.214	8.80	0.044538	-2.553	5.17	0.000703	-2.553	6.09
42	4400.0	-0.000000	0.453634	0.598	9.28	0.026436	-2.722	6.13	0.001049	-0.321	3.37
43	4800.0	0.000000	0.308408	-0.446	11.11	0.025016	2.539	8.52	0.001484	0.308	5.04
44	5200.0	0.000000	0.336989	-1.706	13.27	0.035199	2.408	7.24	0.000521	1.209	5.07
45	5600.0	0.000000	0.442455	-2.623	12.82	0.019014	2.316	7.50	0.000396	-2.493	2.37
46	6000.0	0.0	0.560728	2.933	11.55	0.016136	2.583	7.27	0.000300	-2.751	2.08
47	6400.0	0.000000	0.587482	2.930	11.56	0.016918	2.589	7.27	0.000315	-2.752	2.08

TABLE 5.1C FOURIER TRANSFORM OF U BY ROWS FOR CASE A TIME STEP 64 TIME = 945.0
MAGNITUDE AND HPS IN MKS UNITS ANGLE IN RADIANS
HPS IS THE HORIZONTAL PHASE SPEED CALCULATED FROM THE CHANGE IN THE PHASE ANGLE IN THE LAST 180.00 SECONDS

J	Z	CONSTANT	FUNDAMENTAL MAGNITUDE	FUNDAMENTAL ANGLE	FUNDAMENTAL H P S	SECOND HARMONIC MAGNITUDE	SECOND HARMONIC ANGLE	SECOND HARMONIC H P S	THIRD HARMONIC MAGNITUDE	THIRD HARMONIC ANGLE	THIRD HARMONIC H P S
2	200.0	0.000008	0.019202	1.868	9.98	0.000008	1.822	9.11	0.000000	2.269	6.86
3	600.0	0.000036	0.016159	2.557	8.43	0.000015	2.902	7.14	0.000000	2.974	6.27
4	1000.0	0.000069	0.026671	-2.650	5.52	0.000050	-2.996	6.59	0.000000	-1.918	4.90
5	1400.0	0.000094	0.044614	-2.107	5.24	0.000151	-2.653	5.85	0.000002	-1.863	5.11
6	1800.0	0.000108	0.062561	-1.699	4.88	0.000430	-2.381	5.25	0.000003	-2.954	6.72
7	2012.5	0.000087	0.076753	-1.545	4.46	0.000720	-2.319	5.11	0.000015	2.403	-1.36
8	2037.5	0.000119	0.078157	-1.544	4.37	0.000745	-2.409	5.20	0.000025	2.349	5.72
9	2062.5	0.000260	0.077886	-1.541	4.33	0.000711	-2.596	5.40	0.000028	2.122	5.41
10	2087.5	0.000568	0.075494	-1.515	4.42	0.000602	-2.825	5.69	0.000029	1.359	5.67
11	2112.5	0.001065	0.071572	-1.433	4.74	0.000392	-3.131	6.20	0.000032	1.564	5.39
12	2137.5	0.001727	0.068859	-1.264	5.49	0.000113	1.894	8.96	0.000032	1.237	5.20
13	2162.5	0.002462	0.072641	-1.018	6.25	0.000502	0.122	-0.31	0.000033	0.822	5.20
14	2187.5	0.003123	0.087668	-0.790	6.40	0.001200	-0.195	3.49	0.000036	0.409	5.33
15	2212.5	0.003515	0.113963	-0.662	6.25	0.002111	-0.423	4.75	0.000044	-0.059	5.63
16	2237.5	0.003442	0.148414	-0.633	6.15	0.003246	-0.634	5.20	0.000057	-0.469	5.80
17	2262.5	0.002729	0.187871	-0.672	6.15	0.004615	-0.848	5.48	0.000078	-0.816	5.99
18	2287.5	0.001278	0.230022	-0.753	6.23	0.006233	-1.070	5.73	0.000107	-1.113	6.13
19	2312.5	-0.000918	0.273185	-0.861	6.38	0.008102	-1.303	5.96	0.000149	-1.407	6.32
20	2337.5	-0.003753	0.316177	-0.985	6.56	0.010209	-1.546	6.17	0.000205	-1.696	6.46
21	2362.5	-0.007027	0.357971	-1.117	6.76	0.012496	-1.794	6.42	0.000281	-1.995	6.64
22	2387.5	-0.010473	0.397576	-1.250	6.97	0.014852	-2.044	6.64	0.000372	-2.305	6.82
23	2412.5	-0.013799	0.434001	-1.382	7.16	0.017114	-2.295	6.85	0.000477	-2.618	6.97
24	2437.5	-0.016733	0.466159	-1.508	7.33	0.019092	-2.545	7.05	0.000588	-2.944	7.15
25	2462.5	-0.019066	0.493235	-1.628	7.47	0.020602	-2.793	7.25	0.000689	3.007	7.31
26	2487.5	-0.020669	0.514692	-1.739	7.60	0.021509	-3.039	7.46	0.000771	2.664	7.49
27	2512.5	-0.021510	0.530334	-1.843	7.70	0.021752	3.002	7.66	0.000822	2.307	7.67
28	2537.5	-0.021638	0.540502	-1.937	7.78	0.021344	2.765	7.88	0.000835	1.935	-1.37
29	2562.5	-0.021163	0.545758	-2.024	7.85	0.020366	2.535	8.10	0.000818	1.540	-1.13
30	2587.5	-0.020231	0.546924	-2.103	7.91	0.018943	2.316	8.33	0.000774	1.112	-0.80
31	2612.5	-0.018994	0.544886	-2.174	7.96	0.017218	2.115	8.55	0.000730	0.645	-0.41
32	2637.5	-0.017596	0.540542	-2.238	8.00	0.015329	1.939	8.77	0.000707	0.156	0.05
33	2662.5	-0.016152	0.534610	-2.296	8.05	0.013409	1.798	8.94	0.000718	-0.323	0.56
34	2687.5	-0.014755	0.527745	-2.348	8.09	0.011586	1.707	9.06	0.000752	-0.755	1.15
35	2712.5	-0.013471	0.520427	-2.396	8.14	0.010027	1.686	9.06	0.000767	-1.123	3.43
36	2737.5	-0.012364	0.512974	-2.439	8.20	0.009057	1.739	8.93	0.000743	-1.469	5.58
37	2762.5	-0.011444	0.505316	-2.479	8.26	0.008937	1.833	8.76	0.000658	-1.923	6.64
38	2787.5	-0.011105	0.497504	-2.517	8.35	0.009741	1.929	8.71	0.000226	-1.901	7.53
39	3000.0	-0.014692	0.418176	-2.884	9.16	0.010301	0.756	11.10	0.000402	0.240	5.77
40	3400.0	-0.025843	0.430346	2.481	10.42	0.028029	-0.110	9.52	0.000412	-0.119	7.16
41	3800.0	-0.033262	0.562064	1.681	9.78	0.027362	-0.621	7.47	0.000266	2.816	2.15
42	4200.0	-0.033214	0.694072	0.890	10.48	0.022786	-3.127	12.30	0.001331	2.797	4.59
43	4600.0	-0.031714	0.831088	0.113	11.18	0.027952	-2.539	3.70	0.000726	-2.398	4.86
44	5000.0	-0.036237	0.776662	-0.655	10.79	0.010589	-1.876	3.81	0.001072	-0.405	3.43
45	5400.0	-0.043660	0.720075	-1.549	11.28	0.019490	1.720	0.19	0.000725	0.500	4.84
46	5800.0	-0.046872	0.748845	-2.351	11.23	0.006673	0.751	8.59	0.000097	-2.938	2.30
47	6200.0	-0.046870	0.003005	3.021	9.97	0.000001	0.822	6.74	0.000000	-0.891	3.09

TABLE 5.1D FOURIER TRANSFORM OF RHO BY ROWS FOR CASE A TIME STEP 136 TIME = 2025.0
MAGNITUDE AND HPS IN MKS UNITS ANGLE IN RADIANS
HPS IS THE HORIZONTAL PHASE SPEED CALCULATED FROM THE CHANGE IN THE PHASE ANGLE IN THE LAST 180.00 SECONDS

		CONSTANT	FUNDAMENTAL			SECOND HARMONIC			THIRD HARMONIC		
J	Z		MAGNITUDE	ANGLE	H P S	MAGNITUDE	ANGLE	H P S	MAGNITUDE	ANGLE	H P S
2	200.0	-0.000000	0.000050	0.485	11.41	0.000000	-2.910	-0.45	0.000000	0.0	0.0
3	600.0	-0.000000	0.000144	0.506	11.46	0.000001	-2.164	-1.08	0.000000	2.041	6.25
4	1000.0	-0.000000	0.000220	0.599	10.98	0.000001	-1.541	-1.73	0.000000	2.004	6.45
5	1400.0	-0.000000	0.000265	0.672	10.49	0.000002	-1.563	-2.06	0.000000	0.730	-0.34
6	1800.0	-0.000000	0.000260	0.652	10.50	0.000003	-1.340	11.55	0.000001	0.948	0.16
7	2012.5	-0.000000	0.000250	0.621	10.56	0.000005	-1.073	10.93	0.000001	0.943	-0.03
8	2037.5	-0.000002	0.000268	0.595	10.67	0.000004	-1.465	11.57	0.000001	1.334	0.56
9	2062.5	-0.000001	0.000276	0.562	10.90	0.000003	-1.453	11.27	0.000001	1.291	0.56
10	2087.5	-0.000002	0.000278	0.570	10.93	0.000005	-1.367	10.65	0.000001	1.227	0.49
11	2112.5	-0.000004	0.000293	0.607	10.70	0.000008	-1.553	10.84	0.000001	1.062	0.68
12	2137.5	-0.000005	0.000324	0.592	10.61	0.000008	-1.843	-1.70	0.000001	0.901	1.65
13	2162.5	0.000000	0.000345	0.513	10.97	0.000005	-1.984	-0.80	0.000001	0.943	2.60
14	2187.5	0.000005	0.000328	0.464	11.56	0.000007	-0.681	10.14	0.000001	1.217	2.11
15	2212.5	-0.000003	0.000295	0.616	11.31	0.000019	-0.886	9.80	0.000002	1.019	2.00
16	2237.5	-0.000031	0.000353	0.909	9.92	0.000030	-1.323	9.64	0.000003	0.605	2.16
17	2262.5	-0.000062	0.000524	0.919	9.22	0.000032	-1.658	9.32	0.000004	0.224	2.29
18	2287.5	-0.000062	0.000711	0.695	9.21	0.000027	-2.557	8.90	0.000005	-0.062	1.65
19	2312.5	-0.000015	0.000820	0.382	9.43	0.000018	2.863	5.25	0.000005	-0.224	-0.51
20	2337.5	0.000054	0.000786	0.054	9.93	0.000007	1.610	8.73	0.000005	-0.307	7.78
21	2362.5	0.000080	0.000581	-0.193	11.73	0.000010	-0.317	11.48	0.000004	-0.399	7.04
22	2387.5	0.000007	0.000293	0.154	14.69	0.000024	-0.302	10.23	0.000002	0.485	4.78
23	2412.5	-0.000166	0.000551	1.151	10.95	0.000061	-0.485	9.45	0.000007	1.341	2.31
24	2437.5	-0.000375	0.001154	1.086	10.55	0.000116	-1.039	9.37	0.000013	1.081	1.58
25	2462.5	-0.000523	0.001729	0.838	10.66	0.000165	-1.706	9.36	0.000016	0.987	1.21
26	2487.5	-0.000529	0.002140	0.541	10.82	0.000260	-2.344	9.23	0.000019	1.100	0.89
27	2512.5	-0.000379	0.002360	0.217	11.01	0.000315	-2.910	9.18	0.000026	0.906	0.86
28	2537.5	-0.000146	0.002452	-0.110	11.21	0.000326	2.858	9.34	0.000033	0.412	1.10
29	2562.5	0.000062	0.002463	-0.401	11.39	0.000299	2.357	9.66	0.000037	-0.092	1.35
30	2587.5	0.000180	0.002403	-0.630	11.52	0.000279	1.847	9.90	0.000037	-0.526	1.58
31	2612.5	0.000210	0.002263	-0.802	11.61	0.000278	1.387	9.89	0.000036	-0.793	1.57
32	2637.5	0.000211	0.002119	-0.927	11.66	0.000314	0.957	9.66	0.000033	-1.265	1.71
33	2662.5	0.000180	0.001984	-1.050	11.78	0.000312	0.481	10.02	0.000021	-1.789	1.75
34	2687.5	0.000221	0.001920	-1.102	11.65	0.000293	0.258	9.82	0.000037	-2.856	1.85
35	2712.5	0.000164	0.001880	-1.235	11.77	0.000248	-0.551	10.45	0.000034	1.758	0.73
36	2737.5	0.000166	0.001836	-1.283	11.89	0.000081	0.502	7.63	0.000047	2.339	1.25
37	2762.5	0.000145	0.001981	-1.431	12.19	0.000232	-1.533	11.18	0.000109	1.465	1.20
38	2787.5	-0.000030	0.001531	-1.473	12.21	0.000325	2.540	-0.92	0.000057	1.022	-1.36
39	3000.0	0.000013	0.001444	-1.688	11.63	0.000091	-0.691	8.79	0.000014	-1.160	2.08
40	3400.0	0.000014	0.001434	-1.925	11.16	0.000167	-0.613	8.31	0.000006	-0.406	3.34
41	3800.0	0.000018	0.000773	-2.158	9.84	0.000153	-0.740	8.26	0.000011	0.209	5.40
42	4200.0	-0.000002	0.000411	2.181	11.91	0.000119	-0.414	6.94	0.000006	0.134	7.19
43	4600.0	-0.000009	0.001052	1.364	12.58	0.000054	0.595	4.02	0.000001	1.164	5.75
44	5000.0	0.000005	0.001045	0.982	12.55	0.000046	0.267	4.27	0.000002	2.758	3.20
45	5400.0	0.000003	0.000673	0.166	13.12	0.000024	-1.173	7.60	0.000002	3.057	5.98
46	5800.0	-0.000000	0.000861	-1.184	10.14	0.000028	-0.593	3.81	0.000004	-2.631	5.71
47	6200.0	-0.000000	0.001200	-1.490	10.06	0.000197	-1.451	-0.30	0.000004	-2.651	5.10

TABLE 5.1E   FOURIER TRANSFORM OF W BY ROWS FOR CASE A   TIME STEP 136   TIME = 2025.0
MAGNITUDE AND HPS IN MKS UNITS   ANGLE IN RADIANS
HPS IS THE HORIZONTAL PHASE SPEED CALCULATED FROM THE CHANGE IN THE PHASE ANGLE IN THE LAST 180.00 SECONDS

		CONSTANT	FUNDAMENTAL			SECOND HARMONIC			THIRD HARMONIC		
J	Z		MAGNITUDE	ANGLE	H P S	MAGNITUDE	ANGLE	H P S	MAGNITUDE	ANGLE	H P S
2	400.0	0.000000	0.033260	-1.031	11.19	0.000119	-2.441	10.77	0.000003	-1.122	1.02
3	800.0	0.000000	0.064462	-0.980	10.93	0.000186	-2.707	10.83	0.000015	-0.919	1.16
4	1200.0	0.000000	0.089530	-0.942	10.74	0.000573	-2.766	11.31	0.000053	-0.849	1.37
5	1600.0	0.000000	0.101775	-0.940	10.72	0.001387	-2.583	10.82	0.000178	-0.843	1.36
6	2000.0	-0.000000	0.094772	-0.993	11.00	0.003233	-2.383	10.24	0.000804	-0.871	1.17
7	2025.0	-0.000000	0.094696	-0.999	11.03	0.003483	-2.374	10.23	0.000898	-0.871	1.16
8	2050.0	-0.000000	0.093434	-1.005	11.06	0.003722	-2.367	10.22	0.000995	-0.868	1.14
9	2075.0	-0.000000	0.092151	-1.011	11.09	0.003949	-2.358	10.22	0.001100	-0.865	1.12
10	2100.0	0.000000	0.090803	-1.019	11.13	0.004134	-2.338	10.20	0.001209	-0.861	1.11
11	2125.0	-0.000000	0.089152	-1.028	11.19	0.004339	-2.307	10.15	0.001326	-0.854	1.09
12	2150.0	0.000000	0.087103	-1.037	11.24	0.004619	-2.285	10.09	0.001461	-0.845	1.05
13	2175.0	0.000000	0.084986	-1.045	11.28	0.004868	-2.287	10.08	0.001614	-0.842	1.01
14	2200.0	0.000000	0.083114	-1.057	11.33	0.004903	-2.286	10.12	0.001761	-0.847	0.98
15	2225.0	0.000000	0.080901	-1.087	11.45	0.004772	-2.223	10.10	0.001882	-0.846	0.97
16	2250.0	-0.000000	0.076765	-1.133	11.70	0.004929	-2.085	9.94	0.001986	-0.826	0.97
17	2275.0	-0.000000	0.069349	-1.180	12.05	0.005748	-1.977	9.75	0.002117	-0.786	0.98
18	2300.0	-0.000000	0.059176	-1.188	12.35	0.006941	-1.975	9.65	0.002307	-0.743	1.00
19	2325.0	-0.000000	0.049385	-1.115	12.30	0.007973	-2.030	9.57	0.002566	-0.713	1.01
20	2350.0	0.000000	0.043721	-0.979	11.84	0.008577	-2.086	9.42	0.002896	-0.704	0.97
21	2375.0	-0.000000	0.041205	-0.903	11.46	0.008682	-2.123	9.27	0.003277	-0.735	0.90
22	2400.0	-0.000000	0.035788	-0.963	11.57	0.008050	-2.107	9.26	0.003633	-0.806	0.86
23	2425.0	-0.000000	0.022518	-1.184	12.44	0.007036	-1.852	9.42	0.003886	-0.895	0.86
24	2450.0	0.0	0.005253	-2.902	20.31	0.009253	-1.337	9.18	0.004021	-0.976	0.91
25	2475.0	0.000000	0.032386	1.948	0.33	0.016623	-1.199	9.04	0.004046	-1.035	1.03
26	2500.0	-0.000000	0.068814	1.646	4.38	0.025902	-1.333	9.13	0.003921	-1.034	1.19
27	2525.0	0.000000	0.108615	1.414	6.86	0.033970	-1.562	9.32	0.003779	-0.891	1.27
28	2550.0	-0.000000	0.149293	1.220	8.05	0.038970	-1.833	9.57	0.004224	-0.626	1.15
29	2575.0	-0.000000	0.188598	1.059	8.70	0.040856	-2.138	9.88	0.005605	-0.456	1.05
30	2600.0	-0.000000	0.224946	0.928	9.10	0.040816	-2.477	10.23	0.007507	-0.440	1.05
31	2625.0	-0.000000	0.257902	0.819	9.36	0.040166	-2.845	10.59	0.009575	-0.519	1.08
32	2650.0	-0.000000	0.287894	0.728	9.54	0.039376	3.056	10.95	0.011659	-0.657	1.12
33	2675.0	0.000000	0.315523	0.649	9.67	0.038024	2.680	11.29	0.013640	-0.820	1.15
34	2700.0	0.000000	0.341221	0.579	9.77	0.035582	2.301	11.62	0.014997	-0.985	1.17
35	2725.0	0.0	0.364767	0.517	9.85	0.032619	1.928	-1.96	0.015479	-1.126	1.19
36	2750.0	0.000000	0.385956	0.462	9.91	0.030347	1.562	-1.69	0.014975	-1.218	1.22
37	2775.0	-0.000000	0.404073	0.411	9.96	0.029270	1.262	-1.62	0.013830	-1.224	1.29
38	2800.0	0.000000	0.424415	0.366	10.00	0.030426	1.070	-1.77	0.013001	-1.141	1.36
39	3200.0	0.000000	0.577294	0.029	9.98	0.045434	0.641	8.45	0.008715	-0.133	3.14
40	3600.0	-0.000000	0.466454	-0.089	9.42	0.059430	0.612	8.40	0.003167	-0.065	7.32
41	4000.0	0.000000	0.108947	-0.188	7.99	0.062902	0.286	9.14	0.000607	-3.008	3.12
42	4400.0	0.000000	0.306990	2.791	16.03	0.031984	-0.054	5.86	0.001649	2.977	5.48
43	4800.0	0.000000	0.505749	2.576	13.11	0.021609	-1.112	6.38	0.000602	-2.654	6.28
44	5200.0	0.000000	0.407618	2.115	12.40	0.035783	-1.461	6.95	0.000923	2.441	-0.50
45	5600.0	0.000000	0.337301	0.870	10.58	0.028200	-1.420	6.79	0.001129	2.458	-0.55
46	6000.0	-0.000000	0.598526	0.018	10.39	0.029882	-1.745	8.40	0.000260	2.482	-0.77
47	6400.0	-0.000000	0.629068	0.013	10.39	0.031338	-1.744	8.40	0.000273	2.482	-0.77

TABLE 5.1F FOURIER TRANSFORM OF U BY ROWS FOR CASE A TIME STEP 136 TIME = 2025.0
MAGNITUDE AND HPS IN MKS UNITS ANGLE IN RADIANS
HPS IS THE HORIZONTAL PHASE SPEED CALCULATED FROM THE CHANGE IN THE PHASE ANGLE IN THE LAST 180.00 SECONDS

		CONSTANT	FUNDAMENTAL			SECOND HARMONIC			THIRD HARMONIC		
J	Z		MAGNITUDE	ANGLE	H P S	MAGNITUDE	ANGLE	H P S	MAGNITUDE	ANGLE	H P S
2	200.0	0.000036	0.066329	0.933	11.19	0.000128	-0.065	10.77	0.000003	1.613	1.04
3	600.0	0.000175	0.059199	1.042	10.64	0.000079	-0.814	11.62	0.000009	2.020	1.36
4	1000.0	0.000430	0.044154	1.138	10.16	0.000409	-0.440	11.62	0.000031	1.887	1.52
5	1400.0	0.000693	0.015760	1.046	10.59	0.000867	-0.090	10.45	0.000101	1.910	1.36
6	1800.0	0.000994	0.024210	-1.689	6.38	0.002014	0.129	9.81	0.000517	1.869	1.12
7	2012.5	0.001024	0.050175	-1.790	8.17	0.003629	0.126	9.94	0.001124	1.877	1.04
8	2037.5	0.000795	0.053239	-1.847	8.61	0.004065	0.097	10.12	0.001268	1.907	0.96
9	2062.5	0.000785	0.053923	-1.849	9.11	0.003853	0.150	10.26	0.001379	1.917	0.94
10	2087.5	0.001701	0.057892	-1.774	8.80	0.003377	0.445	9.75	0.001427	1.929	0.97
11	2112.5	0.002602	0.069013	-1.769	8.34	0.004130	0.634	9.08	0.001538	1.975	0.89
12	2137.5	0.001721	0.079753	-1.881	8.53	0.005014	0.408	9.13	0.001784	1.991	0.72
13	2162.5	-0.000189	0.080176	-1.960	9.54	0.004161	0.033	9.60	0.002004	1.930	0.61
14	2187.5	0.001447	0.077537	-1.763	9.70	0.000387	0.403	10.86	0.001932	1.849	0.66
15	2212.5	0.010120	0.111867	-1.479	8.23	0.005924	2.121	7.59	0.001557	1.913	0.84
16	2237.5	0.020701	0.187020	-1.585	7.61	0.012116	1.564	8.23	0.001440	2.297	0.85
17	2262.5	0.019406	0.272924	-1.915	7.49	0.017425	0.945	8.80	0.002013	2.524	0.84
18	2287.5	-0.003785	0.338931	-2.313	7.35	0.020754	0.394	9.05	0.002797	2.457	0.92
19	2312.5	-0.039890	0.350072	-2.711	7.04	0.019277	-0.034	8.75	0.003557	2.311	0.89
20	2337.5	-0.060848	0.279641	-3.053	6.33	0.013074	-0.373	7.81	0.004358	2.110	0.74
21	2362.5	-0.041400	0.136358	-3.005	22.37	0.005761	-1.072	7.26	0.005185	1.782	0.60
22	2387.5	0.013307	0.195987	-1.712	12.17	0.011854	-3.101	9.58	0.005752	1.460	0.37
23	2412.5	0.061403	0.481839	-1.801	10.76	0.034462	2.440	9.29	0.005577	0.954	-0.21
24	2437.5	0.049691	0.780362	-2.144	10.55	0.082369	1.660	9.02	0.004647	0.610	-1.05
25	2462.5	-0.047386	1.046608	-2.535	10.54	0.133053	1.327	8.98	0.003224	0.226	7.44
26	2487.5	-0.205719	1.272478	-2.924	10.60	0.170128	0.791	9.12	0.001853	-1.473	-0.42
27	2512.5	-0.364713	1.447629	3.012	10.70	0.185032	0.198	9.47	0.007702	-2.654	0.55
28	2537.5	-0.466438	1.543504	2.723	10.79	0.194571	-0.456	9.98	0.015427	-3.108	0.68
29	2562.5	-0.489382	1.542908	2.492	10.86	0.216486	-1.057	10.39	0.021616	2.759	0.88
30	2587.5	-0.451108	1.465332	2.305	10.91	0.243124	-1.534	10.59	0.025516	2.356	1.06
31	2612.5	-0.388853	1.355325	2.154	10.93	0.262382	-1.926	10.65	0.029088	1.949	1.19
32	2637.5	-0.330960	1.252007	2.030	10.93	0.267751	-2.309	10.65	0.034024	1.542	1.26
33	2662.5	-0.286510	1.167667	1.925	10.92	0.257080	-2.729	10.67	0.038340	1.192	1.24
34	2687.5	-0.245562	1.100631	1.830	10.91	0.243870	3.097	10.78	0.036425	0.784	1.17
35	2712.5	-0.199892	1.025972	1.739	10.88	0.229621	2.666	10.87	0.029568	0.323	1.18
36	2737.5	-0.158157	0.946151	1.647	10.88	0.207047	2.331	10.89	0.020359	-0.366	0.89
37	2762.5	-0.115170	0.852140	1.536	10.96	0.158507	2.069	10.83	0.016120	-1.533	0.62
38	2787.5	-0.083856	0.771575	1.440	11.05	0.101772	2.079	10.51	0.020322	-2.408	1.64
39	3000.0	-0.083024	0.434014	1.262	10.24	0.022960	2.333	9.29	0.009503	-2.229	1.96
40	3400.0	-0.072471	0.304007	-0.779	12.05	0.012836	2.651	6.02	0.004942	-0.561	3.18
41	3800.0	-0.049458	0.759448	-1.239	11.05	0.021894	1.263	5.19	0.003236	-0.427	3.70
42	4200.0	-0.043419	0.837818	-1.485	11.07	0.040325	-0.209	9.45	0.000873	-0.734	7.21
43	4600.0	-0.065705	0.405140	-2.088	11.39	0.032035	-0.152	7.85	0.001076	2.301	3.61
44	5000.0	-0.068386	0.489377	2.230	13.23	0.017731	0.404	1.67	0.000746	-1.798	3.10
45	5400.0	-0.053705	0.901079	1.722	11.68	0.010162	-2.370	0.04	0.000134	-0.975	4.04
46	5800.0	-0.048761	0.901783	1.352	11.23	0.010384	-0.765	1.88	0.000762	2.058	0.31
47	6200.0	-0.048762	0.006427	0.894	11.33	0.003016	1.732	10.95	0.000000	-2.639	1.96

TABLE 5.1G FOURIER TRANSFORM OF RHO BY ROWS FOR CASE A TIME STEP 232 TIME = 3465.0
MAGNITUDE AND HPS IN MKS UNITS ANGLE IN RADIANS
HPS IS THE HORIZONTAL PHASE SPEED CALCULATED FROM THE CHANGE IN THE PHASE ANGLE IN THE LAST 180.00 SECONDS

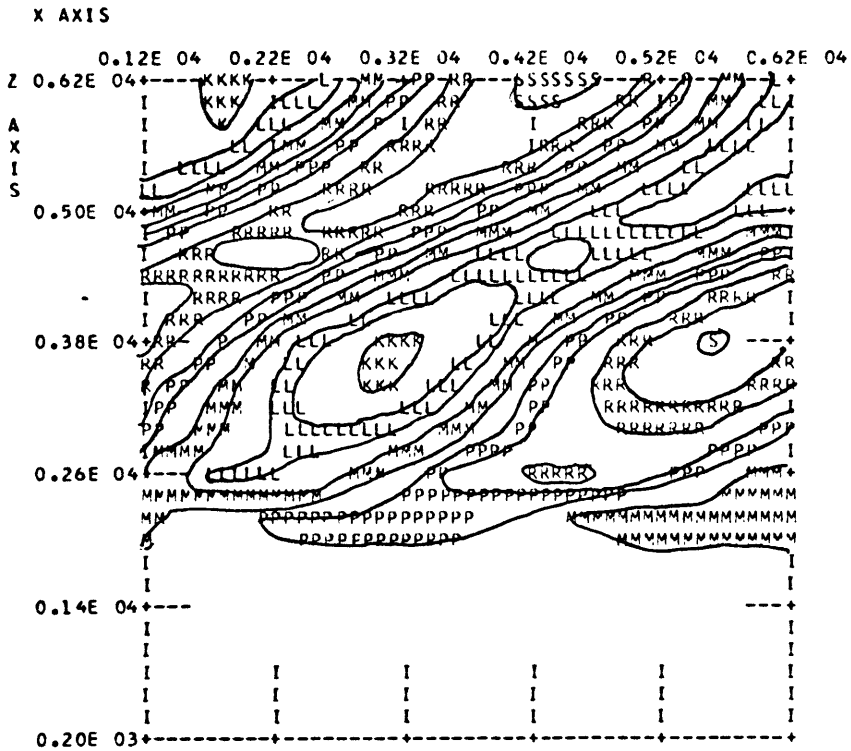
J	Z	CONSTANT	FUNDAMENTAL MAGNITUDE	FUNDAMENTAL ANGLE	FUNDAMENTAL H P S	SECOND HARMONIC MAGNITUDE	SECOND HARMONIC ANGLE	SECOND HARMONIC H P S	THIRD HARMONIC MAGNITUDE	THIRD HARMONIC ANGLE	THIRD HARMONIC H P S
2	200.0	-0.000000	0.000031	-2.428	13.54	0.000001	-1.712	6.51	0.000000	-3.136	7.59
3	600.0	-0.000000	0.000093	-2.317	12.52	0.000002	-2.413	9.61	0.000000	1.149	-0.60
4	1000.0	-0.000000	0.000160	-2.137	11.06	0.000004	-2.619	10.34	0.000001	0.939	6.78
5	1400.0	-0.000000	0.000216	-2.008	10.37	0.000010	-2.779	10.90	0.000001	0.626	6.12
6	1800.0	-0.000000	0.000239	-1.896	10.35	0.000021	-2.755	10.97	0.000002	0.065	6.62
7	2012.5	-0.000001	0.000246	-1.821	10.41	0.000030	-2.775	10.83	0.000004	-0.030	6.53
8	2037.5	-0.000003	0.000268	-1.796	10.09	0.000034	-2.776	11.00	0.000005	-0.248	7.06
9	2062.5	-0.000002	0.000285	-1.843	10.44	0.000038	-2.753	11.03	0.000006	-0.154	7.45
10	2087.5	-0.000004	0.000294	-1.814	10.38	0.000041	-2.739	10.98	0.000009	0.005	-1.45
11	2112.5	-0.000004	0.000325	-1.831	10.25	0.000048	-2.713	10.87	0.000010	0.254	-0.89
12	2137.5	-0.000003	0.000331	-1.874	10.79	0.000053	-2.743	11.13	0.000010	0.525	-0.10
13	2162.5	-0.000008	0.000357	-1.806	10.32	0.000057	-2.680	10.97	0.000007	0.820	1.14
14	2187.5	0.000004	0.000398	-1.937	10.73	0.000070	-2.813	11.01	0.000006	1.507	1.17
15	2212.5	-0.000010	0.000335	-1.868	11.93	0.000063	-2.905	11.98	0.000003	0.417	3.35
16	2237.5	-0.000064	0.000462	-1.519	9.42	0.000057	-2.384	11.05	0.000011	-1.341	6.91
17	2262.5	-0.000024	0.000684	-1.898	8.96	0.000102	-2.459	10.05	0.000014	-1.909	-1.41
18	2287.5	0.000059	0.000616	-2.330	9.70	0.000126	-2.942	10.33	0.000009	-0.242	5.53
19	2312.5	-0.000000	0.000374	-2.046	14.43	0.000092	-3.072	12.13	0.000032	-0.331	6.60
20	2337.5	-0.000024	0.000592	-1.899	11.60	0.000114	-2.739	11.11	0.000052	-0.524	6.96
21	2362.5	0.000037	0.000526	-2.125	14.11	0.000161	-3.080	10.67	0.000080	-0.262	6.36
22	2387.5	-0.000162	0.000445	-0.951	11.98	0.000178	3.115	9.44	0.000102	-0.310	6.14
23	2412.5	-0.000407	0.000915	-1.236	10.67	0.000100	-2.272	-0.74	0.000079	-0.816	6.82
24	2437.5	-0.000122	0.001362	-2.087	10.90	0.000385	-1.379	8.76	0.000048	-1.065	-1.23
25	2462.5	0.000353	0.001426	-2.440	10.64	0.000475	-2.606	8.95	0.000043	-1.440	5.54
26	2487.5	-0.000200	0.001206	-1.503	12.08	0.000626	2.530	10.37	0.000105	-0.891	3.83
27	2512.5	-0.001436	0.002040	-1.163	11.60	0.000028	-1.641	10.77	0.000012	-0.814	2.64
28	2537.5	-0.001015	0.001819	-1.828	11.87	0.000633	-1.321	8.99	0.000064	-2.692	2.75
29	2562.5	0.000228	0.002294	-2.377	11.57	0.000369	-2.613	7.11	0.000163	-2.136	-1.40
30	2587.5	0.000382	0.002473	-2.393	11.07	0.000276	2.727	1.56	0.000034	-2.294	-1.05
31	2612.5	0.000380	0.002457	-2.460	11.42	0.000227	-2.193	-2.03	0.000079	2.051	4.39
32	2637.5	0.000467	0.002327	-2.566	11.39	0.000364	-2.141	11.06	0.000039	-1.642	1.43
33	2662.5	0.000431	0.002268	-2.636	11.22	0.000262	-2.558	12.79	0.000042	-1.936	0.57
34	2687.5	0.000353	0.001931	-2.665	10.66	0.000170	-2.808	-0.04	0.000065	-2.695	2.51
35	2712.5	0.000229	0.001617	-2.729	10.67	0.000048	-2.237	-0.91	0.000039	-1.154	3.33
36	2737.5	0.000076	0.001559	-2.765	10.59	0.000136	2.122	1.72	0.000067	-2.051	3.89
37	2762.5	0.000076	0.001693	-2.795	10.63	0.000253	0.983	7.59	0.000145	3.085	4.63
38	2787.5	0.000020	0.001255	-2.673	10.60	0.000458	0.383	11.31	0.000010	-2.862	7.29
39	3000.0	0.000004	0.001314	3.043	11.13	0.000101	0.396	7.52	0.000032	-0.163	2.96
40	3400.0	0.000002	0.001434	2.546	11.92	0.000125	-0.248	8.20	0.000025	0.208	5.17
41	3800.0	0.000009	0.000999	1.782	10.85	0.000127	-0.495	7.58	0.000024	1.048	6.26
42	4200.0	-0.000011	0.001019	0.688	8.94	0.000167	-0.421	0.98	0.000014	-2.092	1.42
43	4600.0	-0.000012	0.001471	0.112	9.94	0.000157	-2.674	7.61	0.000028	-1.927	6.36
44	5000.0	0.000018	0.001358	-0.238	10.19	0.000273	-2.801	6.72	0.000011	-1.835	5.91
45	5400.0	0.000020	0.000760	-0.690	10.41	0.000200	3.048	7.34	0.000006	-0.390	5.70
46	5800.0	-0.000000	0.000443	-2.600	12.86	0.000165	2.669	8.70	0.000002	0.732	4.54
47	6200.0	0.000000	0.000844	-2.967	11.64	0.000165	0.367	-0.72	0.000002	0.568	4.33

TABLE 5.1H FOURIER TRANSFORM OF W BY ROWS FOR CASE A TIME STEP 232 TIME = 3465.0
MAGNITUDE AND HPS IN MKS UNITS ANGLE IN RADIANS
HPS IS THE HORIZONTAL PHASE SPEED CALCULATED FROM THE CHANGE IN THE PHASE ANGLE IN THE LAST 180.00 SECONDS

J	Z	CONSTANT	FUNDAMENTAL MAGNITUDE	FUNDAMENTAL ANGLE	H P S	SECOND HARMONIC MAGNITUDE	SECOND HARMONIC ANGLE	H P S	THIRD HARMONIC MAGNITUDE	THIRD HARMONIC ANGLE	H P S
2	400.0	0.000000	0.022029	2.478	11.78	0.000578	1.750	-1.71	0.000051	1.013	1.79
3	800.0	0.0	0.045959	2.563	11.34	0.001443	1.950	11.21	0.000072	-0.935	7.46
4	1200.0	0.000000	0.068817	2.678	10.89	0.003720	1.953	11.15	0.000266	-1.689	7.58
5	1600.0	-0.000000	0.086449	2.827	10.50	0.009993	1.951	11.33	0.000803	-1.688	5.32
6	2000.0	-0.000000	0.092523	3.053	10.14	0.025434	1.930	11.44	0.002812	-1.719	4.05
7	2025.0	-0.000000	0.093643	3.072	10.11	0.027271	1.930	11.44	0.003098	-1.722	4.01
8	2050.0	-0.000000	0.093614	3.091	10.08	0.029863	1.929	11.45	0.003371	-1.728	4.01
9	2075.0	0.000000	0.093585	3.112	10.05	0.030549	1.929	11.44	0.003652	-1.742	4.09
10	2100.0	0.000000	0.093475	3.134	10.01	0.032340	1.927	11.45	0.003975	-1.763	4.22
11	2125.0	0.000000	0.093250	-3.125	9.97	0.034192	1.926	11.45	0.004395	-1.786	4.38
12	2150.0	-0.000000	0.092985	-3.100	9.93	0.036195	1.924	11.44	0.004930	-1.795	4.47
13	2175.0	-0.000000	0.092306	-3.073	9.90	0.038265	1.922	11.45	0.005589	-1.786	4.38
14	2200.0	-0.000000	0.091704	-3.038	9.83	0.040438	1.924	11.44	0.006257	-1.766	4.01
15	2225.0	-0.000000	0.090779	-3.008	9.79	0.043068	1.918	11.43	0.006818	-1.722	3.23
16	2250.0	0.000000	0.088137	-2.959	9.78	0.045542	1.902	11.49	0.007518	-1.672	2.52
17	2275.0	-0.000000	0.087578	-2.862	9.56	0.047116	1.904	11.60	0.008245	-1.686	2.33
18	2300.0	-0.000000	0.091160	-2.787	9.22	0.049253	1.926	11.58	0.008445	-1.752	2.66
19	2325.0	-0.000000	0.092959	-2.746	8.99	0.052479	1.935	11.50	0.008245	-1.795	2.88
20	2350.0	0.000000	0.093908	-2.681	8.76	0.055049	1.935	11.56	0.007912	-1.847	2.90
21	2375.0	0.000000	0.096960	-2.649	8.52	0.057596	1.949	11.63	0.007390	-1.960	3.03
22	2400.0	0.000000	0.096025	-2.653	8.33	0.062180	1.942	11.51	0.007482	-2.002	3.18
23	2425.0	0.000000	0.092140	-2.555	8.05	0.068398	1.870	11.46	0.008648	-1.972	3.24
24	2450.0	-0.000000	0.093376	-2.389	7.72	0.070429	1.829	-2.06	0.010141	-1.996	3.33
25	2475.0	0.000000	0.089908	-2.303	7.17	0.072553	1.925	-2.19	0.010844	-2.017	3.53
26	2500.0	0.000000	0.078944	-2.165	5.91	0.085155	1.908	11.19	0.011496	-2.007	3.83
27	2525.0	-0.000000	0.089010	-1.795	5.20	0.097336	1.703	11.13	0.012819	-1.894	3.57
28	2550.0	0.000000	0.135772	-1.512	11.19	0.093868	1.548	11.37	0.015268	-1.614	2.93
29	2575.0	-0.000000	0.194778	-1.387	11.59	0.077634	1.527	10.95	0.019237	-1.465	2.41
30	2600.0	0.000000	0.248028	-1.336	11.48	0.062070	1.487	8.29	0.023045	-1.517	2.32
31	2625.0	-0.000000	0.293402	-1.328	11.41	0.047923	1.253	7.24	0.026818	-1.504	2.40
32	2650.0	-0.000000	0.332056	-1.337	11.29	0.036714	0.942	7.36	0.031985	-1.397	2.37
33	2675.0	0.000000	0.365143	-1.353	11.12	0.032148	0.862	7.30	0.039219	-1.347	2.30
34	2700.0	-0.000000	0.395325	-1.374	10.98	0.035867	1.098	6.76	0.045415	-1.354	2.36
35	2725.0	0.0	0.421914	-1.400	10.89	0.045471	1.357	6.43	0.049753	-1.378	2.48
36	2750.0	0.0	0.441505	-1.426	10.81	0.058941	1.528	6.51	0.050507	-1.379	2.63
37	2775.0	0.0	0.456587	-1.457	10.77	0.067144	1.648	6.30	0.046426	-1.380	2.72
38	2800.0	-0.000000	0.476672	-1.497	10.75	0.067673	1.700	6.04	0.041598	-1.427	2.68
39	3200.0	-0.000000	0.584843	-1.859	10.36	0.079499	1.409	7.63	0.014264	-2.934	3.03
40	3600.0	0.000000	0.428839	-2.227	9.79	0.082634	1.266	8.10	0.008186	-2.962	2.94
41	4000.0	-0.000000	0.245210	2.844	10.80	0.060905	1.128	9.22	0.010616	-2.123	1.00
42	4400.0	0.000000	0.458783	1.753	11.49	0.035994	1.793	4.27	0.006126	-1.910	5.47
43	4800.0	-0.000000	0.587039	1.411	10.86	0.040272	-1.937	10.83	0.003146	0.238	1.63
44	5200.0	0.0	0.467129	1.111	10.44	0.062902	-1.435	8.28	0.005243	0.830	2.87
45	5600.0	-0.000000	0.191029	0.364	10.26	0.066952	-1.842	8.13	0.004020	1.586	2.79
46	6000.0	0.0	0.326986	-1.482	12.24	0.088253	-1.948	8.75	0.001934	1.802	3.08
47	6400.0	-0.000000	0.342752	-1.491	12.25	0.092549	-1.948	8.75	0.002028	1.802	3.08

TABLE 5.11 FOURIER TRANSFORM OF U BY ROWS FOR CASE A TIME STEP 232 TIME = 3465.0
MAGNITUDE AND HPS IN MKS UNITS ANGLE IN RADIANS
HPS IS THE HORIZONTAL PHASE SPEED CALCULATED FROM THE CHANGE IN THE PHASE ANGLE IN THE LAST 180.00 SECONDS

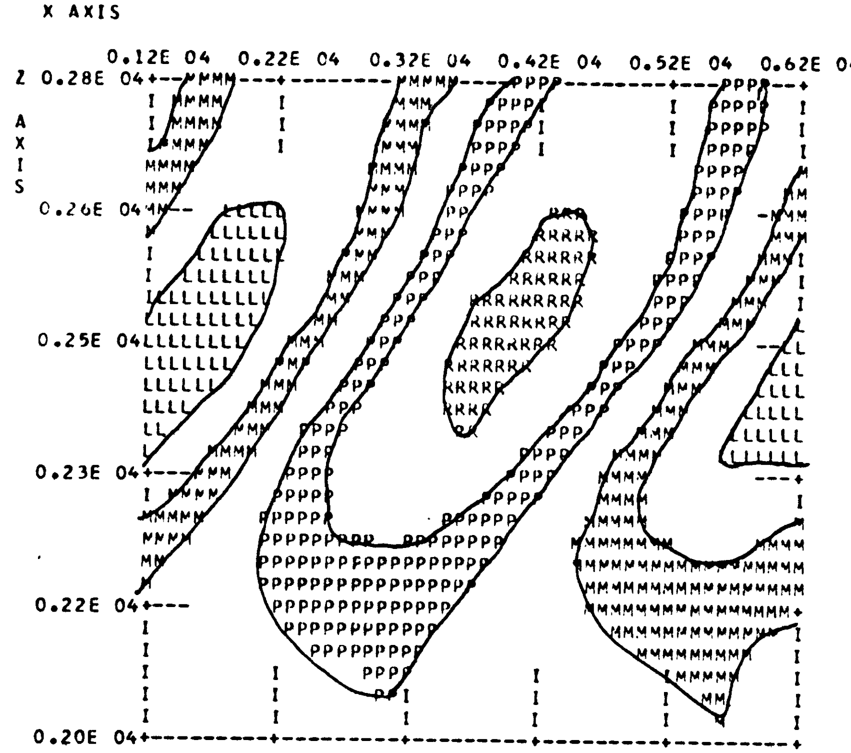
J	Z	CONSTANT	FUNDAMENTAL MAGNITUDE	FUNDAMENTAL ANGLE	H P S	SECOND HARMONIC MAGNITUDE	SECOND HARMONIC ANGLE	H P S	THIRD HARMONIC MAGNITUDE	THIRD HARMONIC ANGLE	H P S
2	200.0	0.000010	0.043931	-1.841	11.78	0.000624	-2.176	-1.71	0.000042	-2.521	1.79
3	600.0	0.000069	0.045920	-1.672	10.89	0.000925	-1.836	10.56	0.000086	1.317	-0.99
4	1000.0	0.000215	0.043219	-1.383	9.87	0.002383	-1.972	11.11	0.000180	0.823	6.92
5	1400.0	0.000448	0.036912	-0.875	9.07	0.006576	-1.977	11.48	0.000433	1.061	4.21
6	1800.0	0.000771	0.041826	0.054	8.72	0.016447	-2.012	11.52	0.001653	1.017	3.71
7	2012.5	0.000387	0.056370	0.429	8.71	0.026039	-1.998	11.51	0.003329	0.989	3.65
8	2037.5	0.000597	0.060543	0.498	8.39	0.026668	-2.003	11.45	0.003575	0.957	4.04
9	2062.5	0.000582	0.065914	0.505	8.40	0.028252	-2.014	11.42	0.003720	0.835	4.78
10	2087.5	0.001178	0.067264	0.563	8.77	0.030019	-2.024	11.48	0.004355	0.741	5.42
11	2112.5	0.001099	0.075469	0.616	8.57	0.031000	-2.022	11.44	0.005661	0.746	6.17
12	2137.5	0.002458	0.077556	0.652	9.11	0.033579	-2.031	11.35	0.007079	0.875	7.55
13	2162.5	0.002681	0.089198	0.805	8.85	0.034679	-2.045	11.54	0.008733	1.035	0.14
14	2187.5	0.001434	0.107808	0.747	8.50	0.036363	-1.978	11.31	0.008934	1.149	0.64
15	2212.5	0.010655	0.096755	0.926	9.69	0.044500	-2.099	11.03	0.008295	1.493	0.91
16	2237.5	0.003784	0.172078	1.135	8.10	0.043290	-2.313	-0.39	0.010407	1.541	0.93
17	2262.5	-0.040805	0.280266	0.719	6.67	0.025449	-1.969	-0.42	0.009637	0.909	2.10
18	2287.5	-0.047859	0.245025	0.252	4.30	0.040215	-1.510	10.86	0.007783	-0.231	5.00
19	2312.5	-0.007596	0.132949	0.384	-2.24	0.054870	-1.859	10.27	0.005727	-1.153	6.27
20	2337.5	-0.019475	0.198359	0.710	10.95	0.042627	-1.997	0.17	0.007484	-1.345	6.06
21	2362.5	-0.030159	0.134282	0.130	0.63	0.044380	-1.661	-1.61	0.013782	-1.288	5.91
22	2387.5	0.009717	0.041440	2.730	11.69	0.078228	-2.073	10.44	0.004326	-0.584	5.88
23	2412.5	-0.086817	0.328821	1.356	11.59	0.135325	-2.687	8.28	0.015796	0.971	3.44
24	2437.5	-0.293862	0.505497	1.030	10.75	0.059492	-3.075	4.74	0.020067	0.609	4.64
25	2462.5	-0.262207	0.282284	1.635	14.67	0.126223	-0.750	9.29	0.009560	0.428	-1.25
26	2487.5	-0.125728	0.527886	2.065	12.36	0.220007	-2.122	9.37	0.008509	0.921	6.14
27	2512.5	-0.443107	1.058622	1.252	12.61	0.390781	-3.127	9.03	0.025479	1.618	2.32
28	2537.5	-1.027265	1.823413	0.914	11.81	0.268667	2.162	8.68	0.062072	2.020	1.02
29	2562.5	-1.194835	2.022932	0.853	11.09	0.293194	0.861	-2.02	0.063192	1.789	-0.38
30	2587.5	-0.965993	1.755861	0.812	10.91	0.283230	0.900	-1.31	0.052781	0.973	0.69
31	2612.5	-0.763024	1.457789	0.682	11.08	0.338274	1.318	11.32	0.050239	1.324	2.93
32	2637.5	-0.643569	1.236120	0.552	10.74	0.305057	1.177	11.09	0.080767	1.853	1.96
33	2662.5	-0.509413	1.062669	0.446	10.07	0.095734	0.649	-2.20	0.099449	1.624	2.20
34	2687.5	-0.371105	0.984276	0.331	10.04	0.155322	-1.797	-1.94	0.082247	1.346	2.93
35	2712.5	-0.242398	0.893857	0.193	10.36	0.249055	-1.859	10.26	0.058774	1.122	4.16
36	2737.5	-0.158909	0.701976	-0.000	10.66	0.282603	-1.893	10.46	0.008269	1.241	5.70
37	2762.5	-0.098760	0.644834	-0.278	11.30	0.194588	-1.582	11.25	0.057165	-1.760	2.02
38	2787.5	-0.051998	0.742568	-0.449	11.82	0.061551	-0.564	0.19	0.078322	-1.439	2.95
39	3000.0	-0.024610	0.427067	-0.856	11.47	0.025411	2.743	11.39	0.036801	-1.494	3.39
40	3400.0	-0.060390	0.524008	-2.412	12.02	0.012877	2.062	1.01	0.005598	2.990	3.29
41	3800.0	-0.041922	0.857772	-2.841	11.00	0.029793	0.787	4.17	0.006642	1.544	5.13
42	4200.0	-0.050110	0.816780	3.126	10.63	0.045210	-0.216	10.50	0.004381	-2.762	1.43
43	4600.0	-0.073762	0.418057	2.496	10.51	0.080546	0.703	10.17	0.007068	-2.616	-0.29
44	5000.0	-0.068501	0.435620	0.917	12.07	0.035541	1.587	3.55	0.002570	-2.071	3.33
45	5400.0	-0.041758	0.743636	0.288	11.17	0.029071	-0.817	9.59	0.003116	-0.383	2.88
46	5800.0	-0.028273	0.852391	0.013	10.99	0.021439	0.028	1.52	0.001952	1.017	2.76
47	6200.0	-0.028277	0.005960	-1.127	12.80	0.000016	0.069	10.86	0.000000	-0.931	6.93



CORRESPONDENCE BETWEEN PRINTER SYMBOLS AND CONTOUR LEVELS

K - -0.14129E-02 TO -0.11560E-02
L - -0.89913E-03 TO -0.64224E-03
M - -0.38534E-03 TO -0.12845E-03
P - 0.12845E-03 TO 0.38534E-03
R - 0.64224E-03 TO 0.89913E-03
S - 0.11560E-02 TO 0.14129E-02

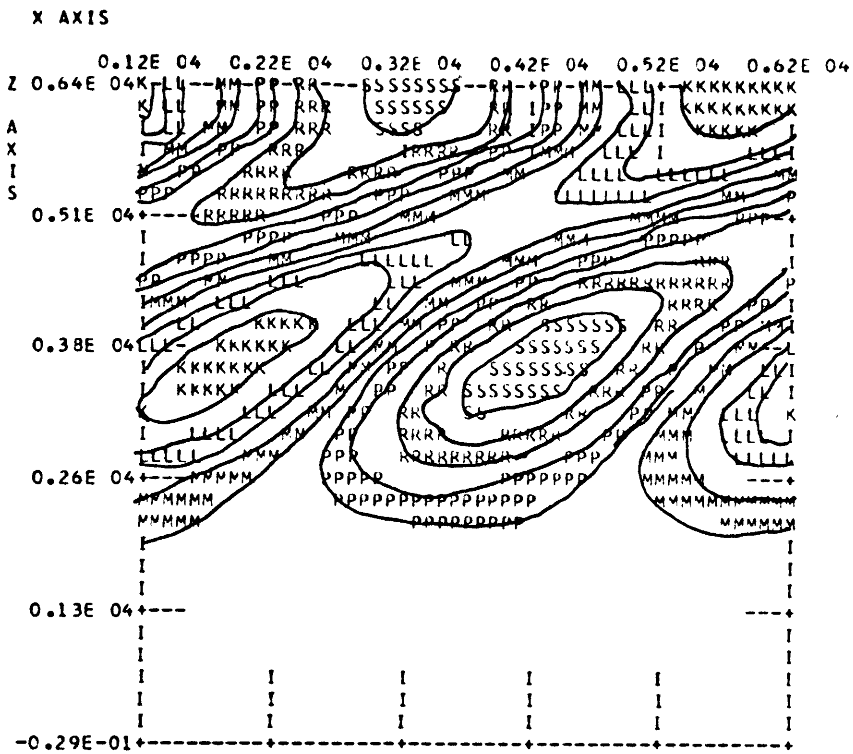
FIGURE 5.1A CONTOUR PLOT OF RHO FOR CASE A
MKS UNITS TIME STEP 64 TIME = 945.0



CORRESPONDENCE BETWEEN PRINTER SYMBOLS AND CONTOUR LEVELS

K - -0.14129E-02 TO -0.11560E-02
L - -0.89913E-03 TO -0.64224E-03
M - -0.38534E-03 TO -0.12845E-03
P - 0.12845E-03 TO 0.38534E-03
R - 0.64224E-03 TO 0.89913E-03
S - 0.11560E-02 TO 0.14129E-02

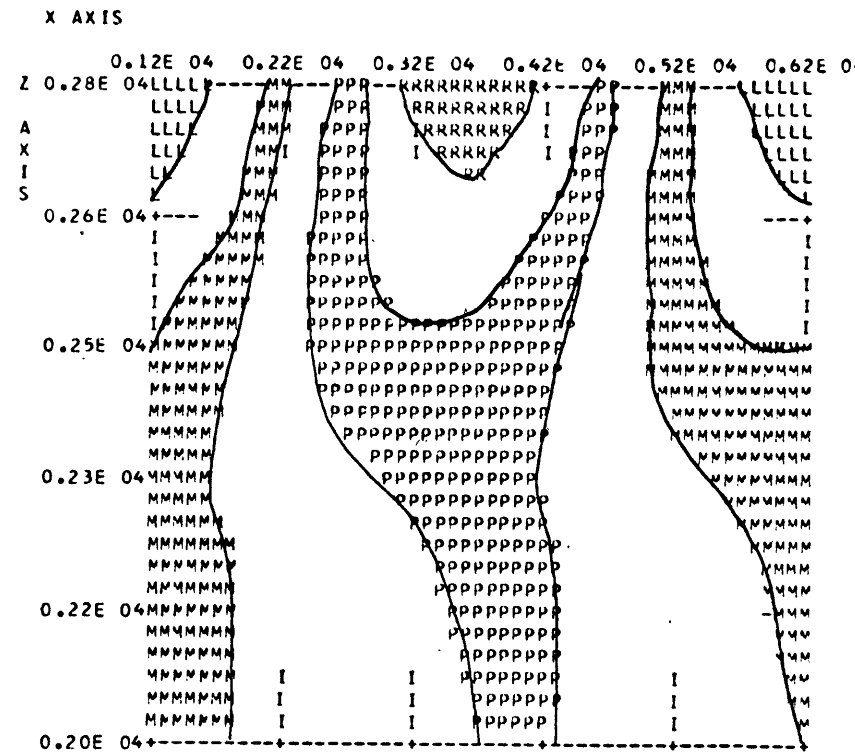
FIGURE 5.1B CONTOUR PLOT OF RHO FOR CASE A
EXPANDED REGION
MKS UNITS TIME STEP 64 TIME = 945.0



CORRESPONDENCE BETWEEN PRINTER SYMBOLS AND CONTOUR LEVELS

K -	-0.59733	TO	-0.48872
L -	-0.38012	TO	-0.27151
M -	-0.16291	TO	-0.54303E-01
P -	0.54302E-01	TO	0.16291
R -	0.27151	TO	0.38012
S -	0.48872	TO	0.59733

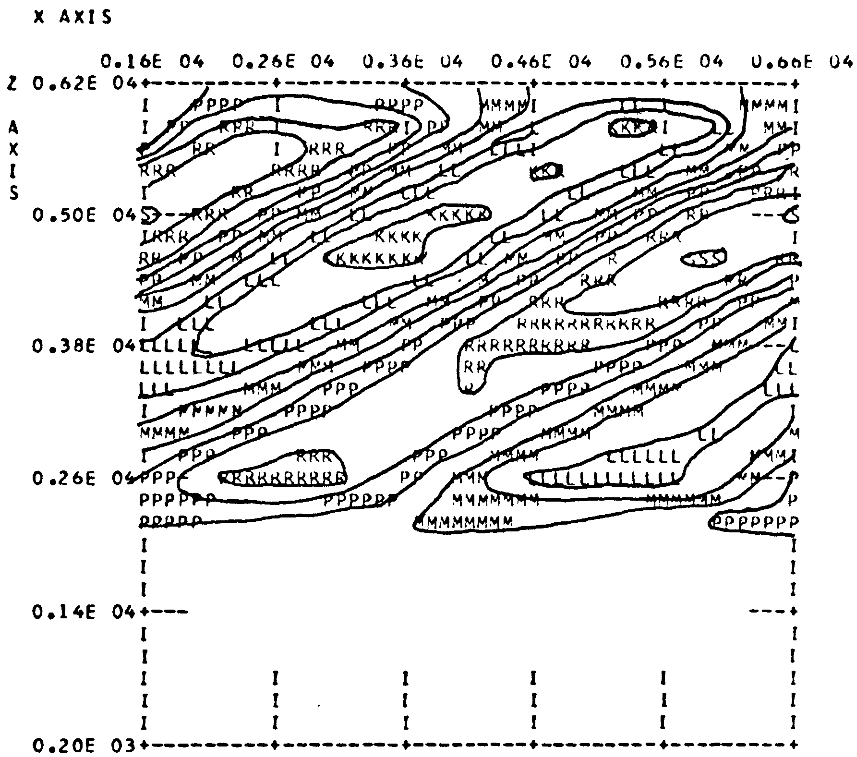
FIGURE 5. 1C CONTOUR PLOT OF W FOR CASE A
MKS UNITS TIME STEP 64 TIME = 945.0



CORRESPONDENCE BETWEEN PRINTER SYMBOLS AND CONTOUR LEVELS

K -	-0.59733	TO	-0.48872
L -	-0.38012	TO	-0.27151
M -	-0.16291	TO	-0.54303E-01
P -	0.54302E-01	TO	0.16291
R -	0.27151	TO	0.38012
S -	0.48872	TO	0.59733

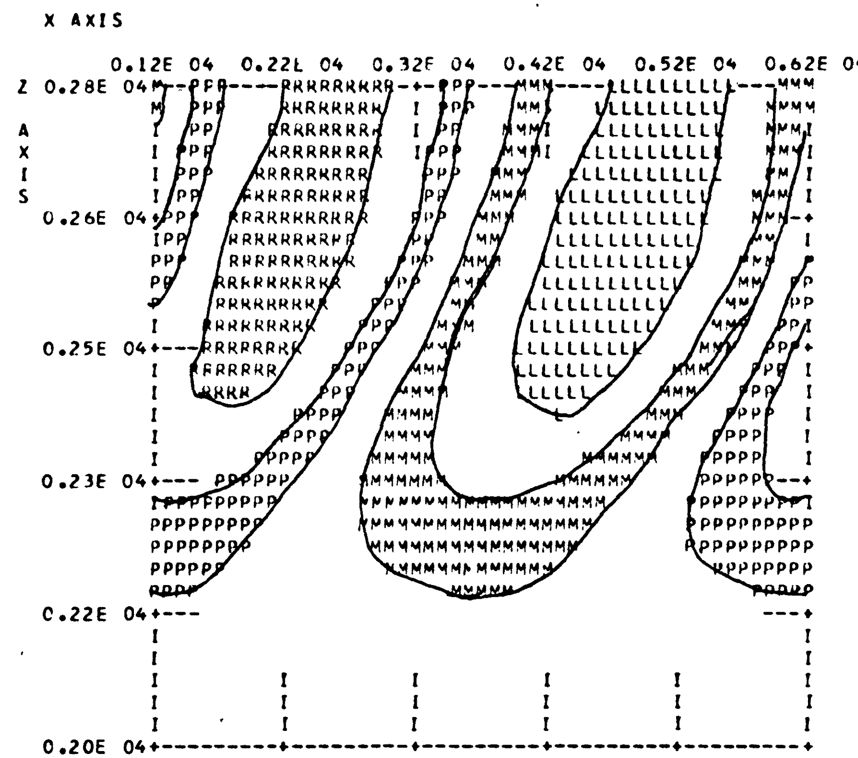
FIGURE 5. 1D CONTOUR PLOT OF W FOR CASE A
EXPANDED REGION
MKS UNITS TIME STEP 64 TIME = 945.0



CORRESPONDENCE BETWEEN PRINTER SYMBOLS AND CONTOUR LEVELS

K - -0.87995 TO -0.71996
L - -0.55997 TO -0.39998
M - -0.23999 TO -0.79995E-01
P - 0.79996E-01 TO 0.23999
R - 0.39998 TO 0.55997
S - 0.71996 TO 0.87995

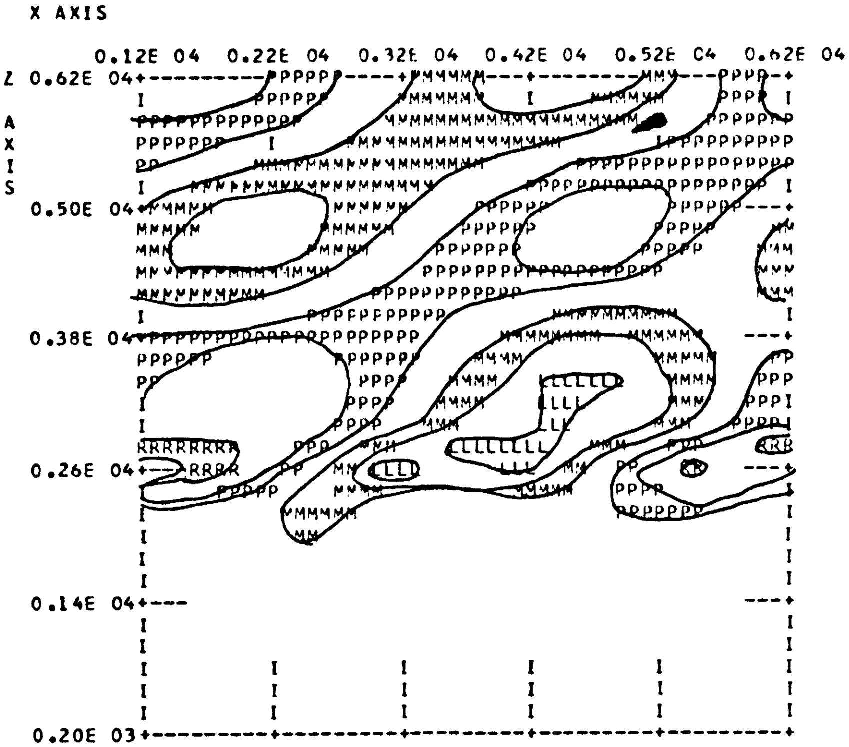
FIGURE 5. 1E CONTOUR PLOT OF U FOR CASE A
MKS UNITS TIME STEP 64 TIME = 945.0



CORRESPONDENCE BETWEEN PRINTER SYMBOLS AND CONTOUR LEVELS

K - -0.87995 TO -0.71996
L - -0.55997 TO -0.39998
M - -0.23999 TO -0.79995E-01
P - 0.79996E-01 TO 0.23999
R - 0.39998 TO 0.55997
S - 0.71996 TO 0.87995

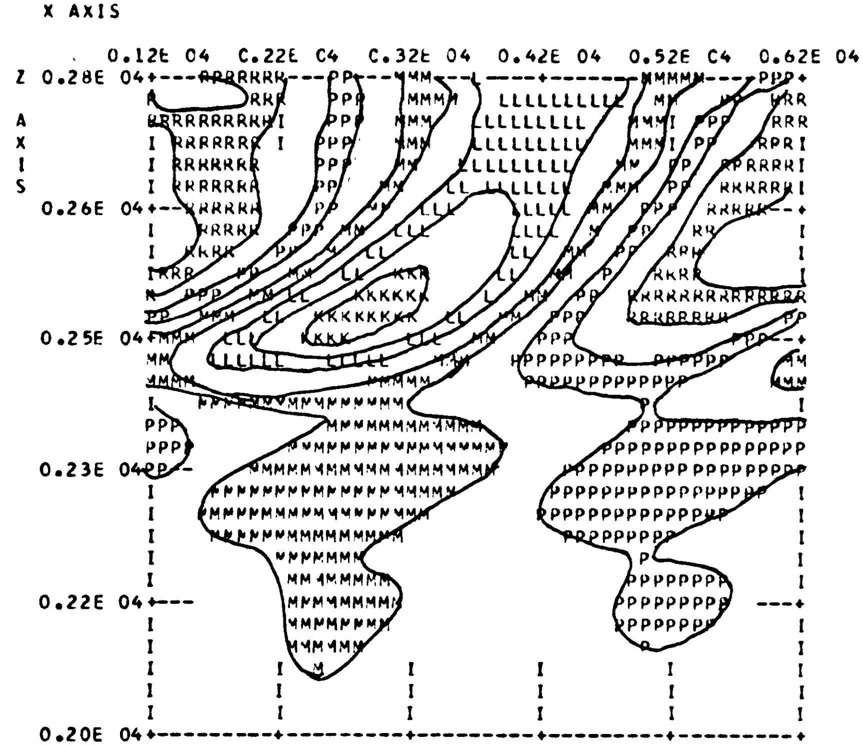
FIGURE 5.1F CONTOUR PLOT OF U FOR CASE A
EXPANDED REGION
MKS UNITS TIME STEP 64 TIME = 945.0



CORRESPONDENCE BETWEEN PRINTER SYMBOLS AND CONTOUR LEVELS

K - -0.30056E-02 TO -0.24592E-02
L - -0.19127E-02 TO -0.13662E-02
M - -0.81972E-03 TO -0.27324E-03
P - 0.27324E-03 TO 0.81971E-03
R - 0.13662E-02 TO 0.19127E-02
S - 0.24591E-02 TO 0.30056E-02

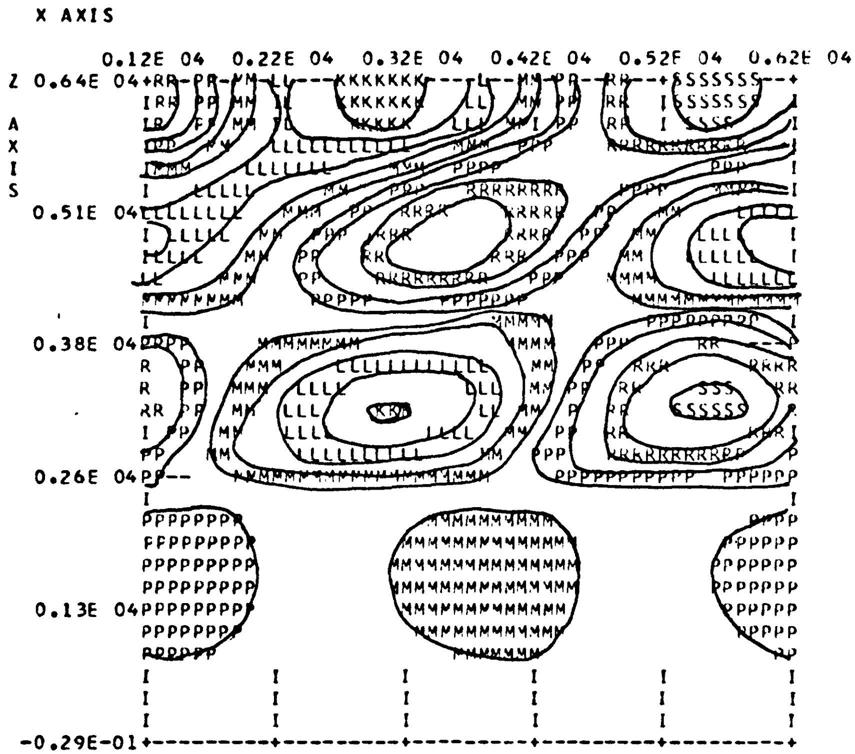
FIGURE 5.1G CONTOUR PLOT OF RHO FOR CASE A
MKS UNITS TIME STEP 136 TIME = 2025.0



CORRESPONDENCE BETWEEN PRINTER SYMBOLS AND CONTOUR LEVELS

K - -0.30056E-02 TO -0.24592E-02
L - -0.19127E-02 TO -0.13662E-02
M - -0.81972E-03 TO -0.27324E-03
P - 0.27324E-03 TO 0.81971E-03
R - 0.13662E-02 TO 0.19127E-02
S - 0.24591E-02 TO 0.30056E-02

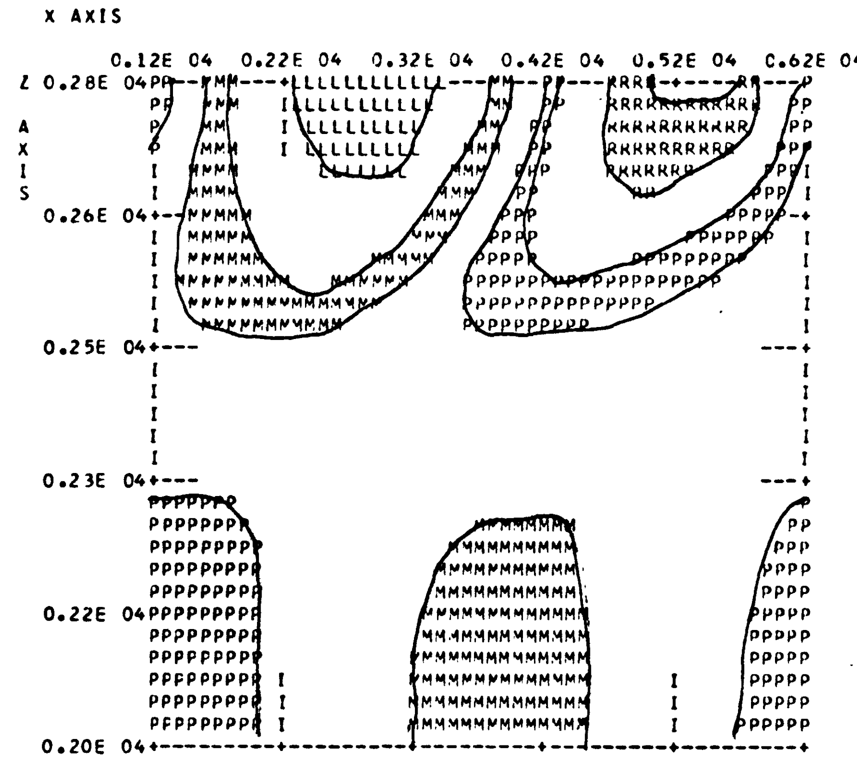
FIGURE 5.1H CONTOUR PLOT OF RHO FOR CASE A
EXPANDED REGION
MKS UNITS TIME STEP 136 TIME = 2025.0



CORRESPONDENCE BETWEEN PRINTER SYMBOLS AND CONTOUR LEVELS

K	-	-0.63421	TO -0.51890
L	-	-0.40359	TO -0.28828
M	-	-0.17297	TO -0.57655E-01
P	-	0.57655E-C1	TO 0.17297
R	-	0.28828	TO 0.40359
S	-	0.5189C	TO 0.63421

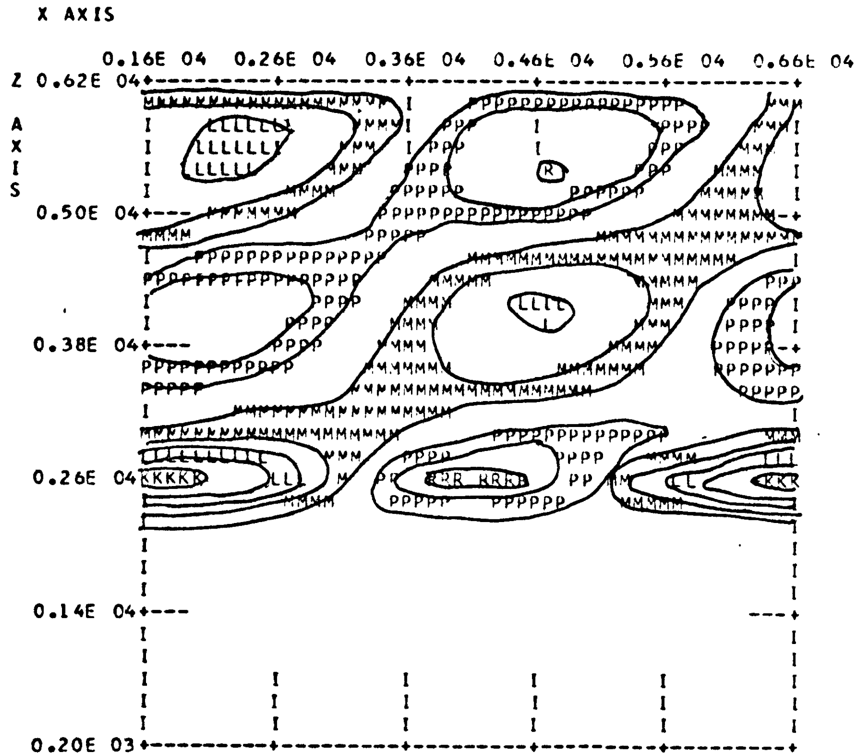
FIGURE 5.1I CONTOUR PLOT OF w FOR CASE A
MKS UNITS TIME STEP 136 TIME = 2025.0



CORRESPONDENCE BETWEEN PRINTER SYMBOLS AND CONTOUR LEVELS

K	-	-0.63421	TO -0.51890
L	-	-0.40359	TO -0.28828
M	-	-0.17297	TO -0.57655E-01
P	-	0.57655E-C1	TO 0.17297
R	-	0.28828	TO 0.40359
S	-	0.5189C	TO 0.63421

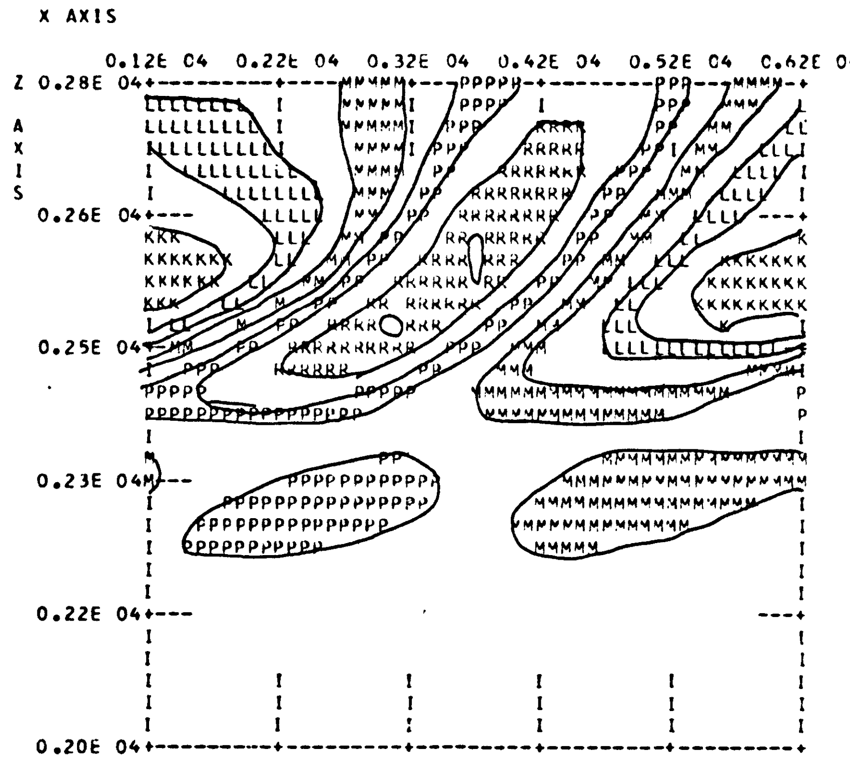
FIGURE 5.1J CONTOUR PLOT OF w FOR CASE A
EXPANDED REGION
MKS UNITS TIME STEP 136 TIME = 2025.0



CORRESPONDENCE BETWEEN PRINTER SYMBOLS AND CONTOUR LEVELS

K -	-1.8211	TO	-1.4900
L -	-1.1589	TO	-0.82777
M -	-0.49666	TO	-0.16555
P -	0.16556	TO	0.49667
R -	0.82778	TO	1.1589
S -	1.4900	TO	1.8211

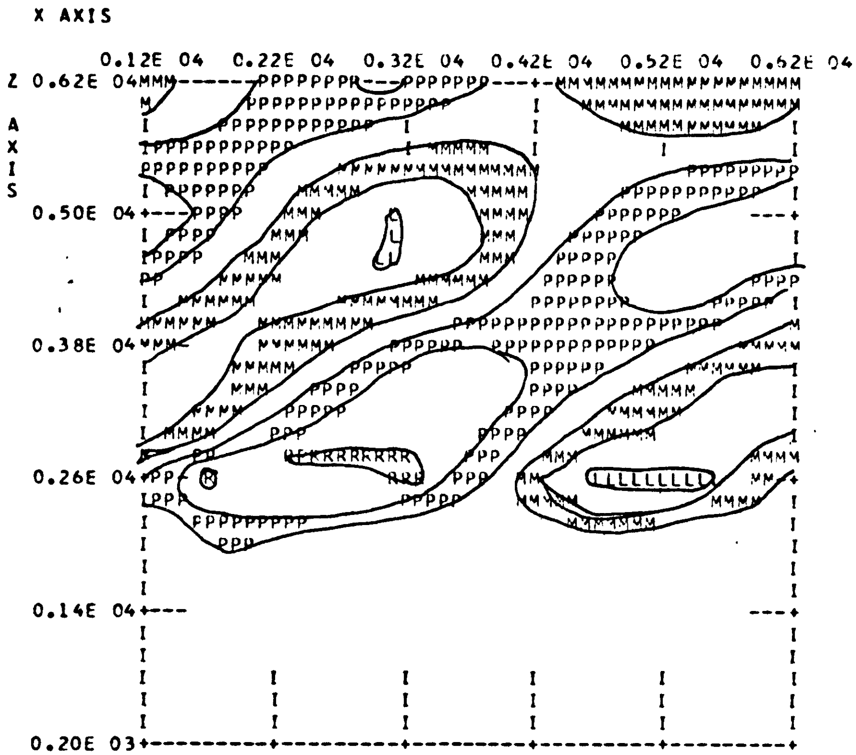
FIGURE 5.1K CONTOUR PLOT OF U FOR CASE A
MKS UNITS TIME STEP 136 TIME = 2025.0



CORRESPONDENCE BETWEEN PRINTER SYMBOLS AND CONTOUR LEVELS

K -	-1.8211	TO	-1.4900
L -	-1.1589	TO	-0.82777
M -	-0.49666	TO	-0.16555
P -	0.16556	TO	0.49667
R -	0.82778	TO	1.1589
S -	1.4900	TO	1.8211

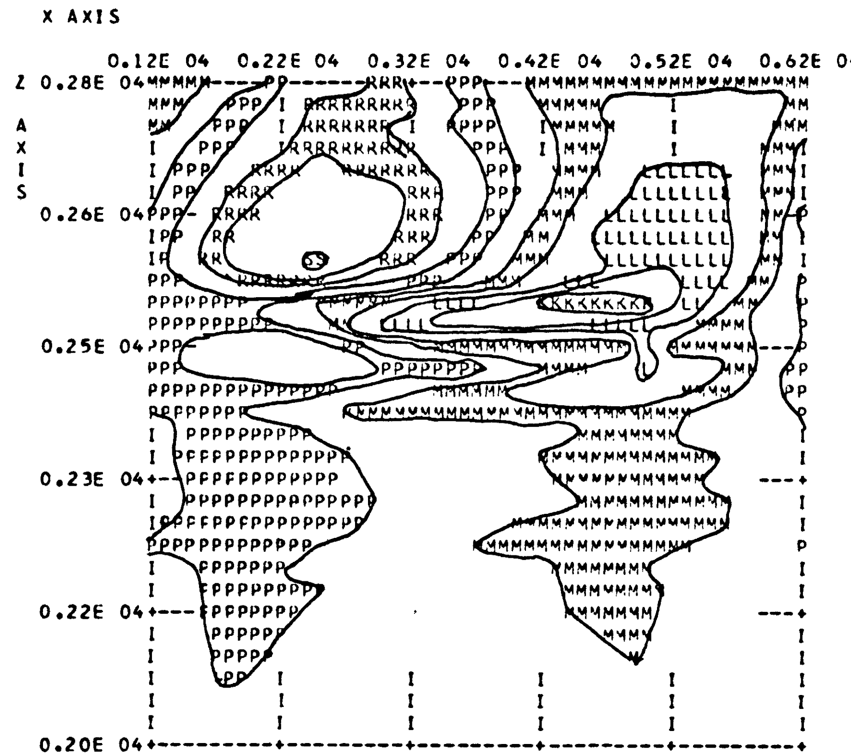
FIGURE 5.1L CONTOUR PLOT OF U FOR CASE A
EXPANDED REGION
MKS UNITS TIME STEP 136 TIME = 2025.0



CORRESPONDENCE BETWEEN PRINTER SYMBOLS AND CONTOUR LEVELS

K - -0.33387E-02 TO -0.27316E-02
L - -0.21246E-02 TO -0.15176E-02
M - -0.91054E-03 TO -0.30352E-03
P - 0.30351E-03 TO 0.91054E-03
R - 0.15176E-02 TO 0.21246E-02
S - 0.27316E-02 TO 0.33386E-02

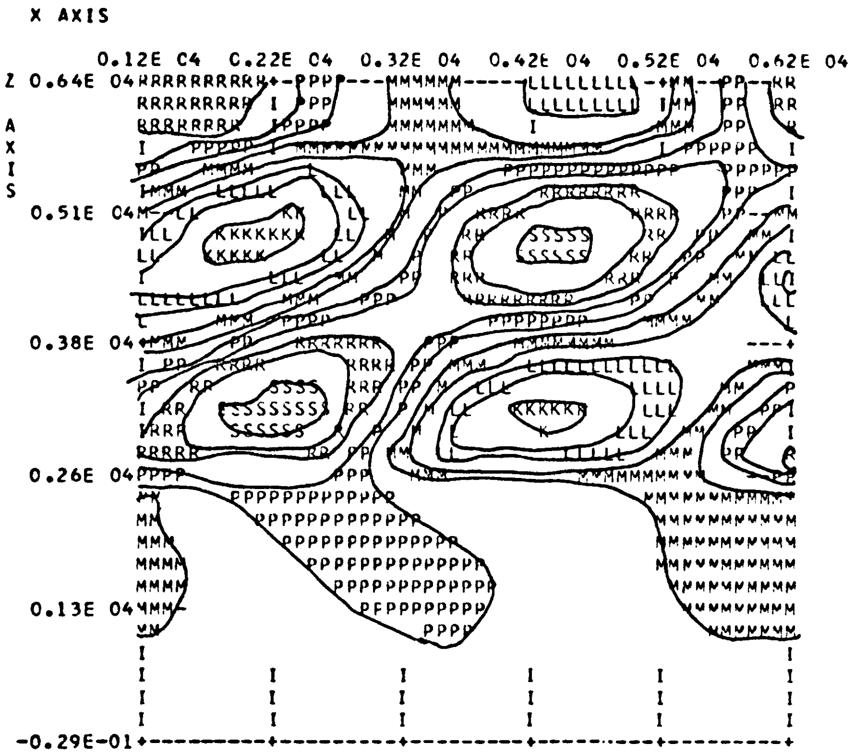
FIGURE 5.1M CONTOUR PLOT OF RHO FOR CASE A
MKS UNITS TIME STEP 232 TIME = 3465.0



CORRESPONDENCE BETWEEN PRINTER SYMBOLS AND CONTOUR LEVELS

K - -0.33387E-02 TO -0.27316E-02
L - -0.21246E-02 TO -0.15176E-02
M - -0.91054E-03 TO -0.30352E-03
P - 0.30351E-03 TO 0.91054E-03
R - 0.15176E-02 TO 0.21246E-02
S - 0.27316E-02 TO 0.33386E-02

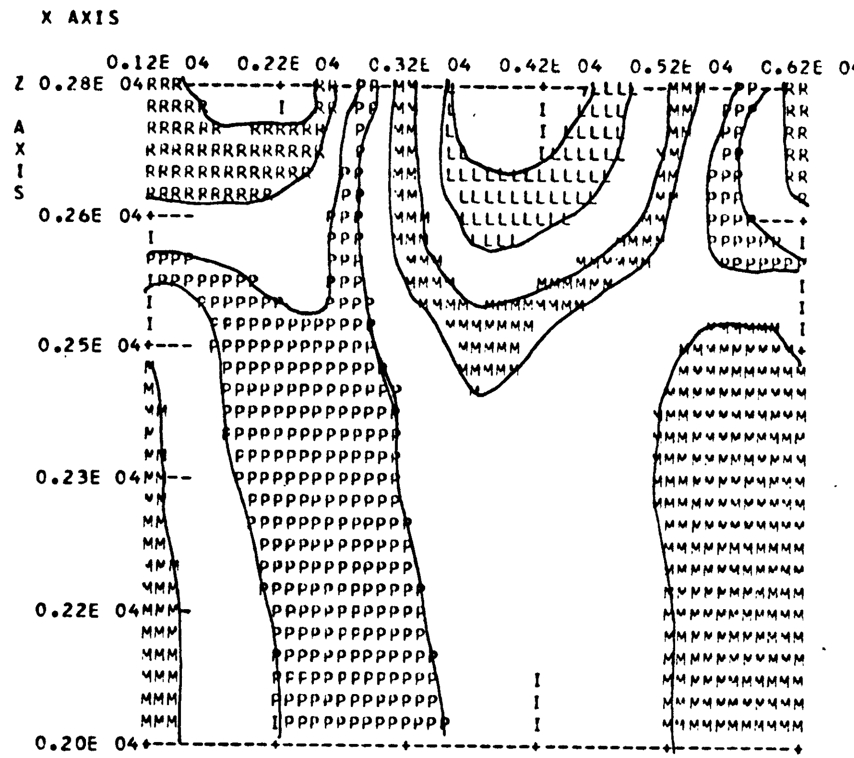
FIGURE 5.1N CONTOUR PLOT OF RHO FOR CASE A
EXPANDED REGION
MKS UNITS TIME STEP 232 TIME = 3465.0



CORRESPONDENCE BETWEEN PRINTER SYMBOLS AND CONTOUR LEVELS

K -	-0.59322	TO	-0.48536
L -	-0.37750	TO	-0.26965
M -	-0.16179	TO	-0.53929E-01
P -	0.53929E-01	TO	0.16179
R -	0.26964	TO	0.37750
S -	0.48536	TO	0.59322

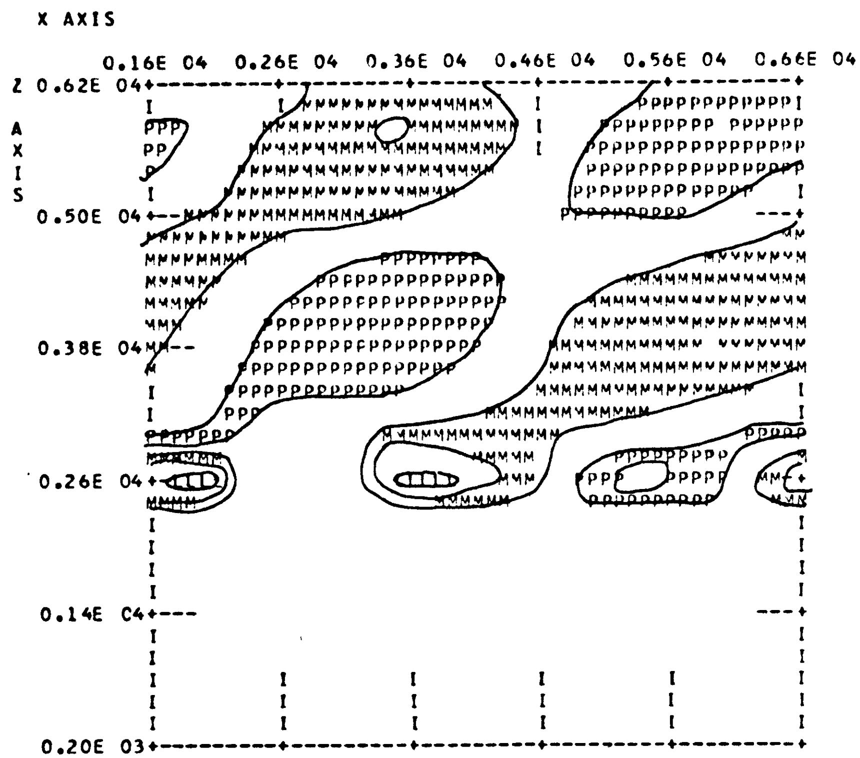
FIGURE 5.1 P CONTOUR PLOT OF W FOR CASE A
MKS UNITS TIME STEP 232 TIME = 3465.0



CORRESPONDENCE BETWEEN PRINTER SYMBOLS AND CONTOUR LEVELS

K -	-0.59322	TO	-0.48536
L -	-0.37750	TO	-0.26965
M -	-0.16179	TO	-0.53929E-01
P -	0.53929E-01	TO	0.16179
R -	0.26964	TO	0.37750
S -	0.48536	TO	0.59322

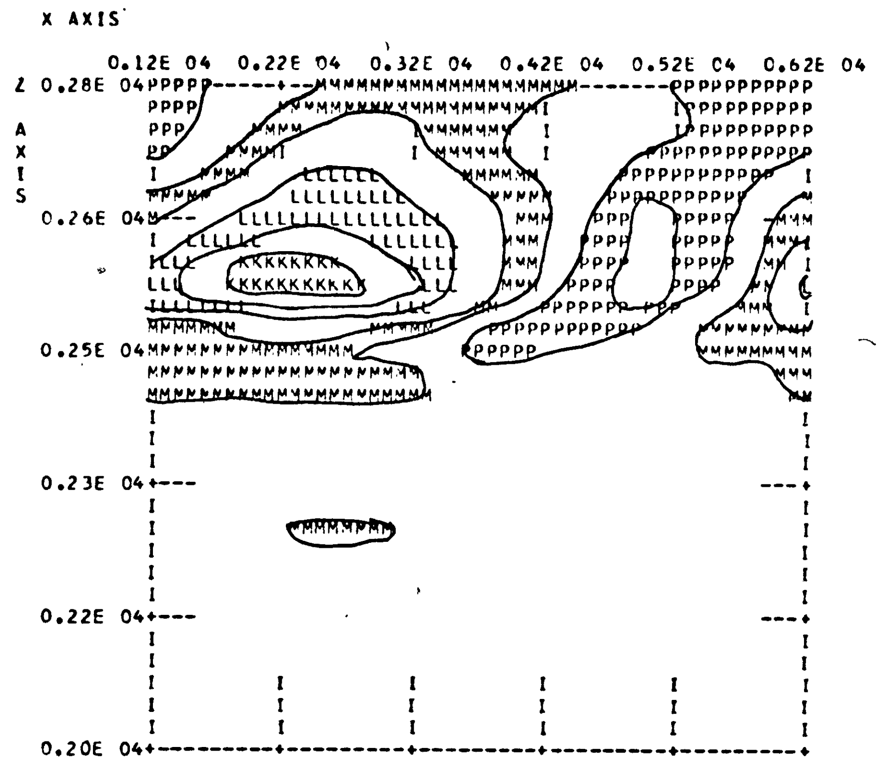
FIGURE 5.1Q CONTOUR PLOT OF W FOR CASE A
EXPANDED REGION
MKS UNITS TIME STEP 232 TIME = 3465.0



CORRESPONDENCE BETWEEN PRINTER SYMBOLS AND CONTOUR LEVELS

K -	-3.0440	TO	-2.4905
L -	-1.5371	TO	-1.3836
M -	-0.83017	TO	-0.27672
P -	0.27673	TO	0.93018
R -	1.3836	TO	1.9371
S -	2.49C5	TO	3.0440

FIGURE 5. 1R CCNTCUR PLCT OF U FOR CASE A
MKS UNITS TIME STEP 232 TIME = 3465.0



CORRESPONDENCF BETWEEN PRINTER SYMBOLS AND CONTOUR LEVELS

K -	-3.0440	TO	-2.4905
L -	-1.9371	TO	-1.3836
M -	-0.83017	TO	-0.27672
P -	0.27673	TO	C.83018
R -	1.3836	TO	1.9371
S -	2.4905	TO	3.0440

FIGURE 5. 1S CONTOUR PLOT OF U FOR CASE A
EXPANDEC REGICN
MKS UNITS TIME STEP 232 TIME = 3465.0

Case B (Low Richardson Number)

This case has the same parameters as case A except that $z_{eb} = z_{wb} = 2400$m, Ri = 0.13, and $z_c = 2622$m. This case terminated at 2430s. Figures 5.2 and tables 5.2 indicate how this case developed. Figures 5.2B,D extend from 2412.5m to 2787.5m, and the axis values of 2562.5m and 2637.5m have both been rounded to 2600m.

From the tables it is clear that in this case the critical level is transmitting almost all of the incident wave. The growth of the second and third harmonics is much faster here than it is in case A, so that by 2310s the third harmonic is too large for the model to be considered reliable. Note that in addition to a decrease in the wind above z_c, in this case there is an increase below z_c. Because case B is the only case run when the program was using a less satisfactory method of handling the boundaries of the expanded region, and to examine the stability question further, case C was run.

TABLE 5.2A FOURIER TRANSFORM OF RHO BY ROWS
FOR CASE B TIME STEP 155 TIME = 2310.0
MAGNITUDE AND HPS IN MKS UNITS ANGLE IN RADIANS
HPS IS THE HORIZONTAL PHASE SPEED CALCULATED FROM THE CHANGE IN THE PHASE ANGLE IN THE LAST 60.00 SECONDS

J	Z	CONSTANT	FUNDAMENTAL MAGNITUDE	FUNDAMENTAL ANGLE	H P S	SECOND HARMONIC MAGNITUDE	SECOND HARMONIC ANGL5	H P S	THIRD HARMONIC MAGNITUDE	THIRD HARMONIC ANGLE	H P S
2	200.0	-0.000002	0.000237	-2.351	11.58	0.000009	1.833	-0.42	0.000001	-0.777	-1.08
3	600.0	-0.000006	0.000679	-2.296	11.57	0.000020	1.343	0.45	0.000007	-1.428	-0.61
4	1000.0	-0.000008	0.001041	-2.172	11.54	0.000011	-1.563	-3.38	0.000019	-2.545	-1.05
5	1400.0	-0.000005	0.001270	-2.000	11.41	0.000109	2.045	1.04	0.000059	1.147	-0.09
6	1800.0	0.000002	0.001345	-1.822	10.91	0.000077	2.563	11.73	0.000058	-0.855	-0.90
7	2200.0	-0.000157	0.001256	-1.865	11.65	0.000131	1.696	22.50	0.000100	0.632	2.50
8	2412.5	0.000463	0.001117	-2.214	7.21	0.000709	2.113	2.84	0.000564	1.684	0.96
9	2437.5	0.000073	0.001311	-1.853	13.15	0.000312	1.107	23.12	0.000582	0.144	2.22
10	2462.5	-0.000114	0.001087	-1.777	14.67	0.000256	1.016	32.13	0.000205	0.223	1.84
11	2487.5	-0.000351	0.001602	-1.701	12.17	0.000183	0.944	26.17	0.000103	-0.112	3.70
12	2512.5	-0.000688	0.002063	-1.714	13.00	0.000343	1.943	15.39	0.000224	-0.413	4.60
13	2537.5	-0.001675	0.002343	-1.466	10.84	0.000067	-1.294	-0.74	0.000020	-1.510	6.46
14	2562.5	-0.001912	0.003914	-1.918	10.19	0.000512	-2.845	7.09	0.000125	0.190	0.77
15	2587.5	-0.002911	0.004324	-2.032	11.25	0.000792	2.917	9.97	0.000309	-0.534	5.13
16	2612.5	-0.002260	0.004487	-2.311	12.66	0.002092	-3.091	11.10	0.000668	0.122	15.15
17	2637.5	-0.001487	0.002996	2.379	16.71	0.001443	-2.597	15.24	0.000555	0.986	12.37
18	2662.5	0.001639	0.004600	2.603	8.65	0.000347	1.689	26.13	0.000715	-0.654	9.74
19	2687.5	-0.001452	0.005247	0.996	14.08	0.000918	-1.815	5.81	0.001052	-2.510	8.52
20	2712.5	0.002913	0.003163	2.381	8.80	0.005247	1.161	11.51	0.001069	0.085	6.82
21	2737.5	-0.000157	0.003344	0.093	15.10	0.002369	-2.263	13.30	0.001125	0.071	16.38
22	2762.5	0.004260	0.006644	1.619	14.02	0.003312	0.432	11.25	0.001785	0.291	4.36
23	2787.5	-0.001603	0.001544	0.061	19.28	0.004944	-1.858	11.57	0.001679	-1.303	6.14
24	3000.0	0.000849	0.001255	1.786	9.81	0.001092	0.860	11.89	0.000245	0.100	5.32
25	3400.0	0.000024	0.001757	1.814	10.92	0.000262	2.506	11.00	0.000237	1.070	5.17
26	3800.0	0.000011	0.000723	2.254	8.48	0.000165	0.219	11.18	0.000067	-1.204	8.20
27	4200.0	-0.000067	0.001470	-1.943	11.95	0.000395	1.039	11.90	0.000100	-3.103	6.61
28	4600.0	-0.000029	0.002485	-1.739	11.99	0.000351	-3.005	10.49	0.000226	2.688	6.55
29	5000.0	0.000069	0.002187	-1.635	11.53	0.000374	-1.900	9.62	0.000004	0.241	10.51
30	5400.0	0.000030	0.000841	-1.456	9.63	0.000152	-1.060	5.49	0.000023	-0.258	10.95
31	5800.0	-0.000259	0.000640	1.213	15.37	0.000134	0.715	20.57	0.000014	-0.391	11.55
32	6200.0	0.000262	0.001178	1.262	13.91	0.000281	2.410	15.88	0.000005	-1.010	9.28

TABLE 5.2B FOURIER TRANSFORM OF W BY ROWS
FOR CASE B TIME STEP 155 TIME = 2310.0
MAGNITUDE AND HPS IN MKS UNITS ANGLE IN RADIANS
HPS IS THE HORIZONTAL PHASE SPEED CALCULATED FROM THE CHANGE IN THE PHASE ANGLE IN THE LAST 60.00 SECONDS

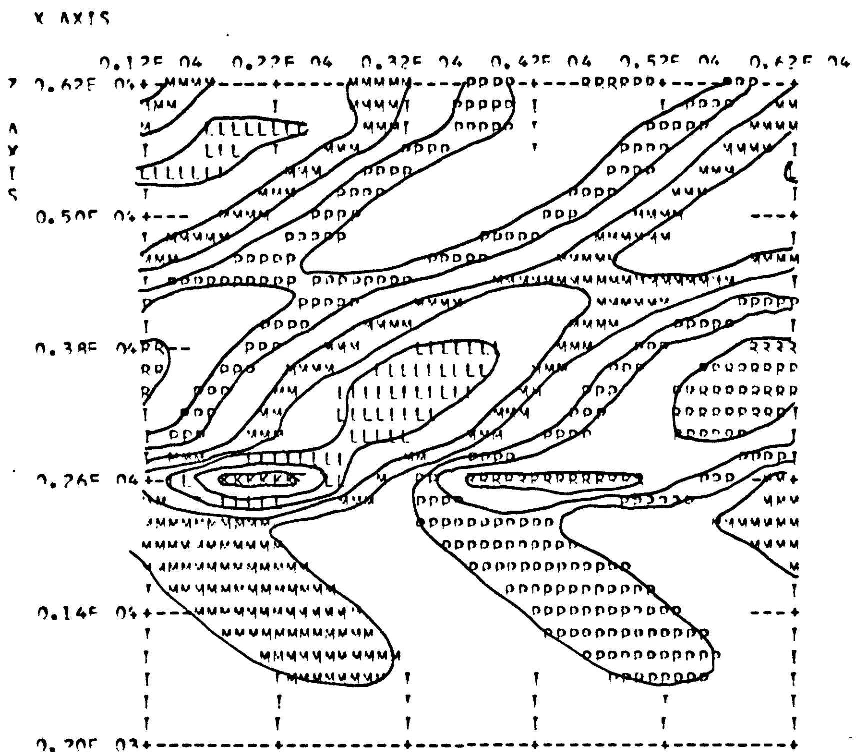
J	Z	CONSTANT	FUNDAMENTAL			SECOND HARMONIC			THIRD HARMONIC		
			MAGNITUDE	ANGLE	H P S	MAGNITUDE	ANGLE	H P S	MAGNITUDE	ANGLE	H P S
2	400.0	0.000000	0.163776	2.595	10.20	0.000751	-0.582	16.95	0.000378	0.818	-2.22
3	800.0	0.0	0.320012	2.665	10.23	0.004836	0.420	8.96	0.001521	-1.159	-0.27
4	1200.0	0.000000	0.455333	2.773	10.05	0.013054	-0.454	24.35	0.008369	-2.144	0.24
5	1600.0	0.000000	0.547622	2.887	9.81	0.031464	1.201	7.70	0.004333	1.904	-3.48
6	2000.0	0.000000	0.548543	3.004	9.75	0.058992	1.159	10.06	0.018394	-1.573	5.70
7	2400.0	0.0	0.461211	-3.137	9.46	0.155589	0.953	10.63	0.021737	-0.696	3.64
8	2425.0	0.000000	0.459630	-3.123	9.43	0.168469	0.963	10.44	0.017004	-0.548	3.40
9	2450.0	-0.000000	0.450012	-3.104	9.43	0.175574	0.966	10.54	0.015600	-0.455	3.35
10	2475.0	-0.000000	0.436338	-3.086	9.46	0.182705	0.957	10.74	0.013946	-0.197	2.51
11	2500.0	0.000000	0.419051	-3.064	9.51	0.189074	0.948	10.92	0.012767	0.140	1.35
12	2525.0	-0.000000	0.398345	-3.039	9.55	0.194505	0.939	11.10	0.015144	0.458	0.19
13	2550.0	0.0	0.373896	-3.007	9.75	0.204885	0.911	11.34	0.017175	0.532	-0.50
14	2575.0	0.000000	0.344985	-2.936	10.33	0.215152	0.764	11.64	0.019692	0.850	-1.13
15	2600.0	0.0	0.317739	-2.817	11.37	0.220861	0.701	11.96	0.028455	1.046	-1.66
16	2625.0	0.000000	0.310343	-2.685	12.32	0.225472	0.599	12.39	0.038055	1.230	-2.90
17	2650.0	0.000000	0.331155	-2.589	14.09	0.238411	0.513	12.70	0.044773	1.339	-4.02
18	2675.0	-0.000000	0.364327	-2.551	14.25	0.249251	0.517	12.66	0.050663	1.193	-3.22
19	2700.0	0.0	0.388896	-2.547	13.86	0.243326	0.585	12.47	0.064195	1.056	-1.82
20	2725.0	-0.000000	0.405387	-2.594	13.46	0.241383	0.655	12.51	0.084851	1.041	-0.53
21	2750.0	-0.000000	0.425799	-2.625	13.41	0.235280	0.639	13.03	0.099293	1.091	-0.33
22	2775.0	-0.000000	0.449460	-2.660	13.24	0.240099	0.643	13.17	0.112993	1.127	0.35
23	2800.0	-0.000000	0.475133	-2.703	12.93	0.230430	0.806	12.75	0.128592	1.154	1.74
24	3200.0	0.000000	0.654285	-2.805	12.51	0.330253	-2.971	9.56	0.080290	1.540	6.07
25	3600.0	0.000000	0.545914	-2.594	13.75	0.149975	-1.996	4.71	0.036841	-2.742	8.05
26	4000.0	0.000000	0.293489	-1.345	12.34	0.098335	2.668	8.28	0.037569	2.919	9.07
27	4400.0	-0.000000	0.697742	-0.285	9.55	0.103124	-2.426	9.55	0.001314	-1.106	0.17
28	4800.0	-0.000000	0.936190	-0.095	9.80	0.155356	-0.676	9.49	0.007562	-0.535	9.01
29	5200.0	-0.000000	0.683073	-0.010	9.07	0.186464	-0.010	11.32	0.007578	-0.902	10.29
30	5500.0	0.0	0.085481	0.321	-2.20	0.141367	0.380	13.85	0.003434	-0.370	11.98
31	6000.0	0.000000	0.472899	3.039	15.18	0.065577	0.522	15.27	0.000346	0.444	18.92
32	6400.0	0.000000	0.499179	3.034	15.16	0.058710	0.623	15.27	0.000362	0.445	18.91

TABLE 5.2C FOURIER TRANSFORM OF U BY ROWS
FOR CASE B TIME STEP 155 TIME = 2310.0
MAGNITUDE AND HPS IN MKS UNITS ANGLE IN RADIANS
HPS IS THE HORIZONTAL PHASE SPEED CALCULATED FROM THE CHANGE IN THE PHASE ANGLE IN THE LAST 60.00 SECONDS

		CONSTANT	FUNDAMENTAL			SECOND HARMONIC			THIRD HARMONIC		
J	Z		MAGNITUDE	ANGLE	H P S	MAGNITUDE	ANGLE	H P S	MAGNITUDE	ANGLE	H P S
2	200.0	0.000839	0.326613	-1.725	10.20	0.000811	1.774	16.95	0.000312	-2.716	25.56
3	600.0	0.004223	0.297535	-1.574	10.22	0.004815	2.925	6.57	0.001418	1.377	-0.11
4	1000.0	0.010257	0.253359	-1.257	9.43	0.011379	1.524	32.94	0.006283	0.429	0.04
5	1400.0	0.018290	0.182073	-0.704	9.01	0.039163	-2.330	4.41	0.009857	-2.247	-0.87
6	1800.0	0.021460	0.140009	0.570	13.19	0.028137	-2.822	12.65	0.018779	1.242	2.69
7	2200.0	0.353553	0.262334	1.298	13.51	0.107997	3.021	10.28	0.014612	3.047	0.07
8	2412.5	-0.008644	0.338836	1.307	14.32	0.130575	-2.909	7.89	0.058065	-1.801	-0.91
9	2437.5	-0.086002	0.451648	1.336	12.45	0.116852	-2.995	11.96	0.051451	-1.327	1.09
10	2462.5	-0.085561	0.557282	1.511	11.18	0.119342	2.987	15.08	0.056349	-1.873	3.75
11	2487.5	-0.055470	0.680667	1.557	10.95	0.107213	2.923	15.63	0.062750	-1.730	3.35
12	2512.5	0.064302	0.789665	1.618	11.04	0.090691	2.875	16.38	0.067457	-2.145	3.58
13	2537.5	0.240375	0.925130	1.639	10.27	0.200156	2.654	15.72	0.046407	-2.032	-1.38
14	2562.5	0.329319	1.287845	1.426	10.27	0.255157	2.402	16.38	0.063547	-1.768	0.27
15	2587.5	0.428438	1.584505	1.278	11.15	0.256571	1.801	17.00	0.133270	-2.090	2.34
16	2612.5	0.351754	1.363894	0.983	12.54	0.406323	1.608	16.59	0.152534	-1.821	16.52
17	2637.5	-0.151260	1.210312	0.332	14.09	0.414660	1.890	15.39	0.107990	-1.649	16.08
18	2662.5	-0.592675	1.135436	-0.221	13.27	0.191272	2.959	11.85	0.122027	3.133	5.61
19	2687.5	-0.775665	0.765542	-0.512	9.81	0.324400	-1.303	7.55	0.209564	-2.944	8.36
20	2712.5	-0.788037	0.696585	-1.371	12.99	0.302777	-1.580	8.94	0.277006	-2.532	6.97
21	2737.5	-1.109176	0.835438	-1.365	15.68	0.140571	0.404	-1.43	0.201373	-2.154	4.12
22	2762.5	-1.365156	0.864445	-1.310	14.39	0.076133	-2.987	15.39	0.188193	-2.149	2.93
23	2787.5	-1.034255	0.912251	-1.496	13.61	0.709772	-1.306	12.22	0.192088	-2.167	5.72
24	3000.0	0.114055	0.342736	-1.140	13.92	0.592965	-0.291	11.71	0.054726	0.287	0.66
25	3400.0	0.054765	0.380292	1.654	7.94	0.297193	2.135	12.19	0.086752	0.823	6.13
26	3800.0	0.033435	1.105459	1.985	10.80	0.204430	-2.237	11.79	0.019296	-1.914	7.35
27	4200.0	0.074130	1.216178	2.135	11.36	0.124699	0.907	10.47	0.033326	2.501	8.09
28	4600.0	-0.014035	0.512933	2.430	12.01	0.220640	2.227	10.32	0.005228	2.153	9.27
29	5000.0	-0.005840	0.611683	-1.466	11.36	0.125578	-2.892	11.57	0.000806	0.068	1.77
30	5400.0	0.085366	1.268341	-1.232	11.45	0.090211	-1.482	9.40	0.003590	-1.142	9.81
31	5800.0	0.558052	1.108462	-1.214	11.02	0.092826	-0.589	11.97	0.002967	-1.356	11.68
32	6200.0	1.030462	0.008722	-1.985	16.22	0.000299	-1.179	13.88	0.000001	-1.004	2.74

This is the most complete text of the thesis available. The following page(s) were not included in the copy of the thesis deposited in the Institute Archives by the author:

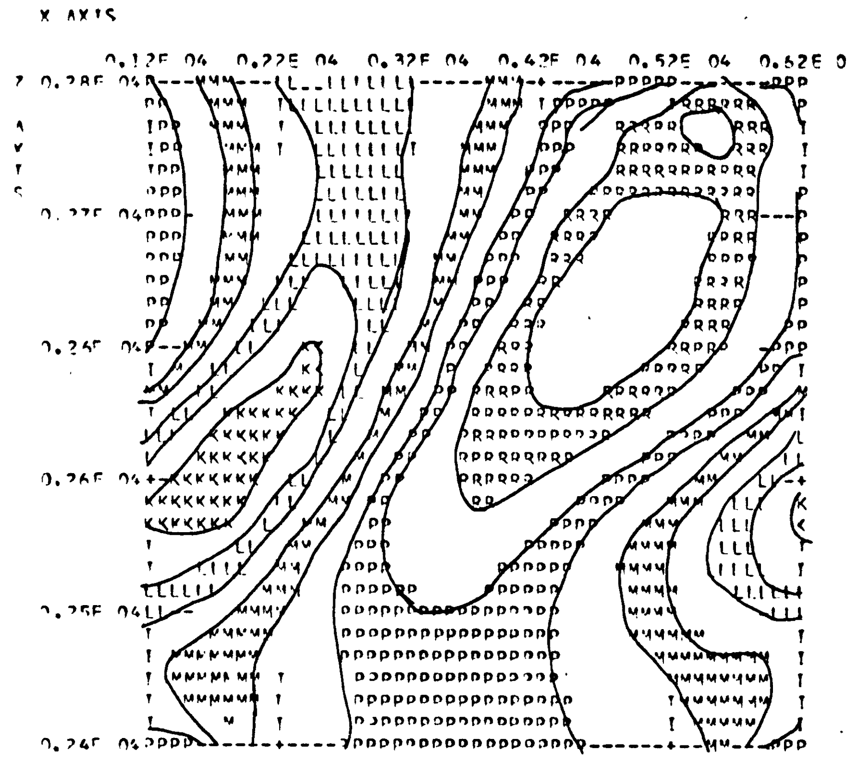
pg 125



CORRESPONDENCE BETWEEN PRINTER SYMBOLS AND CONTOUR LEVELS

K - -0.29465E-02 TO -0.24107E-02
L - -0.18750E-02 TO -0.13393E-02
M - -0.80358E-03 TO -0.26786E-03
P - 0.26786E-03 TO 0.80358E-03
R - 0.13393E-02 TO 0.18750E-02
S - 0.24107E-02 TO 0.29465E-02

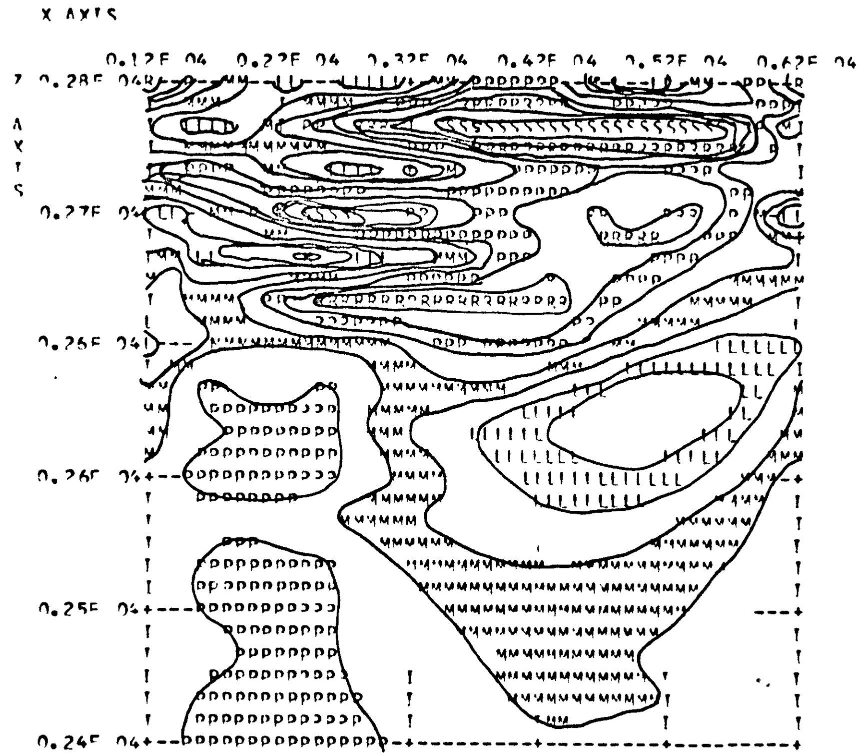
FIGURE 5.2A CONTOUR PLOT OF RHO FOR CASE B
MKS UNITS TIME STEP 75 TIME = 1110.0



CORRESPONDENCE BETWEEN PRINTER SYMBOLS AND CONTOUR LEVELS

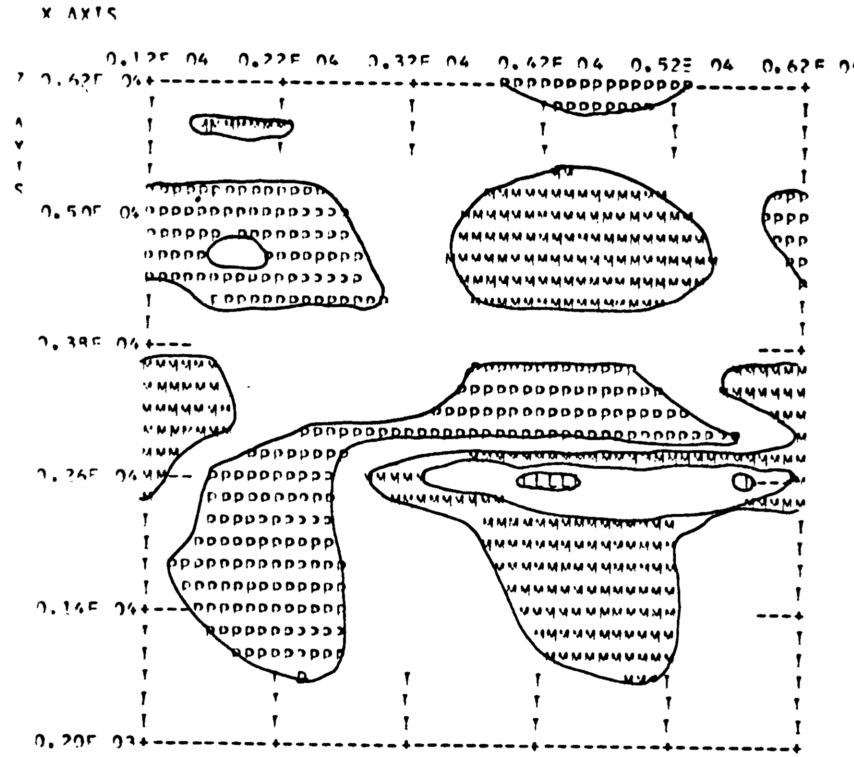
K - -0.29465E-02 TO -0.24107E-02
L - -0.18750E-02 TO -0.13393E-02
M - -0.80358E-03 TO -0.26786E-03
P - 0.26786E-03 TO 0.80358E-03
R - 0.13393E-02 TO 0.18750E-02
S - 0.24107E-02 TO 0.29465E-02

FIGURE 5.2B CONTOUR PLOT OF RHO FOR CASE B
EXPANDED REGION
MKS UNITS TIME STEP 75 TIME = 1110.0



CORRESPONDENCE BETWEEN PRINTER SYMBOLS AND CONTOUR LEVELS
K - -0.90787E-02 TO -0.74280E-02
L - -0.57773E-02 TO -0.41267E-02
M - -0.24760E-02 TO -0.82533E-03
P - 0.82534E-03 TO 0.24760E-02
R - 0.41267E-02 TO 0.57774E-02
S - 0.74280E-02 TO 0.90787E-02

FIGURE 5.2D CONTOUR PLOT OF RHO FOR CASE B
EXPANDED REGION
MKS UNITS TIME STEP 155 TIME = 2310.0



CORRESPONDENCE BETWEEN PRINTER SYMBOLS AND CONTOUR LEVELS
K - -0.90787E-02 TO -0.74280E-02
L - -0.57773E-02 TO -0.41267E-02
M - -0.24760E-02 TO -0.82533E-03
P - 0.82534E-03 TO 0.24760E-02
R - 0.41267E-02 TO 0.57774E-02
S - 0.74280E-02 TO 0.90787E-02

FIGURE 5.2C CONTOUR PLOT OF RHO FOR CASE B
MKS UNITS TIME STEP 155 TIME = 2310.0

Case C (Low Richardson Number, Critical Level in Near Field of the Source)

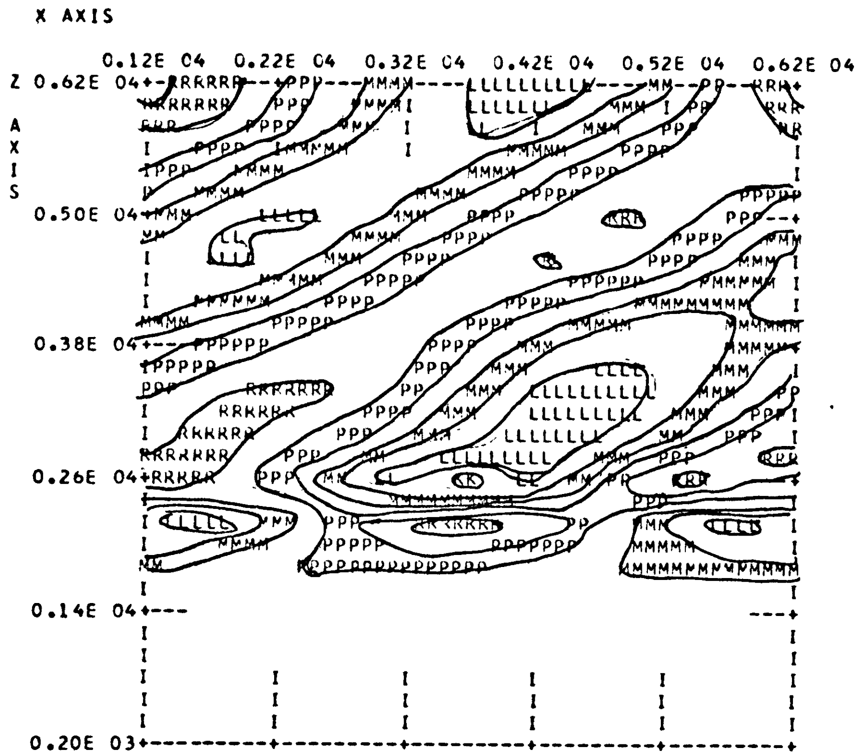
This case had no expanded region in order that the effect of the boundaries of the expanded region in case B might be assessed. The parameters which differ from those of case A are: $h = 1150$m, $z = 25$m, $z_{wb} = 400$m, $z_{wt} = 800$m, $z_s = 1137.5$m, $z_c = 622$m, and $Ri = 0.13$. Since the critical level is in the near field of the source the wave will be only slightly smaller below the critical level than it is above it. Since this was the case at the termination of case B, this case is to some extent a continuation of case B, but the second and third harmonics which were large at the termination of case B are small at the start of case C.

The large changes of u_a that occur near z_{wb} and z_{wt} in table 5.2C do not appear here so it is concluded that they are spurious effects of the vertical spacing change. In this case at 1485s u_a averages about -0.55m/s between h and z_c, about +0.40m/s for the 125m below z_c, and is small below that. By this time the third harmonics are one third the size of the fundamental, and the model blew up shortly thereafter. The blowing up is associated with large changes in u_a which appear just below the source. This localized jet creates values of Ri around 0.05 and several inflection points. This jet develops in only 180s at a height where there was no indication that anything was going to happen. It is not possible to say whether this blowing up was due to the instability of the flow since the Richardson number was less than 0.25 or was due to the unreliability of the model when the third harmonics are large.

Case D (High Richardson Number)

This case has the same parameters as case A except that $z_{wb} = z_{eb} = 1200m$, $z_c = 2089m$, and $Ri = 2.12$. The slowing down of the density and horizontal motion perturbations as the wave nears a critical level becomes more pronounced at higher Richardson numbers. The stretching out of the pattern is seen to be greater in figure 5.3B than in figure 5.1F. The horizontal phase speeds for ρ, w and u are plotted in figure 5.3C as functions of height. $\bar{u}$ has been plotted instead of $\bar{u} + u_a$ because the extremum of u_a at 2025s is only -0.24. This minimum is located about 300m above z_c. By the time the program terminated at 3000s the extremum had doubled, and the second harmonics for ρ and u were about 10% of the fundamental and as large as 100% of the fundamental for w near the critical level.

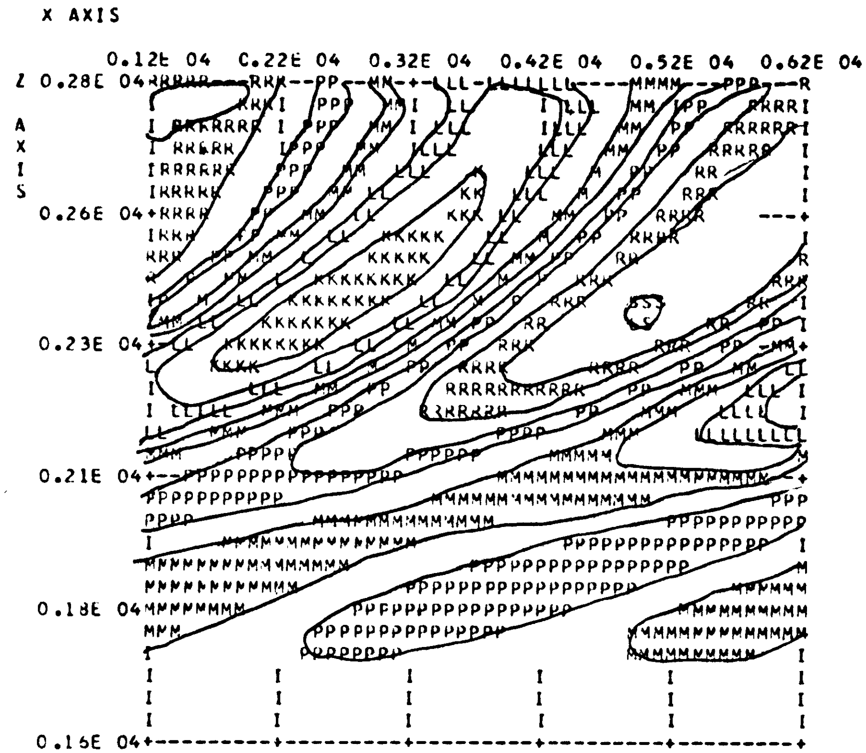
(Note that if the wave were approaching the singular level from the low speed side, the wave speed would increase rather than decrease as the wave approaches z_c)



CORRESPONDENCE BETWEEN PRINTER SYMBOLS AND CONTOUR LEVELS

K - -0.20464E-02 TO -0.16743E-02
L - -0.13023E-02 TO -0.93019E-03
M - -0.55811E-03 TO -0.18604E-03
P - 0.18604E-03 TO 0.55811E-03
R - 0.93018E-03 TO 0.13023E-02
S - 0.16743E-02 TO 0.20464E-02

FIGURE 5. 3A CONTOUR PLOT OF RHO FOR CASE D
MKS UNITS TIME STEP 136 TIME = 2025.0



CORRESPONDENCE BETWEEN PRINTER SYMBOLS AND CONTOUR LEVELS

K - -0.20464E-02 TO -0.16743E-02
L - -0.13023E-02 TO -0.93019E-03
M - -0.55811E-03 TO -0.18604E-03
P - 0.18604E-03 TO 0.55811E-03
R - 0.93018E-03 TO 0.13023E-02
S - 0.16743E-02 TO 0.20464E-02

FIGURE 5. 3B CONTOUR PLOT OF RHO FOR CASE D
EXPANDED REGION
MKS UNITS TIME STEP 136 TIME = 2025.0

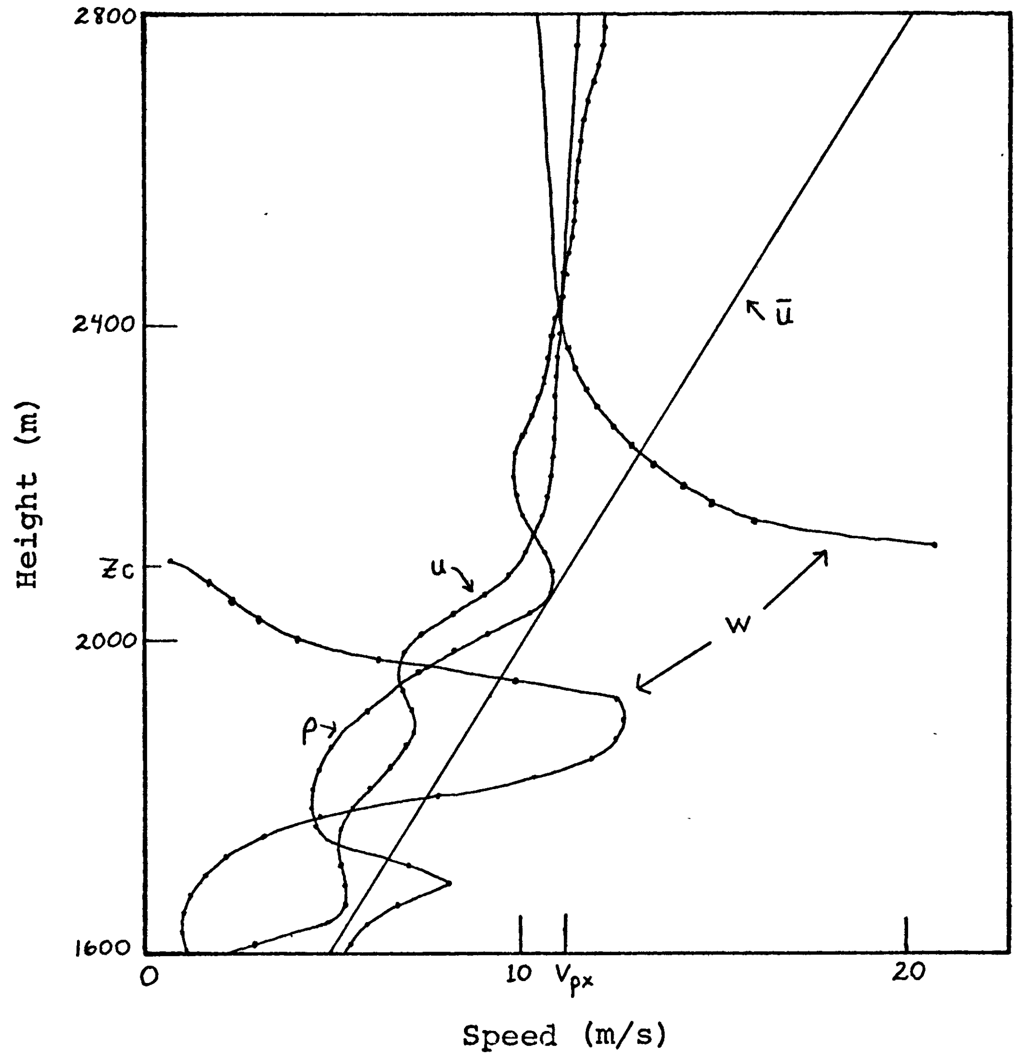


Figure 5.3C. Horizontal phase speed for ρ, u, and w, and the original wind speed, u, as functions of height for case D (Ri = 2.12) at 2025s. Only the expanded region is shown. v_{px} is the horizontal phase speed of the source. z_c is the height of the critical level for a wave whose horizontal phase speed is v_{px}.

Case E (Negative Shear)

In this case the wave approached the singular level from the low speed side rather than the high speed side. The parameters which differ from those for case A are $\overline{u}_b$ = 20m/s, $\overline{u}_t$ = 0, and z_c = 2356m. The transforms at 3465s for this case are contained in tables 5.3, which should be compared with table 5.1G,H.I. The primary differences between cases A and E are that u_a above z_c is positive here, and that the second harmonic for u is largest above z_c in case A and below z_c is case E.

Case F (High Viscosity)

$\mu = 0.02 = K$ has been used in all the other cases, and while these values are about 1000 times greater than the molecular values, they are still less than the commonly quoted eddy values. Since the two damping terms seemed to have little effect in other cases, this case with $\mu = 1.0 = K$ was run. All the other parameters are the same as in case A.

This case blew up at about 2500s due to the generation of large values for the third harmonics at z_{et}. Evidently the manner in which the finite difference analog of the Laplacian operator in equation 4.2-4A,B, treats the vertical spacing change generates large spurious values of the third harmonic. Until the third harmonic becomes large enough to make the model unreliable the results of this case are practically identical to case A. Therefore it is concluded that the viscosity and thermal conduction play a very small part in critical level phenomenon. This is confirmed by the finding that the rates of viscous energy dissipation in other cases would be negligible even with the large values of μ and K used here.

Case G (Small Amplitude)

The only change from case A here was that a source one fifth the magnitude of that for case A was used: $s_s = 0.225$. As might be expected, the magnitudes of the second harmonics and u_a are about 1/25 the size that they are in case A.

TABLE 5.3A FOURIER TRANSFORM OF RHO BY ROWS FOR CASE E TIME STEP 232 TIME = 3465.0
MAGNITUDE AND HPS IN MKS UNITS ANGLE IN RADIANS
HPS IS THE HORIZONTAL PHASE SPEED CALCULATED FROM THE CHANGE IN THE PHASE ANGLE IN THE LAST 180.00 SECONDS

		CONSTANT	FUNDAMENTAL			SECOND HARMONIC			THIRD HARMONIC		
J	Z		MAGNITUDE	ANGLE	H P S	MAGNITUDE	ANGLE	H P S	MAGNITUDE	ANGLE	H P S
2	200.0	-0.000000	0.000107	0.209	11.56	0.000023	-3.110	11.63	0.000000	2.278	4.96
3	600.0	-0.000000	0.000254	0.124	11.49	0.000016	2.117	11.12	0.000001	3.017	4.34
4	1000.0	0.000000	0.000242	-0.145	11.52	0.000031	-0.945	11.16	0.000002	3.096	2.79
5	1400.0	0.000000	0.000130	-1.040	12.14	0.000021	-1.826	11.76	0.000006	2.469	3.77
6	1800.0	-0.000001	0.000197	-2.559	11.12	0.000033	2.591	11.19	0.000012	2.746	2.54
7	2012.5	-0.000002	0.000303	-2.899	11.07	0.000045	1.769	10.54	0.000012	2.869	2.04
8	2037.5	-0.000021	0.000375	-2.882	11.27	0.000042	0.900	11.46	0.000013	2.787	2.59
9	2062.5	-0.000074	0.000404	-2.902	10.99	0.000157	0.833	-2.15	0.000009	2.766	3.08
10	2087.5	-0.000011	0.000386	-2.789	10.91	0.000355	1.296	11.59	0.000015	1.995	5.27
11	2112.5	0.000010	0.000372	-2.267	10.76	0.000568	1.907	11.41	0.000027	2.265	6.89
12	2137.5	-0.000006	0.000496	3.047	11.06	0.000601	2.693	11.30	0.000023	2.307	0.75
13	2162.5	-0.000034	0.000713	3.065	11.34	0.000548	3.129	11.26	0.000029	2.134	2.19
14	2187.5	-0.000035	0.000879	3.122	11.09	0.000532	-2.811	11.26	0.000031	2.457	2.09
15	2212.5	-0.000086	0.001033	3.020	11.15	0.000396	-2.530	11.58	0.000025	2.427	2.09
16	2237.5	-0.000097	0.001452	3.021	11.61	0.000289	-1.910	11.27	0.000032	1.660	3.06
17	2262.5	-0.000029	0.001621	-2.051	11.50	0.000496	-1.453	10.55	0.000058	1.847	3.14
18	2287.5	-0.000221	0.001407	-2.207	10.52	0.000941	-1.643	10.80	0.000096	2.894	2.91
19	2312.5	-0.001136	0.001426	-2.981	9.95	0.001119	-1.740	10.85	0.000116	-3.120	2.16
20	2337.5	-0.001763	0.001554	2.758	11.21	0.000257	-1.930	11.61	0.000028	1.959	0.41
21	2362.5	-0.000084	0.001267	-2.908	12.23	0.000198	1.399	10.96	0.000043	0.001	0.60
22	2387.5	0.000617	0.002461	-2.275	11.50	0.000092	-2.179	10.22	0.000027	-2.009	2.80
23	2412.5	0.000404	0.002829	-2.102	10.92	0.000153	2.412	11.65	0.000041	1.078	4.93
24	2437.5	0.000633	0.002613	-1.906	10.94	0.000217	3.069	10.55	0.000059	1.037	3.51
25	2462.5	0.000297	0.002532	-1.932	10.87	0.000190	2.959	-2.16	0.000063	1.232	3.10
26	2487.5	0.000371	0.002177	-1.780	10.51	0.000209	2.624	11.42	0.000059	1.060	3.16
27	2512.5	0.000291	0.001933	-1.731	10.63	0.000224	3.003	10.64	0.000058	0.991	3.12
28	2537.5	0.000209	0.001692	-1.696	10.75	0.000212	-3.091	10.71	0.000059	1.238	2.78
29	2562.5	0.000126	0.001510	-1.697	10.87	0.000163	-3.059	11.19	0.000043	1.406	2.41
30	2587.5	0.000084	0.001374	-1.676	10.81	0.000144	3.041	-1.85	0.000031	1.241	2.78
31	2612.5	0.000284	0.001265	-1.675	10.68	0.000151	3.097	11.57	0.000034	1.080	3.49
32	2637.5	0.000084	0.001158	-1.576	10.63	0.000142	-3.070	10.49	0.000034	1.203	3.59
33	2662.5	0.000067	0.001051	-1.562	10.91	0.000131	-2.282	10.39	0.000032	1.297	3.22
34	2687.5	0.000045	0.000956	-1.573	11.05	0.000106	-2.892	10.30	0.000030	1.462	2.62
35	2712.5	0.000032	0.000973	-1.585	11.22	0.000079	-3.018	10.99	0.000020	1.463	2.11
36	2737.5	0.000050	0.000869	-1.570	11.26	0.000069	-3.009	10.94	0.000018	1.302	3.90
37	2762.5	0.000008	0.000853	-1.577	11.27	0.000045	3.041	10.89	0.000014	1.003	4.71
38	2787.5	-0.000004	0.000721	-1.556	11.17	0.000070	-2.227	11.61	0.000017	1.176	3.31
39	3000.0	-0.000006	0.000766	-1.298	10.64	0.000025	-2.773	9.89	0.000012	1.280	3.38
40	3400.0	-0.000012	0.000819	-0.272	10.43	0.000014	-2.844	12.69	0.000005	1.753	2.11
41	3800.0	-0.000002	0.000843	-0.771	10.69	0.000013	2.900	-0.11	0.000002	1.362	2.60
42	4200.0	-0.000002	0.000764	-0.423	10.34	0.000012	-0.594	3.02	0.000001	1.625	2.06
43	4600.0	0.000001	0.000580	0.021	9.14	0.000010	1.313	2.62	0.000001	-1.706	6.87
44	5000.0	-0.000001	0.000524	0.840	7.98	0.000011	1.151	3.80	0.000000	-2.531	0.74
45	5400.0	-0.000005	0.000703	1.513	11.47	0.000014	-2.816	3.92	0.000001	1.786	1.30
46	5800.0	0.000000	0.000267	1.915	12.03	0.000011	-1.536	3.00	0.000001	2.455	1.31
47	6200.0	-0.000000	0.001111	2.053	11.91	0.000052	-0.168	0.47	0.000001	2.635	0.74

TABLE 5.3B FOURIER TRANSFORM OF W BY ROWS FOR CASE E TIME STEP 232 TIME = 3465.0
MAGNITUDE AND HPS IN MKS UNITS ANGLE IN RADIANS
HPS IS THE HORIZONTAL PHASE SPEED CALCULATED FROM THE CHANGE IN THE PHASE ANGLE IN THE LAST 180.00 SECONDS

		CONSTANT	FUNDAMENTAL			SECOND HARMONIC			THIRD HARMONIC		
J	Z		MAGNITUDE	ANGLE	H P S	MAGNITUDE	ANGLE	H P S	MAGNITUDE	ANGLE	H P S
2	400.0	0.000003	0.045620	1.661	11.38	0.004807	-1.802	11.44	0.000085	-1.215	3.15
3	800.0	-0.000000	0.045015	1.510	11.35	0.005024	2.258	11.05	0.000163	-1.707	3.95
4	1200.0	0.0	0.046017	1.101	11.30	0.005283	0.932	11.06	0.000411	-2.105	4.80
5	1600.0	-0.000003	0.033700	-0.573	11.38	0.006407	-1.443	-2.12	0.001040	2.426	7.12
6	2000.0	0.000003	0.078808	-1.462	11.58	0.003733	-2.990	10.70	0.004430	2.157	0.55
7	2025.0	0.000003	0.082500	-1.500	11.50	0.010073	-2.990	10.75	0.004800	2.141	0.66
8	2050.0	0.000000	0.085250	-1.533	11.61	0.003702	-3.010	10.60	0.005144	2.120	0.83
9	2075.0	-0.000000	0.082517	-1.567	11.63	0.006617	-3.031	9.64	0.005428	2.003	1.06
10	2100.0	-0.000000	0.091057	-1.594	11.65	0.004348	1.628	10.74	0.005700	2.035	1.21
11	2125.0	0.000000	0.094600	-1.613	11.67	0.017464	1.420	11.25	0.006263	1.987	0.72
12	2150.0	0.000000	0.095197	-1.634	11.68	0.032051	1.606	11.19	0.006560	1.931	0.36
13	2175.0	0.000000	0.094063	-1.671	11.71	0.044950	1.951	11.15	0.007048	1.831	0.58
14	2200.0	-0.000000	0.092350	-1.711	11.70	0.054277	2.170	11.16	0.007062	1.735	0.80
15	2225.0	-0.000000	0.087039	-1.735	11.83	0.060560	2.419	11.16	0.009038	1.644	0.96
16	2250.0	-0.000000	0.079045	-1.773	11.91	0.072203	2.700	11.03	0.010818	1.565	1.12
17	2275.0	-0.000000	0.066006	-1.796	12.31	0.094046	2.879	10.02	0.014168	1.553	1.28
18	2300.0	0.000000	0.047112	-1.634	13.16	0.110578	2.962	10.00	0.016807	1.593	1.35
19	2325.0	0.000000	0.023038	-1.136	13.61	0.104371	2.978	10.93	0.014918	1.675	1.37
20	2350.0	0.000000	0.003308	0.083	12.88	0.084652	3.023	10.75	0.010392	1.921	1.63
21	2375.0	0.000000	0.052075	2.558	10.48	0.070306	3.033	10.93	0.007401	2.235	2.08
22	2400.0	-0.000000	0.110520	2.601	10.70	0.060782	2.952	10.92	0.004870	2.440	1.86
23	2425.0	-0.000000	0.173710	2.776	10.94	0.053557	2.770	10.88	0.001306	2.347	1.07
24	2450.0	-0.000000	0.226671	2.835	10.95	0.049665	2.567	10.93	0.002057	-0.206	3.83
25	2475.0	0.0	0.262085	2.806	10.82	0.047736	2.369	11.03	0.004300	-0.366	3.62
26	2500.0	-0.000000	0.287637	2.935	10.80	0.049756	2.220	11.06	0.005825	-0.454	3.52
27	2525.0	-0.000000	0.305713	2.975	10.80	0.050045	2.147	11.01	0.007017	-0.457	3.34
28	2550.0	-0.000000	0.319180	3.000	10.82	0.051716	2.113	10.90	0.007530	-0.416	3.13
29	2575.0	-0.000000	0.322617	3.040	10.82	0.050905	2.084	11.03	0.007433	-0.322	3.02
30	2600.0	-0.000000	0.328012	3.070	10.92	0.050020	2.055	11.00	0.007180	-0.410	3.05
31	2625.0	-0.000000	0.344894	3.100	10.81	0.049652	2.040	11.00	0.007062	-0.434	3.13
32	2650.0	0.000000	0.350447	3.122	10.82	0.049124	2.043	11.02	0.006936	-0.437	3.16
33	2675.0	-0.000003	0.355042	-3.120	10.84	0.048040	2.055	10.94	0.006603	-0.424	3.12
34	2700.0	0.000000	0.359142	-3.106	10.86	0.046260	2.074	10.36	0.006317	-0.406	3.07
35	2725.0	-0.000002	0.363187	-3.086	10.80	0.044178	2.080	10.81	0.005900	-0.398	3.03
36	2750.0	-0.000003	0.367276	-3.063	10.00	0.042033	2.102	10.76	0.005406	-0.390	3.00
37	2775.0	-0.000003	0.371432	-3.041	10.02	0.039777	2.097	10.74	0.005059	-0.386	3.01
38	2800.0	0.000000	0.380036	-3.010	10.02	0.027732	2.084	10.72	0.005025	-0.402	3.12
39	3200.0	-0.000003	0.427817	-2.703	10.80	0.014222	1.800	10.67	0.003451	-0.515	3.92
40	3600.0	0.000003	0.450533	-2.410	10.74	0.011457	1.712	-0.80	0.001743	-0.587	3.92
41	4000.0	0.000000	0.437166	-2.135	0.88	0.003505	1.262	9.11	0.000547	-0.382	4.62
42	4400.0	0.0	0.391329	-1.805	2.52	0.005656	0.061	-0.34	0.000357	-0.056	3.02
43	4800.0	0.000000	0.312027	-1.284	2.18	0.007041	0.441	0.88	0.000138	0.003	-0.67
44	5200.0	-0.000000	0.309382	-0.440	2.78	0.005955	0.288	-1.41	0.000320	1.452	0.51
45	5600.0	-0.000003	0.477251	0.261	10.50	0.001225	2.138	5.00	0.000084	0.350	2.58
46	6000.0	0.000000	0.727706	0.596	10.91	0.002628	3.036	3.51	0.000078	0.606	1.91
47	6400.0	-0.000000	0.763626	0.591	10.91	0.002747	3.032	3.50	0.000082	0.600	1.90

TABLE 5.3C FOURIER TRANSFORM OF U BY ROWS FOR CASE E TIME STEP 232 TIME = 3465.0
MAGNITUDE AND HPS IN MKS UNITS ANGLE IN RADIANS
HPS IS THE HORIZONTAL PHASE SPEED CALCULATED FROM THE CHANGE IN THE PHASE ANGLE IN THE LAST 180.00 SECONDS

J	Z	CONSTANT	FUNDAMENTAL MAGNITUDE	FUNDAMENTAL ANGLE	H P S	SECOND HARMONIC MAGNITUDE	SECOND HARMONIC ANGLE	H P S	THIRD HARMONIC MAGNITUDE	THIRD HARMONIC ANGLE	H P S
2	200.0	-0.000221	0.000097	-2.659	11.38	0.005188	0.554	11.44	0.000070	1.534	3.14
3	600.0	-0.000672	0.038131	3.087	11.46	0.000737	-2.129	11.20	0.000078	0.580	4.58
4	1000.0	-0.000687	0.063292	0.045	11.66	0.007442	2.331	-2.17	0.000216	0.396	5.60
5	1400.0	-0.000302	0.121760	0.496	11.47	0.011713	0.514	11.13	0.000994	-1.468	6.77
6	1800.0	-0.001356	0.127816	0.063	11.69	0.012090	-1.170	11.12	0.002841	-1.462	0.98
7	2012.5	-0.002287	0.119062	-0.270	11.88	0.018115	-1.257	10.67	0.004151	-1.625	1.55
8	2037.5	-0.002741	0.123342	-0.389	12.16	0.008547	3.098	11.00	0.004672	-1.714	2.28
9	2062.5	0.002113	0.135124	-0.380	12.17	0.054091	2.509	-2.04	0.004223	-1.899	3.48
10	2087.5	0.014086	0.131643	-0.282	12.04	0.143454	3.030	11.62	0.006339	-2.243	6.47
11	2112.5	0.023612	0.206666	-0.200	12.26	0.233475	-2.575	11.43	0.007322	-2.095	-0.96
12	2137.5	0.022725	0.263854	-1.072	12.42	0.281202	-1.927	11.30	0.006132	-2.496	0.98
13	2162.5	0.031055	0.126253	-1.641	12.33	0.281785	-1.443	11.25	0.011132	-2.621	2.07
14	2187.5	0.053695	0.137016	-1.780	11.34	0.252071	-1.004	11.33	0.015500	-2.430	1.07
15	2212.5	0.085130	0.166576	-2.468	11.14	0.272784	-0.459	11.00	0.017649	-2.481	1.92
16	2237.5	0.132357	0.317555	-2.509	11.27	0.395227	-0.344	10.69	0.025921	-2.349	1.92
17	2262.5	0.111789	0.436241	-2.826	9.08	0.461862	-0.534	10.69	0.044879	-2.023	2.10
18	2287.5	0.010837	0.692657	2.962	10.74	0.309352	-0.632	11.10	0.036000	-1.226	2.89
19	2312.5	0.252845	0.857546	2.076	11.75	0.132556	1.646	11.10	0.031429	0.650	1.67
20	2337.5	0.067179	0.384029	-2.303	12.22	0.361516	2.004	10.83	0.072764	0.859	1.27
21	2362.5	1.425561	1.758431	-1.735	11.37	0.258030	2.180	11.07	0.056475	0.859	1.10
22	2387.5	1.314019	2.188002	-1.525	11.04	0.105634	2.708	11.27	0.038200	1.485	2.20
23	2412.5	1.035393	1.002017	-1.378	10.96	0.218260	3.012	11.12	0.049445	2.081	3.04
24	2437.5	0.734005	1.594130	-1.260	10.80	0.205633	3.105	11.16	0.043632	2.316	2.98
25	2462.5	0.510041	1.204243	-1.113	10.65	0.174041	3.043	10.86	0.031022	2.240	2.83
26	2487.5	0.370873	0.812760	-0.910	10.57	0.126770	-3.091	10.07	0.022091	2.058	2.59
27	2512.5	0.232283	0.680717	-0.760	10.70	0.074148	-2.806	9.62	0.014276	2.275	1.65
28	2537.5	0.134461	0.535011	-0.622	10.75	0.032946	-3.020	10.37	0.007812	2.865	0.69
29	2562.5	0.081228	0.446326	-0.400	10.65	0.030781	2.297	9.84	0.002852	-1.769	-1.35
30	2587.5	0.065775	0.407888	-0.320	10.43	0.031884	2.251	9.01	0.004117	-0.355	7.13
31	2612.5	0.061327	0.387550	-0.181	10.34	0.015910	2.210	9.44	0.002963	0.079	0.00
32	2637.5	0.048794	0.356807	-0.063	10.38	0.012126	1.067	-1.57	0.002033	-0.662	2.50
33	2662.5	0.029458	0.317117	0.018	10.53	0.024655	0.771	-0.84	0.003764	-1.154	3.94
34	2687.5	0.014000	0.296337	0.061	10.77	0.036977	0.871	-1.26	0.005601	-1.105	4.33
35	2712.5	0.007226	0.272856	0.076	11.00	0.041005	0.907	-1.05	0.005907	-0.895	4.41
36	2737.5	0.004700	0.275034	0.095	11.14	0.041295	1.077	-2.17	0.006943	-0.882	3.30
37	2762.5	0.003676	0.282113	0.113	11.23	0.042160	1.401	11.31	0.004917	-0.842	2.01
38	2787.5	0.004077	0.288182	0.117	11.28	0.046262	1.487	11.43	0.001772	-0.120	0.71
39	3000.0	0.006251	0.266746	0.421	10.32	0.027752	1.455	10.23	0.001507	-0.577	1.71
40	3400.0	0.009066	0.261446	0.063	9.64	0.003077	1.270	9.47	0.001556	-0.942	3.87
41	3800.0	0.009424	0.258007	1.535	10.93	0.000715	1.106	-0.87	0.001072	-1.065	3.57
42	4200.0	0.003185	0.314039	2.067	10.72	0.005959	1.742	1.56	0.000216	-1.225	5.63
43	4600.0	0.005370	0.402048	2.423	10.63	0.002894	-2.524	0.91	0.000197	-0.535	3.82
44	5000.0	0.003680	0.520008	2.734	10.40	0.001072	0.164	0.70	0.000273	-1.640	1.06
45	5400.0	0.007423	0.610100	2.964	10.37	0.007105	-0.677	1.54	0.000261	1.299	5.30
46	5800.0	0.011647	0.603410	3.127	10.76	0.002251	-0.382	6.29	0.000020	-1.051	4.55
47	6200.0	0.011640	0.003030	1.301	10.92	0.000013	1.436	10.87	0.000000	-2.062	7.47

Case H (Large Amplitude)

This case is the same as case A except that the source is five times larger: $s_s = 5.625$m/s. The magnitudes of the wave motions are over 20% of u at the source, so that the second and third harmonics increase in size rapidly. By 1665s the third harmonic was more than 10% of the fundamental and the model blew up shortly thereafter. This is the only case in which the incident wave contained a significant amount of second harmonic before it reached the shear layer.

Case I (Source at the critical level)

The parameters which differ from case A are: $\tau = 516$s, $v_{px} = 9.69$, and $z_s = z_c = 2387.5$. Row transforms for this case may be found in table 5.4. The line vorticity source normally is associated with a minimum for u and a maximum for w, but here a maximum for u and a minimum for w occur about 100m below the source. The energy going downward from the source is reflected by the ground and a standing wave is evident below the source. Note that the wave which propagates upward has a larger w/u ratio than the wave generated by the same source in other cases, and that the wave magnitudes above the shear layer are about one fourth their values in case A.

TABLE 5.4A FOURIER TRANSFORM OF RHO BY ROWS FOR CASE I TIME STEP 88 TIME = 1305.0
MAGNITUDE AND HPS IN MKS UNITS ANGLE IN RADIANS
HPS IS THE HORIZONTAL PHASE SPEED CALCULATED FROM THE CHANGE IN THE PHASE ANGLE IN THE LAST 180.00 SECONDS

J	Z	CONSTANT	FUNDAMENTAL			SECOND HARMONIC			THIRD HARMONIC		
			MAGNITUDE	ANGLE	H P S	MAGNITUDE	ANGLE	H P S	MAGNITUDE	ANGLE	H P S
2	200.0	0.000000	0.000109	-1.711	8.10	0.000000	-2.088	8.88	0.000000	-2.854	4.26
3	600.0	-0.000003	0.000438	-1.764	8.52	0.000009	0.658	1.34	0.000001	1.177	-0.17
4	1000.0	-0.000005	0.000722	-1.936	10.70	0.000016	0.502	8.67	0.000001	3.052	1.31
5	1400.0	-0.000004	0.000851	-1.891	11.04	0.000020	1.109	6.08	0.000001	2.348	3.82
6	1800.0	0.000000	0.000854	-1.808	10.94	0.000030	1.172	8.58	0.000005	3.046	1.13
7	2012.5	-0.000005	0.000724	-1.876	12.56	0.000031	1.117	8.77	0.000010	1.752	-1.18
8	2037.5	-0.000012	0.000705	-1.675	10.94	0.000128	1.063	8.73	0.000026	2.386	1.45
9	2062.5	-0.000059	0.000776	-1.543	9.59	0.000117	0.600	8.58	0.000010	1.765	2.19
10	2087.5	-0.000075	0.000896	-1.539	8.81	0.000102	0.152	0.69	0.000007	1.485	3.70
11	2112.5	-0.000038	0.000977	-1.669	8.90	0.000068	-0.397	0.96	0.000001	0.243	5.64
12	2137.5	0.000026	0.000992	-1.640	9.82	0.000029	-1.284	2.01	0.000009	-2.245	-0.19
13	2162.5	0.000066	0.000935	-1.947	11.44	0.000029	2.536	6.68	0.000021	-2.822	0.66
14	2187.5	0.000044	0.000844	-1.892	12.35	0.000080	1.598	8.04	0.000035	2.988	1.72
15	2212.5	-0.000035	0.000834	-1.662	11.54	0.000129	1.042	8.57	0.000044	2.601	2.42
16	2237.5	-0.000120	0.000972	-1.474	10.25	0.000149	0.564	8.95	0.000044	2.268	2.78
17	2262.5	-0.000139	0.001153	-1.468	9.50	0.000123	0.097	9.55	0.000037	2.000	3.12
18	2287.5	-0.000081	0.001229	-1.590	9.35	0.000067	-0.400	11.31	0.000024	1.913	3.61
19	2312.5	0.000079	0.001215	-1.722	9.66	0.000032	-1.976	3.29	0.000013	2.858	2.75
20	2337.5	0.000185	0.001083	-1.915	10.89	0.000168	1.961	8.49	0.000072	2.826	2.45
21	2362.5	0.000030	0.000973	-0.939	10.42	0.000061	0.422	11.18	0.000055	1.980	3.08
22	2387.5	-0.000000	0.001658	-0.899	9.74	0.000019	-1.516	2.52	0.000061	2.059	3.12
23	2412.5	-0.000000	0.001797	-1.316	9.91	0.000091	3.038	6.18	0.000068	2.079	3.15
24	2437.5	0.000398	0.001549	-1.907	10.12	0.000097	0.840	8.27	0.000079	0.985	3.22
25	2462.5	0.000336	0.001100	-2.570	10.52	0.000105	0.544	8.20	0.000045	0.970	3.30
26	2487.5	0.000147	0.000586	3.001	11.30	0.000102	-0.436	9.29	0.000047	1.602	2.65
27	2512.5	0.000008	0.000307	1.689	13.95	0.000117	-1.223	9.81	0.000060	1.557	3.01
28	2537.5	-0.000047	0.000371	0.305	14.24	0.000127	-1.940	9.94	0.000058	1.339	3.51
29	2562.5	-0.000031	0.000471	-0.530	12.33	0.000128	-2.685	9.88	0.000046	1.129	3.99
30	2587.5	0.000009	0.000462	-1.166	12.17	0.000123	2.799	9.85	0.000036	1.054	4.21
31	2612.5	0.000031	0.000373	-1.736	12.33	0.000113	1.954	10.07	0.000031	1.111	4.08
32	2637.5	0.000021	0.000176	-2.480	13.16	0.000095	1.060	10.49	0.000030	1.085	3.86
33	2662.5	-0.000003	0.000116	1.645	19.77	0.000066	0.018	11.13	0.000024	0.877	3.74
34	2687.5	-0.000018	0.000247	0.644	20.62	0.000050	-1.582	-1.13	0.000015	0.494	3.73
35	2712.5	-0.000014	0.000310	0.023	18.50	0.000076	-3.012	13.52	0.000004	-0.013	3.81
36	2737.5	0.000002	0.000311	-0.531	14.44	0.000103	2.280	-1.11	0.000006	2.739	-0.98
37	2762.5	0.000013	0.000246	-0.825	12.43	0.000126	1.427	-1.23	0.000025	1.671	4.98
38	2787.5	0.000002	0.000125	-1.102	11.04	0.000077	0.132	-1.20	0.000011	1.050	4.10
39	3000.0	-0.000001	0.000174	-0.067	6.07	0.000017	2.718	4.47	0.000003	1.779	2.54
40	3400.0	-0.000000	0.000202	0.028	7.18	0.000020	1.869	6.66	0.000006	3.075	-0.53
41	3800.0	0.000000	0.000233	0.457	4.88	0.000004	0.218	10.78	0.000004	2.532	-0.74
42	4200.0	0.000001	0.000207	0.377	7.74	0.000026	2.260	4.22	0.000002	1.779	-1.16
43	4600.0	0.000000	0.000039	-1.537	16.19	0.000018	0.895	10.69	0.000001	1.326	7.62
44	5000.0	-0.000001	0.000128	-3.042	16.61	0.000014	2.478	6.89	0.000000	-0.246	1.45
45	5400.0	-0.000002	0.000233	-2.815	8.75	0.000013	2.366	2.34	0.000000	-1.591	3.93
46	5800.0	-0.000002	0.000390	-2.455	6.74	0.000008	0.949	4.83	0.000000	2.904	3.95
47	6200.0	0.000002	0.000448	-2.315	6.00	0.000010	0.434	-1.92	0.000000	1.747	6.80

TABLE 5.4B FOURIER TRANSFORM OF W BY ROWS FOR CASE I TIME STEP 88 TIME = 1305.0
MAGNITUDE AND HPS IN MKS UNITS ANGLE IN RADIANS
HPS IS THE HORIZONTAL PHASE SPEED CALCULATED FROM THE CHANGE IN THE PHASE ANGLE IN THE LAST 180.00 SECONDS

J	Z	CONSTANT	FUNDAMENTAL MAGNITUDE	FUNDAMENTAL ANGLE	H P S	SECOND HARMONIC MAGNITUDE	SECOND HARMONIC ANGLE	H P S	THIRD HARMONIC MAGNITUDE	THIRD HARMONIC ANGLE	H P S
2	400.0	-0.000000	0.113045	2.488	11.40	0.003326	-1.475	9.50	0.000049	2.714	2.19
3	800.0	0.000000	0.180255	2.669	10.21	0.003428	-0.866	6.08	0.000165	1.343	3.45
4	1200.0	0.0	0.221806	2.820	9.29	0.006421	-0.324	6.09	0.000235	0.853	4.57
5	1600.0	0.000000	0.219957	2.885	9.15	0.009503	-0.176	7.29	0.001388	1.254	2.81
6	2000.0	-0.000000	0.168887	3.034	8.85	0.008626	-0.005	7.73	0.002869	1.330	2.31
7	2025.0	0.000000	0.167745	3.043	8.83	0.008216	-0.013	7.85	0.003024	1.342	2.29
8	2050.0	-0.000000	0.164050	3.044	8.86	0.008099	0.055	7.78	0.003489	1.394	2.19
9	2075.0	-0.000000	0.158690	3.044	8.94	0.009500	0.154	7.58	0.004264	1.420	2.11
10	2100.0	0.000000	0.151844	3.053	8.99	0.012045	0.130	7.59	0.005064	1.384	2.13
11	2125.0	0.000000	0.144671	3.082	8.97	0.014717	0.009	7.75	0.005719	1.321	2.23
12	2150.0	-0.000000	0.138638	3.126	8.84	0.016563	-0.138	7.97	0.006043	1.262	2.37
13	2175.0	-0.000000	0.133969	-3.112	8.67	0.016981	-0.252	8.20	0.005945	1.248	2.45
14	2200.0	0.000000	0.129010	-3.084	8.54	0.016175	-0.278	8.39	0.005678	1.328	2.37
15	2225.0	-0.000000	0.121502	-3.073	8.49	0.015652	-0.176	8.44	0.005892	1.494	2.14
16	2250.0	0.000000	0.110391	-3.057	8.47	0.017334	-0.031	8.36	0.006953	1.620	1.94
17	2275.0	-0.000000	0.096592	-3.003	8.38	0.020936	0.018	8.36	0.008360	1.637	1.87
18	2300.0	-0.000000	0.082694	-2.875	8.05	0.024212	-0.019	8.43	0.009270	1.594	1.88
19	2325.0	-0.000000	0.070716	-2.667	7.36	0.025086	-0.069	8.48	0.009085	1.559	1.90
20	2350.0	0.000000	0.059550	-2.405	6.33	0.022983	-0.047	8.38	0.008216	1.636	1.93
21	2375.0	0.000000	0.044750	-1.982	4.43	0.020163	0.029	8.04	0.007627	1.838	1.97
22	2400.0	0.000000	0.025665	-1.354	0.80	0.017973	0.093	7.29	0.007540	2.065	2.06
23	2425.0	0.0	0.027098	-0.135	21.61	0.013732	0.126	4.22	0.008063	2.285	2.18
24	2450.0	0.000000	0.054436	0.032	19.28	0.007859	0.143	2.00	0.008878	2.344	2.33
25	2475.0	-0.000000	0.075114	-0.080	18.28	0.003802	0.540	0.50	0.008911	2.256	2.47
26	2500.0	0.000000	0.084177	-0.219	17.75	0.004491	1.373	-1.85	0.007938	2.188	2.50
27	2525.0	0.000000	0.083037	-0.317	17.87	0.007282	1.421	11.44	0.006677	2.227	2.37
28	2550.0	0.000000	0.076755	-0.335	19.24	0.009526	1.206	11.40	0.005930	2.366	2.10
29	2575.0	0.000000	0.072276	-0.266	20.16	0.010606	0.926	11.53	0.005731	2.491	1.85
30	2600.0	0.000000	0.073429	-0.172	19.52	0.010207	0.646	-2.12	0.005615	2.557	1.68
31	2625.0	0.000000	0.079221	-0.130	18.50	0.008445	0.450	-1.78	0.005446	2.614	1.53
32	2650.0	0.000000	0.083409	-0.141	17.65	0.006387	0.506	-1.47	0.005456	2.690	1.45
33	2675.0	0.000000	0.083589	-0.166	17.05	0.006384	0.814	-1.60	0.005742	2.737	1.66
34	2700.0	0.000000	0.080558	-0.169	16.73	0.008160	0.856	-1.68	0.005974	2.713	2.12
35	2725.0	-0.000000	0.077417	-0.134	16.80	0.009028	0.661	-1.59	0.005781	2.650	2.34
36	2750.0	-0.000000	0.076773	-0.079	17.23	0.007873	0.446	-1.52	0.005193	2.628	2.20
37	2775.0	-0.000000	0.078627	-0.033	17.58	0.005340	0.542	-1.89	0.004778	2.663	1.96
38	2800.0	0.000000	0.081832	-0.008	17.58	0.004655	1.146	10.92	0.004713	2.699	1.82
39	3200.0	-0.000000	0.088003	0.232	15.12	0.007778	1.282	11.48	0.002676	2.785	6.93
40	3600.0	0.000000	0.111744	0.435	20.61	0.003266	1.220	9.04	0.000993	1.843	7.44
41	4000.0	-0.000000	0.128959	0.400	1.48	0.008665	2.017	0.23	0.000543	0.700	-1.02
42	4400.0	-0.000000	0.126211	0.368	3.11	0.002439	1.110	11.26	0.000252	-0.353	-0.02
43	4800.0	0.000000	0.147539	0.271	2.72	0.005770	-2.957	12.80	0.000195	-1.379	-1.41
44	5200.0	-0.000000	0.151234	-0.141	4.85	0.002492	-0.912	8.24	0.000134	-1.651	-0.44
45	5600.0	-0.000000	0.153901	-0.752	8.67	0.007479	-0.451	5.82	0.000056	-0.289	-1.38
46	6000.0	0.000000	0.182222	-1.145	11.35	0.003535	-1.458	3.45	0.000114	0.419	7.13
47	6400.0	-0.000000	0.150910	-1.275	12.29	0.004217	-2.602	6.04	0.000074	0.409	7.20

TABLE 5.4C FOURIER TRANSFORM OF U BY ROWS FOR CASE I TIME STEP 88 TIME = 1305.0
MAGNITUDE AND HPS IN MKS UNITS ANGLE IN RADIANS
HPS IS THE HORIZONTAL PHASE SPEED CALCULATED FROM THE CHANGE IN THE PHASE ANGLE IN THE LAST 180.00 SECONDS

J	Z	CONSTANT	FUNDAMENTAL MAGNITUDE	FUNDAMENTAL ANGLE	H P S	SECOND HARMONIC MAGNITUDE	SECOND HARMONIC ANGLE	H P S	THIRD HARMONIC MAGNITUDE	THIRD HARMONIC ANGLE	H P S
2	200.0	0.000511	0.225443	-1.832	11.40	0.003590	0.881	9.50	0.000041	-0.820	2.19
3	600.0	0.001811	0.133918	-1.327	8.48	0.002237	2.784	2.12	0.000134	-2.507	1.99
4	1000.0	0.003817	0.089919	-0.816	5.63	0.004127	2.539	7.56	0.000096	2.827	5.95
5	1400.0	0.005200	0.038577	0.813	1.81	0.003250	2.515	7.22	0.000962	-2.198	1.84
6	1800.0	0.000083	0.133127	1.318	8.98	0.002184	-1.788	3.88	0.001201	-2.128	2.11
7	2012.5	0.009482	0.126293	1.449	9.80	0.009496	-0.669	2.26	0.001615	-1.885	1.81
8	2037.5	0.021564	0.136955	1.809	7.89	0.010101	-2.087	4.80	0.006569	-1.814	1.34
9	2062.5	0.031798	0.190713	1.876	6.93	0.028747	3.026	6.71	0.010413	-1.995	1.41
10	2087.5	0.019239	0.243234	1.675	7.12	0.044662	2.394	7.13	0.010885	-2.345	1.96
11	2112.5	-0.009206	0.284272	1.383	7.96	0.054519	1.875	6.50	0.009770	-2.667	3.03
12	2137.5	-0.031937	0.295112	1.151	9.40	0.051495	1.365	5.14	0.006252	-3.089	4.14
13	2162.5	-0.026737	0.257828	1.098	11.58	0.034577	0.780	4.56	0.001945	1.494	6.68
14	2187.5	0.008884	0.211979	1.404	11.50	0.016860	-0.593	5.70	0.007363	-0.127	-0.35
15	2212.5	0.051963	0.261717	1.844	9.36	0.030362	-2.245	7.56	0.013246	-0.754	0.36
16	2237.5	0.070193	0.378762	1.884	8.43	0.051174	-3.057	8.12	0.017810	-1.318	0.95
17	2262.5	0.049293	0.495995	1.695	8.44	0.064759	2.606	8.61	0.018816	-1.809	1.37
18	2287.5	0.016087	0.593493	1.487	8.85	0.058651	2.098	8.75	0.013003	-2.322	1.59
19	2312.5	0.019581	0.654948	1.413	9.50	0.025709	1.324	4.75	0.005223	2.173	1.19
20	2337.5	0.110491	0.666976	1.583	10.17	0.039533	-1.079	9.22	0.015068	0.566	1.87
21	2362.5	0.184906	0.860068	1.930	9.88	0.058555	-1.310	9.81	0.023103	0.138	2.12
22	2387.5	0.184905	0.927329	2.565	9.57	0.045130	-1.220	9.84	0.023322	0.051	2.21
23	2412.5	0.184904	0.987023	2.753	9.79	0.076342	-0.798	8.69	0.024102	-0.071	2.33
24	2437.5	0.119161	0.913504	2.158	10.10	0.104442	-0.683	8.10	0.012643	-0.656	2.71
25	2462.5	0.025081	0.708933	1.599	10.36	0.081502	-0.967	8.01	0.010580	-2.797	4.75
26	2487.5	0.002077	0.461319	0.917	10.87	0.060330	-1.582	8.19	0.015572	2.352	4.69
27	2512.5	0.023735	0.272863	-0.042	12.29	0.049270	-2.430	8.71	0.017777	1.595	4.01
28	2537.5	0.047908	0.218146	-1.285	14.37	0.050327	2.986	9.33	0.015728	1.051	4.20
29	2562.5	0.048708	0.228812	-2.314	14.18	0.052571	2.206	9.79	0.010279	0.753	4.45
30	2587.5	0.027506	0.227718	3.123	12.86	0.051827	1.426	10.14	0.005356	0.903	4.11
31	2612.5	-0.002803	0.199220	2.374	12.61	0.045172	0.562	10.50	0.004954	1.152	3.47
32	2637.5	-0.008876	0.132181	1.606	13.02	0.037501	-0.507	11.19	0.005619	0.702	3.67
33	2662.5	-0.005442	0.069458	0.208	15.86	0.034604	-1.685	-2.07	0.005082	-0.047	4.18
34	2687.5	0.002830	0.107251	-1.272	18.62	0.031476	-2.918	11.92	0.003472	-1.390	5.37
35	2712.5	0.004874	0.142446	-2.016	18.54	0.033152	2.007	-1.53	0.005781	-2.938	6.29
36	2737.5	-0.001268	0.141888	-2.653	18.42	0.038176	0.765	-1.04	0.008328	2.439	4.63
37	2762.5	-0.010441	0.127725	3.050	17.60	0.046434	-0.535	-0.50	0.006271	1.870	3.96
38	2787.5	-0.015217	0.093670	2.707	15.56	0.053835	-1.275	-0.24	0.002894	1.371	3.68
39	3000.0	-0.015580	0.041676	-2.788	-0.16	0.003310	-2.430	2.34	0.001844	2.203	2.18
40	3400.0	-0.013354	0.056496	3.114	22.72	0.005288	0.538	0.08	0.001952	2.739	-1.22
41	3800.0	-0.013469	0.024972	2.038	5.00	0.007268	-1.538	-0.52	0.000788	1.995	7.39
42	4200.0	-0.013138	0.019809	-0.358	15.34	0.008443	1.480	1.26	0.000410	0.766	-1.46
43	4600.0	-0.012865	0.040622	1.554	4.64	0.008190	-0.328	-1.29	0.000193	0.051	-0.93
44	5000.0	-0.012817	0.125103	0.404	12.14	0.008118	2.242	6.46	0.000069	-1.329	4.93
45	5400.0	-0.014300	0.193477	-0.001	17.98	0.005687	2.128	0.07	0.000116	-2.450	7.39
46	5800.0	-0.016544	0.139421	-0.409	22.31	0.007188	-0.771	6.45	0.000065	-2.605	0.69
47	6200.0	-0.016141	0.091403	-1.881	8.13	0.004654	-1.145	0.86	0.000038	0.042	7.06

From the results of this finite difference model for these nine cases we will now procede to determine what may be concluded about such things of geophysical interest as wave absorption and transmission at singular levels, changes in the wind, and the sources of gravity waves.

One of the most important questions is how much of the wave's energy and momentum is absorbed by the wind, and where this absorption takes place. Although previous researchers have considered this problem (Hazel, 1967; Jones,1968; Lindzen, 1968), the linearized equations do not contain any terms which are capable of generating a change in the wind. So most of these workers have assumed that the wave's horizontal momentum density flux is attenuated by the factor given by Booker and Bretherton (1967),

$$f = \left| \exp(-2\pi(Ri - 0.25)^{\frac{1}{2}}) \right| \qquad 5.2\text{-}1$$

at the theoretical critical level z_c, and that the absorption took place in a thin layer. In this study no such assumption need be made because the proper nonlinear terms are included. Instead it is found that this absorption occurs over a height range of one hundred meters or more. This is shown by the wind speed changes in figure 5.4A. Three of the cases in which u_a was greater than 0.5m/s have been plotted. Note that the large changes in the shear are always decreases and are close to z_c. The

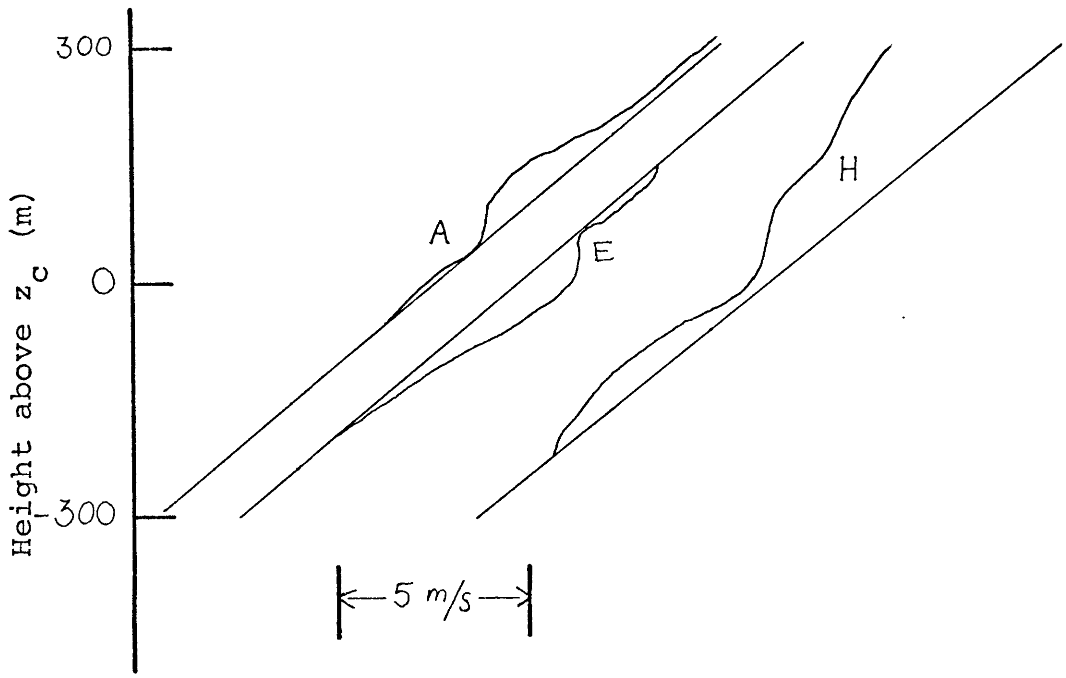


Figure 5.4A. Wind speed near the critical level. The straight lines are u, the original wind. The total wind is shown at 4185s for case A (Reference), at 3465s for case E (Negative shear), and at 1665s for case H (Large Amplitude). The plot for case E has been inverted, so that it is as if the wave was incident from below.

increases in the shear are more smoothly distributed and extend one to two hundred meters from the critical level.

Many of the qualitative features of the changes in the wind speed can be predicted from the momentum and energy relations of the linear theory, and a slight digression to present the needed equations will be made. The equation for the conservation of horizontal momentum is

$$\frac{\partial \bar{\rho} u}{\partial t} = - \vec{\nabla} \cdot (\bar{\rho} U \vec{V}) - \vec{a}_x \frac{\partial p}{\partial x} \qquad 5.2\text{-}2$$

where $\vec{\nabla} \cdot (\bar{\rho} \vec{v}) = 0$ has been used, and the vertical flux of horizontal momentum is

$$FHM_z = \bar{\rho} Uw \qquad 5.2\text{-}3$$

where U is the total horizontal motion. Ignoring the internal energy, viscous losses, etc., conservation of energy is

$$\frac{\partial}{\partial t} (\bar{\rho} \vec{V} \cdot \vec{V}/2) + g \rho w = - \vec{\nabla} \cdot (\vec{V}(\bar{\rho} \vec{V} \cdot \vec{V}/2 + p)) \qquad 5.2\text{-}4$$

Now $\vec{V} \cdot \vec{V} = (\bar{u} + U_a)^2 + 2(\bar{u} + u_a)u + u^2 + w^2$

where $\bar{u} + u_a$ is the total wind and u contains all the oscillatory motion. The $\bar{u}^2$ term is constant with respect to time and is not of interest. The term with only one oscillatory factor will average to zero, and of the three terms remaining after averaging over x, the $2\bar{u}u_a$ term will dominate the others if u_a is the same order of magnitude as u.

Writing

$$g\rho w = g\frac{\partial}{\partial t}\left[\int_0^t \rho\, w dt_1\right]$$

this term can be seen to be a potential energy term. Generally more of the wave energy is contained in this term than in the oscillatory kinetic energy. (In the linear theory this term can be shown to be $\bar{\rho}(\omega_B w/\Omega)^2$ (Claerbout and Madden, 1968). Also, in Lagrange coordinates this potential energy term would be the gravitational potential energy of a fluid parcel, but here w refers to a fixed position, so this identification is not possible.)

The vertical energy flux density is

$$FE_z = pw + (\bar{\rho}/2)\, w\left[(\bar{u} + u_a)^2 + 2u(\bar{u} + u_a) + u^2 + w^2\right] \qquad 5.2\text{-}5$$

Averaging over x

$$FHM_z = \bar{\rho}\langle uw\rangle$$

$$FE_z = \bar{\rho}(\bar{u} + u_a)\,\langle wu\rangle + \langle pw\rangle$$

and applying analytic relations from the simple linear theory:

$$FHM_z = (-m\tilde{\rho}/2k) \; |w|^2$$

$$FE_z = (-m\tilde{\rho}/2k) \; |w|^2 \; (\omega/k + u_a)$$

where the factor of one half appears because peak-to-peak amplitudes are being used. Since ω/k is greater than u_a for all cases of interest we may conclude that FE_z is always in the same direction as FHM_z and in the opposite direction to the vertical phase velocity $v_{pz} = \omega/m$. The vertical group velocity is

$$v_{gz} = \partial\omega/\partial m = - m\Omega/(m^2 + k^2)$$

and the vertical phase and group velocities have the same direction only when Ω is negative.

So for case A, above the shear layer, Ω, m, v_{pz} and v_{gz} are negative and FHM_z and FE_z are positive. That m is negative may be confirmed from tables 5.1. The wave is carrying negative momentum and negative energy downward, and the absorption of both is in accordance with the negative values of u_a just above z_c in figure 5.4A. For case E the wind is zero above the shear layer and Ω, m, and v_{pz} are positive, v_{gz} is negative, and FHM_z and FE_z are negative. The wave is carrying positive horizontal momentum and positive energy downward, and the positive values for u_a

around z_c show that the momentum and energy are being absorbed there.

Thus the changes in the wind indicate that a portion of the wave's energy and momentum are being absorbed near the critical level. The actual mechanism by which the wave is absorbed is nonlinear, and our insight into non-linear interactions is not sufficient at this time to say why the wave is absorbed farther from the critical level in case A than in case E.

It is difficult to give a quantitative figure for the portion of the incident wave's energy and momentum which is absorbed near the critical level because the incident and reflected wave cannot be easily separated above the shear layer. Some conclusions can be reached about the transmitted energy and momentum, and these will be presented shortly. A further difficulty is that, unlike the linear approximation, the relative phases between the various wave variables is not fixed, and there is no simple way of evaluating the wave pressure, so that we do not know exactly how much energy and momentum has been supplied by the source. It is noted that the phase angles above the shear layer in case A show that a partial standing wave is present, while there is less sign of a standing wave in case E. The ratio of the energy change to the horizontal momentum change for a change in the wind speed is u, and in the linear approximation, the ratio of the energy flux to the momentum flux of the wave is ω/k, so that the wave must be absorbed right at z_c if both momentum and energy are to be conserved in such an absorption. Reflections will not help create an energy/momentum balance because the incident, transmitted, and reflected wave all have the same energy/momentum ratio. For case E the absorption is roughly symmetrical around z_c and, the reflected wave and the transmitted wave are fairly small; all of which is in reasonable agreement with this linear sketch.

In this model, however, the phase relationships are not fixed, and the incident, transmitted, and reflected waves may all have different energy/momentum ratios. In addition, these ratios may be functions of height. In case A the wave absorption is not symmetrical about z_c, so no wholly linear explanation can be offered. Evidently the phase relationships are such that a reflected wave is necessary to conserve momentum and energy. The presence of a reflected wave might explain why the wave magnitudes are larger above the shear layer in case A than they are in case E.

In case B it is quite clear that large reflections are present. By 2310s in fact, the reflected wave is larger than the incident wave, and the net transport of energy and momentum is in the opposite direction from case A, so over-reflection is present. That the vertical phase velocity v_{pz} is indeed positive may be seen from the phase angles in table 5.2. v_{pz} was downward in the early stages of this case and reversed about 2000s, so the critical level went from under reflecting through total reflecting to over reflecting about this time. The wave below the critical level is largely a standing wave by 2310s, so little energy is being transmitted at this time and the wind need supply only the excess needed by the over-reflected wave.

The vertical flux of horizontal momentum density, also called the Reynolds stress, is plotted in figure 5.4B as a function of height for cases A and E. The time rate of change of the horizontal momentum density is the negative vertical derivative of the Reynolds stress, so the rate of momentum gain in the height range shown is proportional to the change in the Reynolds stress between the top and bottom of the graph. It is clear that most of the incident momentum flux is absorbed near the critical level. Of course the incident flux shown in this manner is the difference between the downgoing and upgoing flux, so no conclusions about the size of the reflected wave may be drawn. The changes in these curves within the shear layer represent the shifting of momentum locally near the critical level. In particular, the large negative spike for case A represents the shifting of some negative momentum upwards a few tens of meters. This shift has been confirmed by comparing values of u_a at the preceding and following time steps.

The vertical energy flux has not been analyzed

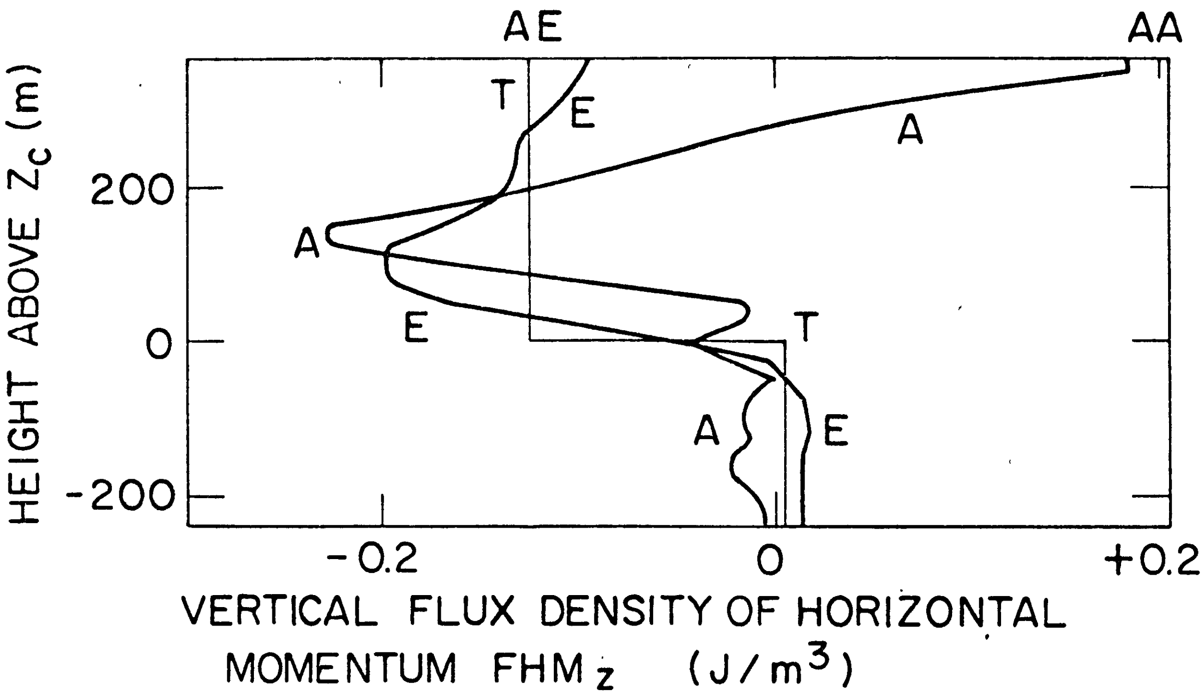


Figure 5.4B Vertical flux of Horizontal momentum density (Reynolds stress) at 3465s for Ri = 0.53. In case A the wave is incident from the high wind speed side, and in case E the wave is incident from the low wind speed side. AA and AE are the averages of FHM_z above the shear layer for cases A and E respectively. T is the hypothetical case of a wave incident with FHM_z = AE attenuated by f at z_c (see equation 5.2-1).

similarly to the momentum flux because p is not available, but the transmission of the wave through the singular level can be shown by plotting the ratio of the kinetic energy density of the oscillatory motion below the shear layer to that above the shear layer. This is done as a function of time in figure 5.5.

Case F is not shown because it was essentially identical to case A. The cause of the fluctuations in cases A and G is not clear. The wave kinetic energy is not going into the kinetic energy associated with u_a. The total kinetic energy in the bottom 2000m is so small, however, that it could easily be accounted for by small changes in the few hundred meters around the critical level.

There is no apparent dependence of energy transmission on the wave amplitude. Cases A and G are quite similar, but there is no way of being certain that the large amplitude case would continue to give like results if it had run longer. There does seem to be considerable dependence on whether or not the wave approaches the singular level from the high speed or the low speed side. Case E develops quite differently from case A. Further, it is noted that the second harmonics are generated on the side of the critical level away from the source in case E, and if the second harmonic had been propagating below the shear layer instead of evanescent there, the energy ratio would be higher than it is.

Energy transmission is strongly dependent on the

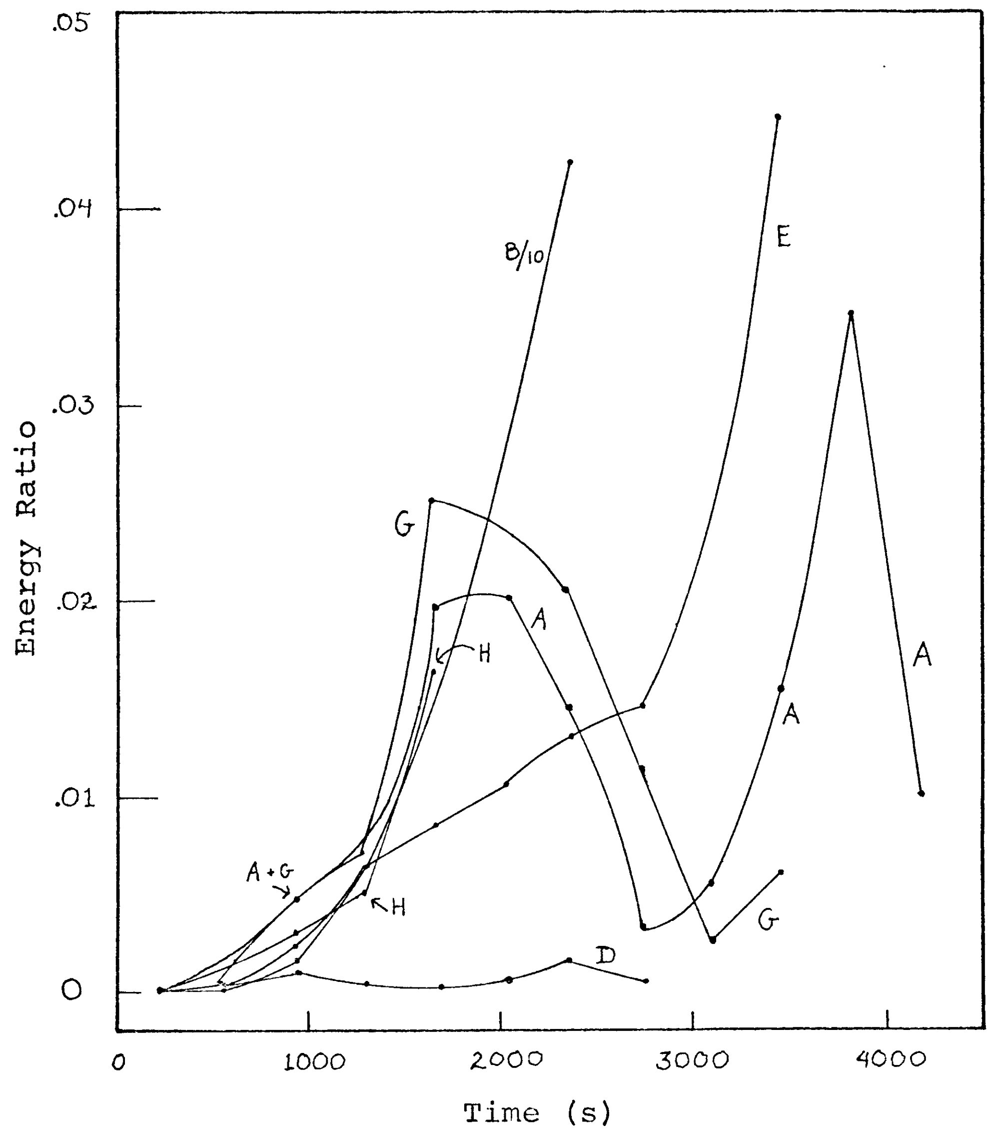


Figure 5.5. The energy ratio as a function of time. The energy ratio is the average oscillatory kinetic energy density below the shear layer divided by the average oscillatory kinetic energy density between 3600m and 5600m. The values for case B (Ri = 0.13) have had to be divided by ten to fit on this graph. Case F (High viscosity) is identical to case A (Reference, Ri = 0.53). Ri = 2.12 for case D. Case E is the negative shear case. Cases G and H are the low and high amplitude cases, respectively.

Richardson number as a comparison of cases B, A, and D will show. These values are plotted on figure 5.6 in comparison with f, the exponential factor determined from the linear theory by Booker and Bretherton. This factor was derived for the Reynolds stress or momentum density flux, but the energy density has nearly the same expression in the linear case, so it seems appropriate to apply it here also. The value for case D is higher than it should be for the steady state because f is so small that the little energy that gets through the critical level in the transient stages when the wave arrives at the shear layer is enough to cause the energy ratio to exceed f.

The value plotted for case A is an average value. The value plotted for case B is the final value, and there is no reason to expect that, barring instability, the energy ratio would not have reached unity in this case. Had a point for case E been plotted, it would have been slightly above f. Since there is no sign that a steady state had been reached in case E, it must be concluded that f underestimates the amount of energy transmitted when the wave approaches from the low speed sides.

It is of interest for source considerations to know how rapidly higher harmonics are generated. This information can not be obtained from the linear theory. Figures 5.7 show how the constant and the second harmonic for u developed in time with respect to the fundamental. The values plotted are for a single row about one hundred

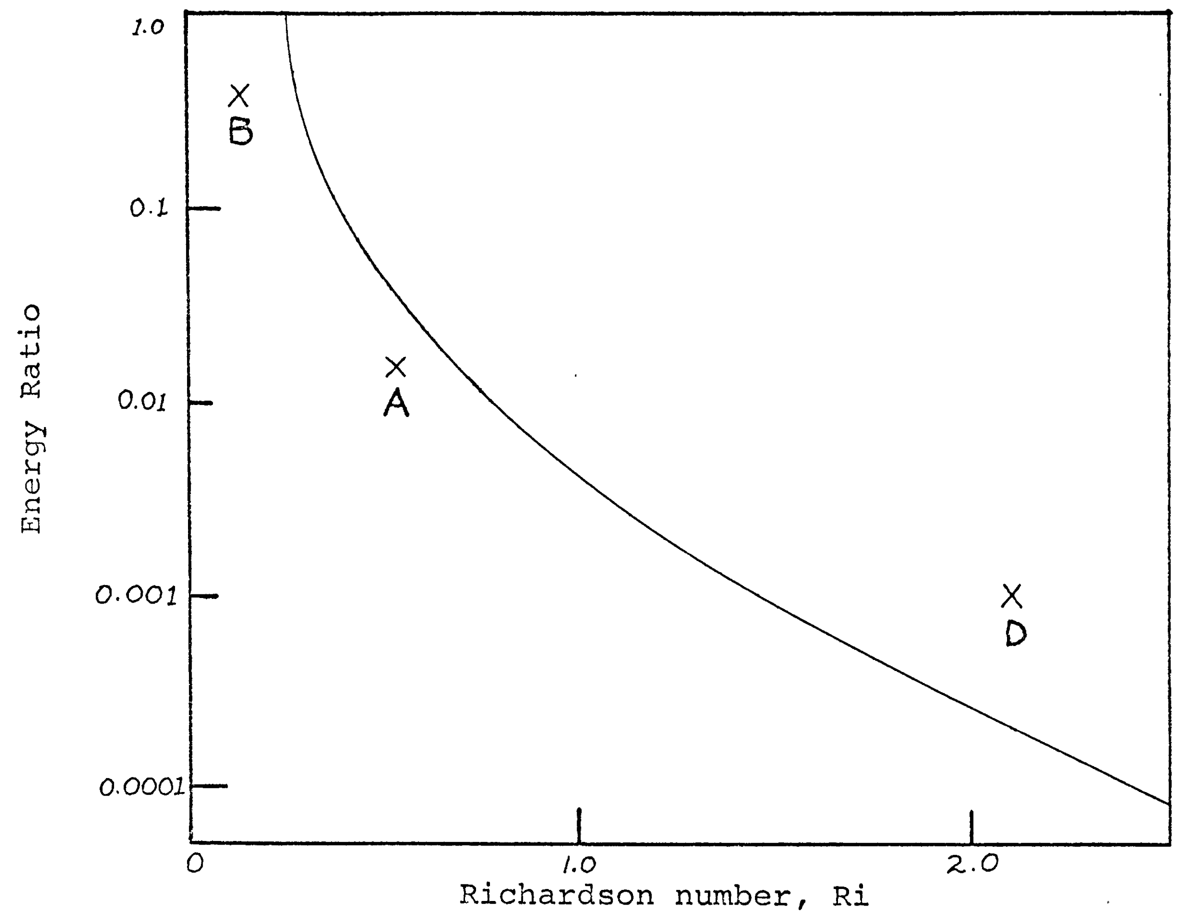


Figure 5.6. Oscillatory kinetic energy transmission as a function of Richardson number. The curve is a plot of f (see figure 5.4B). The x's mark values from figure 5.5 for cases B, A, and D. The values for cases A and D are average values. The maximum value has been used for case B.

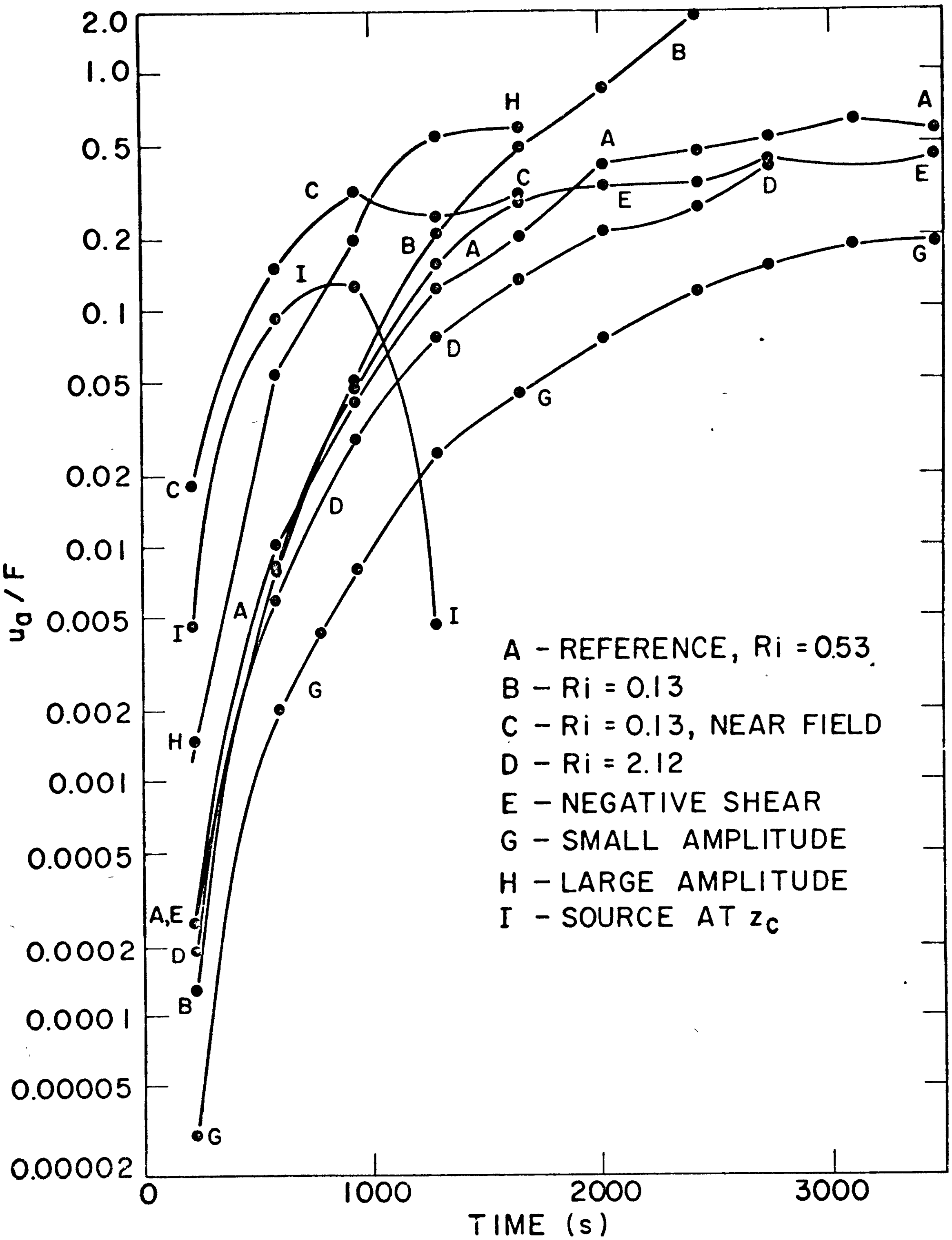


Figure 5.7A u_a/F, the absolute value of the ratio of the constant term in u to the fundamental in u, as a function of time. This quantity is evaluated at a row of points about 100m above z_c.

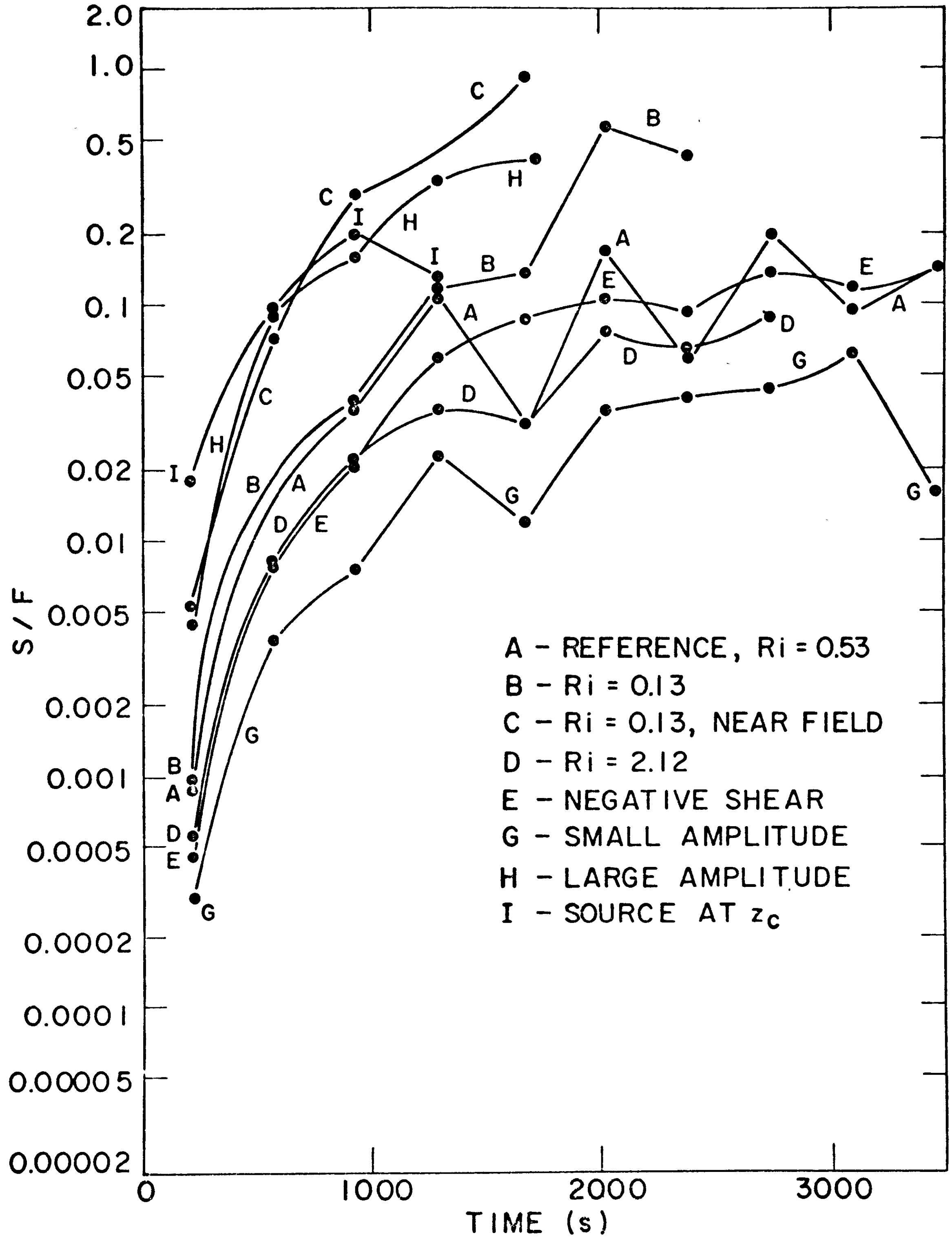


Figure 5.7B S/F, the absolute value of the ratio of the second harmonic in u to the fundamental in u, as a function of time. This quantity is evaluated at a row of points about 100m above z_c.

meters above z_c.

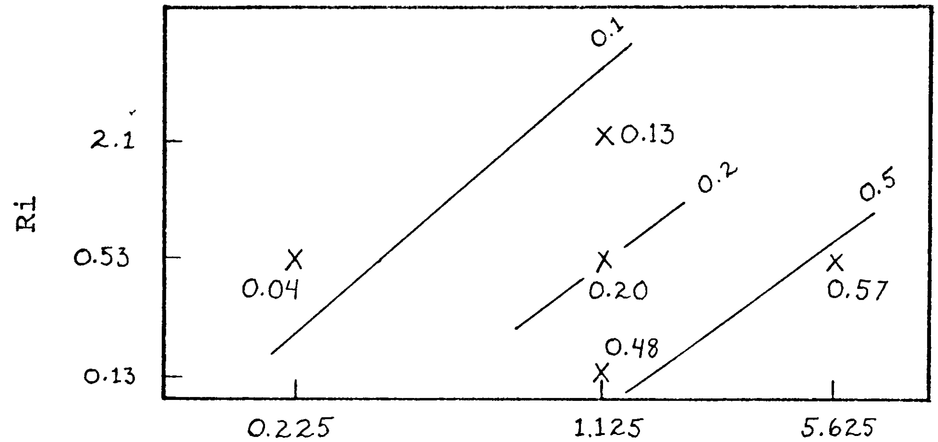
If averages over several rows had been used, the curves would be much smoother, and the lowest points would be moved up.

The dependence of the ratios on source magnitude is clear, but there seems to be no dependence on Richardson number as long as it is greater than 0.25. These cases, with the possible exception of case H, show a definite approach to steady state. The initial growth rate seems quite similar for all cases, but the S/F ratio in the cases with Ri less than 0.25 show no sign of decreasing. Presumably this is a manifestation of the basic instability of the flow.

In figures 5.7 the curves for case E are lower than they might be. In all the other cases the height about 100m above z_c coincides with the maximum for u_a, S, and F; while in case E S has its maximum below z_c and u_a reaches its greatest magnitude right at z_c.

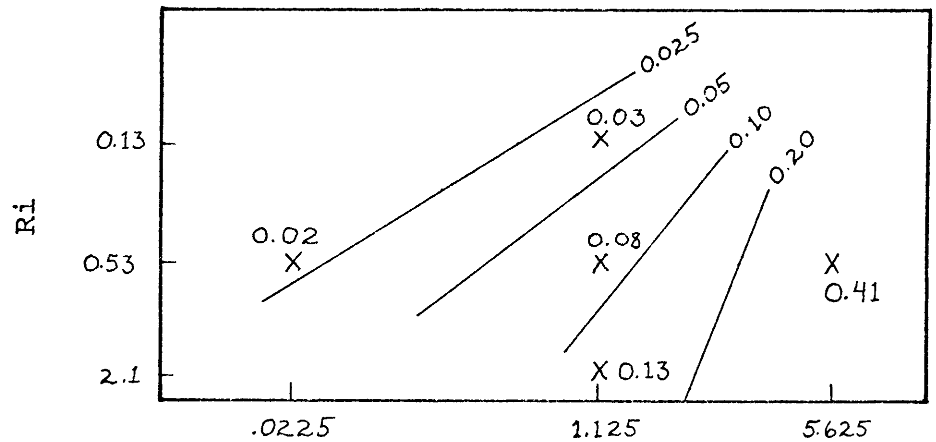
In figures 5.8 values from figures 5.7 have been plotted as functions of Ri and source (or wave) amplitude. It may be presumptious to have included the point for Ri = 0.13 since there is no indication that a steady state has been reached in this case.

Although the exact mechanism of internal gravity wave generation has not been investigated, if the source is localized and is nearly at zero frequency with respect to the air around it some idea of the magnitudes of the motions



Amplitude (source strength) (m/s)

Figure 5.8A. Contour plot of u_a/F at 1665s as a function of Richardson number and amplitude (see figure 5.7A).



Amplitude (source strength) (m/s)

Figure 5.8B. Contour plot of S/F at 1665s as a function of Richardson number and amplitude (see figure 5.7B).

at the source needed to produce observed magnitudes of the motions at the ground can be obtained from case I. For the parameters used in this case, u near the ground is about one fourth of the size of the horizontal motions near the source.

5.3 General conclusions

It has been shown that an internal gravity wave is largely absorbed by the wind at a singular level when the important nonlinear terms are included if the Richardson number is greater than 0.25. When the Richardson number is less than 0.25 the incident wave is transmitted practically unattenuated. The factor $f = |\exp(-2\pi(Ri - 0.25)^{\frac{1}{2}})$ (Booker and Bretherton, 1967) gives a reasonable idea of what attenuation to expect for the Reynolds tress, and of what the ratio of the oscillatory kinetic energy density on the side away from the source will be to that on the side near the source. Only cases in which the energy transmitted through the singular level was trapped by a solid boundary beyond it were considered.

The model used does not permit easy separation of any reflected wave from the incident wave, but there are indications that some reflection takes place for Ri greater than 0.25. For Ri less than 0.25, the reflected wave became larger than the incident wave, which was clearly evident. The excess energy for this over-reflection is supplied by the wind. In the stable cases the wind absorbed much of the horizontal momentum and energy of the incident wave, and this absorption took place in a layer a few hundred meters high.

The linear predictions concerning wave behavior hold to within a few hundred meters of the simgular level for a horizontal wavelength of five thousand meters. The nonlinear terms become important inside this region and change the character of the waves. w decreases as predicted by linear theory but does not go to zero. ρ and u stop increasing and start decreasing before the critical level is reached. The nonlinear terms also allow the wave to generate wind changes and higher harmonics. These wind changes absorb most of the momentum and energy of the incident wave for Ri greater than 0.25.

When Ri is less than 0.25 most of the wave passed through the critical level and over-reflections later developed. When the singular level was overlain by an evanescent region, Jones (1968) found that over-reflections may occur for Richardson numbers equal to or less than 0.25. Since the rigid surface used in this work totally reflects the incident wave as does the boundary with the evanescent layer, it is not surprising that over-reflection is observed here for Ri = 0.13.

There are great differences in the interaction of the waves and the wind depending on the relative velocity of the waves to the wind as the singular level is approached. The sign of the energy and momentum changes of the wind are as predicted from linear theory. Slow waves decrease the energy and momentum of the wind while fast waves increase these values. Other differences which cannot be predicted by linear theory involve the details of the interaction. The fast wave's momentum is absorbed symmetrically around the critical level and its second harmonics are generated on the side of the singular level away from the source. If these harmonics are able to propagate out of the shear layer they may carry a significant amount of energy. The slow wave is absorbed about a hundred meters before it reaches the singular level, and the second harmonics are generated on the side near the source. Also, it appears that the reflected wave is larger for these waves than it is for the fast waves.

It has been observed in this work that the wave's frequency and horizontal phase velocity are not constant. The change of the horizontal phase speed with height and the consequent shearing of the wave pattern accompany the decrease of the vertical wavelength near the critical level. The changes of the horizontal phase speed with height can also result in moving the actual critical level several hundred meters from the original critical level.

Inclusion of the nonlinear terms in the equations allows the different wave variables to travel at different apparent phase speeds. Figure 5.3C shows that the phase speeds can be less than half the phase speed of the source, so the approximation that the horizontal phase speed is constant near a singular level is invalid.

The viscous and heat conduction terms have been shown to be unimportant. Even with large eddy values for the coefficients the effect of these terms was not significant. As energy continues to be absorbed near the critical level, of course, the generation of higher harmonics will take place and eventually the wavelength will be short enough that viscous dissipation will occur, but the basic critical level behavior is apparently independent of viscosity.

This model was run only for simulated times of fifteen Brunt periods or less so it is not possible to say definitely what might develop over much longer periods. In figure 5.5 case D has certainly reached a steady state, and cases A and G may have done so. In figures 5.7 cases A, E, E, and G appear to be exponentially approaching constant values, so extensions of these results to much longer times could be expected to give reasonable results for these cases.

The momentum absorption described by Lindzen (1968) is in qualitative agreement with the results here. However the absorption of the incident wave over a broad area here acts to decrease the shear markedly near the critical level and to increase it slightly elsewhere. The high shear zones predicted by Lindzen did not develop, but Lindzen considered very long times, and continued absorption of momentum as in this study could lead to such zones in time.

Appendix A

Inclusion of Heat Conduction

From Eckart (1960, pp. 9, 10)

$$\frac{D\rho}{Dt} - \frac{1}{c^2}\frac{Dp}{Dt} - w\left(\frac{\omega_B^2 \bar{\rho}}{g}\right) = - \frac{\bar{\rho}}{C_p T}\frac{Dq}{Dt}$$

$$\frac{Dq}{Dt} = \frac{1}{\bar{\rho}} \vec{\nabla} \cdot (K \vec{\nabla} T)$$

where q is defined to be the thermal energy per unit mass and $\bar{\rho} + \rho \cong \bar{\rho}$ has been used. The first equation is conservation of heat energy and the second is heat flow by conduction. Conduction is here used to include convection by eddies of a scale smaller than the scale of interest for the gravity waves. Assume that K is independent of position and time. Let $T = \bar{T} + T'$ where T is the total temperature and $\bar{T} = T(z)$ is the mean temperature. Assume that the mean quantities satisfy

$$\bar{p} = \bar{\rho} R \bar{T}$$

and that the density and temperature perturbations are independent of pressure so that

$$\bar{p} = (\bar{\rho} + \rho) R (\bar{T} + T')$$

To first order

$$T' = - T \rho / \bar{\rho}$$

Assuming that T satisfies the basic equations in the absence of any perturbations:

$$\frac{D\rho}{Dt} - \frac{1}{C^2}\frac{Dp}{Dt} - w\left(\frac{\omega_B^2\bar{\rho}}{g}\right) = \frac{K}{C_p T}\nabla^2\left(\rho\frac{T}{\bar{\rho}}\right)$$

This is close to the equation Hazel (1967) uses.

$C_p = \frac{7}{2}R$ is the specific heat capacity at constant pressure. K is called the thermal conductivity by Eckart and has units of watts/(m°K). $K = K/(C_p\bar{\rho})$ is called the coefficient of thermometric conductivity by Chandrasekhar (1961, p.18) and had units of m^2/s.

If one keeps all terms throughout, one gets

$$\frac{D\rho}{Dt} - \frac{1}{c^2}\frac{Dp}{Dt} - w\left(\frac{\omega_B^2\bar{\rho}}{g}\right) = \frac{K}{C_p T}\nabla^2\left[\frac{T\rho}{\bar{\rho}} - \frac{p}{R}\right]$$

From the linear theory $p/R \simeq \frac{1}{20}T\rho$, so the neglect of the pressure term is reasonable. The approximation that $\nabla^2\left(\rho\frac{T}{\bar{\rho}}\right) = \frac{T}{\bar{\rho}}\nabla^2\rho$ is also a fairly good one.

Appendix B

Boundary Conditions for Poisson's Equation

In this appendix the equations analogous to 4.4-3 are obtained for the model in which the finite difference region is overlain by an infinitely high region in which the wind is constant. Let the subscript r denote the variables in the region extending from the ground to some height $z = h$, and let the subscript u denote the variables in the region from h to infinity. Since there is no shear in the upper region, analytic solutions exist and we may assume $\exp(-i\omega t+ikx+imz)$ dependence for the wave variables. It is assumed that the ambient pressure and density are continuous at the boundary between the two regions. The group velocity and energy flow in the upper region should be upward, so m is taken to be positive if $\bar{u}_u > v_{ph}$ and negative otherwise, where $v_{ph} = \omega/k$ is the horizontal phase speed. This choice results from the fact that the vertical phase and group velocities are in oppositie directions if $u_u < v_{ph}$ and in the same direction if $u_u > v_{ph}$.

The boundary conditions at the interface of the two regions (at the top of the finite difference region) are that the pressure p and the ratio w/Ω be continuous. The pressure is not readily available in the finite difference region, but if it is assumed that the simple linear approximation to the complete horizontal momentum equation

$$-i\Omega\beta + \bar{u}_z\Upsilon + ikp = 0$$

$$-i\Omega\beta \quad + \bar{u}_z \gamma \quad + ikp = 0$$

is valid just below the boundary, the boundary conditions are:

$$\left.\begin{aligned} \frac{\gamma_r}{\Omega_r} &= \frac{\gamma_u}{\Omega_u} \\ \Omega_r \beta_r + i\,\bar{u}_z \gamma &= \Omega_u \beta_u \end{aligned}\right\} \text{ at } z = h$$

where a subscript r is not needed on the wind shear $\bar{u}_z$ since a shear exists only in region r.

The neglect of the nonlinear and viscous terms in the equation for the continuity of pressure at the boundary is probably valid if the critical level is not near the boundary. In any event the inclusion of these terms is not feasible computationally.

The angular frequency ω used in forming the intrinsic frequencies Ω_r and Ω_u is that specified for the source. This would appear reasonable since the source is usually quite close to the top boundary. The value used for the wind $\bar{u}$ in Ω_r is that at the top of region r.

The momentum variables in region u are related by equation 4.2-3D and by its use these variables may be eliminated from equations B-1:to give

$$\frac{\beta_r}{\gamma_r} = -C_a \qquad \text{at } z = h \qquad\qquad \text{B-2A}$$

where

$$C_a = \frac{m}{K}\left(\frac{\Omega_y}{\Omega_r}\right)^2 + \frac{i\bar{U}_z}{\Omega_r} \qquad \text{B-2B}$$

In obtaining an equation in ψ from B-2, the expression for H_J in 4.4-4 cannot be used as it is because we are now working in the (k,m) domain, not in the (k,z) domain. From the z dependence it may be concluded that at the upper boundary G_J and e^{kz} are upgoing terms and e^{-kz} is a downgoing term. Thus 4.4-7 becomes

$$\frac{\dfrac{\partial \psi_J}{\partial z}}{\dfrac{\partial \psi_J}{\partial x}} C_a = \frac{im_r\left[G_J(k,h) - Ae^{kh} + Be^{-kh}\right]}{ik\left[G_J(k,h) + Ae^{kh} + Be^{-kh}\right]} \qquad \text{B-3}$$

where m_r is the vertical wavenumber at the top of region r and is found from the simple dispersion relation

$$m_r^2 = k^2\left[(\omega_B/\Omega_r)^2 - 1\right]$$

with the assumption that the shear is zero. In the cases when the shear at the top of region r is nonzero, the error introduced by this assumption is not significant.

Equation B-3 may now be solved with the equation resulting from setting $\psi_J = 0$ at $z = 0$ with the result:

$$A = \frac{\exp(-kh)\ G_J(k,0) - G_J(k,h)}{C_b \exp(kh) - \exp(-kh)} \qquad \text{B-4A}$$

$$B = G_J(k,0) - A \qquad \text{B-4B}$$

where

$$C_b = \frac{kC_a + m_r}{kC_a - m_r} \qquad \text{B-4C}$$

From this point on the solution is the same as in the case of a rigid surface at $z = h$ except that B-4 are used instead of 4.4-3.

Note that for $|\Omega_u| =\ \ $, $C_b = 1$ and the boundary condition just derived becomes identical to that for a rigid surface as would be expected. At the other extreme, when the wind is continuous at the boundary and there is no shear in the lower region, $C_b = \infty$ and $A = 0$ which agrees with what is expected for a region r of infinite height.

Appendix C

The Boussinesq Approximation

The term "Boussinesq approximation" has been associated with a number of approximations which, in different circumstances, amount to assuming that the density is constant to a certain degree. Some of the history and a discussion of the Boussinesq approximation in the study of thermal convective motions is given by Spiegel and Veronis (1960). In general any approximation in which the density is considered constant in the inertial (acceleration) term in Newton's law but not in the buoyancy (gravity) term is a Boussinesq approximation. Here the dropping of a term involving derivatives of the ambient density in the vorticity equation in section 4.2 will be justified.

Let B be the absolute value of the ratio of the neglected term to the buoyancy term:

$$B = \left| \frac{\beta \gamma \frac{d}{dz}\left[\bar{\rho}^{-2} \frac{d\bar{\rho}}{dz}\right]}{g \frac{\partial \rho}{\partial x}} \right|$$

If B is small the neglected term is insignificant. $\bar{\rho} = \bar{\rho}_0 \exp(-z/H)$ is a good approximation for the height ranges in which we are interested, where the scale height H is about 8km. Eliminating ρ from the expression for B by using 4.2-3E with the linear and adiabatic assumptions:

$$B = \left| \frac{\Omega}{\omega_B} \left(\frac{u}{H\,\omega_B} \right) \right|$$

Ω is always less than ω_B because we wish to deal only with waves which will propagate vertically, so that factor in B is less than unity. $H\omega_B$ is about 100m/s, so that as long as u is small with respect to this speed the neglect of the term in question is valid. The linear theory predicts that Ω will approach zero at twice the rate that u approaches infinity as a critical level is neared, so B is certainly small near a critical level no matter what the size of u. No conditions have been found in this entire project when the neglected term was of significant size.

Appendix D

Importance of various terms

In this appendix the complete basic equations will be examined as a singular level is approached in order to determine which terms are important at various distances from the singular level. Since the primary interest here is in the region near the singular level, the complete equations may be simplified by neglecting terms which are shown to have little effect in this region. If large terms must be dropped to make the equations tractable, their importance and possible effect may be estimated. Also this analysis will help in understanding the validity of the various assumptions which have been made in previous works.

Two parameters will be necessary for this analysis. ϵ will represent the relationship between the magnitude of the perturbation quantities and the magnitude of the ambient quantities in a region far from a critical level, at the ground for example. The variation of the magnitude of the perturbation variables with distance from the critical level will be contained in a second parameter δ.

Analysis of the basic equations as written in the usual variables with MKS units is difficult because of the different magnitudes of the quantities involved. Identification of the important terms is faciliated by the introduction of dimensionless variables of order one.

The normalization factors needed to accomplish this change of variables can be grouped into numerical coefficients for each term in the equation and the magnitude of each term seen at once.

The basic equations to be considered will include the nonlinear, viscous, thermal conduction, and Coriolis terms. The effect of the curvature of the Earth has not been included. Convection cells and turbulence on the same scale as the gravity waves are not included either, but the average effect of these random phenomena with smaller scales has been taken into account by the use of 'eddy' values for the viscosity and thermal conductivity.

Let the MKS perturbation pressure, density, and velocity be represented by p', ρ', and $\vec{v}\,' = \vec{a}_x u' + \vec{a}_y v' + \vec{a}_z w'$. $\vec{a}_x$, $\vec{a}_y$, $\vec{a}_z$ are the unit vectors in a Cartesian coordinate system with the positive x' direction eastward, the positive y' direction northward, and the positive z' direction upward. Time is t'. The ambient or time-independent pressure, density, temperature, and fluid flow are denoted $\bar{p}'$, $\bar{\rho}'$, T', and $\bar{u}'$. It is assumed that the background fluid flow or mean wind is in the x' direction only. The total pressure, density and velocity are (MKS units):

$$p' = \bar{p}' + p'$$

$$\rho_T' = \bar{\rho}' + \rho'$$

$$\vec{V} = \vec{a}_x \bar{u}' + \vec{v}\,' = \vec{a}_x(\bar{u}' + u') + \vec{a}_y v' + \vec{a}_z w' .$$

In all the other sections of this work the dimensional variables with MKS units are represented by unprimed quantities. In this appendix, however, the final equations will include only nondimensional variables, and for ease of notation it is desirable that these nondimensional quantities be unprimed. For this reason the dimensional quantities are represented by primed variables at this point. The following differential operators are defined:

$$\nabla' = \vec{a}_x \frac{\partial}{\partial x'} + a_y \frac{\partial}{\partial y'} + a_z \frac{\partial}{\partial z'}$$

$$\frac{D}{Dt'} = \frac{\partial}{\partial t'} + \vec{V}' \cdot \vec{\nabla}'$$

Under the assumptions that:

1. the curvature of the Earth can be neglected;
2. there are no sources or sinks of heat;
3. the mean wind $\bar{u}'$, ambient density $\bar{\rho}'$, and ambient temperature T' are functions of height, z', only;

the basic equations are:

$$\rho_T' \left[\frac{D\vec{V}'}{Dt'} + 2\,\omega_{RD} \times \vec{V}'\right] + \vec{\nabla}' P' + g_D \rho_T' - \mu_D\left[\frac{1}{3}\vec{\nabla}'(\vec{\nabla}' \cdot \vec{V}') + (\vec{\nabla}')^2 \vec{V}'\right] = 0 \qquad \text{D-1A,B,C}$$

$$\frac{D}{Dt'}\rho_T' + \rho_T' \vec{\nabla}' \cdot \vec{V}' = 0 \qquad \text{D-1D}$$

$$\frac{D}{Dt'}\rho_T' - \frac{1}{c_D^2}\frac{D}{Dt'}p' - \frac{\rho_T' K_D}{T'}\left(\vec{\nabla}'\right)^2\left(\frac{\rho' T'}{\rho_T'}\right) = 0 \qquad \text{D-1E}$$

where μ_D is the dynamic viscosity and K_D is the coefficient of thermometric conductivity. The other constants, with average values for the lowest 10 km of the atmosphere, are:

g_D = acceleration due to gravity = 9.8m/s^2

c_D = speed of sound = 320m/s

ω_{RD} = radian frequency of the Earth's rotation = 2π/day = 0.73x10^{-4}/s.

Quantities with a subscript D are constants with MKS units. If the above equations are unfamiliar, either Eckart (1960) or Lamb (1945) may be consulted for their derivation. The extension to include the heat conduction term is presented in Appendix A.

The next step is to rewrite equations D-1 using dimensionless variables of order one. First the parameters ϵ and δ must be defined and the normalization factors introduced. Although only three normalization factors are strictly necessary, it has been found easier to use seven normalization factors and inter-relate them later. Because of the way in which the temperature enters the equations, the normalization factor for the temperature will never appear and therefore need not be defined.

Let ϵ be defined by the equation

$$\epsilon = |u'| \;/\; \bar{u}_a$$

with the stipulations that $|u'|$ is the magnitude of the perturbation horizontal motion in a region far from a critical level and that $\bar{u}_a$ is typical or average mean wind speed. In general $|u'|$ will be taken to be the value at the ground and $\bar{u}_a$ will be taken to be half the maximum mean wind speed. ϵ is a dimensionless quantity of order 10^{-2} which is independent of time and position. Since its magnitude depends on the size of the wave perturbation, its value may change from case to case.

Let δ be defined by the equation

$$\delta = \left|\frac{\Omega'}{\omega'}\right|^{1/2} = \left|\frac{\omega' - k'\bar{u}'}{\omega'}\right|^{1/2} = \left|1 - \frac{k'\bar{u}'}{\omega'}\right|^{1/2}$$

where ω' is the radian frequency of the wave perturbation, k' is the horizontal wavenumber, and $\Omega' = \omega' - k'\bar{u}'$ is the intrinsic or Doppler frequency. It will be shown later that δ^2 can be thought of as a normalized, dimensionless distance from the critical level. δ is a dimensionless parameter which is independent of the magnitude of the perturbations. It depends on the height through the mean wind, and varies in value from near unity far from a singular level to zero right at a singular level.

Five of the normalization factors can be defined at once:

$$\tau_D = 600s$$

$$v_D = 40m/s$$

$$L_D = v_D \tau_D = 24km$$

$$\bar{p}_D = 5x10^4 n/m^2$$

$$\bar{\rho}_D = 0.6kg/m^3$$

The D subscript indicates that these quantities are constants with MKS units. τ_D is approximately equal to the Brunt period for the standard 6.5^o/km lapse rate. v_D is half of a typical maximum jet stream speed, and L_D, in addition to being the product indicated, is a typical wavelength for gravity waves. $\bar{p}_D$ and $\bar{\rho}_D$ are average values of the ambient pressure and density for the troposphere.

The factors $\bar{p}_D$ and $\bar{\rho}_D$ are not convenient for normalizing the perturbation pressure and density. Therefore two additional normalization factors

$$p_D = 0.25x10^4 n/m^2$$

$$\rho_D = 0.05kg/m^3$$

are introduced. These magnitudes were selected in the following manner: Because of the definition of ϵ and because the horizontal motion and vertical motion of a gravity wave are of the same order of magnitude far from a singular level, This is the most complete text of the thesis available. The following page(s) were not included in the copy of the thesis deposited in the Institute Archives by the author:

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one can write $|u'| = \epsilon v_D$ and $|w'| = \epsilon v_D$. Similarly we wish to be able to write $|p'| = \epsilon p_D$ and $|\rho'| = \epsilon \rho_D$, but $\bar{p}_D$ and $\bar{\rho}_D$ do not allow this. For a small amplitude wave the magnitudes of the perturbation variables are

$$|p'| = 5\text{n/m}^2$$
$$|u'| = 0.08\text{m/s}$$
$$|w'| = 0.04\text{m/s}$$
$$|\rho'| = 0.12\times10^{-3}\text{kg/m}^3$$

where the pressure is a value measured in Cambridge and the other values are calculated from the value for the pressure using the relationships given by the linear theory (Mines, 1960).

It was assumed for these calcuations that there was no mean wind, and $\omega' = 2\pi/900\text{s}$ and $k' = 2\pi/24\text{km}$ were used for the radian frequency and the horizontal wave-number. This gives $\epsilon = 0.002$, and $p_d = |p'| / \epsilon$ and $\rho_D = |\rho'| / \epsilon$ are used to obtain the values given.

The case just described with $\epsilon = 0.002$ is the main case which will be considered. The other is that of a fairly large amplitude wave for which $\epsilon = 0.02$ will be used. The relationships between the magnitudes of the wave variables are of course independent of their magnitude since they are derived from the linear theory. They do depend on the frequency and wavelength, however. The values used in the calculation of p_D and ρ_D above have been chosen so that these normalization factors will suffice for almost all gravity waves of interest. As long as the frequency and wavelength are not changed too drastically, a change in perturbation magnitude can be expressed by changing only the value of ϵ without adjusment of the normalization factors.

Before the new dimensionless variables or order one can be defined it is necessary to consider how the dimensional variables depend on δ, that is, how they behave as a singular level is approached. Since ω'/k' is the horizontal phase speed, it can be seen from the definition of the intrinsic frequency Ω' that Ω' and thus δ approach zero with decreasing distance from a critical level. A number of researchers (Bretherton and Garrett, 1968; Eliassen and Palm, 1960; and Claerbout, 1967) have analyzed in detail how the various quantitites associated with the wave perturbation vary as Ω' approaches zero. They concluded that the simplified linear equations predict that the vertical wavelength λ'_z will vary as δ^{-2}, that the pressure p' and vertical motion w' will vary as δ, and that the horizontal motions u' and v' and the density ρ' will vary as δ^{-1}. Inclusion of the proper dependence of δ is the definition of the new variables insures that they remain of order one as the critical level is neared. This dependence cannot be expected to hold very close to the singular level, but it is the only guide available.

The new dimensionless variables of order one are defined by the following equations:

$$x = x'/L_D \qquad y = y'/L_D \qquad z = z'/L_D \qquad t = t'/L_D$$

$$\bar{p} = \bar{p}'/\bar{p}_D \qquad \bar{\rho} = \bar{\rho}'/\bar{\rho}_D \qquad \bar{u} = \bar{u}'/v_D \qquad u = u'\delta/\epsilon v_D$$

$$p = p'/\epsilon\delta p_D \qquad \rho = \rho'\delta/\epsilon\rho_D \qquad v = v'\delta/\epsilon v_D \qquad w = w'/\epsilon\delta v_D$$

$$p = \bar{p} + p \qquad \rho_T = \bar{\rho} + \rho$$

$$\vec{v} = \vec{a}_x u + \vec{a}_y v + \vec{a}_z w \qquad \vec{V} = \vec{a}_x \bar{u} + \vec{v}$$

By analogy with the dimensional variables, P is the total pressure, p is the ambient pressure, $\bar{p}$ is the wave perturbation pressure, and so on for the others.

There is one last step before the equation can be rewritten in the new variables and that is to make certain that all the derivatives are of order one. Since the wavelength is assumed to be of the order of L_D,

$$\frac{\partial p'}{\partial x'} = \left(\frac{2\pi}{L_D}\right) \frac{\partial p}{\partial x} \left(\epsilon \delta p_D\right)$$

$$\frac{\partial p'}{\partial z'} = \left(\frac{2\pi}{\delta^2 L_D}\right) \frac{\partial p}{\partial z} \left(\epsilon \delta p_D\right)$$

where the additional factor of δ^{-2} in the z derivative is introduced to account for the shortening of the vertical wavelength which the linear theory predicts. The operator $\left(\frac{\partial}{\partial t'} + a' \frac{\partial}{\partial x'}\right)$ is analogous to the Doppler frequency so we have

$$\left(\frac{\partial}{\partial t'} + \bar{u}'\frac{\partial}{\partial x'}\right)p' = \frac{2\pi\delta^2}{\tau_D}\left(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\right)p \;(\epsilon\,\delta\rho_D)$$

For the variation of the ambient quantities, the relevant distance is the scale height H_D = 8km, so

$$\frac{\partial \bar{\rho}'}{\partial z'} = \frac{\bar{P}_D}{H_D}\frac{\partial \bar{\rho}}{\partial z}$$

The derivatives of the pressure have been presented as examples, the other variables are treated simiarly.

In order that the equations may be written more compactly, the following notation is defined:

$$\vec{v}_d = \vec{a}_x u/\delta + \vec{a}_y v/\delta + \vec{a}_z w\,\delta \qquad \text{D-2A}$$

$$\vec{\nabla}_d = \vec{a}_x\frac{\partial}{\partial x} + a_y\frac{\partial}{\partial y} + a_z\,\delta^{-2}\frac{\partial}{\partial z} \qquad \text{D-2B}$$

$$D_d = \delta^2\left(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\right) + \vec{v}_d\cdot\nabla_d \qquad \text{D-2C}$$

$\vec{\nabla}$ retains its usual meaning

$$\vec{\nabla} = \vec{a}_x\frac{\partial}{\partial x} + a_y\frac{\partial}{\partial y} + a_z\frac{\partial}{\partial z}$$

$\vec{\nabla}_d$ will be used on perturbation quantities, and $\vec{\nabla}$ on ambient quantities, but note that $\vec{\nabla}_d\cdot\vec{v}_d = \vec{\nabla}\cdot\vec{v}/\delta$ and $v_d\cdot\vec{\nabla}_d = \vec{v}\cdot\vec{\nabla}/\delta$. The basic equations can now be rewritten in terms of the new variables.

$$\left[\bar{\rho}_D \rho + \frac{\epsilon \rho_D}{\delta} \rho\right]\left[\left(\frac{2\pi \epsilon V_D}{\tau_D}\right) D_d \vec{V}_d + \vec{a}_x \left(\frac{\epsilon \delta V_D^2}{H_D}\right) \frac{d\bar{u}}{dz} w\right.$$

$$+ \left(\sqrt{2} \epsilon V_D \sqrt{2}\, \omega_{RD}\right)\left(\vec{a}_y + \vec{a}_z\right) \times \vec{V}_d + \left(\sqrt{2}\, V_D \omega_{RD}\right)\left(\vec{a}_y - \vec{a}_z\right) \bar{u}$$

D-3A,B,C

$$\left. + \vec{a}_z g_D\right] + \frac{P_D}{H_D} \vec{\nabla} \bar{P} + \left(\frac{2\pi \epsilon \delta P_D}{L_D}\right) \vec{\nabla}_d P$$

$$- \left(\frac{2\pi}{L_D}\right)^2 \epsilon \mu_D V_D \left[\frac{1}{3\delta} \vec{\nabla}_d \left(\vec{\nabla} \cdot \vec{V}\right) + \left(\vec{\nabla}_d\right)^2 \vec{V}_d\right.$$

$$\left. - \frac{\mu_D V_D}{H_D^2} \bar{a}_x \frac{d^2 \bar{u}}{dz^2}\right] = 0$$

$$\left(\frac{2\pi \epsilon \rho_D}{\delta \tau_D}\right) D_d \rho + \left(\frac{\epsilon \delta V_D \bar{\rho}_D}{H_D}\right) \frac{d\bar{\rho}}{dz} w$$

D-3D

$$+ \left[\bar{\rho}_D \bar{\rho} + \frac{\epsilon \rho_D}{\delta} \rho\right] \left(\frac{2\pi \epsilon V_D}{\delta L_D}\right) \vec{\nabla} \cdot \vec{V} = 0$$

$$\left(\frac{2\pi}{\tau_D}\right) D_d \left[\frac{\epsilon \rho_D}{\delta} \rho - \frac{\epsilon \delta P_D}{C_D^2} p \right] - \left(\frac{\epsilon \delta V_D \bar{\rho}_D \omega_{BD}^2}{g_D} \right) w$$

$$- \left(\frac{2\pi}{L_D}\right)^2 \left(\frac{\epsilon \rho_D K_D}{\delta} \right) \left(\nabla_d \right)^2 \rho \qquad \text{D-3E}$$

$$- \left(\frac{\epsilon \rho_D K_D}{\delta H_D^2} \right) \frac{\rho \bar{\rho}}{T} \frac{d^2}{dz^2} \left(\frac{T}{\bar{\rho}} \right) = 0$$

where

$$\omega_{BD}^2 = \frac{-g_D'}{\bar{\rho}'} \left[\frac{d\bar{\rho}'}{dz'} - \frac{1}{C_D^2} \frac{\partial \bar{p}'}{\partial z'} \right]$$

defines the Brunt frequency, and the latitude has been taken to be 45° for simplicity. The complete dependence on ϵ and δ in the above equations is not apparent due to the presence of ϵ and δ in equations D-2.

In the absence of perturbations, the following zero order in ϵ equations relating the ambient variables result:

$$\frac{\partial \bar{p}}{\partial x} - \frac{\mu_D V_D}{\bar{\rho}_D H_D} \frac{d^2 \bar{u}}{dz^2} = 0 \qquad \text{D-4A}$$

$$\frac{\partial \bar{p}}{\partial} + \frac{\bar{\rho}_D V_D \omega_{RD} H_D}{\sqrt{2}\, \bar{p}_D} \bar{\rho} \bar{u} = 0 \qquad \text{D-4B}$$

$$\frac{\partial \bar{p}}{\partial z} + \frac{g_D \bar{\rho}_D H_D}{\bar{p}_D} \bar{\rho} - \frac{\bar{\rho}_D V_D \omega_{RD} H_D}{\sqrt{2}\, \bar{p}_D} \bar{\rho} \bar{u} = 0 \qquad \text{D-4C}$$

From the values already given for the constants with D subscript:

$$g_D \bar{\rho}_D H_D / p_D = 1$$

$$\bar{\rho}_D v_D \omega_{RD} H_D / \sqrt{2}\, \bar{p}_D = 2x10^{-4}$$

It is evident that the effect of the Coriolis term in the hydrostatic relation will be very slight and that the north-south pressure gradient will be much smaller than the vertical pressure gradient.

To find the magnitude of the east-west pressure gradient a value for μ_D is needed. This brings up the euqation of whether the molecular (laminar flow) or eddy (turbulent flow) values are the appropriate ones to use. Since the same problem applies to the conduction coefficients as well, they will be included in this discussion.

The molecular values for the dynamic viscosity μ_D, the kinematic viscosity $\nu_D = \mu_D/\bar{\rho}'$, and the thermal conductivity κ_D may be found in the U. S. Standard Atmosphere (1962). Both μ_D and κ_D decrease slightly with height. The coefficient of thermometric conductivity is defined by

$$K_D = \kappa_D/C_p \bar{\rho}$$

where C_p is the specific heat capacity at constant pressure. Since $\bar{\rho}$ increases with height faster than μ_D and κ_D decrease, ν_D and K_D decrease with height. The change of all four coefficients with height in the troposphere is small enough that the most convenient two are often taken to be constant. Average values for the lowest 10km of the atmosphere are $\mu_D = 1.6 \times 10^{-5}$kg/ms, $\nu_D = 2.2 \times 10^{-5}$m^2/s, $\kappa_D = 0.023$watt/m^{o}K, and $K_D = 3.5 \times 10^{-5}$m^2/s. These values would be appropriate for still air or purely laminar motion.

In actuality the atmosphere is quite turbulent on scales smaller than those of interest for internal gravity waves. Therefore heat is transmitted by convective cells in addition to conduction, and motion is retarded by the formation of turbulence in addition to molecular viscosity.

These additional processes are many times more efficient than the molecular level processes. Since these random phenomena are indeed present, their effect will be included insofar as possible. For gravity waves the average effect of many convection cells of different scales may be adequately described by using a new, 'eddy' value for the coefficient of thermometric conductivity. The eddy values for the viscosity and conductivity are determined by actual measurements. We will use

$$\mu_D = 0.1 \text{kg/ms}$$

$$K_D = 1 \text{m}^2/\text{s}$$

(Sutton, 1953, p. 264; Sutton, 1955, p. 31, 211, 214) which are approximately 10^4 times the molecular values.

Using this value for μ_D

$$\mu_D v_D / \bar{p}_D H_D = 10^{-8}$$

and the east-west pressure gradient is also much smaller than the vertical gradient. For our purposes the ambient pressure $\bar{p}'$ may be taken to be a function of height only.

Equations D-3 will now be rewritten with equations D-4 subtracted out and each equation multiplied by an appropriate factor of dimensional constants so that the largest term is of order one if the ϵ and δ are not included. Although a common factor of ϵ could have been removed, it has been retained to indicate that these are the perturbation equations. The various factors of dimensional constants which are formed are dimensionless and have been replaced by numbers.

$$\left[\epsilon\bar{\rho} + \frac{\epsilon^2}{12\delta}\rho\right]\left[D_d \vec{V}_d + \vec{a}_x \left(\frac{\delta}{2}\right)\frac{d\bar{u}}{dz} W + 0.01\left(\vec{a}_y + \vec{a}_z\right)\times \vec{V}_d\right] + \vec{a}_z\left(\frac{2\epsilon}{\delta}\right)\rho + 2.6\,\epsilon\delta\,\vec{\nabla}_d\, p - 10^{-6}\epsilon\left[\frac{1}{3\delta}\vec{\nabla}_d\left(\vec{\nabla}\cdot\vec{V}\right) + \left(\vec{\nabla}_d\right)^2 \vec{V}_d\right] = 0 \qquad \text{D-5A,B,C}$$

$$\frac{\epsilon}{12\delta} D_d \rho + \frac{\epsilon\delta}{2}\frac{d\bar{\rho}}{dz} W + \left[\bar{\rho} + \frac{\epsilon}{12\delta}\rho\right]\left(\frac{\epsilon}{\delta}\right)\vec{\nabla}\cdot\vec{V} = 0 \qquad \text{D-5D}$$

$$\epsilon D_d\left[\frac{1}{\delta}\rho - \frac{\delta}{2} p\right] - \frac{\epsilon\delta}{2} W - 0.65\times 10^{-5}\left(\frac{\epsilon}{\delta}\right)\left(\vec{\nabla}_d\right)^2 \rho + 1.5\times 10^{-6}\left(\frac{\epsilon}{\delta}\right)\left(\frac{\rho\bar{\rho}}{T}\right)\frac{d^2}{dz^2}\left(\frac{T}{\bar{\rho}}\right) = 0 \qquad \text{D-5E}$$

where the following expressions have been used:

$$\omega_{BD} = 2\pi/\tau_D \qquad \frac{\rho_D}{\bar{\rho}_D} = 1/12$$

$$\frac{L_D}{2\pi H_D} = 1/2 \qquad \frac{\sqrt{2}\,\omega_{BD}\tau_D}{2\pi} = 0.01$$

$$\frac{g_D \tau_D \rho_D}{2\pi V_D \bar{\rho}_D} = 2 \qquad \frac{P_D}{\bar{\rho}_D V_D^2} = 2.6$$

$$\frac{2\pi \mu_D}{\bar{\rho}_D V_D L_D} = 10^{-6} \qquad \frac{P_D}{\rho_D C_D^2} = 1/2$$

$$\frac{\tau_D V_D \omega_{BD}^2 \bar{\rho}_D}{2\pi g_D \rho_D} = 1/2 \qquad \frac{2\pi \tau_D K_D}{L_D^2} = 6.5\text{x}10^{-6}$$

$$\frac{\tau_D K_D}{2\pi H_D^2} = 1.5\text{x}10^{-6}$$

The dependence of equations D-5 on ϵ and δ is not explicit because of the definitions in equations D-2.

The next step is to consider all the terms which have the same dependence on δ and to discard those whose magnitude, including ϵ, is considerably smaller than some other term. By keeping the largest terms for every power of δ there is no possibility that a term is being discarded which might become important near a singular level. To make the ϵ and δ dependence completely explicit the vector equation will be written out as three scalar equations.

The x momentum equation will be treated in detail as an illustration. First D-5A is written out with the terms grouped according to their dependence on δ :

$$\delta\left[\epsilon\bar{\rho}\left(\frac{\partial}{\partial t}+\bar{u}\frac{\partial}{\partial x}\right)u+\frac{\epsilon}{2}\bar{\rho}\frac{d\bar{u}}{dz}W+0.01\,\epsilon\bar{\rho}W+2.6\,\epsilon\frac{\partial p}{\partial x}\right]$$

$$+\left[\frac{\epsilon^2}{12}\rho\left(\frac{\partial}{\partial t}+\bar{u}\frac{\partial}{\partial x}\right)u+\frac{\epsilon^2}{24}\rho\frac{d\bar{u}}{dz}W+\frac{0.01\,\epsilon^2\rho W}{12}\right]+$$

$$\delta^{-1}\left[-0.01\,\epsilon\bar{\rho}v-10^{-6}\epsilon\left\{\frac{1}{3}\frac{\partial}{\partial x}\left(\vec{\nabla}\cdot\vec{V}\right)+\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right\}\right]+$$

$$\delta^{-2}\left[\epsilon^2\bar{\rho}\left(\vec{V}\cdot\vec{\nabla}\right)\vec{V}-\frac{0.01\,\epsilon^2}{12}\rho v\right]+$$

$$\delta^{-3}\left[\frac{\epsilon^3}{12}\rho\left(\vec{V}\cdot\vec{\nabla}\right)\vec{V}\right]+\delta^{-5}\left[-10^{-6}\epsilon\,\frac{\partial^2 u}{\partial z^2}\right]=0$$

Of the four terms which vary as δ , it is seen that the third is one percent of the largest, so it will be neglected. Likewise, the third of the three terms which have no δ dependence, the second of the two terms which vary as δ^{-1}, and the second of the two terms which vary as δ^{-2} may be discarded.

It develops that there are no terms which may be neglected in equations D-5D and D-5E, so these equations are just written out with the ϵ and δ dependence made explicit.

$$\delta\epsilon\left[\bar{\rho}\left(\frac{\partial}{\partial t}+\bar{u}\frac{\partial}{\partial x}\right)u+\frac{\bar{\rho}}{z}\frac{d\bar{u}}{dz}w+2.6\frac{\partial p}{\partial x}\right]+$$

$$\frac{\epsilon^2}{12}\left[\rho\left(\frac{\partial}{\partial t}+\bar{u}\frac{\partial}{\partial x}\right)u+\frac{\rho}{z}\frac{d\bar{u}}{dz}w\right]-\frac{0.01\epsilon}{\delta}\left[\bar{\rho}v\right]$$

D-6A

$$+\frac{\epsilon^2}{\delta^2}\left[\bar{\rho}\left(\vec{V}\cdot\vec{\nabla}\right)u\right]+\frac{\epsilon^3}{12\delta^3}\left[\rho\left(\vec{V}\cdot\vec{\nabla}\right)u\right]$$

$$-\frac{10^{-6}\epsilon}{\delta^5}\left[\frac{\partial^2 u}{\partial z^2}\right]=0$$

$$\delta\epsilon\left[\bar{\rho}\left(\frac{\partial}{\partial t}+\bar{u}\frac{\partial}{\partial x}\right)v+2.6\frac{\partial p}{\partial y}\right]+\frac{\epsilon^2}{12}\left[\rho\left(\frac{\partial}{\partial t}+\bar{u}\frac{\partial}{\partial x}\right)v\right]$$

D-6B

$$+\frac{0.01\epsilon}{\delta}\left[\bar{\rho}u\right]+\frac{\epsilon^2}{\delta^2}\left[\bar{\rho}\left(\vec{V}\cdot\vec{\nabla}\right)v\right]+\frac{\epsilon^3}{12\delta^3}\left[\rho\left(\vec{V}\cdot\vec{\nabla}\right)v\right]$$

$$-\frac{10^{-6}\epsilon}{\delta^5}\left[\frac{\partial^2 v}{\partial z^2}\right]=0$$

$$\delta^3\epsilon\left[\bar{\rho}\left(\frac{\partial}{\partial t}+\bar{u}\frac{\partial}{\partial x}\right)w\right]-10^{-6}\delta\epsilon\left[\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right]$$

D-6C

$$+\epsilon^2\left[\bar{\rho}\left(\vec{V}\cdot\vec{\nabla}\right)w\right]+\frac{2\epsilon}{\delta}\left[p+1.3\frac{\partial p}{\partial z}\right]-$$

$$\frac{0.01\,\epsilon^2}{12\,\delta^2}\left[\rho u\right]-\frac{10^{-6}\epsilon}{\delta^3}\left[\frac{1}{3}\frac{\partial}{\partial z}\left(\vec{\nabla}\cdot\vec{V}\right)+\frac{\partial^2 w}{\partial z^2}\right]=0$$

$$\frac{\epsilon\delta}{2}\left[\frac{1}{6}\left(\frac{\partial}{\partial t}+\bar{u}\frac{\partial}{\partial x}\right)\rho+\frac{d\bar{\rho}}{dz}w\right]$$

D-6D

$$+\frac{\epsilon}{\delta}\left[\bar{\rho}\,\vec{\nabla}\cdot\vec{V}\right]+\frac{\epsilon^2}{12\delta^2}\left[\left(\vec{V}\cdot\vec{\nabla}\right)\rho+\rho\,\vec{\nabla}\cdot\vec{V}\right]=0$$

$$\frac{\delta^3\epsilon}{2}\left[\left(\frac{\partial}{\partial t}+\bar{u}\frac{\partial}{\partial x}\right)p\right]+\delta\epsilon\left[\left(\frac{\partial}{\partial t}+\bar{u}\frac{\partial}{\partial x}\right)\rho-\frac{w}{z}\right]$$

D-6E

$$-\frac{\epsilon^2}{2}\left[\left(\vec{V}\cdot\vec{\nabla}\right)p\right]-\frac{0.65\times10^{-5}\epsilon}{\delta}\left[\frac{\partial^2\rho}{\partial x^2}+\frac{\partial^2\rho}{\partial y^2}+\right.$$

$$\left.\frac{\bar{\rho}\rho}{4T}\frac{d^2}{dz^2}\left(\frac{T}{\bar{\rho}}\right)\right]+\frac{\epsilon^2}{\delta^2}\left[\left(\vec{V}\cdot\vec{\nabla}\right)\rho\right]$$

$$-\frac{0.65\times10^{-5}\epsilon}{\delta^5}\left[\frac{\partial^2\rho}{\partial z^2}\right]=0$$

Equations D-6 will now be examined as δ approaches zero so that the variation in magnitude of the various terms may be assessed as a critical level is neared. This could be done using only values of δ, but δ is hard to relate to the physical situation. It has been found helpful to express δ in terms of an actual distance from the critical level. To do this it is assumed that the mean wind is a linear function of height:

$$\bar{u}'(z') = \bar{u}_o' + \bar{u}_z' z'$$

where $\bar{u}_o'$ and $\bar{u}_z'$ are constants. $\bar{u}_z'$ is the wind shear and has units of s^{-1}. Let the origin of the coordinate system be at the critical level, then $\delta = 0$ for $z = 0$ and from the definition of δ:

$$\bar{u}_o' = \omega'/k'$$

$$\delta^2 = |b'z'|$$

with $b' = k'\bar{u}_z'/\omega'$. For the case with $\omega' = 2\pi/900s$, $k' = 2\pi/L_D$, and $\bar{u}_z' = 2v_D/H_D$, $b' = 3/H_D$. This gives $|z'| = \delta^2(2667m)$. b has the units of m^{-1} and is the ratio of the wind shear to the wave's horizontal phase speed.

By equating the magnitudes of two terms we wish to compare, and solving for δ^2 and then for $|z'|$ using the value of b' above, the crossover distance is obtained. This distance, denoted z_a', is the distance from the critical level to the height where the two terms are of equal magnitude. The term which has the lower (including sign) power of δ will be the larger in that region which is within z_a' of the critical level, and the other term will be the larger elsewhere. By taking the δ dependence of the terms into account, one may get an idea of how far from the crossover distance a term may be of significant size in comparison to the other. For many typical cases where b' is not too different from $3/H_D$ this method allows comparison of the terms in a way which relates to they physical scale of the problem.

If z_a' is small enough, the term with the lower power of δ can be considered negligible for all regions of interest. How small is small enough? Since internal gravity waves are on the scale of kilometers to tens of kilometers, and since the atmosphere becomes increasingly random as the scale decreases, events on scales of less than a meter are probably not relevant. The predictions concerning the behavior of the magnitudes of the perturbation variables is based on the linear, inviscid, irrotational approximation, and by the time the one meter scale is reached these neglected terms will be seen to be large, so that the predictions on which this analysis is based can not be expected to hold for that scale in any event. Therefore, if a value for z_a' of less than one meter is obtained, the term with the lower power of δ is considered to be negligibly small for all regions of interest.

Equation D-6A will be treated in detail as an example. The first term is called the linear term even though the Coriolis term (the third term) and the viscous term (the last term) are also linear. The first term is the only one retained in the simple linear approximation. The fourth term, which does not contain the perturbation density, is called the nonlinear term. The second and fifth terms are called the nonlinear density terms and they will be seen to be considerably less important than the nonlinear term which does not contain the density.

The linear term is seen to be the largest for $\delta = 1$, so the other five terms are first compared to it. Each line below represents the comparison of two terms. The first equation on the line is the equation of the two magnitudes, the second gives the value for δ^2 which the first implies, and the last equation gives the crossover distance using the value of b' given above.

For the case where $\epsilon = 0.002$:

$\delta\epsilon = \epsilon^2/12$	$\delta^2 = 2.8\text{x}10^{-8}$	$z_a' = 7.4\text{x}10^{-5}\text{m}$
$\delta\epsilon = 0.01\,\delta^{-1}\epsilon$	$\delta^2 = 0.01$	$z_a' = 27\text{m}$
$\delta\epsilon = \delta^{-2}\epsilon^2$	$\delta^2 = 0.016$	$z_a' = 43\text{m}$
$\delta\epsilon = \delta^{-3}\epsilon^3/12$	$\delta^2 = 5.9\text{x}10^{-4}$	$z_a' = 1.5\text{m}$
$\delta\epsilon = 10^{-6}\,\delta^{-5}\epsilon$	$\delta^2 = 0.01$	$z_a' = 27\text{m}$

The first line is the comparison of the linear term and the first nonlinear density term. It is obvious that this nonlinear density term is never large enough to be of interest. The other nonlinear density term is compared to the linear term on the fourth line and it appears that it may be important since the crossover distance is over one meter. If this second nonlinear density term is compared to the nonlinear term:

$$\delta^{-2}\epsilon^{2} = \delta^{-3}\,\epsilon^{3}/12 \qquad \delta^{2} = 2.8\times10^{-8} \quad z_a' = 7.4\times10^{-5}\text{m}.$$

So this nonlinear density term bears the same relation to the nonlinear term and the other does to the linear term. For the case with $\epsilon = 0.02$ the values of z_a' are increased by a factor of 100, but z_a' is still so small that the nonlinear density terms need not be considered further.

From the second, third and fifth lines we see that the Coriolis, nonlinear, and viscous terms all become larger than the linear term in the region between ten and fifty meters from the critical level. It is of interest to compare these three terms with each other:

$$0.01\,\delta^{-1}\epsilon = \delta^{-2}\epsilon^{2} \qquad \delta^{2} = 0.04 \qquad z_a' = 107\text{m}$$

$$0.01\,\delta^{-1}\epsilon = 10^{-6}\,\delta^{-5}\epsilon \qquad \delta^{2} = 0.01 \qquad z_a' = 27\text{m}$$

$$\delta^{-2}\epsilon^{2} = 10^{-6}\,\delta^{-5}\epsilon \qquad \delta^{2} = 0.0063 \quad z_a' = 16\text{m}$$

Since all the terms still being considered as first power in ϵ except the nonlinear term, only the comparisons involving the nonlinear term need to be recalculated for the large amplitude case where $\epsilon = 0.02$:

$$\delta\epsilon = \delta^{-2}\epsilon^{2} \qquad \delta^{2} = 0.073 \qquad z_a' = 195\text{m}$$

$$0.01\,\delta^{-1}\epsilon = \delta^{-2}\epsilon^{2}\delta^{2} = 4 \qquad z_a' = 10700\text{m}$$

$$\delta^{-2}\epsilon^{2} = 10^{-6}\delta^{-5}\epsilon \quad \delta^{2} = 0.0013 \qquad z_a' = 3.4\text{m}$$

Before making definite conclusions about the size of the various terms, the factors evaluated just after equations 3.1-5 must be examined to see how these numbers depend on the specific wave parameters chosen. While the values used were chosen to be representative and are adequate for most comparisons, here the nonlinear, viscous and Coriolis terms are very nearly the same size, and the wave period and wavelength do make a difference. It is seen that the factor containing ω_{RD} will be increased for waves of longer period, and that the viscous factor contains the ratio $1/L_D v_D = \tau_D/L_D^2$. The period of the wave considered was taken to be about the Brunt period because it is simpler and because the primary interest here is in waves of twenty minutes or less.

The figures above show that the Coriolis term is not too important since it is already smaller than the non-linear term when it becomes as large as the linear term. For $\epsilon = 0.002$ the values are close enough that the period of the wave is quite important. In this case the Coriolis term will be of the same importance as the nonlinear term for a period of about twice the Brunt period, and will completely dominate the nonlinear term for a one hour period. In the large amplitude case, the nonlinear term remains significantly greater than the Coriolis term for all periods of interest.

Due to its δ^{-5} dependence the viscous term passes from insignificance to dominance within a small region around the crossover height. Thus the viscous effect will not extend as far from its crossover point with the linear term as will the Coriolis or nonlinear effects. The period and wavelength can be adjusted so that the viscous term is the most important of the three, but generally it is of secondary importance. If the nonlinear term is larger, the behavior of the wave may be altered so that the viscous term never becomes significant. This is unlikely if the larger term is the Coriolis term because the inclusion of the Coriolis force does not alter the basic nature of the singularity. Note that only the viscous term involving the z derivative appears. This is due to the shortening of the vertical wavelength which enhances the values of the vertical derivative with respect to the horizontal derivatives. If the molecular value for the viscosity had been used the viscous terms would have been completely negligible.

In both the large and the small amplitude cases, the nonlinear term is the most important of the additional terms. This term reaches one tenth of the magnitude of the linear term 200m from the critical level for $\epsilon = 0.002$ and 900m from the critical level for $\epsilon = 0.02$. In the large amplitude case, the viscous force as well as the Coriolis effect is dominated by the nonlinearities.

In conclusion, for the x momentum equation, the linear, inviscid approximation is valid to within a kilometer or so of a critical level. The exact distance will of course depend on the wave parameters such as wavelength and magnitude. Of the neglected terms, the nonlinear term is the most important. For small amplitudes, the viscous force may be important if the nonlinearities do not alter the linear predictions concerning the behavior of the perturbation variables as a critical level is approached. For a large amplitude wave the nonlinear term is much larger than the viscous as well as the Coriolis term.

Comparison of equations D-6B and D-6A shows that the two are analogous except for the lack of a y component of the mean wind. Since the magnitudes of the corresponding terms are identical, the conclusions reached for the x momentum equation hold for the y momentum equation as well.

In equation D-6C the first and fourth terms are the only ones retained in the simple linear approximation. These terms are the largest for $\delta = 1$, but since the fourth term, called the gravity term, increases with decreasing δ while the other decreases, all of the other terms will be compared to the gravity term. For $\epsilon = 0.002$:

$2\delta^{-1}\epsilon = \delta^3\epsilon$	$\delta^2 = 1.4$	$z_a' = 3730\text{m}$
$2\delta^{-1}\epsilon = 10^{-6}\delta\epsilon$	$\delta^2 = 2\text{x}10^6$	$z_a' = 5.3\text{x}10^9\text{m}$
$2\delta^{-1}\epsilon = \epsilon^2$	$\delta^2 = 10^6$	$z_a' = 2.7\text{x}10^9\text{m}$
$2\delta^{-1}\epsilon = 0.01\,\delta^{-2}\epsilon^2/12$	$\delta^2 = 0.69\text{x}10^{-12}$	$z_a' = 18\text{x}10^{-9}\text{m}$
$2\delta^{-1}\epsilon = 10^{-6}\delta^{-3}\epsilon$	$\delta^2 = 0.5\text{x}10^{-6}$	$z_a' = 0.0013\text{m}$

For $\epsilon = 0.02$ the two comparisons involving ϵ^2 terms are recalculated:

$2\delta^{-1}\epsilon = {}^2$	$\delta^2 = 10^4$	$z_a' = 2.7\text{x}10^7\text{m}$
$2\delta^{-1}\epsilon = 0.01\,\delta^{-2}\epsilon^2/12$	$\delta^2 = 0.69\text{x}10^{-10}$	$z_a' = 18\text{x}10^{-7}\text{m}$

One viscosity term, the second term in D-6C, is small with respect to the gravity term for $\delta = 1$ and grows more so as δ decreases. The other viscosity term, the last term, has a crossover distance, with respect to the gravity term, of about a millimeter, so the viscosity is entirely negligible for this equation. The crossover distance of the Coriolis term is also very small so that this term is also insignificant. The largest of the nonlinear terms, the third term, does not increase with decreasing as the gravity term, does, so it, too, need not be included. The first term, which being linear is usually included anyway, is less than ten percent of the gravity term for the region within 1700m of the critical level. So, within a kilometer or so of a critical level, only the two quantities comprising the gravity term need be retained in the vertical momentum equation.

The continuity equation, D-6D, has only three different δ dependencies, so for $\epsilon = 0.002$:

$$\delta^{-1}\epsilon = \delta\epsilon/2 \qquad \delta^2 = 2 \qquad z_a' = 5300\text{m}$$

$$\delta^{-1}\epsilon = \delta^{-2}\epsilon^2/12 \qquad \delta^2 = 2.8\text{x}10^{-8} \qquad z_a' = 7.4\text{x}10^{-5}\text{m}$$

and for $\epsilon = 0.02$ the second comparison becomes:

$$\delta^{-1}\epsilon = \delta^{-2}\epsilon^2/12 \qquad \delta^2 = 2.8\text{x}10^{-6} \qquad z_a' = 7.4\text{x}10^{-3}\text{m}$$

The nonlinear term is clearly negligible for all regions of interest. If the first term is also neglected the continuity equation become $\vec{\nabla}\cdot\vec{v}' = 0$, which is the equation usually used for an incompressible fluid. The first term contains the change of density with time and position, and becomes less than one tenth of the second term for the region within 500m of the critical level. These distances are calculated using the values of b' discussed above.

The expression of the incompressibility of a fluid is sometimes written $\vec{\nabla}' \cdot (\bar{\rho}' \vec{v}') = 0$ when the fluid is stratified. By writing the equation this way the more usual incompressibility equation is extended by the addition of $w' \, (d\bar{\rho}'/dz')$. This extension takes into account the fact that fluid parcels at different heights were originally of different densities and includes the effect of vertical motion in changing the density at a given location. Change of density with time or with horizontal position continues to be excluded from consideration. To obtain this extended equation of incompressibility from 3.1-6D the third term and the first of the two quantities in the first term are to be neglected. The second quantity in the first term is retained. The third term has already been seen to be negligible. When the first quantity in the first term is compared to the second term, one gets:

$$\delta^{-1}\epsilon = \delta\epsilon/12 \qquad \delta^2 = 12 \qquad z_a' = 32{,}500\text{m}$$

It may easily be calculated that the magnitude of this first quantity is less than ten percent of the second term for the region within 3250m of the singular level.

Thus by using $\vec{\nabla}' \cdot (\bar{\rho}' \vec{v}') = 0$ as the expression of incompressibility, the continuity equation can be considered to be the incompressibility equation over a much larger region than if $\vec{\nabla}' \cdot \vec{v}' = 0$ were used to express the incompressibility. Since both expressions allow the stream function-vorticity formulation to be adopted, this is a significant point.

For the heat transfer equation, D-6E, only the first two terms are retained in the simplest linear approximation. The second term, called the linear density term, decreases in magnitude as the critical level is neared slower than does the first term, the linear pressure term, so that the linear density term is used for the comparisons. For $\epsilon = 0.002$:

$\delta\epsilon = \delta^3\epsilon/2$	$\delta^2 = 2$	$z_a' = 5300$m
$\delta\epsilon = \epsilon^2/2$	$\delta^2 = 10^{-6}$	$z_a' = 0.0027$m
$\delta\epsilon = 6.5\text{x}10^{-6}\delta^{-1}\epsilon$	$\delta^2 = 6.5\text{x}10^{-6}$	$z_a' = 0.017$m
$\delta\epsilon = \delta^{-2}\epsilon^2$	$\delta^2 = 0.016$	$z_a' = 43$m
$\delta\epsilon = 6.5\text{x}10^{-6}\delta^5\epsilon$	$\delta^2 = 0.019$	$z_a' = 50$m

and for $\epsilon = 0.02$

$\delta\epsilon = \epsilon^2/2$	$\delta^2 = 10^{-4}$	$z_a' = 0.27$m
$\delta\epsilon = \delta^{-2}\epsilon^2$	$\delta^2 = 0.073$	$z_a' = 195$m

From these figures it may be concluded that the non-linear pressure term is entirely negligible. The linear pressure term is equal to one tenth of the linear density term at 530m from the critical level, so none of the terms involving pressure is likely to play an important role in determining the wave's behavior near a critical level. This is equivalent to saying that the speed of sound is nearly infinite, or that this equation becomes nearly that for an incompressible medium near the singular level. It should be noted, however, that the second quantity in the linear density term involves the Brunt frequency (the Brunt frequency disappeared when numbers were substituted for dimensional quantities) and that the sound speed cannot be taken to be infinite in the calculation of the Brunt frequency because the ambient pressure is involved in that calculation, and the ambient pressure is not affected by the presence of a singular level as is the perturbation pressure.

For the small amplitude case the nonlinear density term and the larger conduction term have nearly the same crossover distance, but for the large amplitude case the nonlinear term has a much larger crossover distance. Comparison of these two terms gives, for the case $\epsilon = 0.002$:

$$\delta^{-2}\epsilon^{2} = 6.5\times10^{-6}\,\delta^{-5}\epsilon \qquad \delta^{2} = 0.022 \qquad z_a' = 58\text{m}$$

$$\delta^{-2}\epsilon^{2} = 6.5\times10^{-6}\,\delta^{-5}\epsilon \qquad \delta^{2} = 0.0047 \qquad z_a' = 12\text{m}$$

As in the x momentum equation some of the additional terms are nearly equal in importance and the variation of the wave parameters must be considered. The ratio τ_D/L_D^2 can be made large enough so that the thermal conduction term dominates the nonlinear term in the small amplitude case and is of about equal importance in the large amplitude case. From the numbers above it is seen that the conduction is probably unimportant in the large amplitude case. Thus either the nonlinear density term or the conduction term must be kept, depending on the wave parameters, and, in general and for many cases, both terms must be retained. Only the conduction term involving the vertical derivative is important, and the conduction term can be dropped completely if the molecular value for the thermometric conductivity is used. It is a valid approximation to neglect the terms involving the wave pressure within a kilometer or so of a critical level.

In this section the linear, inviscid, adiabatic, irrotational predictions concerning the variation of the perturbation variables as a function of the proximity to a critical level have been used to determine which terms are important at different distances from the critical level. Small and large amplitude waves were considered for a typical value of b, the wind shear - horizontal phase speed ratio. Values of δ^2 have been given to facilitate consideration of other cases.

Two of the equations, it was shown, not only remain linear, but become simpler near the critical level.

$$g_D \rho' + \frac{\partial p'}{\partial z'} = 0$$

is a good approximation for the vertical momentum equation within a kilometer or so of a critical level, as is

$$\vec{\nabla}' \cdot (\tilde{\rho}' \vec{v}') = 0$$

for the continuity equation. Within a few hundred meters, the more usual expression of incompressibility, $\vec{\nabla}' \cdot \vec{v}' = 0$ is a good approximation.

The other equations, however, become increasingly complicated with proximity to a singular level. Only the nonlinear density terms in the horizontal momentum equation and the nonlinear pressure term in the heat transfer equation can be generally neglected. Within a kilometer or so of a critical level the linear pressure term in the heat transfer equation is unimportant. In general the Coriolis, viscous, and nonlinear terms must be kept in the x momentum equation. According to the choice of wave parameters each can be the most important of the three, although the nonlinear term is the largest for most of the cases of interest. Likewise in the heat transfer equation, either the conduction or nonlinear density term may be the largest, but the nonlinear term is more important for a majority of interesting cases.

The damping terms are large enough to be significant with respect to the nonlinear terms only if the eddy rather than the molecular values are used for the viscosity and conductivity. Also, due to the predicted wavelength shortening, only the terms involving the vertical derivatives are important.

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