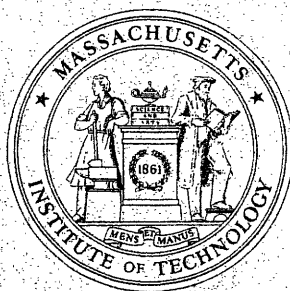


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**MASSACHUSETTS INSTITUTE  
OF TECHNOLOGY**

INVERSE OPTIMIZATION: AN APPLICATION TO  
THE CAPACITATED PLANT LOCATION PROBLEM

by

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## Abstract

Inverse optimization refers to the fact that each time a Lagrangean derived from a given mathematical programming problem is solved, it produces an optimal solution to some problem with a different right hand side. This paper reports on the application of inverse optimization to the capacitated plant location problem including the study of implied mappings of dual variables into the space of demand vectors. A new parametric method based on inverse optimization and subgradient optimization is also presented.

Inverse Optimization: An Application to Capacitated  
Plant Location Problems

1. Introduction

Lagrangian techniques have had wide application in discrete optimization; see Shapiro [13] for an extensive survey. Nevertheless, there is an implied property of Lagrangian analysis that has not been widely recognized and exploited. We develop and discuss this property for a very general case in the introduction and specialize our results to the capacitated plant location problem in the remainder of the paper.

Consider the family of (primal) mathematical programming problems

$$\begin{aligned} v(d) = \min f(x) \\ \text{s.t. } g(x) = d \\ x \in X \subseteq \mathbb{R}^n \end{aligned} \quad P(d)$$

where  $f$  is a real valued function,  $g$  is a function mapping vectors in  $\mathbb{R}^n$  into vectors in  $\mathbb{R}^m$ , and  $X$  is an arbitrary non-empty set. The constraints  $g(x) = d$  are soft and do not have to be satisfied exactly. The  $m$ -vector  $d$  can be viewed as demand to be satisfied, for example, by an electric utility or a manufacturing firm. Alternatively, it could be that the decision maker is interested in a parametric analysis of  $P(d)$  for  $d$  in some set of interest. By contrast, the constraints  $x \in X$  are hard constraints, often of a logical nature, that cannot be violated; for example, constraints that force a fixed cost to be incurred at a particular location if a plant is built there.

The Lagrangian approach is to place prices on the demand constraints and add them to the objective function. Specifically, we define the Lagrangian

function

$$L(u;d) = ud + Z(u) \quad (1)$$

where

$$Z(u) = \underset{x \in X}{\text{minimum}} \{f(x) - ug(x)\}. \quad (2)$$

The Lagrangean is a concave function and  $L(u;d) \leq v(d)$ . An implied dual problem to  $P(d)$  is to find the best lower bound to  $v(d)$ ; namely,

$$\begin{aligned} w(d) &= \max L(u;d) \\ \text{s.t. } u &\in \mathbb{R}^m \end{aligned} \quad D(d)$$

The purpose of the Lagrangean approach is imbedded in the following conditions.

Definition 1: For a given  $m$ -vector  $d$ , the solutions  $\bar{x} \in X$  and  $\bar{u} \geq 0$  are said to satisfy the global optimality conditions if

- (i)  $L(\bar{u};d) = \bar{u}d + f(\bar{x}) - \bar{u}g(\bar{x})$
- (ii)  $g(\bar{x}) = d$

Theorem 1: If  $\bar{x} \in X$  and  $\bar{u}$  satisfy the global optimality conditions for a given vector  $d$ , then  $\bar{x}$  is optimal in the primal problem  $P(d)$  and  $\bar{u}$  is optimal in the dual problem  $D(d)$ . Moreover,  $v(d) = w(d) = L(\bar{u};d)$ .

Proof: See chapter 5 of Shapiro [15]. ||

A strategy for solving a specific primal problem  $P(d)$  implied by Theorem 1 is to select a dual vector  $u$  such that the solution  $x(u)$  computed from the Lagrangean calculation (1) satisfies, along with  $u$ , the global optimality conditions for the given  $d$ . Without additional assumptions about  $f$ ,  $g$ ,  $X$ , however, there is no guarantee that such a  $u$  exists. If not, then we say

there is a duality gap between  $P(d)$  and  $D(d)$ . The following theorem makes precise the observation that each time we compute the Lagrangean function, we find an optimal solution to  $P(d)$  for some  $d$ . This is the idea of inverse optimization.

Theorem 2 (Inverse Optimization): For any  $u \in \mathbb{R}^m$ , let  $x(u)$  denote an optimal solution to the problem  $Z(u) = \underset{x \in X}{\text{minimum}} \{f(x) - ug(x)\}$ . The solution  $x(u)$  is optimal in the primal problem  $P(d(u))$  where  $d(u) = g(x(u))$ . Moreover,  $v(d(u)) = f(x(u))$ .

Proof: The proof is immediate by appeal to Theorem 1 and the global optimality conditions. The solution  $x(u)$  satisfies  $g(x(u)) = d(u)$  by construction. It also satisfies

$$L(u; d(u)) = ud(u) + f(x(u)) - ug(x(u))$$

by the definition of  $x(u)$ . ||

According to the principle of inverse optimization, the function  $Z(u)$  in the Lagrangean (1) induces a mapping of the  $m$ -space of prices into the set

$$D^* = \{d \mid v(d) \text{ is finite}\}.$$

For any point  $u$ , the image of this mapping is the set

$$D(u) = \{d \mid d_i = g_i(x(u)) \text{ for any } x(u) \text{ optimal in (2) for all } i\}$$

Clearly,  $D(u) \subseteq D^*$  implying

$$\bigcup_u D(u) \subseteq D^*. \quad (3)$$

We say  $Z$  spans  $P(d)$  if  $d \in D(u)$  for some  $u$ . A major purpose of this paper is to analyze the mathematical properties of the sets  $D(u)$  and  $\bigcup_u D(u)$  for the capacitated plant location problem.

Our underlying goal is to use inverse optimization to develop new approaches to right hand side parametric analyses for discrete optimization problems. The nonconvex structure of these problems suggests that we need to be more flexible and less exacting in the parametric analyses than is possible for in linear programming. Some results for the parametric analysis of the capacitated plant location problem are presented in this paper.

One of the concerns of inverse optimization is how to proceed if there is strict inequality in the set relation (3), or in other words, if  $Z$  does not span  $P(d)$  for all  $d \in D^*$ . The difficulty is due to the presence of duality gaps between  $P(d)$  and  $D(d)$  for some  $d$  which can be overcome, at least in theory, by the application of group theoretic methods (see Bell and Shapiro [2], Shapiro [15])) to strengthen the Lagrangean (1). Again, we show how this can be done for the capacitated plant location problem.

The plan of this paper is as follows. The following section contains an application of the inverse optimization approach to the capacitated plant location problem. In the section after that, we develop the group theoretic methods for filling in duality gaps for the capacitated plant location problem, thereby permitting inverse optimization to find optimal solutions to all problems. Section four contains a new method for the parametric analysis of capacitated plant location problems based on inverse optimization and subgradient optimization. There follows a numerical example. The final section contains a brief discussion of areas of future research.

## 2. Inverse Optimization and Capacitated Plant Location Problems.

We consider the plant location problem

$$r(d) = \min \sum_{i=1}^m \sum_{j=1}^n (c_{ij} + v_i)x_{ij} + \sum_{i=1}^m f_i y_i$$

$$\text{s.t.} \quad \sum_{i=1}^m x_{ij} = d_j \quad \text{for } j = 1, \dots, n$$

R(d)

$$\sum_{j=1}^n x_{ij} - K_i y_i \leq 0 \quad \text{for } i = 1, \dots, m$$

$$x_{ij} \geq 0 \text{ and integer, } y_i = 0 \text{ or } 1$$

where  $c_{ij}$  is the non-negative cost per unit to ship an item from site  $i$  to the customer at location  $j$ , and  $d_j$  is the demand for the item by customer  $j$ . The production cost associated with each plant site  $i$  is shown in Figure 1. Any positive production at site  $i$  involves an initial fixed cost of  $f_i$  after which there is a variable cost of production at the rate  $v_i$  up to the capacity limit  $K_i$ . We say site  $i$  is closed if  $y_i = 0$  and open if  $y_i = 1$ . The family of problems R(d) of interest are those with demand vectors in the set

$$D^* = \{d \mid d_j \geq 0 \text{ and } \sum_{j=1}^n d_j \leq \sum_{i=1}^m K_i\} \quad (4)$$

We assume the quantities  $d_j$  and  $K_i$  are non-negative integers. This implies that the transportation variables will naturally take on integer values in any basic feasible solution to the linear program that results if the  $y_i$  variables are fixed. We have required the  $d_j$  and the  $K_i$  to be integer, and explicitly constrained the  $x_{ij}$  to take on integer values, in order to facilitate the analysis in the following section where we derive procedures for resolving duality



production  
cost

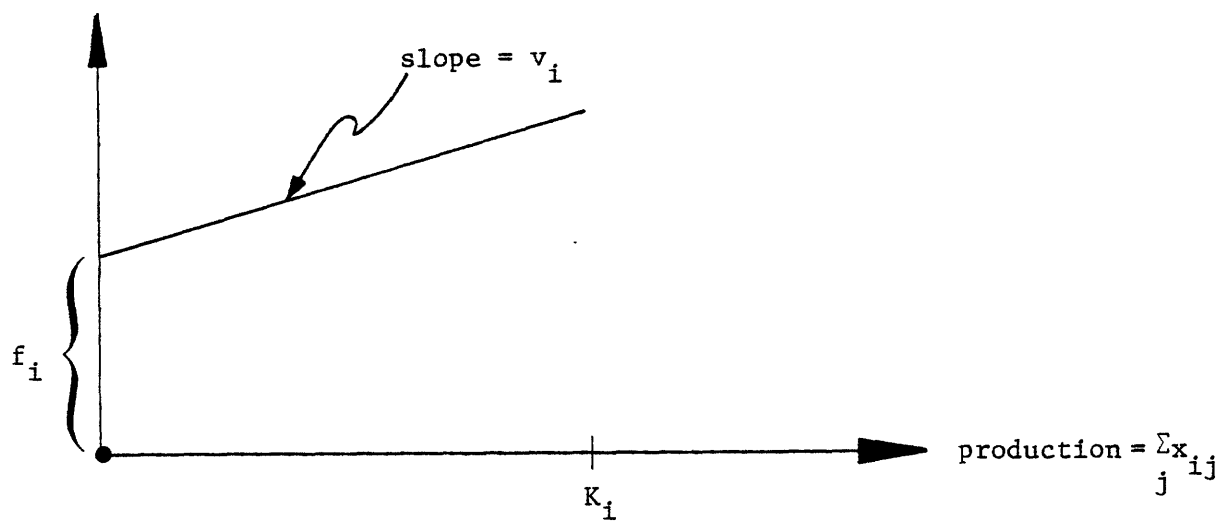


Figure 1

gaps. There is very little loss of mathematical generality in requiring the  $d_j$  and the  $K_i$  to be integer since if they were rational they could be converted to integers by multiplying by a sufficiently large number to change the scale of the problem. As a practical matter, the selection of a scale for problem  $R(d)$  has an important effect on its analysis and solution. This point is discussed again in the following section.

For future reference, let  $R^C(d)$  denote the ordinary linear programming relaxation of  $R(d)$  that results if we omit the integrality restrictions on the variables  $x_{ij}$  and  $y_i$ . Let  $r^C(d)$  denote the corresponding minimal objective function value.

We let  $u_j$  denote the dual variable associated with demand  $d_j$  and let  $L^0(u;d)$  denote the Lagrangean function that results if we dualize on the demand constraints in the plant location problem  $R(d)$ . This dualization permits us to separately evaluate each site since

$$L^0(u;d) - ud = Z^0(u) = \sum_{i=1}^m Z_i^0(u) \quad (5)$$

where

$$Z_i^0(u) = \text{minimum} \sum_{j=1}^n (c_{ij} + v_i - u_j)x_{ij} + f_i y_i$$

$$\text{s.t.} \quad \sum_{j=1}^n x_{ij} - K_i y_i \leq 0 \quad (6)$$

$$x_{ij} \geq 0 \text{ and integer, } y_i = 0 \text{ or } 1$$

The calculation of (6) is easy as shown by the following analysis. We define for each  $i$

$$C_i(u) = \text{minimum}_{j=1, \dots, n} \{c_{ij} + v_i - u_j\} \quad (7)$$

$$J_i(u) = \{j | c_{ij} + v_i - u_j = C_i(u)\}$$

Theorem 3. If  $C_i(u) \geq 0$  or if  $C_i(u) < 0$  and  $\frac{-f_i}{C_i(u)} > K_i$ , then  $y_i = 0$  and  $x_{ij} = 0$  for all  $j$  is the unique optimal solution to (6). If  $C_i(u) < 0$  and  $\frac{-f_i}{C_i(u)} < K_i$ , then the optimal solutions in (6) are all those satisfying  $y_i = 1$ ,  $\sum_{j \in J_i(u)} x_{ij} = K_i$ , and  $x_{ij}$  non-negative integer for  $j \in J_i(u)$ . If  $C_i(u) < 0$  and  $\frac{-f_i}{C_i(u)} = K_i$ , then the solutions of both the previous cases are optimal in (6).

Proof: The dual variables  $u_j$  are, in effect, the unit profit or return values for the customer served at location  $j$ . The quantity  $-Z_i^0(u)$  measures the net profit to be realized at site  $i$ , and the quantity  $C_i(u)$  measures the net unit profit of the most favored customers there, who are the ones contained in the set  $J_i(u)$ . If  $C_i(u) \geq 0$ , then there is no way to make a positive net profit at site  $i$  and therefore the optimal solution is  $y_i = 0$  and  $x_{ij} = 0$  for  $j=1, \dots, n$ , with net profit  $Z_i^0(u) = 0$ . Even if  $C_i(u) < 0$ , a positive net profit cannot be realized by opening up the plant at site  $i$  if the fixed cost  $f_i > 0$  is too high; that is, if  $f_i + C_i(u)K_i > 0$  or  $\frac{-f_i}{C_i(u)} > K_i$ . On the other hand, a positive net profit can be realized in the opposite case, namely,  $\frac{-f_i}{C_i(u)} < K_i$ , and then we select  $y_i = 1$  and  $\sum_{j \in J_i(u)} x_{ij} = K_i$  to maximize the net profit. Finally, if

$\frac{-f_i(u)}{C_i(u)} = K_i$ , then net profit is zero in both of the last two cases, and either of the two indicated solutions is optimal. ||

Theorem 3 provides us with the simple logic needed to compute optimal solutions for the  $Z_i^0$  functions. Any optimal solution  $x_{ij}(u)$ ,  $y_i(u)$  for all  $i$  and  $j$  is, according to the inverse optimization theorem 2, optimal in the capacitated plant location problem  $R(d(u))$  where

$$d_j(u) = \sum_{i=1}^m x_{ij}(u) \quad \text{for } j = 1, \dots, n \quad (8)$$

Note that  $u_j \leq 0$  implies  $d_j(u) = 0$  since  $c_{ij}$  and  $v_i$  are non-negative and  $f_i$  is positive implying  $j \notin J_i(u)$  for any  $i$ . Some additional analysis is needed to completely characterize the set  $D(u)$  of all demand vectors  $d$  for which we know an optimal solution to  $R(d)$ . The characterization must take into account the possible multiplicity of customers  $j \in J_i(u)$  that can most profitably be served by plant  $i$  when  $\frac{-f_i}{C_i(u)} < K_i$ , and the indifference about whether or not to open a plant at site  $i$  when  $\frac{-f_i}{C_i(u)} = K_i$ . We omit further details.

A direct consequence of Theorem 3 is that any primal problem  $R(d)$  for which inverse optimization with the function  $Z^0$  can find an optimal solution must satisfy  $\sum_{j=1}^n d_j = \sum_{i \in I} K_i$  for some set  $I \subseteq \{1, \dots, m\}$ . This difficulty is related to an inherent limitation of our approach thus far.

Theorem 4: For any  $u$ , any solution  $x_{ij}(u)$ ,  $y_i(u)$  for all  $i, j$  that is optimal in (6) is optimal in the linear programming relaxation  $R^C(d(u))$  where  $d(u)$  is defined in (8).

Proof: We use the result that feasible solutions to a pair of primal and dual linear programming problems are optimal if and only if complementary

slackness holds (see chapter two in Shapiro [15]). For the linear programming relaxation of the capacitated plant location problem, this result translates as follows. Suppose the variables  $x_{ij}$ ,  $y_i$  for all  $i$  and  $j$  are a feasible primal solution, and the non-negative dual variables  $u_j$ ,  $\Pi_i$  for all  $i$  and  $j$  satisfy  $u_j - \Pi_i \leq c_{ij} + v_i$  and  $\Pi_i \geq 0$ . Then, for all  $i$  and  $j$ ,  $x_{ij}$ ,  $y_i$ ,  $u_j$ ,  $\Pi_i$  are optimal in the linear programming primal and dual capacitated plant location problems if and only if

$$(a) \quad u_j \left( \sum_{i=1}^m x_{ij} - d_j \right) = 0 \quad \text{for } j = 1, \dots, n$$

$$(b) \quad \left( \sum_{j=1}^n x_{ij} - K_i y_i \right) \Pi_i = 0 \quad \text{for } i = 1, \dots, m$$

$$(c) \quad u_j - \Pi_i = c_{ij} + v_i \quad \text{if } x_{ij} > 0$$

$$(d) \quad f_i - \Pi_i K_i \begin{cases} \geq 0 & \text{if } y_i = 0 \\ = 0 & \text{if } 0 < y_i < 1 \\ \leq 0 & \text{if } y_i = 1 \end{cases}$$

For a given  $u$ , we show that a solution  $x_{ij}(u)$  for all  $i$  and  $j$  that is optimal in (6) satisfies these conditions for  $d(u)$  given by (8) when we define

$$\Pi_i(u) \begin{cases} = -C_i(u) & \text{if } C_i(u) < 0 \\ = 0 & \text{if } C_i(u) \geq 0 \end{cases}$$

where  $C_i(u)$  is defined in (7). Condition (a) holds trivially since  $\sum_{i=1}^m x_{ij}(u) - d_j(u) = 0$  for all  $j$  by definition. Condition (b) holds since  $\sum_{j=1}^n x_{ij}(u) = K_i y_i(u)$  for all  $i$  according to Theorem 3. Condition (c) also holds according to Theorem 3 since  $x_{ij}(u) > 0$  implies  $j \in J_i(u)$  and therefore  $-\Pi_i = C_i(u) = c_{ij} + v_i - u_j$ . Finally, condition (d) holds according to Theorem 3 since  $y_i(u) = 1$  only if  $\frac{-f_i}{C_i(u)} \leq K_i$  or  $f_i - \Pi_i(u)K_i \leq 0$  whereas  $y_i(u) = 0$  only if  $\frac{-f_i}{C_i(u)} \geq K_i$  or  $f_i - \Pi_i(u)K_i \geq 0$ . ||

Thus, inverse optimization of the family of capacitated plant location problems relative to the function  $Z^0(u)$  produces optimal solutions only to those problems  $R(d)$  for which there is an optimal integer solution to the linear programming relaxation  $R^C(d)$ . In spite of this shortcoming, the approach has some advantages over the simplex method applied to the linear programming relaxations. Optimal solutions to the function  $Z^0(u)$  are easier to compute by the rules of Theorem 3 and they are always integer even when there are alternative optimal fractional solutions to the linear programming relaxation. Moreover, we have no consistent way of knowing a priori which demand vectors  $d$  produce linear programming relaxations with optimal integer solutions. In the following section, the inverse optimization approach is extended to permit the calculation of optimal solutions to  $R(d)$  for all  $d \in D^*$ .

### 3. Resolution of Duality Gaps.

The extension of inverse optimization to the calculation of optimal solutions to  $R(d)$  for all  $d \in D^*$  defined in (4) is intimately related to the resolution of duality gaps between some primal problems  $R(d)$  and their duals. Let  $\bar{d}$  be a demand vector not spanned by  $Z^0$  for any  $u$ . We use  $\bar{d}$  to strengthen  $Z^0$  and  $L^0$  in the analysis of the specific problem  $R(\bar{d})$ , which also permits us to extend the family of problems for which inverse optimization can find an optimal solution. The construction is derived from a linear programming representation of the dual problem

$$\begin{aligned} w^0(\bar{d}) &= \max L^0(u; \bar{d}) \\ \text{s.t. } &u \in R^m. \end{aligned} \quad (9)$$

Specifically, let  $x_{ij}^t, y_i^t$ , for all  $i$  and  $j$  and for  $t = 1, \dots, T$  denote the collection of solutions satisfying

$$\sum_{j=1}^n x_{ij}^t - K_i y_i^t \leq 0 \quad i = 1, \dots, m$$

$$x_{ij}^t \geq 0 \text{ and integer, } y_i = 0 \text{ or } 1.$$

These are the feasible solutions in (6) for all  $i$ . Define

$$C^t = \sum_{i=1}^m \left\{ \sum_{j=1}^n (c_{ij} + v_i) x_{ij}^t + f_i y_i^t \right\}$$

and

$$d_j^t = \sum_{i=1}^m x_{ij}^t \quad \text{for } j = 1, \dots, n \quad (10)$$

Let  $d^t$  denote the  $n$ -vector with components  $d_j^t$ . It can be shown (see Magnanti, Shapiro and Wagner [ 9 ] or Shapiro [ 13 ]) that the dual problem (9) is equivalent to a linear programming problem whose linear programming dual is

$$\begin{aligned}
 w^0(\bar{d}) &= \min \sum_{t=1}^T (C^t) \lambda_t \\
 \text{s.t.} \quad &\sum_{t=1}^T (d_j^t) \lambda_t = \bar{d}_j \quad \text{for } j = 1, \dots, n \\
 &\sum_{t=1}^T \lambda_t = 1
 \end{aligned} \tag{11}$$

$$\lambda_t \geq 0$$

It can easily be shown that  $w^0(\bar{d})$  obeys the inequalities  $r(\bar{d}) \geq w^0(\bar{d}) \geq r^c(\bar{d})$  (see Theorem 1 of Geoffrion [6]), and we have a duality gap  $r(\bar{d}) > w^0(\bar{d})$ .

The resolution of a duality gap begins with the following result.

Theorem 5 (Bell and Shapiro [ 2 ]). Suppose problem (11) is solved by the simplex method and let  $\lambda_t^*$  for all  $t$  denote the optimal solution found by the method. Since  $R(\bar{d})$  is not spanned by  $Z^0$  for any  $u$ , it must be that two or more  $\lambda_t^*$  are positive, say  $\lambda_1^* > 0, \dots, \lambda_K^* > 0, \lambda_{K+1}^* = \dots = \lambda_T^* = 0$ , where  $K \geq 2$ . Moreover,  $d^k \neq \bar{d}$  for  $k = 1, \dots, K$ .

Proof: To show more than one  $\lambda_t^*$  must be positive, we assume the contrary and show a contradiction. Thus, suppose  $\lambda_1^* = 1$  and let  $u^*$  denote the computed optimal  $n$ -vector of shadow prices on the demand rows, and let  $\theta^*$  denote the scalar shadow price on the convexity row. Since  $\lambda_1^* = 1$ , we have  $d^1 = \bar{d}$ . By the linear programming optimality conditions



$$\begin{aligned} C^1 - u^* d^1 - \theta^* &= 0 \\ C^t - u^* d^t - \theta^* &\geq 0 \quad \text{for all } t \end{aligned}$$

or

$$C^1 - u^* d^1 \leq C^t - u^* d^t \quad \text{for all } t.$$

Using the definitions (10), this last condition can be rewritten as

$$\begin{aligned} &\sum_{i=1}^m \left\{ \sum_{j=1}^n (c_{ij} + v_i - u_i^*) x_{ij}^1 + f_i y_i^1 \right\} \\ &\leq \sum_{i=1}^m \left\{ \sum_{j=1}^n (c_{ij} + v_i - u_i^*) x_{ij}^t + f_i y_i^t \right\} \quad \text{for all } t. \end{aligned}$$

In words, this condition says that the solution  $x_{ij}^1, y_i^1$  for all  $i$  and  $j$  is optimal in  $Z^0(u^*)$  and by the inverse optimization Theorem 2, the solution is optimal in  $R(\bar{d})$ , which is impossible.

To show  $d^k \neq \bar{d}$  for  $k = 1, \dots, K$ , we again assume the contrary and show a contradiction. Suppose  $d^1 = \bar{d}$  and consider

$$\sum_{k=1}^K d^k \lambda_k^* = \bar{d}$$

or

$$\sum_{k=2}^K d^k \lambda_k^* = \bar{d}(1 - \lambda_1^*).$$

Since  $0 < \lambda_1^* < 1$ , we can divide by  $1 - \lambda_1^*$  which gives us

$$\sum_{k=2}^K d^k \left( \frac{\lambda_k^*}{1-\lambda_1^*} \right) = d^1.$$

The last equation contradicts the fact that  $\lambda_1, \dots, \lambda_K$  are basic variables in an optimal basic solution and therefore their columns must be linearly independent. ||

The construction given by Bell and Shapiro [2] is number theoretic in nature and derived from the vectors  $d^1, \dots, d^K$  identified in Theorem 5. The vectors are used to determine a finite abelian group  $\mathcal{G}$  and a homomorphism  $g$  mapping  $I^n$  onto  $\mathcal{G}$ , where  $I^n$  is the group of integer  $n$ -vectors under ordinary addition. The homomorphism has the property that for  $k = 1, \dots, K$ ,

$$g(d^k) = \sum_{i=1}^m \sum_{j=1}^n x_{ij}^k \varepsilon_j \neq \sum_{j=1}^n d_j \varepsilon_j = g(\bar{d}) \quad (12)$$

where  $\varepsilon_j = g(e_j)$  and  $e_j$  is the  $j^{\text{th}}$  unit vector in  $n$ -space.

The homomorphism is used to define a stronger Lagrangean by incorporating the group image of the demand constraints into the calculation. For any group element  $\delta$  and any demand vector  $d$  such that  $g(d) = \delta$ , and any dual vector  $u$ , the new Lagrangean is

$$L^{\mathcal{G}}(u; d) = ud + Z^{\mathcal{G}}(u; \delta) \quad (13)$$

where

$$\begin{aligned}
z^g(u; \delta) = \min & \sum_{i=1}^m \sum_{j=1}^n (c_{ij} + v_i - u_i)x_{ij} + \sum_{i=1}^m f_i y_i \\
\text{s.t.} & \sum_{i=1}^m \sum_{j=1}^n x_{ij} \varepsilon_j = \delta \\
& \sum_{j=1}^n x_{ij} - K_i y_i \leq 0 \quad \text{for } i = 1, \dots, m
\end{aligned} \tag{14}$$

$$x_{ij} \geq 0 \text{ and integer, } y_i = 0 \text{ or } 1.$$

Note that unlike the Lagrangean  $L^0$ , this Lagrangean does not separate into  $m$  individual calculations, one for each plant site. By the inverse optimization Theorem 2, for any  $u$  and  $\delta$ , a solution  $x_{ij}(u; \delta)$ ,  $y_i(u; \delta)$  for all  $i$  and  $j$  is optimal in  $R(d(u))$  where  $d_j(u) = \sum_{j=1}^n x_{ij}(u; \delta)$ .

Problem (14) is a fixed-charge group optimization problem that can be solved for all  $\delta \in \mathcal{G}$  by an algorithm developed by Northup and Sempolinski [11]. The computation is somewhat more complex than that required for the unconstrained or zero-one group optimization problems (see Gorry, Northup and Shapiro [8]). The algorithm of Northup and Sempolinski is a generalization of one devised by Glover [ 7 ] (see also Denardo and Fox [ 4 ] or Shapiro [ 15 ]).

Relative to the specific problem  $R(\bar{d})$ , the Lagrangean calculation  $L^g$  given in (13) has been strengthened over the Lagrangean  $L^0$  given in (15) because the solutions  $x_{ij}^k$ ,  $y_i^k$  for all  $i$  and  $j$ , and  $k = 1, \dots, K$ , that are optimal in  $L^0$  at dual optimality, have been made infeasible according to (12). It may still be, however, that there is a duality gap between  $R(\bar{d})$  and its new dual

$$\begin{aligned}
w^g(\bar{d}) = \max & L^g(u; \bar{d}) \\
\text{s.t.} & u \in R^m.
\end{aligned}$$

If so, the Lagrangean can be strengthened still further by reapplication of the same procedures. In this way, we can ultimately find an optimal solution to  $R(d)$  for all  $d \in D^*$ . The reader is referred to Bell and Shapiro [2] and Bell [1] for more details.

The importance of the scale used to measure the integer demand vector  $d$  to the analysis of the capacitated plant location problem  $R(d)$  is brought into focus by problem (11). If a coarse scale were used to measure the demands  $\bar{d}_j$ , then we would expect the  $d_j^t$  clustered around it to be smaller in magnitude than they would be if the  $\bar{d}_j$  were measured according to a refined scale. The magnitudes of the  $d_j^t$  tend to determine the magnitude of the order of the group  $\mathcal{G}$  induced by an optimal basis for (11) and hence the computational effort required to optimize the fixed-charge group optimization problem (14). Of course, any group  $\mathcal{G}$  can be used in the construction of (14). For example, we could use instead a group induced by a derived basis for (11) with  $d_j^t$  replaced by  $[\frac{d_j^t}{\lambda_j}]$  for all  $j$ , where  $\lambda_j > 0$  and  $[ \ ]$  denotes "integer part of". Alternatively, the procedures of Bell [1] would permit the construction of small groups in many cases even when the demands  $d_j$  are large in magnitude.

#### 4. Parametric Analyses and Inverse Optimization.

The use of inverse optimization methods in parametric analyses of integer programming and other combinatorial optimization problems is a rich area of research. In this section, we give some details on one approach for the capacitated plant location problem. The approach is based on subgradient optimization using the function  $Z^0$  defined in (6).

Suppose we are interested in an optimal solution to the capacitated plant location problem  $R(d)$  for  $d$  equal to and near the target demand vector  $\bar{d}$ . We begin by using the simplex method to solve the ordinary linear programming relaxation  $R^c(\bar{d})$  with minimal objective function value  $r^c(\bar{d})$ . Suppose further that the linear programming solution is fractional. We then use subgradient optimization to generate a sequence of dual solutions  $\{u^k\}$  and a corresponding sequence of demand vectors  $\{d^k\}$  defined by (8) for which we know an optimal solution to  $R(d^k)$ .

Specifically, as long as  $d^k \neq \bar{d}$ , we choose the dual sequence  $\{u^k\}$  by the rule

$$u^{k+1} = u^k + \rho_k \frac{(r^c(\bar{d}) - u^k \bar{d} - Z^0(u^k))}{\|\bar{d} - d^k\|^2} (\bar{d} - d^k) \quad (15)$$

$$k = 1, 2, \dots$$

where the sequence of demand vectors  $\{d^k\}$  is given by (8),  $0 < \epsilon_1 < \rho_k < 2 - \epsilon_2 < 2$  and  $\|\cdot\|$  denotes Euclidean norm. The formula (15) is a specialization of the general subgradient optimization rule (see Shapiro [15, p. 184]) to the capacitated plant location problem. The vector  $\bar{d} - d^k$  is a subgradient of  $L^0(u^k; \bar{d}) = u^k \bar{d} + Z^0(u^k)$ . Unlike other applications of subgradient optimization, we know in this case that the maximum value of  $w^0(\bar{d})$  of  $L^0(u; \bar{d})$

is at least as great as  $r^c(\bar{d})$  and therefore we can guarantee that  $\lim L^0(u^k; \bar{d}) \geq r^c(\bar{d})$  (e.g., see Theorem 6-2 of Shapiro [15]). Recall that  $w^0(\bar{d})$  is the maximal dual objective function value and  $r^c(\bar{d})$  is the minimal objective function value of the ordinary linear programming relaxation.

The procedure is completely specified once we have given the starting dual vector  $u^0$ , and values for the relaxation parameters  $\rho_k$ . An optimal dual vector from  $R^c(\bar{d})$  may not be a good choice because in the case when  $w^0(\bar{d}) = r^c(\bar{d})$ , we would have  $u^k = u^0$  for all  $k$ . One possibility is to take  $u^0 = u^* + \rho^*(\bar{d} - d^*)$  for some  $\rho^*$  where  $d^*$  is computed from  $Z^0(u^*)$  by (8). The choice of  $u^0$  and the  $\rho_k$  depends heavily on experimentation still to be gained.

There are only a finite number of distinct demand vectors that can arise in the sequence  $\{d^k\}$  derived from the sequence of dual vectors given by (15). This is because the set of feasible solutions to  $R(\bar{d})$  is finite. For a special case, the following theorem gives a characterization of those that can arise infinitely often.

**Theorem 6:** Suppose  $d^k \neq \bar{d}$  in the sequence of demand vector  $\{d^k\}_{k=0}^{\infty}$ , and let  $d^\ell$  denote a vector occurring infinitely often. Suppose further that  $w^0(\bar{d}) = r^c(\bar{d})$ . Then the vector  $\bar{d} - d^\ell$  is a subgradient of  $L^0(u; \bar{d})$  at an optimal solution for the dual problem (9).

**Proof:** In the special case when  $w^0(\bar{d}) = r^c(\bar{d})$ , the sequence of dual solutions  $\{u^k\}_{k=1}^{\infty}$  converges to a solution  $u^*$  that is optimal in the dual problem (9) (see Shapiro [15; pp. 185-186]). Since for  $u^k = u^\ell$

$$L(u^k; \bar{d}) = C^\ell + u^k(\bar{d} - d^\ell)$$

for  $C^\ell$  defined in (10), we have

$$\begin{aligned} w^0(\bar{d}) &= \lim_k L(u^k; \bar{d}) = L(u^*; \bar{d}) \\ &= c^\ell + u^*(\bar{d} - d^\ell) \end{aligned}$$

which is what we wanted to show. ||

## 5. Numerical Example.

Consider a capacitated plant location problem with three plant sites and two customers. The capacities of the three potential plants are  $K_1 = 8$ ,  $K_2 = 7$ ,  $K_3 = 6$ , with associated fixed-charges  $f_1 = 10$ ,  $f_2 = 8$ ,  $f_3 = 12$ . Table 1 gives the unit transportation costs from site  $i$  to customer  $j$ . The quantities  $v_1 = v_2 = v_3 = 0$ .

	j=1	j=2
i = 1	5	6
i = 2	6	7
i = 3	7	3

Unit Transportation Costs

Table 1

The circled demand vectors in Figure 2 are the ones that can be spanned by  $Z^0$  for some  $u$ ; by Theorem 4, these correspond to capacitated plant location problems for which there is an optimal integer solution to the linear programming relaxations.

We extend the inverse optimization analysis by considering the capacitated plant location problem with the demand vector  $(d_1, d_2) = (10, 2)$  not spanned in Figure 2. An optimal basis for problem (11) for this problem is given by

$$\begin{pmatrix} 8 & 15 & 8 \\ 0 & 0 & 6 \\ 1 & 1 & 1 \end{pmatrix}$$



The implied optimal solution to the linear programming relaxation of this problem is  $x_{11} = 8$ ,  $x_{21} = 2$ ,  $x_{32} = 2$ ,  $y_1 = 1$ ,  $y_2 = 6/21$ ,  $y_3 = 7/21$ . The optimal linear programming shadow prices are  $(\Pi_1^*, \Pi_2^*) = (7.14, 5.00)$ . To try to fill in the gaps in Figure 2, we use the above basis to construct a homomorphism  $g$  mapping  $Z^2$  onto  $Z_{42}$ , the cyclic group of order 42, which we use to aggregate the demand equations. The homomorphism is specified by  $g(e_1) = \varepsilon_1 = 6$  and  $g(e_2) = \varepsilon_2 = 7$  where  $\varepsilon_1$  and  $\varepsilon_2$  satisfy, along with  $\varepsilon_3$ , the group equations

$$\begin{aligned} 8\varepsilon_1 + 1\varepsilon_3 &\equiv 0 \pmod{42} \\ 15\varepsilon_1 + 1\varepsilon_3 &\equiv 0 \pmod{42} \\ 8\varepsilon_1 + 6\varepsilon_2 + 1\varepsilon_3 &\equiv 0 \pmod{42} \end{aligned}$$

See chapter eight of Shapiro [15] for more details about group constructions.

The resulting fixed-charge group optimization (14) is

$$Z^g(u_1, u_2) =$$

$$\begin{aligned} \min (5-u_1)x_{11} + (6-u_1)x_{21} + (7-u_1)x_{31} + (6-u_2)x_{10} + (7-u_2)x_{22} + (8-u_2)x_{32} \\ + 10y_1 + 8y_2 + 12y_3 \end{aligned}$$

$$\text{s.t.} \quad 6x_{11} + 6x_{21} + 6x_{31} + 7x_{12} + 7x_{22} + 7x_{32} \equiv \delta \pmod{42}$$

$$x_{11} + x_{12} - 8y_1 \leq 0$$

$$x_{21} + x_{22} - 7y_2 \leq 0$$

$$x_{31} + x_{32} - 6y_3 \leq 0$$

$$x_j \geq 0, \text{ integer, } y_i = 0 \text{ or } 1.$$

Figure 3 shows the demand vectors spanned by computing  $Z^g(7,6)$  for all 42 group right hand sides. There are 50 demand vectors shown in the figure because we include some obvious alternative optima. Although a considerable number of unspanned demand vectors in Figure 2 are spanned in Figure 3, the target demand vector (10,2) is not spanned. Reoptimization of  $Z^g$  at other dual solutions might produce an optimal solution spanning the target.

Finally, we illustrate the parametric procedure outlined in section using the function  $Z^0$ . Figure 4 shows the demand vectors spanned by seven steps with the procedure with the relaxation parameter  $\rho = 1$ . Table 2 gives some additional data. We remark that for this simple example, subgradient optimization converges rapidly to an optimal solution to the dual of the ordinary linear programming relaxation.

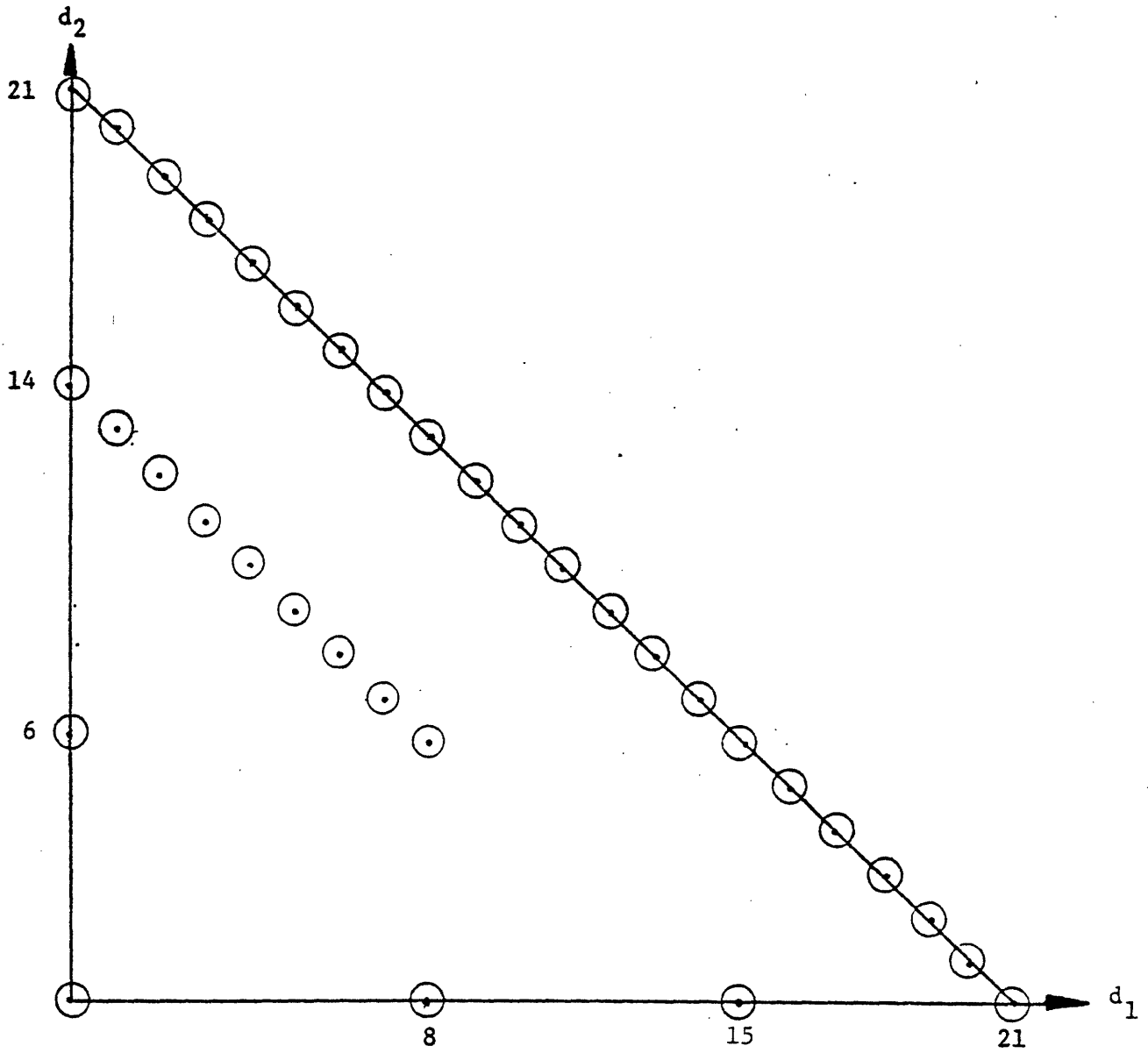


Figure 2

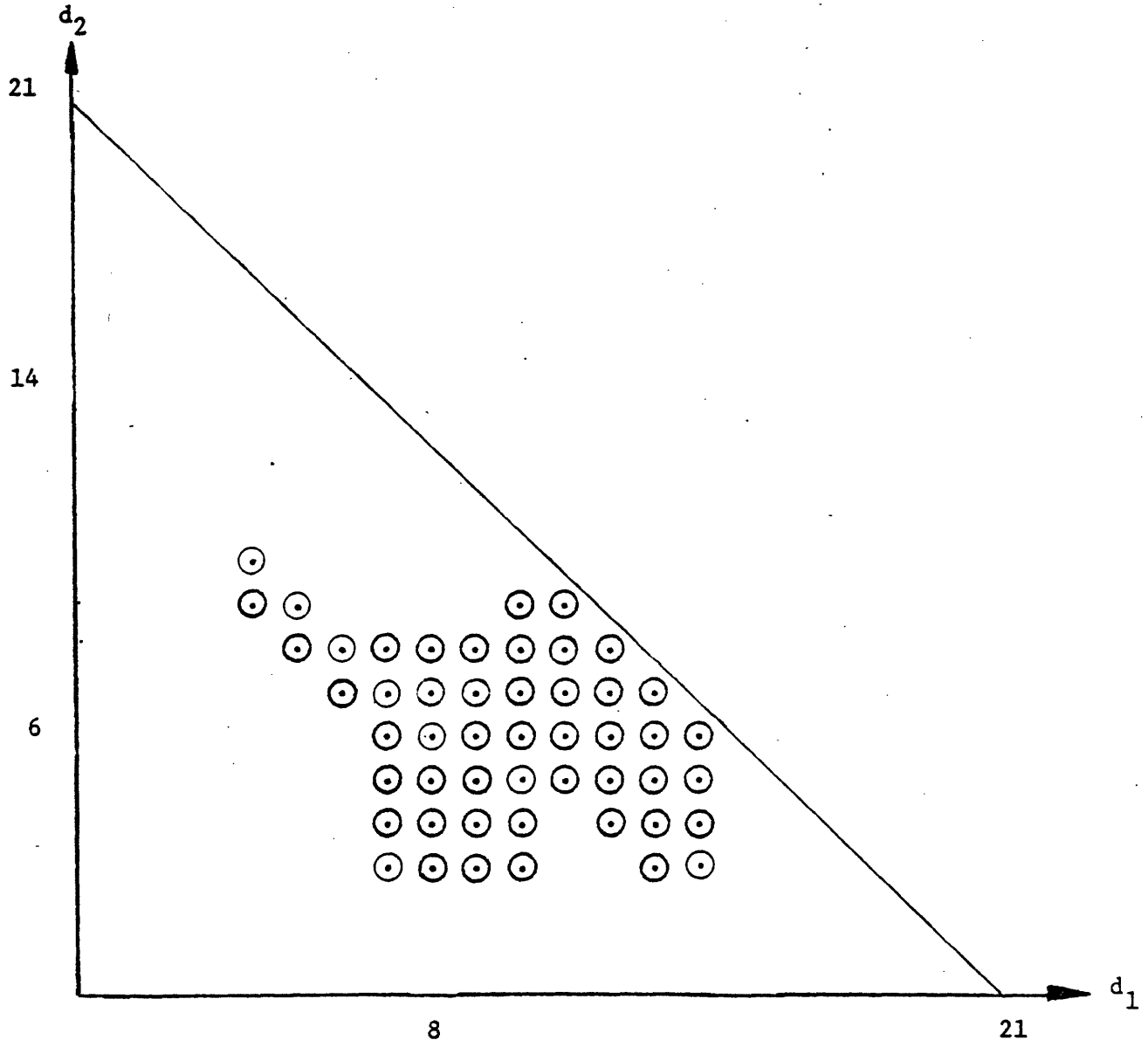


Figure 3

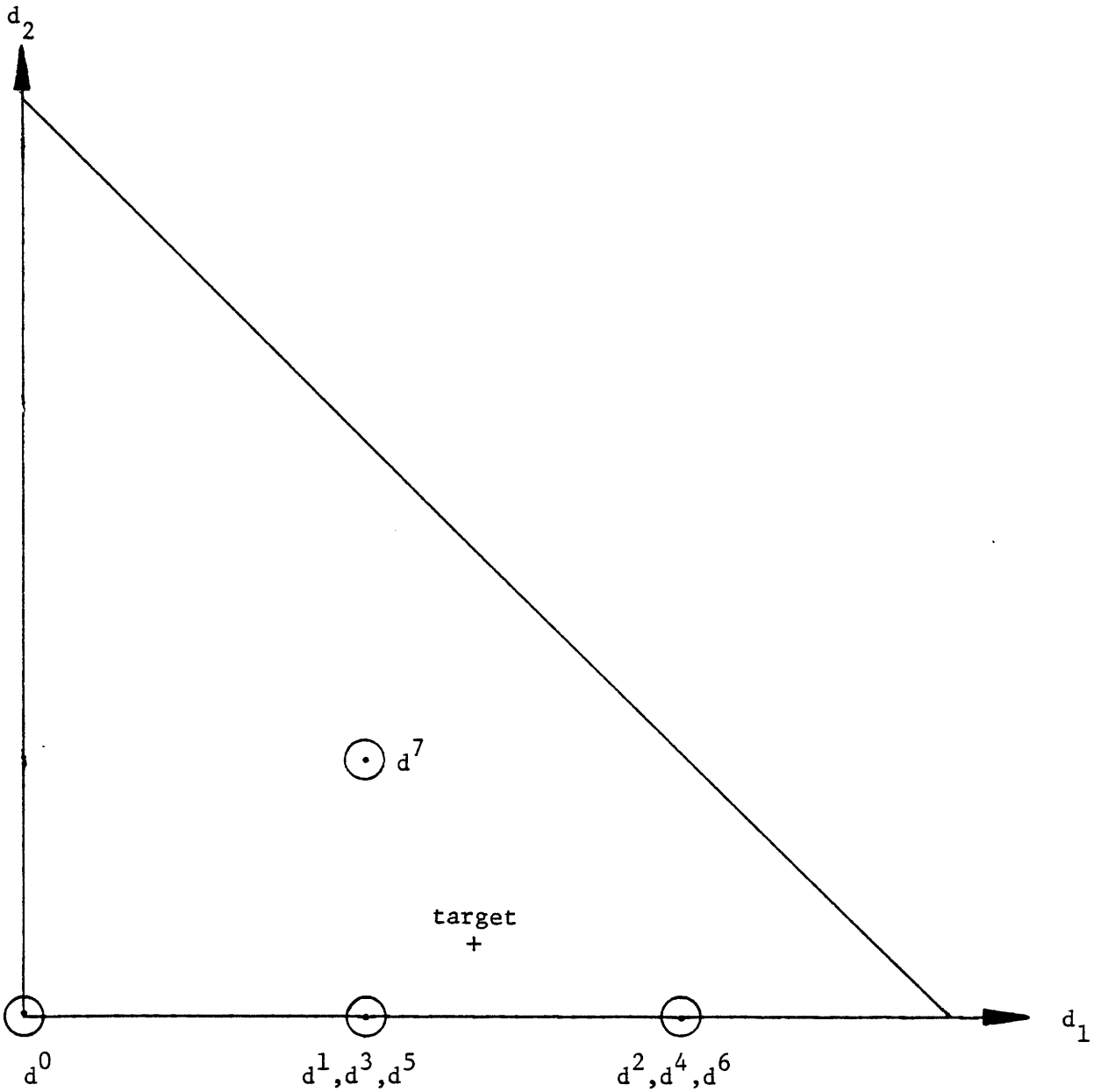


Figure 4

Table 2

$k$	$u^k$	$J_1(u^k)$	$J_2(u^k)$	$J_3(u^k)$	$y^k$	$L(u^k)$	$d^k$	$\gamma^k = \bar{d} - d^k$
0	(0,0)	1	1,2	2	(0,0,0)	0	0 0	10 2
1	(7.14,1.43)	1	1	1	(1,0,0)	67.14	8 0	2 2
2	(8.93,3.22)	1	1	1	(1,1,0)	61.79	15 0	-5 2
3	(6.77,4.08)	1	1	2	(1,0,0)	71.70	8 0	2 2
4	(7.42,4.73)	1	1	2	(1,1,0)	72.36	15 0	-5 2
5	(7.09,4.86)	1	1	2	(1,0,0)	73.90	8 0	2 2
6	(7.19,4.96)	1	1	2	(1,1,0)	73.97	15 0	-5 2
7	(6.99,5.04)	1	1	2	(1,0,1)	73.82	8 6	2 -4

## 5. Conclusions and Areas of Future Research.

We have presented the conceptual background for the inverse optimization approach to analyzing the capacitated plant location problem and more general mathematical programming problems. Several areas of future research are suggested by our results so far. For the capacitated plant location problem, there remains considerable theoretical and empirical research to be done on the use of inverse optimization in parametric analyses. A related topic to be investigated is the specialization of integer programming shadow price results (Shapiro [14]) to the capacitated plant location problem.

The fixed-charge group optimization algorithm devised by Northup and Sempolinski [11] for computing  $Z^g$  needs implementation and testing to measure its efficiency as a function of the number of plant sites  $m$  and the size of the group  $g$ . Moreover, this algorithm is applicable to a wider class of fixed-charge problems than the capacitated plant location problem. The explicit inclusion of the constraints  $\sum_{j=1}^n x_{ij} - K_i y_i \leq 0$  in the Lagrangean is highly desirable for these problems as long as it does not impose severe computational limits.

The capacitated plant location model can be extended in a number of directions to incorporate multiple time periods, multiple commodities, more general concave cost curves for capacity expansion and endogenous demand. Examples of such model extensions can be found in Bloom [3], Erlenkotter [5] and Kazmi and Shapiro [10]. The practicality of inverse optimization for more general models is another area of future research. An example is a multi-item production/inventory mixed integer programming model given in chapter eight of Shapiro [15].

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