Extension of gyrokinetics to transport time scales

by

Félix Ignacio Parra Díaz

Aeronautical Engineer, Universidad Politécnica de Madrid (2004)
S.M., Massachusetts Institute of Technology (2007)

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Author ....
Department of Aeronautics and Astronautics
May 21, 2009

Certified by ....
Peter J. Catto
Senior Research Scientist
Thesis Supervisor

Certified by ....
Manuel Martínez-Sánchez
Full Professor
Thesis Committee Chair

Certified by ....
Jeffrey P. Freidberg
Full Professor
Thesis Committee Member

Accepted by ....
David L. Darmofal
Associate Department Head
Chair, Committee on Graduate Students

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Abstract

In the last decade, gyrokinetic simulations have greatly improved our theoretical understanding of turbulent transport in fusion devices. Most gyrokinetic models in use are δf simulations in which the slowly varying radial profiles of density and temperature are assumed to be constant for turbulence saturation times, and only the turbulent electromagnetic fluctuations are calculated. Due to the success of these models, new massive simulations are being built to self-consistently determine the radial profiles of density and temperature. However, these new codes have failed to realize that modern gyrokinetic formulations, composed of a gyrokinetic Fokker-Planck equation and a gyrokinetic quasineutrality equation, are only valid for δf simulations that do not reach the longer transport time scales necessary to evolve radial profiles. In tokamaks, due to axisymmetry, the evolution of the axisymmetric radial electric field is a challenging problem requiring substantial modifications to gyrokinetic treatments. The radial electric field, closely related to plasma flow, is known to have a considerable impact on turbulence saturation, and any self-consistent global simulation of turbulent transport needs an accurate procedure to determine it. In this thesis, I study the effect of turbulence on the global electric field and plasma flows. By studying the current conservation equation, or vorticity equation, I prove that the long wavelength, axisymmetric flow must remain neoclassical and I show that the tokamak is intrinsically ambipolar, i.e., the radial current is zero to a very high order for any long wavelength radial electric field. Intrinsic ambipolarity is the origin of the problems with the modern gyrokinetic approach since the lower order gyrokinetic quasineutrality (if properly evaluated) is effectively independent of the radial electric field. I propose a new gyrokinetic formalism in which, instead of a quasineutrality equation, a current conservation equation or vorticity equation is solved. The vorticity equation makes the time scales in the problem explicit and shows that the radial electric field is determined by the conservation of toroidal angular momentum.

Thesis Supervisor: Peter J. Catto
Title: Senior Research Scientist

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Nomenclature

Miscellaneous

$\overline{(\dots)}$	Gyroaverage holding $\mathbf{r}$, $v_{\|\|}$, $v_\perp$ and t fixed.
$\langle\dots\rangle$	Gyroaverage holding $\mathbf{R}$, E, μ and t fixed.
$\langle\dots\rangle_\psi$	In tokamaks, flux surface average.
$\overline{\nabla}$	Gradient holding E_0, μ_0, φ_0 and t fixed.

Greek letters

ψ	In tokamaks, poloidal magnetic field flux, radial coordinate.
θ	In tokamaks, poloidal angle; in θ-pinches, azimuthal angle.
$\hat{\boldsymbol{\theta}}$	In θ-pinches, unit vector in the azimuthal direction.
ζ, $\hat{\boldsymbol{\zeta}}$	In tokamaks, toroidal angle and unit vector in the toroidal direction.
ϵ	In tokamaks, inverse aspect ratio a/R.
ρ_e, ρ_i	Electron and ion gyroradii, mcv_e/eB and Mcv_i/ZeB.
Ω_e, Ω_i	Electron and ion gyrofrequencies, eB/mc and ZeB/Mc.
δ_e, δ_i	Expansion parameters $\delta_e = \rho_e/L \ll \delta_i = \rho_i/L \ll 1$.
λ_D	Debye length.
ν_{ii}, ν_{ie}, ν_{ee}, ν_{ei}	Ion-ion, ion-electron, electron-electron and electron-ion Braginskii collision frequencies.
ω_*	Drift wave frequency.
$\omega_*^{n,T}$	Drift wave frequency in gyrokinetic equation (3.57) dependent on ∇n_i and ∇T_i.

ϕ Electrostatic potential.

$\langle \phi \rangle$ Electrostatic potential averaged in a gyromotion [see (3.16)].

$\widetilde{\phi}$ Difference between the potential seen by the particle and the potential averaged in a gyromotion [see (3.17)].

$\widetilde{\Phi}$ Indefinite integral of $\widetilde{\phi}$ with vanishing gyroaverage [see (3.18)].

μ Gyrokinetic magnetic moment defined to be an adiabatic invariant to higher order [see (3.33)].

μ_g Gyrokinetic magnetic moment in which the explicit dependence on the potential has been subtracted.

μ_0 Lowest order magnetic moment of the particle $v_\perp^2/2B$.

μ_1 First order correction to the gyrokinetic magnetic moment [see (3.34)].

μ_{10} First order correction to the gyrokinetic magnetic moment in which the explicit dependence on the potential has been subtracted [see (4.21)].

φ Gyrokinetic gyrophase in which the fast time variation has been averaged out [see (3.28)].

φ_0 Lowest order gyrophase of the particle [see (2.2)].

φ_1 First order correction to the gyrokinetic gyrophase [see (3.29)].

φ_{10} First order correction to the gyrokinetic gyrophase in which the explicit dependence on the potential has been subtracted [see (4.22)].

$\overleftrightarrow{\boldsymbol{\pi}}_i$ Ion viscosity. Its definition includes the Reynolds stress because the average velocity has not been subtracted [see (2.6)].

$\boldsymbol{\pi}_{ig||}$ Vector that gives the transport of parallel ion momentum by the $E \times B$ and magnetic drifts and the finite gyroradius drift $\tilde{\mathbf{v}}_1$ [see (4.39)].

$\overleftrightarrow{\boldsymbol{\pi}}_{ig\times}$ Tensor that gives the transport of perpendicular ion momentum by the parallel velocity, the $E \times B$ and magnetic drifts and the finite gyroradius drift $\tilde{\mathbf{v}}_1$ [see (4.40)].

$\overleftrightarrow{\boldsymbol{\pi}}_{iG}$ Effective viscosity for gyrokinetic vorticity equation (4.53).

ϖ Vorticity [see (2.11)].

ϖ_G Gyrokinetic "vorticity" [see (4.51)].

$\varpi_G^{(2)}$ Higher order gyrokinetic "vorticity" [see (5.41)].

Roman letters

a In tokamaks, minor radius.

$\mathbf{B}$, B, $\hat{\mathbf{b}}$ Magnetic field, magnetic field magnitude, and unit vector parallel to the magnetic field.

B_p In tokamaks, magnitude of the poloidal component of the magnetic field.

c Speed of light.

D_{gB} GyroBohm diffusion coefficient $\delta_i \rho_i v_i$.

e Electron charge magnitude.

$\hat{\mathbf{e}}_1$, $\hat{\mathbf{e}}_2$ Orthonormal vectors such that $\hat{\mathbf{e}}_1 \times \hat{\mathbf{e}}_2 = \hat{\mathbf{b}}$ used to define the gyrophase φ_0 [see (2.2)].

E Gyrokinetic kinetic energy in which the fast time variation has been averaged out [see (3.24)].

E_0 Kinetic energy of the particle $v^2/2$.

E_1 First order correction to the gyrokinetic kinetic energy [see (3.25)].

E_2 Second order correction to the gyrokinetic kinetic energy [see (3.26)].

$\mathbf{E}$ Electric field.

f_e Electron distribution function, dependent on $\mathbf{r}$, E_0, μ_0 and φ_0.

$\overline{f}_e$ Gyrophase independent piece of the electron distribution function.

$f_e - \overline{f}_e$ Gyrophase dependent piece of the electron distribution function.

f_i Ion distribution function, usually dependent on the gyrokinetic variables $\mathbf{R}$, E and μ.

$\tilde{f}_i$ Small piece of the ion distribution function that depends on the gyrokinetic gyrophase responsible for classical diffusion [see (3.38)].

f_{ig}	Distribution function found by replacing the gyrokinetic variables $\mathbf{R}$, E and μ in $f_i(\mathbf{R}, E, \mu, t)$ by $\mathbf{R}_g$, E_0 and μ_0 [see (4.3)].
f_{iG}	Distribution function found by replacing the gyrokinetic variables $\mathbf{R}$, E and μ in $f_i(\mathbf{R}, E, \mu, t)$ by $\mathbf{R}_g$, E_0 and μ_g.
f_{i0}	Distribution function found by replacing the gyrokinetic variables $\mathbf{R}$, E and μ in $f_i(\mathbf{R}, E, \mu, t)$ by $\mathbf{r}$, E_0 and μ_0.
$\overline{f}_i$	Gyrophase independent piece of the ion distribution function when written in physical phase space variables $\mathbf{r}$, E_0, μ_0 and φ_0.
$f_i - \overline{f}_i$	Gyrophase dependent piece of the ion distribution function when written in physical phase space variables $\mathbf{r}$, E_0, μ_0 and φ_0.
f_{Me}, f_{Mi}	Lowest order electron and ion distribution functions, assumed to be stationary Maxwellians.
$\mathbf{F}_{ei}$	Electron collisional momentum exchange with ions.
$\mathbf{F}_{iB}$	Change in the perpendicular momentum of the gyromotion due to variations in the magnetic field strength [see (4.42)].
$\mathbf{F}_{iB}^{(2)}$	Higher order version of $\mathbf{F}_{iB}$ [see (5.37)].
$\mathbf{F}_{iC}$	Force due to finite gyroradius effects on collisions [see (4.43)].
$\tilde{F}_{iE}$	Change in the parallel momentum due to the short wavelength components of the electrostatic potential [see (4.41)].
$\tilde{F}_{iE}^{(2)}$	Higher order version of $\tilde{F}_{iE}$ [see (5.36)].
h_i	Correction to the Maxwellian for ions.
h_{i1}, h_{i1}^{tb}, h_{i1}^{nc}	First order correction to the Maxwellian in δ_i, decomposed into two pieces: the piece due to turbulence and the neoclassical contribution.
h_{i2}, h_{i2}^{tb}, h_{i2}^{nc}	Second order correction to the Maxwellian in δ_i, decomposed into two pieces: the piece due to turbulence and the neoclassical contribution.
I	Function $R\mathbf{B} \cdot \hat{\boldsymbol{\zeta}}$; it only depends on ψ to lowest order.
$\overleftrightarrow{\mathbf{I}}$	Unit matrix.
J, J_u	Jacobian of the gyrokinetic transformation [see (3.44) and (3.47)].
$\mathbf{J}$	Current density.

$\mathbf{J}_d$	Current density due to magnetic drifts [see (2.10)].
$\mathbf{J}_{gd}$	Current density due to magnetic drifts calculated integrating $\mathbf{v}_{M0} f_{ig}$ over velocity space instead of $\mathbf{v}_{M0} f_i$ [see (4.47)].
$J_{g\vert\vert}^{(2)}$	Parallel current density calculated integrating $v_{\vert\vert} f_{ig}$ over velocity space instead of $v_{\vert\vert} f_i$ [see (5.39)]. The superindex (2) emphasizes that $J_{\vert\vert} = \mathbf{J} \cdot \hat{\mathbf{b}}$ and $J_{g\vert\vert}^{(2)}$ differ only in higher order terms.
$\tilde{\mathbf{J}}_i$	Polarization current density in gyrokinetic vorticity equation (4.45).
$\tilde{\mathbf{J}}_{i\phi}$	Polarization current density in gyrokinetic vorticity equation (4.53).
$\tilde{\mathbf{J}}_{i\phi}^{(2)}$	Polarization current density in gyrokinetic vorticity equation (5.40).
$k_{\vert\vert}$, $k_\perp$	Wavenumbers parallel and perpendicular to the magnetic field.
L	Characteristic length in the problem.
m, M	Electron and ion masses.
n_e, n_i	Electron and ion densities.
n_{ip}	Gyrokinetic polarization density defined in (3.55).
$n_{ip}^{(2)}$	Higher order gyrokinetic polarization density defined in (5.33).
p_e, p_i	Electron and ion "pressures." They are not the usual definitions because the average velocity is not subtracted.
$p_{e\vert\vert}$, $p_{e\perp}$	Electron parallel and perpendicular "pressures." They are not the usual definitions because the average velocity is not subtracted.
$p_{i\vert\vert}$, $p_{i\perp}$	Ion parallel and perpendicular "pressures." They are not the usual definitions because the average velocity is not subtracted.
$p_{ig\vert\vert}$, $p_{ig\perp}$	Parallel and perpendicular "pressures" calculated integrating $Mv_{\vert\vert}^2 f_{ig}$ and $(Mv_\perp^2/2) f_{ig}$ over velocity space instead of $Mv_{\vert\vert}^2 f_i$ and $(Mv_\perp^2/2) f_i$.
$\overleftrightarrow{\mathbf{P}}_i$	Total ion stress tensor, including contributions due to the average ion velocity.
q	Safety factor.
$\mathbf{r}$	Position of the particle.

r	In θ-pinches, radial coordinate.
$\hat{\mathbf{r}}$	In θ-pinches, unit vector in the radial direction.
R	In tokamaks, the distance between the axis of symmetry and the position of the particle, also used as major radius in estimates of order of magnitude.
$\mathbf{R}$	Position of the gyrocenter [see (3.12)].
$\mathbf{R}_g$	Position of the guiding center $\mathbf{r} + \Omega_i^{-1}\mathbf{v} \times \hat{\mathbf{b}}$.
$\mathbf{R}_1$	First order correction to the gyrocenter position $\Omega_i^{-1}\mathbf{v} \times \hat{\mathbf{b}}$.
$\mathbf{R}_2$	Second order correction to the gyrocenter position [see (3.15)].
T_e, T_i	Electron and ion temperatures.
u	Velocity of the gyrocenter parallel to the magnetic field [see (3.23)].
u_g	Parallel velocity found by replacing the gyrokinetic variables $\mathbf{R}$, E and μ in $u(\mathbf{R}, E, \mu)$ by $\mathbf{R}_g$, E_0 and μ_g. [see (5.29)].
$\mathbf{v}$	Velocity of the particle.
$v_{\|\|}$	Velocity component parallel to the magnetic field.
$v_{\|\|0}$	Velocity parallel to the magnetic field with finite gyroradius modifications [see (4.16)].
$\mathbf{v}_\perp$, $v_\perp$	Velocity perpendicular to the magnetic field and its magnitude.
v_e, v_i	Electron and ion thermal velocities, $\sqrt{2T_e/m}$ and $\sqrt{2T_i/M}$.
$\mathbf{v}_E$	Gyrokinetic $E \times B$ drift [see (3.21)].
$\mathbf{v}_{E0}$	Lowest order gyrokinetic $E \times B$ drift [see (4.18)].
$\mathbf{v}_d$	Gyrokinetic drift composed of $\mathbf{v}_E$ and $\mathbf{v}_M$.
$\mathbf{v}_{de}$	Electron drift [see (3.50)].
$\mathbf{v}_{dg}$	Drift found by replacing the gyrokinetic variables $\mathbf{R}$, E and μ in $\mathbf{v}_d(\mathbf{R}, E, \mu)$ by $\mathbf{R}_g$, E_0 and μ_g [see (5.28)].
$\mathbf{v}_M$	Gyrokinetic magnetic drift [see (3.22)].
$\mathbf{v}_{M0}$	Standard magnetic drift [see (4.17)].
$\tilde{\mathbf{v}}_1$	Drift due to finite gyroradius effects [see (4.19)].

V'	In tokamaks, flux surface volume element $dV/d\psi$.
$\mathbf{V}_e$, $\mathbf{V}_i$	Electron and ion average velocities.
$\mathbf{V}_{ig}$	Ion average velocity calculated integrating $\mathbf{v} f_{ig}$ over velocity space instead of $\mathbf{v} f_i$.
$V_{ig\vert\vert}^{(2)}$	Ion parallel average velocity calculated integrating $v_{\vert\vert} f_{ig}$ over velocity space instead of $v_{\vert\vert} f_i$ [see (5.34)]. The superindex (2) emphasizes that $V_{i\vert\vert} = \mathbf{V}_i \cdot \hat{\mathbf{b}}$ and $V_{ig\vert\vert}^{(2)}$ differ only in higher order terms.
$\mathbf{V}_{iC}$	Ion average velocity due to finite gyroradius effects on collisions [see (4.33)].
$\mathbf{V}_{iE}$, $\mathbf{V}_{igd}$	Ion average velocities due to the gyrokinetic $E \times B$ and magnetic drifts [see (4.31) and (4.32)].
$\tilde{\mathbf{V}}_i$	Ion average velocity due to finite gyroradius contribution $\tilde{\mathbf{v}}_1$ [see (4.30)].
$V_{i\vert\vert}^{\mathrm{nc}}$	Neoclassical ion parallel velocity.
Z	Ion charge number.

Chapter 1

Introduction

Magnetic confinement is the most promising concept for production of fusion energy and the tokamak is the best candidate among all the possible magnetic confinement devices. However, transport of particles, energy and momentum is still not well understood in tokamaks. The transport is mainly turbulent, and modelling and predicting how it evolves is necessary to build a viable reactor.

The understanding of turbulent transport in tokamaks has greatly improved in the last decade, mainly due to more comprehensive simulations [1, 2, 3, 4, 5, 6]. These simulations employ the gyrokinetic formalism to shorten the computational time. Gyrokinetics is a sophisticated asymptotic method that keeps finite gyroradius effects without solving on the gyrofrequency time scale – a time too short to be of interest in turbulence. Unfortunately, modern formulations of gyrokinetics are still only valid for times shorter than the energy diffusion time or transport time scale. As simulations try to reach longer time scales, the gyrokinetic formalism needs to be extended. This thesis identifies the shortcomings of gyrokinetics at long transport times and solves one of the most pressing issues, namely, the calculation of the axisymmetric radial electric field, and thereby, the transport of momentum.

In this introduction, first I will review the characteristics of turbulence in tokamaks in section 1.1. This review is followed by a brief history of gyrokinetic simulations in section 1.2, with emphasis on new developments. The new codes being built will require a new formulation of gyrokinetics valid for longer time scales. In section 1.3, I will close the introduction by discussing the requirements for future gyrokinetic formalisms and summarizing the rest of the thesis.

1.1 Turbulence in tokamaks

Currently, the main part of the turbulence in the core is believed to be driven by drift waves. These waves propagate in the plasma perpendicularly to density and temperature gradients. They become unstable in tokamaks due to the curvature in the magnetic field and other inhomogeneities. There are several modes that are considered important, but the two of most interest here are the Ion Temperature Gradient mode (ITG) [7] and the Trapped Electron Mode (TEM) [8].

These modes have frequencies much smaller than the ion gyrofrequency. They are unstable at short wavelengths – the fastest growing mode wavelengths are on the order of the ion gyroradius because shorter wavelengths are stabilized by finite gyroradius effects. The measurements in tokamaks suggest turbulence correlation lengths on the order of five to ten gyroradii [9], which agrees with this idea. The anisotropy induced by the magnetic field is reflected in the spatial structure of these modes. The wavelength along the magnetic field, on the order of the characteristic size of the device, is much longer than the perpendicular wavelengths.

The ITG and TEM instabilities provide energy for the turbulence at short wavelengths. By nonlinear beating, part of the turbulence energy is deposited in a radial mode known as zonal flow [10, 11, 12]. The zonal flow is a radial structure in the radial electric field that gives rise to a sheared poloidal and toroidal $E \times B$ flow. It is a robust mode because it does not have any parallel electric field and hence electrons cannot shield it or Landau damp it. Then, any energy deposited in the zonal flow will remain there, leading to a rapid nonlinear growth of this mode. It has an impact on turbulence dynamics because the velocity shear decorrelates the turbulence at the shorter wavelengths. The statistical equilibrium of the turbulence is determined by the feedback between zonal flow and short wavelength fluctuations. The effect of zonal flow is so important that it can suppress the turbulence when the instability is not too strong [13].

The radial electric field is crucial in the saturation of turbulence. Its short wavelength radial structure is the zonal flow, whose importance in turbulence dynamics has already been discussed. It is also quite clear that it plays an important role in the pedestal of high confinement or H-mode plasmas in tokamaks [14], where the shear in the macroscopic radial electric field becomes large. Experimentally, it is observed that the radial electric field shear increases before the turbulent fluctuations are suppressed, radial transport is quenched and the gradient of density increases to form the pedestal [15]. Both zonal flow and transport barriers highlight the importance of the calculation of the radial electric field in any turbulence simulation. In this thesis, I will show that the traditional gyrokinetic approach is unable to provide the correct long wavelength axisymmetric radial electric field. This problem has gone undetected up until now because it is only noticeable at long time scales.

To summarize, the turbulence in a tokamak is characterized by electromagnetic fluctuations with wavelengths as small as the ion gyroradius. On the other hand, the frequency of these fluctuations is much smaller than the ion gyrofrequency, making the timescales of gyromotion and turbulence so disparate that both can be treated independently. It is this scale separation that gyrokinetics exploits by "averaging out" the gyromotion, while keeping the finite gyroradius effects. However, as formulated, the traditional gyrokinetic model does not contain enough physics to provide the self-consistent long wavelength axisymmetric radial electric field. Since the radial electric field affects turbulent transport, we need to extend the gyrokinetic formalism to calculate it.

1.2 Gyrokinetics: history and current challenges

The gyrokinetic model is a more suitable way of writing the Fokker-Planck equation for low frequencies and short perpendicular wavelengths. The idea of gyrokinetics is to define new variables to replace the position $\mathbf{r}$ and velocity $\mathbf{v}$ of the particle. The gyrokinetic variables are constructed such that the the gyromotion is decoupled from the slowly varying electromagnetic fluctuations. This approach is especially convenient in turbulence simulation because retaining the gyromotion is unnecessary. The gyrokinetic variables are the appropriate variables to solve the problem since they retain wavelengths on the order of the ion gyroradius and ignore the high frequencies.

Gyrokinetics had its roots in reduced kinetic techniques used to analyze stability problems with finite gyroradius effects. The early works of Rutherford and Frieman [16] and Taylor and Hastie [17] treated small perpendicular wavelengths in stability calculations for general magnetic field geometries by using an eikonal approximation. Years later, Catto [18, 19] formulated the gyrokinetic approach by introducing the gyrokinetic change of variables.

The gyrokinetic formulation eventually evolved to a nonlinear model. Frieman and Chen [20] developed a nonlinear theory for perturbations of small amplitude over the distribution function in general magnetic field geometry. Their work was extended later for a full distribution function in a slab geometry by Dubin *et al* [21]. Hahm *et al* extended the work of Dubin *et al* to electromagnetic perturbations [22], and toroidal geometry [23]. These nonlinear gyrokinetic equations were found employing a Hamiltonian formulation and Lie transforms [24, 25].

Based on these seminal nonlinear models, Lee developed the first gyrokinetic code for investigation of the drift wave turbulence [26, 27]. This first approach was a primitive Particle-In-Cell (PIC) δf model. The δf models avoid solving for the full distribution function, which would require much computational time. Instead, the distribution function is assumed to be Maxwellian to lowest order, and a nonlinear equation for δf retaining small fluctuations is solved. The assumption is that the time it takes the turbulence to saturate is much shorter than the diffusion time. Then, the density and temperature profiles are given as an input and do not change in time. In these codes, the turbulent fluctuations evolve and saturate, and from their saturated value we can calculate the radial particle and heat fluxes. The modern, less noisy δf models originate in the ideas put forth by Kotschenreuther in [28].

Several δf codes, both continuum, like GS2 [1], GENE [2] and GYRO [3], and PIC, like GEM [4], PG3EQ [5] and GTC [6], have been developed and benchmarked.

It is based on these codes that most of the recent advances in tokamak turbulence theory have occurred.

As it was already pointed out, the δf codes are only useful to compute the particle and heat fluxes once profiles for density and temperature are given. It is necessary to develop a new generation of models capable of self-consistently calculating and evolving those profiles. It is not obvious that it can be done with the current gyrokinetic formalism. For the δf models it was enough to run the codes until the turbulence had saturated, but in order to let the profiles relax to their equilibrium, runs on the order of the transport time scale are needed. This extension is both a costly numerical task and an unsolved physical problem. Models that reach transport time scales need to take into account phenomena that were negligible when looking for the turbulence saturation. In gyrokinetics, corrections to the velocity of the particles small in a ion gyroradius over scale length are neglected. However, as run times become longer, these terms must be retained since a small velocity correction gives a considerable contribution to the total particle motion.

In recent years, several groups have begun to build codes that evolve the full distribution function, without splitting it into a slowly varying Maxwellian and a fast, fluctuating piece. These simulations, known as full f models, are employing the traditional gyrokinetic formulation. In this thesis, I will argue that this approach is inadequate since it is unable to solve for the self-consistent radial electric field that is crucial for the turbulence.

Before getting into details, I will briefly review the four main efforts in this field: GYSELA [29], ELMFIRE [30], XGC [31] and TEMPEST [32]. All these models are electrostatic. ELMFIRE and XGC are PIC simulations, and GYSELA and TEMPEST are continuum codes. GYSELA and XGC calculate the full ion distribution function, but they adopt a fluid model for electrons that assumes an adiabatic response along the magnetic field lines. ELMFIRE and TEMPEST solve kinetically for both ions and electrons. Importantly, all four models find the electrostatic potential from a gyrokinetic Poisson's equation [27]. This gyrokinetic Poisson's equation just imposes that the ion and electron density must be equal. It looks like Poisson's equation because there is a piece of the ion density, known as polarization density, that can be written explicitly as a Laplacian of the potential. Regardless of its appearance, the gyrokinetic Poisson's equation is no more than a lower order quasineutrality condition. It is lower order because the density is calculated from the gyrokinetic equation in which higher order terms have been neglected. This is the most problematic part of these models, as I will demonstrate in this thesis. Interestingly, GYSELA, ELMFIRE and XGC have reported an extreme sensitivity to the initialization.

Operationally, the polarization density depends on the velocity space derivatives of the distribution function (the polarization density is presented in section 3.4). It is difficult to evaluate directly. In GYSELA, XGC and TEMPEST, the wavelengths are taken to be longer than the ion gyroradius and the distribution function is assumed to be close to a Maxwellian to obtain a simplified expression. In an attempt to circumvent this problem, ELMFIRE employs the gyrokinetic variables proposed by Sosenko *et al* [33]. These variables include a polarization drift that largely removes the polarization density. With the polarization velocity, it is possible to use implicit numerical schemes that give the dependence of the ion density with the electrostatic potential.

To summarize, gyrokinetic modelling has been successfully used for studying turbulence in the past decade. Codes based on δf formulations, especially the continuum ones, have provided valuable insights into tokamak anomalous transport. Currently, there is an interest in extending these simulations to transport timescales, and that requires careful evaluation of both physical and numerical issues. As a result, several groups are building and testing full f simulations. In these codes, solving for the axisymmetric radial electric field is crucial because it determines the poloidal and toroidal flows, and those flows strongly affect and, near marginality, control the turbulence level. Unfortunately, the full f community has failed to realize that a straightforward extension of the equations valid for δf codes are unable to provide the long wavelength radial electric field. The objective of this thesis is exposing this problem and proposing a solution.

1.3 Calculating the radial electric field

There are several problems that a gyrokinetic formulation has to face before it is satisfactory for long time scales. The main issue is the missing higher order terms in the gyrokinetic Fokker-Planck equation. The transport of particles, momentum and energy from one flux surface to the next is slow compared to the typical turnover time of turbulent eddies, the characteristic time scale for the traditional gyrokinetic formulation. To see this, recall that the typical structures in the turbulence are of the size of the ion gyroradius $\rho_i = Mcv_i/ZeB$, with $v_i = \sqrt{2T_i/M}$ the ion thermal velocity, Ze, M and T_i the ion charge, mass and temperature, B the magnetic field magnitude, and e and c the electron charge magnitude and the speed of light. Then, eddies are of ion gyroradius size, requiring many eddies – and hence many eddy turnover times – for a particle to diffuse out of the tokamak.

The gyrokinetic Fokker-Planck equation is derived to an order adequate for simulation of turbulence saturation, i.e., for time scales on the order of the eddy turnover time. This equation is too low of an order for transport time scales because flows and fluxes that were neglected as small now have enough time to contribute to the motion of the particles. In other words, a higher order distribution function and hence a higher order Fokker-Planck equation are required. For this reason, the extension of gyrokinetics to transport time scales must draw from the experience developed in neoclassical theory [34, 35]. Not only can neoclassical transport compete with the turbulent fluxes in some limited cases, but the tools and techniques developed in neoclassical theory become extremely useful because they require only a lower order distribution function to determine higher order radial fluxes of particles, energy and momentum. The application of neoclassical tools in gyrokinetic simulations is described in [36] and extended herein.

The physics in which the modern gyrokinetic formulation is especially flawed is the calculation of the long wavelength radial electric field. In this case, the comparison between neoclassical theory and gyrokinetics is striking. In neoclassical theory, the tokamak is intrinsically ambipolar due to its axisymmetry [37, 38], i.e., the plasma remains quasineutral for any value of the radial electric field unless the distribution function is known to higher order than second in an expansion on the ion gyroradius over the scale length. The reason for this is that the radial electric field is related to the toroidal velocity through the $E \times B$ drift. Due to axisymmetry, the evolution of the toroidal velocity only depends on the small off-diagonal terms of the viscosity, making impossible the self-consistent calculation of the radial electric field unless the proper off-diagonal terms are included. The distribution function required to directly obtain the viscosity is higher order than second; the order at which intrinsic ambipolarity is maintained. The axisymmetric radial electric field has only been recently found in the Pfirsch-Schlüter regime [39, 40, 41, 42], and there has been some incomplete work on the banana regime for high aspect ratio tokamaks [43, 44].

In gyrokinetics, however, the electric field is found from a lower order gyrokinetic quasineutrality equation [21, 26] rather than from the transport of toroidal angular momentum. Implicitly, it is assumed that the tokamak is not intrinsically ambipolar in the presence of turbulence. In this thesis, I prove that even turbulent tokamaks are intrinsically ambipolar in the gyrokinetic ordering. Consequently, if the radial electric field is to be retrieved from a quasineutrality equation, the distribution function must be found to a hopelessly high order. The physics that determine the radial electric field, namely, the transport of angular momentum, enters the quasineutrality condition only in higher order terms, making the gyrokinetic quasineutrality equation inadequate for the calculation.

In this thesis, I pay special attention to the evolution of the long wavelength axisymmetric radial electric field in the presence of drift wave turbulence. Employing a current conservation equation or vorticity equation, I assess the feasibility of different methods to find the long wavelength axisymmetric radial electric field. Each method requires the ion Fokker-Planck equation to a different order in $\delta_i = \rho_i/L \ll 1$ and $B/B_p \gtrsim 1$, with ρ_i the ion gyroradius, L a characteristic size in the machine, typically the minor radius a, B the magnitude of the magnetic field and B_p the magnitude of its poloidal component. The different methods explored in this thesis are summarized in table 1.1. In this table, I give the chapter in which the method is presented and the

Method		Order of f_i	Chapter
Gyrokinetic quasineutrality equation		$\delta_i^4 f_{Mi}$	3
Radial transport of toroidal angular momentum	Evaluated directly from f_i	$\delta_i^3 f_{Mi}$	2
	Moment equation	$\delta_i^2 f_{Mi}$	5
	Moment equation and $B/B_p \gg 1$	$(B/B_p)\delta_i^2 f_{Mi}$	5

Table 1.1: Comparison of different methods to obtain the long wavelength axisymmetric radial electric field.

order of magnitude to which the ion distribution function must be known compared to the zeroth order distribution function f_{Mi}. To better explain the classification by required order of magnitude of f_i, I present here a very simplified heuristic study of the first method in table 1.1. This method, known as the gyrokinetic quasineutrality equation, will be rigourously described at the end of chapter 3. In the gyrokinetic quasineutrality equation, used in modern gyrokinetics, the electric field is adjusted so that the ion and electron densities satisfy $Zen_i = en_e$, and the densities are calculated by direct integration of the distribution functions, i.e., $n_i = \int d^3v\, f_i$ and $n_e = \int d^3v\, f_e$. The ion and electron Fokker-Planck equations used to solve for the distribution functions f_i and f_e are approximate. In chapter 3, I will describe in more detail the formalism to obtain these approximate equations. For now, it is enough to consider a heuristic form of the long wavelength limit of these equations in which only the motion of the guiding center $\mathbf{R}$ is considered. Then, schematically the Fokker-Planck equations for ions and electrons are

$$\frac{\partial f_i}{\partial t} + \dot{\mathbf{R}}_i \cdot \nabla f_i + \ldots = C_{i,\text{eff}}\{f_i, f_e\} \tag{1.1}$$

and

$$\frac{\partial f_e}{\partial t} + \dot{\mathbf{R}}_e \cdot \nabla f_e + \ldots = C_{e,\text{eff}}\{f_e, f_i\}, \tag{1.2}$$

where $\mathbf{R}_i$ and $\mathbf{R}_e$ are the ion and electron guiding center positions, $\dot{\mathbf{R}}_i = v_{||i}\hat{\mathbf{b}} + \mathbf{v}_{di} + \ldots$ and $\dot{\mathbf{R}}_e = v_{||e}\hat{\mathbf{b}} + \mathbf{v}_{de} + \ldots$ are the drifts of those guiding centers, and $C_{i,\text{eff}}$ and

$C_{e,\text{eff}}$ are the effective collision operators. The lower order drifts are the parallel velocities $v_{||i} \sim v_i$ and $v_{||e} \sim v_e$, and the perpendicular drifts $\mathbf{v}_{di} = \mathbf{v}_{Mi} + \mathbf{v}_E \sim \delta_i v_i$ and $\mathbf{v}_{de} = \mathbf{v}_{Me} + \mathbf{v}_E \sim \delta_e v_e \sim \delta_i v_i$, with $\mathbf{v}_{Mi}$ and $\mathbf{v}_{Me}$ the magnetic drifts, $\mathbf{v}_E = (c/B)\mathbf{E} \times \hat{\mathbf{b}}$ the $E \times B$ drift and $\mathbf{E}$ the electric field. The ion drifts are expanded in the small parameter $\delta_i = \rho_i / L \ll 1$, and the electron drifts are expanded in the small parameter $\delta_e = \rho_e / L \sim \delta_i \sqrt{m/M} \ll \delta_i$, with $\rho_e = mcv_e/eB$ the electron gyroradius, $v_e = \sqrt{2T_e/m}$ the electron thermal speed, and $T_e \sim T_i$ and m the electron temperature and mass. The next order corrections in δ_i to $\dot{\mathbf{R}}_i$ may be written as $\dot{\mathbf{R}}_i = v_{||i}\hat{\mathbf{b}} + \mathbf{v}_{di} + \dot{\mathbf{R}}_i^{(2)} + \dot{\mathbf{R}}_i^{(3)} + \ldots$, with $\dot{\mathbf{R}}_i^{(2)} = O(\delta_i^2 v_i)$, $\dot{\mathbf{R}}_i^{(3)} = O(\delta_i^3 v_i)$... The operators $C_{i,\text{eff}}$ and $C_{e,\text{eff}}$ are asymptotic expansions in δ_i as well. The next order corrections for $\dot{\mathbf{R}}_e$ are small in the parameter $\delta_e \ll \delta_i$. For this heuristic introduction, I will drop the next order corrections to $\dot{\mathbf{R}}_e$ exploiting the scale separation $\delta_e \ll \delta_i$ to find $\dot{\mathbf{R}}_e \simeq v_{||e}\hat{\mathbf{b}} + \mathbf{v}_{de}$. This is not rigorous, but it does not change the final result and simplifies the derivation. Under all these assumptions, the ion and electron distribution functions can be solved for perturbatively, giving $f_i = f_i^{(0)} + f_i^{(1)} + f_i^{(2)} + \ldots$ and $f_e = f_e^{(0)} + f_e^{(1)} + f_e^{(2)} + \ldots$, with $f_i^{(0)} = f_{Mi}$, $f_i^{(1)} = O(\delta_i f_{Mi})$, $f_i^{(2)} = O(\delta_i^2 f_{Mi})$... and $f_e^{(0)} = f_{Me}$, $f_e^{(1)} = O(\delta_i f_{Me})$, $f_e^{(2)} = O(\delta_i^2 f_{Me})$...

In this thesis I prove that to find the long wavelength radial electric field in axisymmetric configurations using the gyrokinetic quasineutrality equation, it is necessary to solve the Fokker-Planck equations to fourth order because equations (1.1) and (1.2) satisfy the condition

$$\frac{\partial}{\partial t}\langle Zen_i - en_e \rangle_\psi \equiv - \left\langle \nabla \cdot \left[\int d^3v \left(Zef_i \dot{\mathbf{R}}_i - ef_e \dot{\mathbf{R}}_e \right) \right] \right\rangle_\psi$$
$$+ \left\langle \int d^3v \left(ZeC_{i,\text{eff}}\{f_i, f_e\} + eC_{e,\text{eff}}\{f_e, f_i\} \right) \right\rangle_\psi = O(\delta_i^4 en_e v_i / L). \quad (1.3)$$

Here, $\langle \ldots \rangle_\psi$ is the flux surface average. For now, it is only important to know that $\langle \ldots \rangle_\psi$ makes the non-axisymmetric pieces vanish [see chapter 2]. Since the axisymmetric radial electric field adjusts so that the axisymmetric pieces of the ion and electron densities $\langle n_i \rangle_\psi$ and $\langle n_e \rangle_\psi$ satisfy quasineutrality, equation (1.3) requires that terms of order $\delta_i^4 f_{Mi} v_i / L$ be kept in equations (1.1) and (1.2) to obtain the self-consistent radial electric field. Gyrokinetic codes solve a Fokker-Planck equation only through $O(\delta_i f_{Mi} v_i/L)$, leaving the radial electric field as a free parameter in the best case (intrinsic ambipolarity), or finding an unphysical result in the worst scenario. From an ion Fokker-Planck equation of order $\delta_i^4 f_{Mi} v_i/L$ it is possible in principle (but not in practice) to obtain a distribution function good to order $\delta_i^4 f_{Mi}$. For this reason, in table 1.1 the order to which the distribution function is required is $\delta_i^4 f_{Mi}$. Interestingly, equation (1.3) simplifies considerably because of the flux surface average, giving

$$\frac{\partial}{\partial t}\langle Zen_i - en_e\rangle_\psi \equiv -Ze\left\langle \nabla\cdot\left[\int d^3v\left(f_i^{(2)}\dot{\mathbf{R}}_i^{(2)} + f_i^{(1)}\dot{\mathbf{R}}_i^{(3)} + f_i^{(0)}\dot{\mathbf{R}}_i^{(4)}\right)\right]\right\rangle_\psi$$
$$+\left\langle \int d^3v \left(ZeC_{i,\mathrm{eff}}^{(4)}\{f_i^{(1)}, f_i^{(2)}\} - eC_{e,\mathrm{eff}}^{(4)}\{f_i^{(1)}, f_i^{(2)}\}\right)\right\rangle_\psi = O(\delta_i^4 e n_e v_i/L). \quad (1.4)$$

Notice that, after flux surface averaging and integrating over velocity space, the difference of the fourth order pieces of the collision operators only depends on $f_i^{(1)}$ and $f_i^{(2)}$, and the terms

$$\langle \nabla\cdot(J_{||}^{(4)}\hat{\mathbf{b}})\rangle_\psi \equiv \left\langle \nabla\cdot\left[\int d^3v\left(Zef_i^{(4)}v_{||i}\hat{\mathbf{b}} - ef_e^{(4)}v_{||e}\hat{\mathbf{b}}\right)\right]\right\rangle_\psi = 0 \quad (1.5)$$

and

$$\langle \nabla\cdot\mathbf{J}_d^{(3)}\rangle_\psi \equiv \left\langle \nabla\cdot\left[\int d^3v\left(Zef_i^{(3)}\mathbf{v}_{di} - ef_e^{(3)}\mathbf{v}_{de}\right)\right]\right\rangle_\psi = 0 \quad (1.6)$$

exactly vanish in axisymmetric configurations. The term (1.5) is the contribution of the parallel current density $J_{||}$ to the radial current. Since the radial current is perpendicular, this contribution is obviously zero. The term (1.6) is more subtle. It is the contribution of the current density $\mathbf{J}_d$ due to the magnetic drifts. This contribution vanishes in axisymmetric configurations because to lowest order the net radial displacement due to magnetic drifts is zero. In chapter 2, I will show that the radial component of the current density $\mathbf{J}_d$ is related to the parallel and perpendicular pressures $p_{||}$ and $p_\perp$ in the momentum conservation equation, and these pressures finally do not enter in the calculation of the toroidal rotation, the quantity that determines the radial electric field.

Importantly, according to equation (1.4), a distribution function f_i good to $O(\delta_i^2 f_{Mi})$ is enough to calculate the radial electric field although we need the higher order corrections $\dot{\mathbf{R}}_i^{(2)}$, $\dot{\mathbf{R}}_i^{(3)}$ and $\dot{\mathbf{R}}_i^{(4)}$; terms never employed in gyrokinetic codes. In this thesis, I exploit the fact that only the second order correction of f_i is needed to obtain the long wavelength axisymmetric electric field. Moreover, I will not need to compute the higher order terms $\dot{\mathbf{R}}_i^{(2)}$, $\dot{\mathbf{R}}_i^{(3)}$ and $\dot{\mathbf{R}}_i^{(4)}$ explicitly. It is possible to circumvent this calculation by employing the radial transport of toroidal angular momentum, introduced in chapter 2. By doing so we gain two orders in δ_i, i.e., we make explicit that according to (1.4) we only need an ion distribution function good to order $\delta_i^2 f_{Mi}$. It is not necessary to obtain the higher order corrections to $\dot{\mathbf{R}}_i$ because the toroidal angular momentum conservation equation is obtained from a full Fokker-Planck equation where no approximation for small δ_i has been made. Importantly, the procedure used to evaluate the radial transport of toroidal angular momentum makes a considerable difference. As given in table 1.1, direct evaluation from the ion distribution function requires the ion distribution function to $O(\delta_i^3 f_{Mi})$, whereas the moment approach presented in chapter 5 only requires a distribution function good to $O(\delta_i^2 f_{Mi})$. The moment approach works because the radial transport of toroidal angular momentum depends only on the third order gyrophase dependent piece of the ion distribution function. The third order gyrophase dependent piece of the ion distribution function can be expressed as a function of the second order piece by using the full Fokker-Planck equation, and the moment approach is a simple way to write that relation.

The second order distribution function is still an order higher than usual gyrokinetic codes are built for, but there is a possible simplification listed last in table 1.1. The idea is exploiting the usually largish parameter $B/B_p \sim 10$. The new method, described at the end of chapter 5, is advantageous because conventional gyrokinetic Fokker-Planck equations can provide, with only a few modifications in the implementation, the ion distribution function to order $(B/B_p)\delta_i^2 f_{Mi}$, high enough order to self-consistently determine the long wavelength axisymmetric radial electric field for $B/B_p \gg 1$.

Along with the long wavelength axisymmetric radial electric field, I investigate the evolution of the axisymmetric flows in drift wave turbulence. Under the assumptions explained at the beginning of chapter 4, the long wavelength axisymmetric flows remain neoclassical even in turbulent tokamaks. Moreover, it is possible to prove that in the modern gyrokinetic formalism the axisymmetric components of the flows and the radial electric field with radial wavelengths above $\sqrt{\rho_i L}$ are unreliable due to intrinsic ambipolarity. This result is related to the unrealistic higher order distribution functions needed to obtain the long wavelength axisymmetric radial electric field from the quasineutrality equation.

The rest of the thesis is organized as follows. In chapter 2, the calculation of the radial electric field is formulated in terms of a current conservation equation or vorticity equation. I show that, for the axisymmetric radial electric field, the vorticity equation reduces to the radial transport of toroidal angular momentum. The vorticity equation must be evaluated in the presence of turbulence, and the natural formulation for drift wave turbulence is the gyrokinetic formalism, presented in chapter 3. In chapter 4, the gyrokinetic formulation is applied to the vorticity equation. The resulting equation makes explicit the time scales involved in the evolution of the axisymmetric flows and the axisymmetric radial electric field. With this formulation, turbulent tokamaks are proven to be intrinsically ambipolar to the same order as neoclassical theory. In chapter 5, I present a different approach based on the radial transport of toroidal angular momentum that only requires an $O(\delta_i^2 f_{Mi})$ distribution function, and by exploiting an expansion in $B/B_p \gg 1$, I formulate a relatively simple model capable of self-consistently evolving the axisymmetric radial electric field in the core of a tokamak. By employing an example with simplified geometry in chapters 4 and 5, I illustrate the problems that arise from the use of the gyrokinetic quasineutrality equation, and how a new approach can solve them. Finally, in chapter 6, I summarize the findings in this thesis, and I describe a program to gradually implement a new gyrokinetic formulation in current codes that can solve for the axisymmetric radial electric field.

Chapter 2

Vorticity and intrinsic ambipolarity

In this chapter, I study the quasineutrality equation in an axisymmetric configuration. The time derivative of quasineutrality, also known as vorticity equation, makes the time scales in the problem explicit. With this equation, it is possible to show that the radial current is zero to a very high order independently of the axisymmetric, long wavelength radial electric field, i.e., both ions and electrons drift radially in an intrinsically ambipolar manner even in the presence of turbulence. Moreover, if the radial current is calculated to high enough order, I can also show that forcing it to vanish is equivalent to solving the toroidal angular momentum conservation equation.

The chapter is organized as follows. In section 2.1, I explain and justify the assumptions necessary to simplify the problem. In section 2.2, I derive a vorticity equation from the full Fokker-Planck equation. This equation cannot be usefully implemented as it is written in this chapter because some of the terms are difficult to evaluate. This issue will be addressed in chapter 4. However, this vorticity equation is useful because it makes the study of the evolution of the radial electric field easier. In section 2.3, I flux surface average the vorticity equation to determine the radial electric field. The radial electric field adjusts so that the total radial current in the tokamak vanishes, and the flux surface averaged vorticity equation is equivalent to imposing that the total radial current is zero. I show here that the flux surface averaged vorticity equation is the conservation equation for toroidal angular momentum. Then, by estimating the size of the term that contains the transport of angular momentum in the vorticity equation, I can argue that the radial current is zero to a very high order and hence the plasma is intrinsically ambipolar to the same order as neoclassical theory.

2.1 Orderings and assumptions

To simplify the calculations, and for the rest of this thesis, I assume that the electric field is electrostatic, i.e., $\mathbf{E} = -\nabla\phi$, with ϕ the electrostatic potential. In general, the electromagnetic turbulent fluctuations are important and should be considered [45, 46]. However, keeping electromagnetic effects would obscure derivations that are already quite involved. Furthermore, in the Coulomb gauge, the axisymmetric radial electric field is purely derived from the potential. Thus, calculating the axisymmetric radial electric field is fundamentally an electrostatic problem. The electrostatic formulation presented in this thesis will offer a solution that can be extended later to electromagnetic turbulence.

To be consistent with the electrostatic electric field, the magnetic field $\mathbf{B}$ is assumed to be constant in time. In addition, it has a characteristic length of variation much larger than the ion gyroradius. I use an axisymmetric magnetic field,

$$\mathbf{B} = I\nabla\zeta + \nabla\zeta \times \nabla\psi, \tag{2.1}$$

with ψ and ζ the magnetic flux and toroidal angle coordinates. The vector $\nabla\zeta = \hat{\boldsymbol{\zeta}}/R$ with $\hat{\boldsymbol{\zeta}}$ the unit vector in the toroidal direction and R the radial distance to the symmetry axis of the torus. I use a poloidal angle θ as the third coordinate, and employ the unit vector $\hat{\mathbf{b}} = \mathbf{B}/B$ with $B = |\mathbf{B}|$. The coordinates ψ, ζ and θ are shown in figure 2-1. The toroidal magnetic field, $B_\zeta = I/R$, is determined by the function I that only depends on the radial variable ψ to zeroth order.

The results in this thesis are expansions in the small parameter $\delta_i = \rho_i/L \ll 1$,

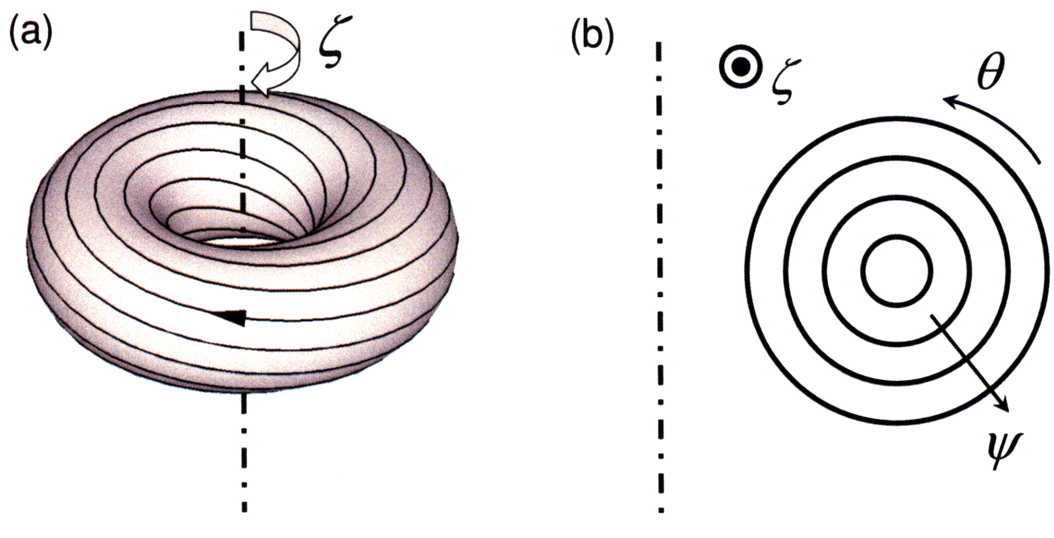

Figure 2-1: Magnetic coordinates in tokamaks. (a) Three-dimensional view of the tokamak where the magnetic field lines are the solid black lines and the axis of symmetry is the chain-dot line. (b) Poloidal plane where the flux surfaces are schematically represented as concentric circles. Here, ζ points out of the paper, ψ labels different flux surfaces, and θ defines the position within the flux surface.

with L the characteristic length in the problem. Collisions and non-neutral effects must be ordered with respect to δ_i. To do so, I will use the characteristic tokamak values given in table 2.1. In this table, the minor and major radii and the magnetic field strength are taken from [47] for Alcator C-Mod and [48] for DIII-D. The electron density n_e and the electron temperature T_e are given in [47] for Alcator C-Mod, and in [49, 50] for DIII-D. The ion temperature T_i is difficult to measure, but in general can be assumed to be of the order of the electron temperature T_e. The average ion velocity V_i is taken from [51, 52] for Alcator C-Mod, and from [49, 53] for DIII-D. The average velocities in DIII-D depend strongly on the neutral beam injection, ranging from 20 km/s when the neutral beams are turned off or their net momentum input is zero, to 200 km/s in cases that the beams drive large rotation. Alcator C-Mod, on the other hand, does not have neutral beam injection and tends to rotate at lower speeds. The rest of the quantities in table 2.1 are calculated from the measured values. These quantities are the electron and ion thermal velocities $v_e = \sqrt{2T_e/m}$ and $v_i = \sqrt{2T_i/M}$; the electron and ion gyrofrequencies $\Omega_e = eB/mc$

and $\Omega_i = ZeB/Mc$; the electron and ion gyroradii, $\rho_e = v_e/\Omega_e$ and $\rho_i = v_i/\Omega_i$; the plasma frequency $\omega_p = \sqrt{4\pi e^2 n_e/m}$ and Debye length $\lambda_D = v_e/\omega_p$; the electron-electron, electron-ion, ion-electron and ion-ion Coulomb collision frequencies $\nu_{ee} = (4\sqrt{2\pi}/3) \times (e^4 n_e \ln\Lambda/\sqrt{m}T_e^{3/2})$, $\nu_{ei} = Z\nu_{ee}$, $\nu_{ie} = (2Z^2 m/M)\nu_{ee}$ and $\nu_{ii} = (4\sqrt{\pi}/3) \times (Z^3 e^4 n_e \ln\Lambda/\sqrt{M}T_i^{3/2})$, and the corresponding mean free paths $\lambda_{ee} = v_e/\nu_{ee}$, $\lambda_{ei} = v_e/\nu_{ei}$, $\lambda_{ie} = v_i/\nu_{ie}$ and $\lambda_{ii} = v_i/\nu_{ii}$.

In table 2.1, $V_i/v_i \sim 0.3$ for shots with neutral beam injection, $V_i/v_i \sim 0.1$ in the pedestal region, and $V_i/v_i \sim 0.03$ in the core in the absence of neutral beam injection. In general, in the core, $V_i \sim V_e < v_i$ can be assumed. Moreover, in the absence of neutral beam injection, the average ion velocity is comparable to $V_i \sim \delta_i v_i$, with $\delta_i = \rho_i/L \sim 5 \times 10^{-3} \ll 1$ and L the characteristic length in the problem, in this case the minor radius a. Ordering $V_i \sim \delta_i v_i \sim \delta_e v_e \sim V_e$, known as the low flow or drift ordering, is then justified. This ordering allows the electric field to compete with the pressure gradient by making the $E \times B$ flow, $(cn_i/B)\mathbf{E} \times \hat{\mathbf{b}}$, and diamagnetic flow, $(c/ZeB)\hat{\mathbf{b}} \times \nabla p_i$, comparable. According to the drift ordering, the electric field is $\mathbf{E} = -\nabla\phi \sim T_e/eL$, giving a total electrostatic potential drop across the core of order T_e/e. The cases with neutral beam injection can be recovered by employing the drift ordering and then sub-expanding in $eL|\nabla\phi|/T_e \gg 1$, giving a velocity $V_i \sim (eL|\nabla\phi|/T_e)\delta_i v_i \gg \delta_i v_i$. In the pedestal, on the other hand, the gradients are large, making $L \ll a$, and the average velocity is closer to the thermal speed, $V_i/v_i \sim 0.1$. Interestingly, it is possible to find an ordering similar to the low flow ordering because the pressure gradient and the electric field must be allowed to compete [54]. In this thesis, I focus in the core, where the drift ordering $V_i \sim \delta_i v_i \sim \delta_e v_e \sim V_e$ is valid, but it is possible that the formalism I will develop could be extended to the pedestal region.

To evaluate the importance of non-neutral effects, I need to compare the turbulence frequencies and wavelengths with the plasma frequency ω_p and the Debye length λ_D. I am interested in the long wavelength, axisymmetric radial electric field and its evolution in the presence of drift wave turbulence. In section 1.1, I explained that the drift wave turbulence spectrum extends from the minor radius to the ion gyroradius.

	Alcator C-Mod		DIII-D	
	Core	Separatrix	Core	Separatrix
B (T)	5	5	2	2
n_e (m^{-3})	3×10^{20}	10^{20}	3×10^{19}	10^{19}
$T_e \sim T_i$ (keV)	2	0.1	2	0.1
V_i (km/s)	20	20	20 – 200	20
v_e (km/s)	27000	5900	27000	5900
v_i (km/s)	620	140	620	140
Ω_e (GHz)	880	880	350	350
Ω_i (GHz)	0.48	0.48	0.19	0.19
ρ_e (μm)	31	6.7	77	17
ρ_i (μm)	920	290	3300	740
ω_p (GHz)	980	560	310	180
λ_D (μm)	28	11	87	33
$\nu_{ee} \sim \nu_{ei}$ (kHz)	150	3600	16	400
ν_{ii} (kHz)	2.4	60	0.26	6.5
ν_{ie} (kHz)	0.16	4.0	0.017	0.44
$\lambda_{ii} \sim \lambda_{ei} \sim \lambda_{ee}$ (m)	260	2.3	2400	22
λ_{ie} (m)	3800	35	35000	320
Minor radius a (m)	0.21		0.67	
Major radius R (m)	0.67		1.66	

Table 2.1: Typical numbers for Alcator C-Mod [47, 51, 52] and DIII-D [49, 48, 50, 53].

Theoretical studies [1, 55] suggest that the turbulent spectrum can also reach wavelengths of the order of the electron gyroradius. Since my research focuses on the long wavelength part of the electric field, I restrict myself to perpendicular wavelengths between the ion gyroradius and the minor radius. In this range, the Debye length is small, $\lambda_D/\rho_i \sim 0.03 \ll 1$. Moreover, the typical frequencies are of the order of the drift wave frequency $\omega_* = k_\perp c T_e / eBL_n \sim k_\perp \rho_i v_i / L \lesssim v_i/L \sim \delta_i \Omega_i \ll \Omega_i$, with $L_n = |\nabla \ln n_e|^{-1}$. The plasma frequency is then very high, $\omega_*/\omega_p \sim 3 \times 10^{-6}$, and the plasma may be assumed quasineutral. There are exceptions in which non-neutral effects have to be considered. I will point them out as they appear, but in general I will ignore these corrections to simplify the derivation. They are easy to implement and they could be added in the future with the electromagnetic corrections.

Collisional mean free paths range from very long in the center of the tokamak, $R/\lambda_{ii} \sim R/\lambda_{ee} \sim R/\lambda_{ei} \sim 10^{-3} \sim \delta_i$, to comparable to the major radius in the pedestal, $R/\lambda_{ii} \sim R/\lambda_{ee} \sim R/\lambda_{ei} \sim 0.01 - 0.1$. Notice that the mean free path is compared to the typical length along the magnetic field, proportional to the major radius R. I will order collisions as $\nu_{ie} \ll \nu_{ii} \sim v_i/L$ and $\nu_{ee} \sim \nu_{ei} \sim v_e/L$, and the low collisionality case can be obtained by then sub-expanding in the small parameter $qR/\lambda_{ii} \sim qR/\lambda_{ee} \sim qR/\lambda_{ei} \ll 1$, where $q \gtrsim 1$ is the safety factor. In this manner, collisions are kept in the derivation. This collisional ordering implies that particles are confined long enough to become a Maxwellian to lowest order. Then, the lowest order ion and electron distribution functions are the stationary Maxwellians f_{Mi} and f_{Me}. They are assumed to be stationary to be consistent with the drift ordering.

To summarize, electromagnetic effects and non-neutral effects are dropped. The typical frequency is assumed to be $\omega_* \lesssim v_i/L$. Collisions are ordered as $\nu_{ie} \ll \nu_{ii} \sim v_i/L$ and $\nu_{ee} \sim \nu_{ei} \sim v_e/L$. The ion and electron distribution functions are stationary Maxwellians, f_{Mi} and f_{Me}, to lowest order. Since L is the characteristic length in the problem, $\nabla f_i \sim f_{Mi}/L$ and $\nabla f_{Me} \sim f_{Me}/L$. The average velocities are ordered as small in δ_i, $V_i \sim \delta_i v_i \sim \delta_e v_e \sim V_e$, and consequently the electric field is $O(T_e/eL)$. In chapters 3 and 4, I will extend these assumptions to the shorter turbulent wavelengths. For this chapter, intended as an introduction to the properties of the long wavelength axisymmetric radial electric field, it will be enough to consider only the longer wavelengths, of order L. Thus, in this chapter, the gradients are $\nabla \sim 1/L$.

Finally, in the derivation it will be useful to split the velocity of the particles into components parallel and perpendicular to the magnetic field, with $v_{||} = \mathbf{v} \cdot \hat{\mathbf{b}}$ the parallel component and $\mathbf{v}_\perp = \mathbf{v} - v_{||}\hat{\mathbf{b}}$ the perpendicular. The perpendicular velocity is determined by its magnitude $v_\perp = |\mathbf{v}_\perp|$ and the gyrophase φ_0, defined such that

$$\mathbf{v}_\perp = v_\perp(\hat{\mathbf{e}}_1 \cos\varphi_0 + \hat{\mathbf{e}}_2 \sin\varphi_0), \tag{2.2}$$

where the unit vectors $\hat{\mathbf{b}}(\mathbf{r})$, $\hat{\mathbf{e}}_1(\mathbf{r})$ and $\hat{\mathbf{e}}_2(\mathbf{r})$ are an orthonormal system such that $\hat{\mathbf{e}}_1 \times \hat{\mathbf{e}}_2 = \hat{\mathbf{b}}$. Notice that $\hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_2$ depend on the position $\mathbf{r}$ because $\hat{\mathbf{b}}$ depends on $\mathbf{r}$. For a general magnetic field, $\hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_2$ may be chosen to be the normal $\hat{\mathbf{e}}_1 = \hat{\mathbf{b}} \cdot \nabla\hat{\mathbf{b}}/|\hat{\mathbf{b}} \cdot \nabla\hat{\mathbf{b}}|$ and binormal $\hat{\mathbf{e}}_2 = \hat{\mathbf{b}} \times \hat{\mathbf{e}}_1$ of the magnetic field line. In a tokamak, they could be defined as $\hat{\mathbf{e}}_1 = \nabla\psi/|\nabla\psi|$ and $\hat{\mathbf{e}}_2 = (\hat{\mathbf{b}} \times \nabla\psi)/|\nabla\psi|$.

The distinction between the gyrophase independent and dependent pieces of the distribution function will be important. I will denote the gyroaverage, or average over the gyrophase φ_0, holding $\mathbf{r}$, $v_{||}$, $v_\perp$ and t fixed, by $\overline{(\ldots)}$. It is important which variables are held fixed because when the gyrokinetic variables are defined, they will have their own distinct gyroaverage. Notice that this average is not weighted with the distribution function, i.e., $\int d^3v f\overline{Q} \neq \int d^3v f Q$, where f is the distribution function and $Q(\varphi_0)$ is some given function of the gyrophase φ_0.

2.2 Vorticity equation

To obtain the electrostatic potential and build in the quasineutrality condition, but also make explicit the time scales that enter the problem, I work with the current conservation or vorticity equation,

$$\nabla \cdot \mathbf{J} = 0, \tag{2.3}$$

where $\mathbf{J} = Zen_i\mathbf{V}_i - en_e\mathbf{V}_e$ is the current density, and $n_i = \int d^3v\, f_i$, $n_e = \int d^3v\, f_e$,

$n_i\mathbf{V}_i = \int d^3v\,\mathbf{v} f_i$ and $n_e\mathbf{V}_e = \int d^3v\,\mathbf{v} f_e$ are the ion and electron densities, and the ion and electron average flows. The functions f_i and f_e are the ion and electron distribution functions, respectively. The parallel current $J_{||} = \mathbf{J}\cdot\hat{\mathbf{b}}$ can be obtained to the requisite order by integrating over the ion and electron distribution functions as discussed in more detail in chapter 4.

The perpendicular current $\mathbf{J}_\perp = \mathbf{J} - J_{||}\hat{\mathbf{b}}$ is given by the perpendicular component of the total momentum conservation equation. The total momentum conservation equation is

$$\frac{\partial}{\partial t}(n_i M\mathbf{V}_i) + \nabla\cdot\left[\int d^3v\ (Mf_i + mf_e)\mathbf{vv}\right] = \frac{1}{c}\mathbf{J}\times\mathbf{B}. \tag{2.4}$$

I neglect the inertia of electrons because their mass is much smaller than the mass of the ions. The total stress tensor can be rewritten as

$$\int d^3v\ (Mf_i + mf_e)\mathbf{vv} = p_\perp(\overleftrightarrow{\mathbf{I}} - \hat{\mathbf{b}}\hat{\mathbf{b}}) + p_{||}\hat{\mathbf{b}}\hat{\mathbf{b}} + \overleftrightarrow{\boldsymbol{\pi}}_i, \tag{2.5}$$

where $p_\perp = p_{i\perp} + p_{e\perp} = \int d^3v\ (Mf_i + mf_e)v_\perp^2/2$ is the total perpendicular "pressure", $p_{||} = p_{i||} + p_{e||} = \int d^3v\ (Mf_i + mf_e)v_{||}^2$ is the total parallel "pressure", and

$$\overleftrightarrow{\boldsymbol{\pi}}_i = M\int d^3v\ f_i(\mathbf{vv} - \overline{\mathbf{vv}}) = M\int d^3v\ f_i\left[\mathbf{vv} - \frac{v_\perp^2}{2}(\overleftrightarrow{\mathbf{I}} - \hat{\mathbf{b}}\hat{\mathbf{b}}) - v_{||}^2\hat{\mathbf{b}}\hat{\mathbf{b}}\right] \tag{2.6}$$

is the ion "viscosity." The electron viscosity is neglected because it is m/M smaller, as I will prove in chapter 4. Here, $\overleftrightarrow{\mathbf{I}}$ is the unit dyad, and $\overline{\mathbf{vv}} = (v_\perp^2/2)(\overleftrightarrow{\mathbf{I}} - \hat{\mathbf{b}}\hat{\mathbf{b}}) + v_{||}^2\hat{\mathbf{b}}\hat{\mathbf{b}}$ is the gyroaverage of $\mathbf{vv}$ holding $\mathbf{r}$, $v_{||}$, $v_\perp$ and t fixed. The definitions of the "pressures" $p_\perp$ and $p_{||}$ and the "viscosity" $\overleftrightarrow{\boldsymbol{\pi}}_i$ differ from the usual in that the average velocity is not subtracted. The usual perpendicular and parallel pressures are $p'_\perp = p_\perp - n_iMV_{i\perp}^2/2$ and $p'_{||} = p_{||} - n_iMV_{i||}^2$, and the usual viscosity is $\overleftrightarrow{\boldsymbol{\pi}}'_i = \overleftrightarrow{\boldsymbol{\pi}}_i - n_iM[\mathbf{V}_i\mathbf{V}_i - (V_{i\perp}^2/2)(\overleftrightarrow{\mathbf{I}} - \hat{\mathbf{b}}\hat{\mathbf{b}}) - V_{i||}^2\hat{\mathbf{b}}\hat{\mathbf{b}}]$. Notice that $\overleftrightarrow{\boldsymbol{\pi}}_i$ contains the turbulent Reynolds stress. In the drift ordering, the pressures and viscosity used here are more convenient than the more common definitions.

Obtaining the perpendicular current $\mathbf{J}_\perp$ from (2.4), substituting it into (2.3) and employing

$$\frac{c}{B}\hat{\mathbf{b}} \times \nabla p_\perp = -\nabla \times \left(\frac{cp_\perp}{B}\hat{\mathbf{b}}\right) + \frac{cp_\perp}{B^2}\hat{\mathbf{b}} \times \nabla B + \frac{cp_\perp}{B}\nabla \times \hat{\mathbf{b}} \tag{2.7}$$

and

$$\nabla \times \hat{\mathbf{b}} = \hat{\mathbf{b}}\hat{\mathbf{b}} \cdot \nabla \times \hat{\mathbf{b}} + \hat{\mathbf{b}} \times \boldsymbol{\kappa}, \tag{2.8}$$

with $\boldsymbol{\kappa} = \hat{\mathbf{b}} \cdot \nabla \hat{\mathbf{b}}$ the curvature of the magnetic field lines, gives the vorticity equation

$$\frac{\partial \varpi}{\partial t} = \nabla \cdot \left[J_{||}\hat{\mathbf{b}} + \mathbf{J}_d + \frac{c}{B}\hat{\mathbf{b}} \times (\nabla \cdot \overleftrightarrow{\boldsymbol{\pi}}_i) \right], \tag{2.9}$$

where

$$\mathbf{J}_d = \frac{cp_\perp}{B}\hat{\mathbf{b}}\hat{\mathbf{b}} \cdot \nabla \times \hat{\mathbf{b}} + \frac{cp_\perp}{B^2}\hat{\mathbf{b}} \times \nabla B + \frac{cp_{||}}{B}\hat{\mathbf{b}} \times \boldsymbol{\kappa} \tag{2.10}$$

is the current due to the magnetic drifts and

$$\varpi = \nabla \cdot \left(\frac{Zen_i}{\Omega_i}\mathbf{V}_i \times \hat{\mathbf{b}} \right) \tag{2.11}$$

is the "vorticity", with $\Omega_i = ZeB/Mc$ the ion gyrofrequency. The quantity ϖ has dimensions of charge density, but it is traditionally called vorticity because for constant magnetic fields it is proportional to the parallel component of the curl of the ion flow, $\varpi \propto \hat{\mathbf{b}} \cdot \nabla \times (n_i \mathbf{V}_i)$.

Vorticity equations like equation (2.9) have been used in the past to solve for the electrostatic potential in fluid models for turbulence [56, 57]. The vorticity ϖ is evolved in time, and the potential can be obtained from ϖ by employing the lowest order result

$$\varpi \simeq \nabla \cdot \left[\frac{Zecn_i}{B\Omega_i}\nabla_\perp \phi + \frac{c}{B\Omega_i}(\nabla \cdot \overleftrightarrow{\mathbf{P}}_i)_\perp \right]. \tag{2.12}$$

To find this equation, I substitute in (2.11) the lowest order perpendicular flow

$$n_i \mathbf{V}_{i\perp} \simeq \frac{cn_i}{B}\hat{\mathbf{b}} \times \nabla \phi + \frac{c}{ZeB}\hat{\mathbf{b}} \times (\nabla \cdot \overleftrightarrow{\mathbf{P}}_i), \tag{2.13}$$

where $\overleftrightarrow{\mathbf{P}}_i = \int d^3v\, M\mathbf{v}\mathbf{v} f_i$. The lowest order perpendicular ion flow (2.13) is obtained from the conservation of ion momentum in the same way that the perpendicular current density $\mathbf{J}_\perp$ was found from the total momentum equation (2.4). In the lowest order result (2.13), I have neglected the ion inertia, $\partial(n_i M \mathbf{V}_i)/\partial t \sim \delta_i p_i/L$, and the ion-electron friction force, $\mathbf{F}_{ei} \sim \sqrt{m/M}\delta_i p_i/L$.

2.3 Radial electric field in the vorticity equation

In chapter 4, I will show that in general $\nabla \cdot \mathbf{J}_d$ dominates or at least is comparable to the other terms in (2.9). However, in axisymmetric configurations, the physics that determines the radial electric field is an exception in that $\nabla \cdot \mathbf{J}_d$ no longer dominates and only the viscosity term $(c/B)\hat{\mathbf{b}} \times (\nabla \cdot \overleftrightarrow{\boldsymbol{\pi}}_i)$ matters.

The radial electric field adjusts so that the radial current and hence the flux surface average of the vorticity equation vanish. To see that the flux surface average of the vorticity equation is equivalent to forcing the radial current to vanish, recall that the vorticity is $\nabla \cdot \mathbf{J} = 0$, and the flux surface average of this equation gives $\langle \mathbf{J} \cdot \nabla\psi \rangle_\psi = 0$, i.e., the total radial current out of a flux surface must be zero. The flux surface average of equation (2.9) is

$$\frac{\partial}{\partial t}\langle \varpi \rangle_\psi = \frac{1}{V'}\frac{\partial}{\partial \psi} V' \left\langle \mathbf{J}_d \cdot \nabla\psi - \frac{cI}{B}(\nabla \cdot \overleftrightarrow{\boldsymbol{\pi}}_i) \cdot \hat{\mathbf{b}} \right\rangle_\psi + \frac{1}{V'}\frac{\partial^2}{\partial \psi^2} V' \langle cR\hat{\boldsymbol{\zeta}} \cdot \overleftrightarrow{\boldsymbol{\pi}}_i \cdot \nabla\psi \rangle_\psi, \quad (2.14)$$

where $\langle \ldots \rangle_\psi = (V')^{-1} \int d\theta \, d\zeta (\ldots)/(\mathbf{B} \cdot \nabla\theta)$ is the flux surface average and $V' \equiv dV/d\psi = \int d\theta \, d\zeta (\mathbf{B} \cdot \nabla\theta)^{-1}$ is the flux surface volume element. To simplify, I have used

$$\langle \nabla \cdot (\ldots) \rangle_\psi = \frac{1}{V'}\frac{\partial}{\partial \psi} V' \langle \nabla\psi \cdot (\ldots) \rangle_\psi, \quad (2.15)$$

$$\hat{\mathbf{b}} \times \nabla\psi = I\hat{\mathbf{b}} - RB\hat{\boldsymbol{\zeta}} \quad (2.16)$$

and $R(\nabla \cdot \overleftrightarrow{\boldsymbol{\pi}}_i) \cdot \hat{\boldsymbol{\zeta}} = \nabla \cdot (R \overleftrightarrow{\boldsymbol{\pi}}_i \cdot \hat{\boldsymbol{\zeta}})$. Equations (2.15) and (2.16) are obtained from the definition of flux surface average and equation (2.1), respectively. The flux surface average of $\mathbf{J}_d \cdot \nabla\psi$ is conveniently rewritten using (2.8), $(\nabla \times \hat{\mathbf{b}}) \cdot \nabla\psi = \nabla \cdot (\hat{\mathbf{b}} \times \nabla\psi)$

and (2.16) to find

$$\mathbf{J}_d \cdot \nabla\psi = \frac{cp_{||}}{B}\nabla \cdot (I\hat{\mathbf{b}}) - \frac{cIp_\perp}{B^2}\hat{\mathbf{b}} \cdot \nabla B, \tag{2.17}$$

where I use that $\nabla \cdot (RB\hat{\boldsymbol{\zeta}}) = 0 = \hat{\boldsymbol{\zeta}} \cdot \nabla B$ due to axisymmetry. The flux surface average of this expression is

$$\langle \mathbf{J}_d \cdot \nabla\psi \rangle_\psi = -\left\langle \frac{cI}{B}[\hat{\mathbf{b}} \cdot \nabla p_{||} + (p_{||} - p_\perp)\nabla \cdot \hat{\mathbf{b}}] \right\rangle_\psi, \tag{2.18}$$

where I have integrated by parts and used $\hat{\mathbf{b}} \cdot \nabla \ln B = -\nabla \cdot \hat{\mathbf{b}}$. Substituting this result into equation (2.14), using the parallel component of (2.4) to write

$$\hat{\mathbf{b}} \cdot \nabla p_{||} + (p_{||} - p_\perp)\nabla \cdot \hat{\mathbf{b}} + (\nabla \cdot \overleftrightarrow{\boldsymbol{\pi}}_i) \cdot \hat{\mathbf{b}} = -\frac{\partial}{\partial t} M n_i \mathbf{V}_i \cdot \hat{\mathbf{b}} \tag{2.19}$$

and employing

$$\langle \varpi \rangle_\psi - \frac{1}{V'}\frac{\partial}{\partial \psi} V' \left\langle \frac{Zen_i I}{\Omega_i} \mathbf{V}_i \cdot \hat{\mathbf{b}} \right\rangle_\psi = -\frac{1}{V'}\frac{\partial}{\partial\psi} V' \langle cn_i RM \mathbf{V}_i \cdot \hat{\boldsymbol{\zeta}} \rangle_\psi \tag{2.20}$$

gives the conservation equation for toroidal angular momentum

$$\frac{\partial}{\partial t}\langle n_i RM \mathbf{V}_i \cdot \hat{\boldsymbol{\zeta}} \rangle_\psi = -\frac{1}{V'}\frac{\partial}{\partial \psi} V' \langle R\hat{\boldsymbol{\zeta}} \cdot \overleftrightarrow{\boldsymbol{\pi}}_i \cdot \nabla\psi \rangle_\psi, \tag{2.21}$$

where I have integrated once in ψ assuming that there are no sources or sinks of momentum. Equation (2.21) shows that setting the total radial current to zero is equivalent to the toroidal angular momentum conservation equation. Equation (2.21) includes both turbulent and neoclassical effects. In a model in which the transport time scale is not reached, as is the usual case in gyrokinetics, there is not enough time for the angular momentum to diffuse from one flux surface to the next, keeping then the long wavelength toroidal velocity constant and equal to its initial value. Consequently, the long wavelength radial electric field, related to the toroidal velocity by the $E \times B$ velocity, must not evolve and must be determined by the initial condition. The vorticity equation makes this fact explicit by including the radial current density $(c/B)[\hat{\mathbf{b}} \times (\nabla \cdot \overleftrightarrow{\boldsymbol{\pi}}_i)] \cdot \nabla\psi$.

Equation (2.21) can be generalized by flux surface averaging the toroidal angular momentum conservation equation with the charge density and full current density retained. Writing the electromagnetic force as the Maxwell stress gives

$$\frac{\partial}{\partial t}\langle n_i RM\mathbf{V}_i \cdot \hat{\boldsymbol{\zeta}}\rangle_\psi = -\frac{1}{V'}\frac{\partial}{\partial \psi}V'\left\langle R\hat{\boldsymbol{\zeta}} \cdot (\overleftrightarrow{\boldsymbol{\pi}}_i + \overleftrightarrow{\boldsymbol{\pi}}_e) \cdot \nabla\psi - \frac{1}{4\pi}R\hat{\boldsymbol{\zeta}} \cdot (\mathbf{EE} + \mathbf{BB}) \cdot \nabla\psi \right\rangle_\psi . \tag{2.22}$$

In equation (2.21), the components of the toroidal angular momentum transport due to the electron viscosity and the Maxwell stress are neglected. In subsequent chapters I will argue that the ion viscosity term $\langle R\hat{\boldsymbol{\zeta}}\cdot \overleftrightarrow{\boldsymbol{\pi}}_i \cdot\nabla\psi\rangle_\psi$ is of order $\delta_i^3 p_i R|\nabla\psi|$. The other terms must be compared to this order of magnitude estimate to asses their importance. In chapter 4, I will show that $\overleftrightarrow{\boldsymbol{\pi}}_e\sim \delta_e^2 p_e \sim (m/M)\delta_i^2 p_i$, negligible compared to $\overleftrightarrow{\boldsymbol{\pi}}_i$ because $m/M = 5\times 10^{-4} \ll \delta_i$. The magnetic portion of the Maxwell stress vanishes because the calculation is electrostatic and the magnetic field is unperturbed and given by (2.1); in particular, it satisfies $\mathbf{B}\cdot\nabla\psi = 0$. The electric part of the Maxwell stress is a non-neutral contribution. Using $\mathbf{E} = -\nabla\phi \sim T_e/eL$, I find $R\hat{\boldsymbol{\zeta}}\cdot\mathbf{EE}\cdot\nabla\psi/4\pi \sim (\lambda_D/L)^2 p_e R|\nabla\psi|$, with $\lambda_D = \sqrt{T_e/4\pi e^2 n_e}$ the Debye length. This is one of the instances when the non-neutral effects may contribute to the final result. In both Alcator C-Mod and DIII-D, the non-neutral transport of toroidal angular momentum $R\hat{\boldsymbol{\zeta}}\cdot\mathbf{EE}\cdot\nabla\psi/4\pi$ is comparable to the viscosity contribution, albeit somewhat smaller. For the rest of the thesis, I will drop the non-neutral contribution to simplify the presentation, but it is important to remark that it could be easily added if necessary.

The fact that $\langle \mathbf{J}\cdot\nabla\psi\rangle_\psi = 0$ is equivalent to equation (2.21) means that whether a configuration is intrinsically ambipolar or not depends then on the size of the radial current density $(c/B)[\hat{\mathbf{b}}\times(\nabla\cdot\overleftrightarrow{\boldsymbol{\pi}}_i)]\cdot\nabla\psi$. If this current density is small, as I will prove it to be in chapter 4, the radial current is effectively zero. Then, if the plasma is quasineutral initially, it stays so independently of the radial electric field, that is, the configuration remains intrinsically ambipolar.

I will finish this chapter with some simple estimates that I will prove in chapters 4 and 5. In neoclassical calculations of the radial electric field [39, 40, 42, 43, 44], the flux surface average $\langle R\hat{\boldsymbol{\zeta}}\cdot \overleftrightarrow{\boldsymbol{\pi}}_i \cdot\nabla\psi\rangle_\psi$ is of order $\delta_i^3 p_i R|\nabla\psi|$. The radial current density $(c/B)[\hat{\mathbf{b}} \times (\nabla\cdot \overleftrightarrow{\boldsymbol{\pi}}_i)] \cdot \nabla\psi$ associated with this piece of the viscosity is tiny, of order $\delta_i^4 e n_e v_i |\nabla\psi|$. The estimate from neoclassical theory is not necessarily applicable to turbulent transport of toroidal angular momentum, but it is suggestive. Indeed, the same order of magnitude is recovered if the transport of toroidal angular momentum is at the gyroBohm level and the average velocities in the tokamak are small compared to the ion thermal velocity by δ_i, i.e., $V_i \sim \delta_i v_i$. The gyroBohm diffusion coefficient is obtained from turbulence fluctuations employing a simple random walk argument as follows. In chapter 1, I pointed out that the typical eddy size is $\Delta_{\text{eddy}} \sim \rho_i$. The eddy turnover time is determined by the average velocity $V_i \sim \delta_i v_i$, giving $\tau_{\text{eddy}} \sim \Delta_{\text{eddy}}/V_i \sim L/v_i$. With these estimates, the gyroBohm diffusion coefficient is calculated to be $D_{gB} = \Delta_{\text{eddy}}^2/\tau_{\text{eddy}} \sim \delta_i \rho_i v_i$. Multiplying this diffusivity by the macroscopic gradient of momentum, $\nabla(n_i M V_{i\zeta}) \sim \delta_i n_e M v_i / L$, I find the same result as the neoclassical calculation, namely, the transport of momentum is $\delta_i^3 p_i$ and the associated radial current density is $\delta_i^4 e n_e v_i$. In chapter 5, I will argue in favor of this estimate more strongly. In any case, in current gyrokinetic formalisms, the flux surface averaged radial current should remain equal to zero independently of the long wavelength radial electric field.

Chapter 3

Derivation of gyrokinetics

Gyrokinetics is a kinetic formalism that is especially adapted to model drift wave turbulence. By defining more convenient variables to replace the position $\mathbf{r}$ and velocity $\mathbf{v}$, it "averages out" the fast gyromotion time scales and keeps finite gyroradius effects, permitting perpendicular wavelengths on the order of the ion gyroradius. Currently, there is no other model that allows simulation of short wavelength turbulence in magnetized plasmas in reasonable computational times.

In this chapter, I derive the gyrokinetic Fokker-Planck equation and the gyrokinetic quasineutrality equation. The gyrokinetic Fokker-Planck equation models the response of the plasma to fluctuations in the electrostatic potential with wavelengths that may be as small as the ion gyroradius. The gyrokinetic quasineutrality equation, or gyrokinetic Poisson's equation, is the equation used so far to determine the self-consistent electrostatic potential. This equation is given here for completeness, but in chapter 4 I will show that it is flawed at long wavelengths.

The rest of the chapter is organized as follows. In section 3.1, I lay out the assumptions I need to derive the gyrokinetic equation. In section 3.2, I find the gyrokinetic variables with a new nonlinear approach, based on the linear derivation of [58, 59]. I have already published this derivation in [60]. In section 3.3, I give the Fokker-Planck equation in these new variables. This gyrokinetic Fokker-Planck equation does not have the fast gyromotion scale, yet it retains finite gyroradius effects, as desired. I finish by deriving the traditional gyrokinetic quasineutrality in section 3.4. The algebraic details of the calculation are relegated to Appendices A - E.

3.1 Orderings

The assumptions are the same as in section 2.1. The characteristic frequency of the processes of interest is assumed to be the drift wave frequency $\omega \sim \omega_* = k_\perp c T_e/eBL_n \sim k_\perp \rho_i v_i/L$, with $L_n^{-1} = |\nabla \ln n_e|$. To treat arbitrary collisionality, the ion-ion collision frequency ν_{ii} is assumed to be of the order of the transit time of ions, $\nu_{ii} \sim v_i/L$. Consequently, the electron-electron and the electron-ion collision frequencies are of the order of the electron transit time, $\nu_{ee} \sim \nu_{ei} \sim v_e/L$. With this assumption, I can treat low collisionality cases by sub-expanding the final results in the parameter $\nu_{ii}L/v_i \sim \nu_{ee}L/v_e \sim \nu_{ei}L/v_e \ll 1$.

The average velocities are assumed to be in the low flow or drift ordering, where the $E \times B$ drift is of order $\delta_i v_i$. Therefore, the electric field is of order $\mathbf{E} = -\nabla\phi \sim T_e/eL$, and the total electrostatic potential drop across the characteristic length L is of order T_e/e. The spatial gradients of the distribution functions are assumed to be $\nabla f_i \sim f_{Mi}/L$ and $\nabla f_e \sim f_{Me}/L$, where f_{Mi} and f_{Me} are the zeroth order distribution functions. Since I am primarily interested in the core plasma in tokamaks, I will assume that the zeroth order distribution functions are stationary Maxwellians, with ion and electron temperatures of the same order, $T_i \sim T_e$. The Maxwellians are stationary to be consistent with the drift ordering.

To include the turbulence, I allow wavelengths perpendicular to the magnetic field that are on the order of the ion gyroradius, $k_\perp \rho_i \sim 1$. At the same time, due to low flow or drift ordering, I assume $\nabla\phi \sim T_e/eL$, $\nabla f_i \sim f_{Mi}/L$ and $\nabla f_e \sim f_{Me}/L$. This ordering requires that the pieces of the potential and the distribution functions with short perpendicular wavelengths ϕ_k, $f_{i,k}$ and $f_{e,k}$ be small in size, in particular

$$\frac{f_{i,k}}{f_{Mi}} \sim \frac{f_{e,k}}{f_{Me}} \sim \frac{e\phi_k}{T_e} \sim \frac{1}{k_\perp L} \lesssim 1, \qquad (3.1)$$

with $k_\perp \rho_i \lesssim 1$. According to (3.1), the pieces of the distribution function that have wavelengths on the order of the ion gyroradius are next order in the expansion in δ_i. Notice that $k_\perp f_{i,k} \sim \nabla_\perp f_{i,k} \sim \nabla f_{Mi} \sim f_{Mi}/L$, and since $\partial/\partial t \sim k_\perp \rho_i v_i/L$, $\partial f_{i,k}/\partial t \sim \delta_i f_{Mi} v_i/L$. I could allow wavelengths on the order of the electron gyroradius following a similar ordering, but I ignore these small wavelengths to simplify the presentation. Unlike the perpendicular wavelengths, the wavelengths along the magnetic field, $k_{||}^{-1}$, are taken to be on the order of the larger scale L. Moreover, except for initial transients, the variations along the magnetic field of f_i, f_e and ϕ are slow, i.e., in general $\hat{\mathbf{b}} \cdot \nabla f_i \sim \delta_i f_{Mi}/L$, $\hat{\mathbf{b}} \cdot \nabla f_e \sim \delta_i f_{Me}/L$ and $\hat{\mathbf{b}} \cdot \nabla \phi \sim \delta_i T_e/eL$.

Both the potential and the distribution function may be viewed as having a piece with slow spatial variations (representing the average value in the plasma) plus some rapid oscillations of small amplitude. The zonal flow, for example, will be included in the small piece if its characteristic wavelength is comparable to the gyroradius, but its amplitude may be larger for longer wavelengths. This ordering implies that the rapid spatial potential fluctuations seen by a particle in its gyromotion are small compared to the average energy of the particle. Then, the gyromotion remains almost circular, and the distribution function of the gyrocenters is equal, to zeroth order, to the distribution function of the particles. The difference, coming from the rapidly oscillating pieces, is small in our ordering. Notice that the δf codes [1, 2, 3, 4, 5, 6] explicitly adopt this treatment for the components of ϕ, f_i and f_e that satisfy $k_\perp \rho_i \sim 1$, and, as in this thesis, they order them as $O(\delta_i)$.

3.2 Gyrokinetic variables

In this section, I derive the gyrokinetic variables for ions, the only species where the finite gyroradius effects matter, since $k_\perp \rho_i \sim 1$. It is possible to find a similar gyrokinetic equation for electrons, but for this thesis the drift kinetic equation is all that is required. I begin by defining the Vlasov operator for ions in the usual $\mathbf{r}$, $\mathbf{v}$

variables for an electrostatic electric field as the following total derivative

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla + \left(-\frac{Ze}{M} \nabla \phi + \Omega_i \mathbf{v} \times \hat{\mathbf{b}} \right) \cdot \nabla_v, \tag{3.2}$$

where $\Omega_i = ZeB/Mc$ is the gyrofrequency. The Fokker–Planck equation for ions is then simply

$$\frac{df_i}{dt} = C\{f_i\}, \tag{3.3}$$

where $C\{f_i\}$ is the relevant Fokker-Planck collision operator for the ions. The ion-electron collision operator is small in $\sqrt{m/M}$, leaving only the ion-ion collision operator.

The objective of gyrokinetics is to change the Fokker-Planck equation to gyrokinetic variables, defined such that the gyromotion time scale disappears from equation (3.3). The nonlinear gyrokinetic variables to be employed are the guiding center location $\mathbf{R}$, the kinetic energy E, the magnetic moment μ, and the gyrophase φ. These variables will be defined to higher order than is customary by employing an extension of the procedure presented in [58, 59]. The general idea is to construct the gyrokinetic variables to higher order by adding in δ_i corrections such that the total derivative of a generic gyrokinetic variable Q is gyrophase independent to the desired order, and we may safely employ

$$\frac{dQ}{dt} \simeq \left\langle \frac{dQ}{dt} \right\rangle, \tag{3.4}$$

where the gyroaverage $\langle \ldots \rangle$ is performed holding $\mathbf{R}$, E, μ and t fixed. This gyrokinetic gyroaverage can be understood as a fast time average where $\tau = -\varphi/\Omega_i$ is the fast gyromotion time and t is the slow time of the turbulent fluctuations (ion gyromotion is such that $d\varphi/dt < 0$, hence the sign difference between τ and φ). The gyrokinetic variable Q is expanded in powers of δ_i,

$$Q = Q_0 + Q_1 + Q_2 + \ldots, \tag{3.5}$$

where Q_0 is the lowest order gyrokinetic variable (kinetic energy, magnetic moment, etc.), and $Q_1 = O(\delta_i Q_0)$, $Q_2 = O(\delta_i^2 Q_0)$... are the order δ_i, δ_i^2 ... corrections. The first correction Q_1 is constructed so that $dQ/dt = \langle dQ/dt\rangle + O(\delta_i^2 \Omega_i Q)$, while the second correction Q_2 is evaluated such that $dQ/dt = \langle dQ/dt\rangle + O(\delta_i^3 \Omega_i Q)$. In principle this process can be continued indefinitely. Any Q_k can be found once the functions Q_1, Q_2 ..., Q_{k-1} are known. The functions Q_1, Q_2 ..., Q_{k-1} are constructed so that

$$\frac{dQ}{dt} \simeq \frac{d}{dt}(Q_0 + \ldots + Q_{k-1}) = \left\langle \frac{d}{dt}(Q_0 + \ldots + Q_{k-1}) \right\rangle + O(\delta_i^k \Omega_i Q_0). \tag{3.6}$$

Adding Q_k to (3.5) means adding dQ_k/dt to (3.6). To lowest order, $dQ_k/dt \simeq -\Omega_i\, \partial Q_k/\partial\varphi$, which to the requisite order leads to an equation for Q_k,

$$\frac{dQ}{dt} \simeq \frac{d}{dt}(Q_0 + \ldots + Q_{k-1}) - \Omega_i \frac{\partial Q_k}{\partial \varphi} = \left\langle \frac{d}{dt}(Q_0 + \ldots + Q_{k-1}) \right\rangle, \tag{3.7}$$

where $\langle \partial Q_k/\partial\varphi\rangle = 0$ is employed. Notice that the gyrophase derivative is holding the gyrokinetic variables fixed, and not $\mathbf{r}$, $v_{||}$ and $v_\perp$ fixed (in some cases these two distinct derivatives with respect to gyrophase are almost equal). Using (3.7), $Q_k = O(\delta_i^k Q_0)$ is found to be periodic in gyrophase and given by

$$Q_k = \frac{1}{\Omega_i} \int^{\varphi} d\varphi' \left[\frac{d}{dt}(Q_0 + \ldots + Q_{k-1}) - \left\langle \frac{d}{dt}(Q_0 + \ldots + Q_{k-1}) \right\rangle \right]. \tag{3.8}$$

More explicitly, through the first two orders, Q_1 and Q_2 are determined to be

$$Q_1 = \frac{1}{\Omega_i} \int^{\varphi} d\varphi' \left(\frac{dQ_0}{dt} - \left\langle \frac{dQ_0}{dt} \right\rangle \right) \tag{3.9}$$

and

$$Q_2 = \frac{1}{\Omega_i} \int^{\varphi} d\varphi' \left[\frac{d}{dt}(Q_0 + Q_1) - \left\langle \frac{d}{dt}(Q_0 + Q_1) \right\rangle \right]. \tag{3.10}$$

By adding Q_1 and Q_2, the total derivative of the gyrokinetic variable $Q = Q_0 + Q_1 + Q_2$ is

$$\frac{dQ}{dt} = \left\langle \frac{d}{dt}(Q_0 + Q_1) \right\rangle + O(\delta_i^3 \Omega_i Q_0). \tag{3.11}$$

In the remainder of this subsection, I present the gyrokinetic variables that result from this process. I begin with the guiding center position expanded as

$$\mathbf{R} = \mathbf{R}_0 + \mathbf{R}_1 + \mathbf{R}_2, \tag{3.12}$$

where $\mathbf{R}_0 = \mathbf{r}$, $|\mathbf{R}_1| = O(\rho_i)$ and $|\mathbf{R}_2| = O(\delta_i \rho_i)$. I construct $\mathbf{R}_1$ and $\mathbf{R}_2$ such that the gyrocenter position time derivative is gyrophase independent to order $\delta_i v_i$,

$$\frac{d\mathbf{R}}{dt} = \left\langle \frac{d\mathbf{R}}{dt} \right\rangle + O(\delta_i^2 v_i). \tag{3.13}$$

The explicit details of the calculation are presented in sections A.1 and A.2 of Appendix A. To first order, I find the usual result [18]

$$\mathbf{R}_1 = \frac{1}{\Omega_i} \mathbf{v} \times \hat{\mathbf{b}}. \tag{3.14}$$

The gyromotion is approximately circular, as sketched in figure 3-1(a), even in the presence of fluctuations with wavelengths of the order of the ion gyroradius. The ordering in (3.1) that bounds the electric field to be $O(T_e/eL)$ leads to this significant simplification. To next order, I obtain

$$\begin{aligned}\mathbf{R}_2 = \frac{1}{\Omega_i} \left[\left(v_{||} \hat{\mathbf{b}} + \frac{1}{4} \mathbf{v}_\perp \right) (\mathbf{v} \times \hat{\mathbf{b}}) + (\mathbf{v} \times \hat{\mathbf{b}}) \left(v_{||} \hat{\mathbf{b}} + \frac{1}{4} \mathbf{v}_\perp \right) \right] \dot{\times} \nabla \left(\frac{\hat{\mathbf{b}}}{\Omega_i} \right) \\ + \frac{v_{||}}{\Omega_i^2} \mathbf{v}_\perp \cdot \nabla \hat{\mathbf{b}} + \frac{\hat{\mathbf{b}}}{\Omega_i^2} \left\{ v_{||} \hat{\mathbf{b}} \cdot \nabla \hat{\mathbf{b}} \cdot \mathbf{v}_\perp + \frac{1}{8} [\mathbf{v}_\perp \mathbf{v}_\perp - (\mathbf{v} \times \hat{\mathbf{b}})(\mathbf{v} \times \hat{\mathbf{b}})] : \nabla \hat{\mathbf{b}} \right\} \\ - \frac{c}{B\Omega_i} \nabla_{\mathbf{R}} \widetilde{\Phi} \times \hat{\mathbf{b}},\end{aligned} \tag{3.15}$$

which is the same as [58] except for the nonlinear term given last. My vector conventions are $\mathbf{xy}:\overleftrightarrow{\mathbf{M}} = \mathbf{y} \cdot \overleftrightarrow{\mathbf{M}} \cdot \mathbf{x}$ and $\mathbf{xy} \dot{\times} \overleftrightarrow{\mathbf{M}} = \mathbf{x} \times (\mathbf{y} \cdot \overleftrightarrow{\mathbf{M}})$. One of the terms in the second order correction $\mathbf{R}_2$ includes the function $\widetilde{\Phi}(\mathbf{R}, E, \mu, \varphi, t)$. It will appear several times during derivations, as will the functions $\langle \phi \rangle$ and $\widetilde{\phi}$. These are functions related to the electrostatic potential that depend on the new gyrokinetic variables. Their definitions are

$$\langle \phi \rangle \equiv \langle \phi \rangle (\mathbf{R}, E, \mu, t) = \frac{1}{2\pi} \oint d\varphi \, \phi(\mathbf{r}(\mathbf{R}, E, \mu, \varphi, t), t), \tag{3.16}$$

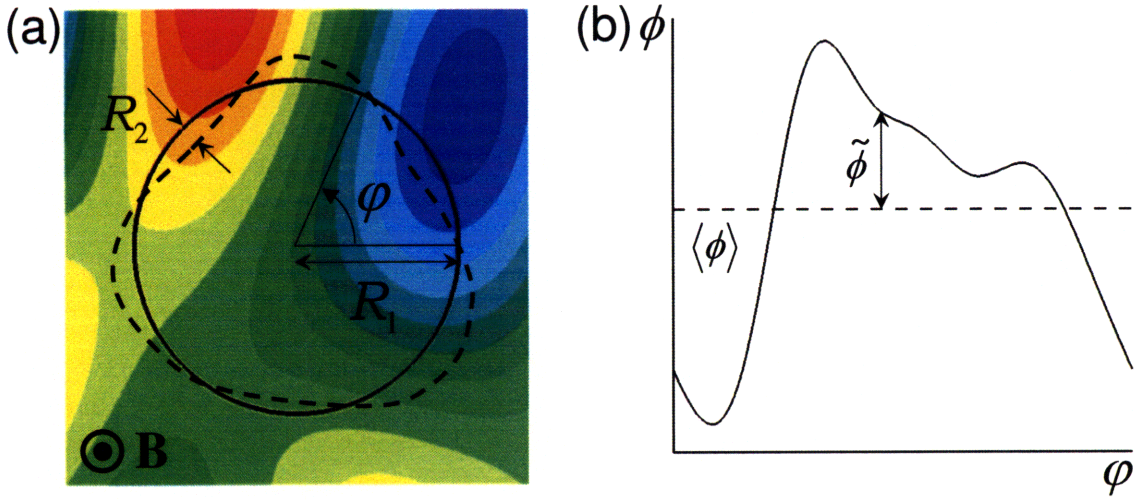

Figure 3-1: (a) Gyrokinetic gyromotion $\mathbf{r} = \mathbf{R} - \mathbf{R}_1 - \mathbf{R}_2$ (dashed line) in a turbulent potential. The solid circle $\mathbf{R} - \mathbf{R}_1$ describes the gyromotion to first order. (b) Potential ϕ along gyromotion as a function of φ. The average $\langle\phi\rangle$ is the dashed line, and the fast time variation $\tilde{\phi}$ is the difference $\phi - \langle\phi\rangle$.

$$\tilde{\phi} \equiv \tilde{\phi}(\mathbf{R}, E, \mu, \varphi, t) = \phi(\mathbf{r}(\mathbf{R}, E, \mu, \varphi, t), t) - \langle\phi\rangle(\mathbf{R}, E, \mu, t) \tag{3.17}$$

and

$$\tilde{\Phi} \equiv \tilde{\Phi}(\mathbf{R}, E, \mu, \varphi, t) = \int^{\varphi} d\varphi' \, \tilde{\phi}(\mathbf{R}, E, \mu, \varphi', t), \tag{3.18}$$

such that $\langle\tilde{\Phi}\rangle = 0$. These are definitions similar to those used by Dubin [21]. I have discussed the subtle differences between them in [61], as summarized in Appendix C. From their definitions (3.16) and (3.17), we see that the function $\langle\phi\rangle$ is the fast time average of the potential in a gyromotion, and $\tilde{\phi}$ is the difference between the potential at the position of the particle and $\langle\phi\rangle$. Both are represented in figure 3-1(b) as a function of the gyrophase φ. The function $\tilde{\Phi}$ is proportional to the fast time integral of $\tilde{\phi}$, with $\tau = -\varphi/\Omega_i$ the fast time variation. The term $-(c/B\Omega_i)\nabla_{\mathbf{R}}\tilde{\Phi} \times \hat{\mathbf{b}}$ in (3.15) is the correction to the gyromotion due to the fast time component of the $E \times B$ drift $-(c/B)\nabla_{\mathbf{R}}\tilde{\phi} \times \hat{\mathbf{b}}$. This fast time contribution integrated over fast times $\tau = -\varphi/\Omega_i$ gives the correction to the gyrocenter position. The other corrections in $\mathbf{R}_2$ are fast time contributions due to magnetic geometry.

It is important to comment on the size of functions $\langle\phi\rangle$, $\tilde{\phi}$ and $\tilde{\Phi}$. Both ϕ and $\langle\phi\rangle$

are of the same order as the temperature for long wavelengths, but small for short wavelengths [recall (3.1)]. However, $\widetilde{\phi}$ is always small as it accounts for the variation in the electrostatic potential that a particle sees as it moves in its gyromotion. Of course, since the potential is small for short wavelengths, the variation observed by the particle is also small. For long wavelengths, even though the potential is comparable to the temperature, the particle motion is small compared to the wavelength, and the variations that it sees in its motion are small. Therefore, $\widetilde{\phi} \sim \delta_i T_e/e$ for all wavelengths in my ordering, making $\widetilde{\Phi}$ small as well.

The Vlasov operator acting on $\mathbf{R}$ gives

$$\frac{d\mathbf{R}}{dt} = \left\langle \frac{d\mathbf{R}}{dt} \right\rangle + O(\delta_i^2 v_i) = u\hat{\mathbf{b}}(\mathbf{R}) + \mathbf{v}_d + O(\delta_i^2 v_i), \tag{3.19}$$

where $\mathbf{v}_d$ is the total drift velocity,

$$\mathbf{v}_d = \mathbf{v}_E + \mathbf{v}_M, \tag{3.20}$$

composed of the $E \times B$ drift

$$\mathbf{v}_E = -\frac{c}{B(\mathbf{R})} \nabla_{\mathbf{R}} \langle \phi \rangle \times \hat{\mathbf{b}}(\mathbf{R}) \tag{3.21}$$

and the magnetic drift

$$\mathbf{v}_M = \frac{u^2}{\Omega_i(\mathbf{R})} \hat{\mathbf{b}}(\mathbf{R}) \times \boldsymbol{\kappa}(\mathbf{R}) + \frac{\mu}{\Omega_i(\mathbf{R})} \hat{\mathbf{b}}(\mathbf{R}) \times \nabla_{\mathbf{R}} B(\mathbf{R}). \tag{3.22}$$

In the preceding equations, u is the gyrocenter parallel velocity defined by

$$\frac{u^2}{2} + \mu B(\mathbf{R}) = E. \tag{3.23}$$

Notice that in (3.19), (3.21), (3.22) and (3.23), all the terms are given as a function of the new gyrokinetic variables, $\mathbf{R}$, E and μ.

Following the same procedure as for the gyrocenter position $\mathbf{R}$, the gyrokinetic

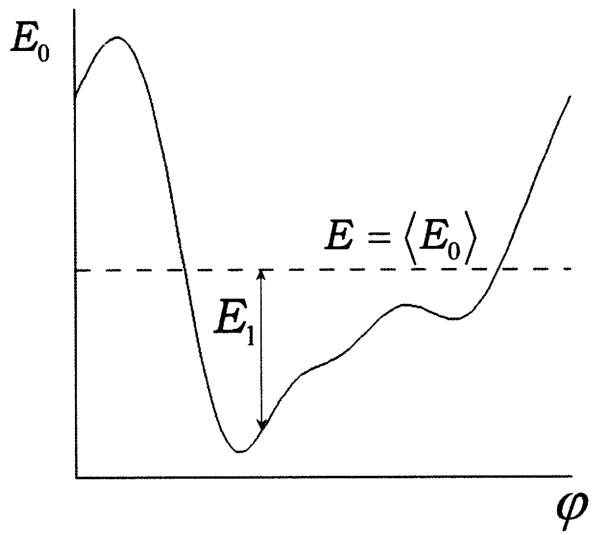

Figure 3-2: Kinetic energy $E_0 = v^2/2$ along the gyromotion. The dashed line is the average value $E = \langle E_0 \rangle$, and the difference $E - E_0$ is the correction E_1. Notice that the variation of the kinetic energy and the variation of the potential have opposite signs because maxima of potential correspond to minima of kinetic energy.

kinetic energy is defined as

$$E = E_0 + E_1 + E_2, \tag{3.24}$$

where $E_0 = v^2/2$, $E_1 = O(\delta_i v_i^2)$ and $E_2 = O(\delta_i^2 v_i^2)$. The details of the calculation are given in sections A.1 and A.2 of Appendix A. I find

$$E_1 = \frac{Ze\tilde{\phi}}{M} \tag{3.25}$$

and

$$E_2 = \frac{c}{B}\frac{\partial \tilde{\Phi}}{\partial t}. \tag{3.26}$$

The Vlasov operator acting on E is shown in section A.2 of Appendix A to give

$$\frac{dE}{dt} = \left\langle \frac{dE}{dt} \right\rangle + O\left(\delta_i^2 \frac{v_i^3}{L}\right) = -\frac{Ze}{M}\left[u\hat{\mathbf{b}}(\mathbf{R}) + \mathbf{v}_M\right] \cdot \nabla_{\mathbf{R}} \langle \phi \rangle + O\left(\delta_i^2 \frac{v_i^3}{L}\right). \tag{3.27}$$

The corrections E_1 and E_2 are necessary because otherwise the kinetic energy would vary in the fast gyromotion time scale. Figure 3-1 shows how the potential changes rapidly along the gyromotion due to the short wavelength turbulent structures. The kinetic energy variations are accordingly rapid, as sketched in figure 3-2. The gyrokinetic energy is the average kinetic energy $E = \langle E_0 \rangle$, where the fast time variation has been extracted.

The gyrokinetic gyrophase is obtained in a similar way as the energy and the gyrocenter position by defining

$$\varphi = \varphi_0 + \varphi_1, \tag{3.28}$$

with φ_0 the original gyrophase. The details are again in Appendix A, in section A.1. Notice that only the first order correction φ_1 is calculated since gyrokinetics will make the gyrophase dependence weak and hence next order corrections unnecessary. The first order correction is

$$\varphi_1 = -\frac{Ze}{MB}\frac{\partial \widetilde{\Phi}}{\partial \mu} - \frac{1}{\Omega_i}\mathbf{v}_\perp \cdot \left[\nabla \ln B + \frac{v_\parallel^2}{v_\perp^2}\hat{\mathbf{b}} \cdot \nabla \hat{\mathbf{b}} - \hat{\mathbf{b}} \times \nabla \hat{\mathbf{e}}_2 \cdot \hat{\mathbf{e}}_1 \right] \\ - \frac{v_\parallel}{4\Omega_i v_\perp^2}[\mathbf{v}_\perp \mathbf{v}_\perp - (\mathbf{v} \times \hat{\mathbf{b}})(\mathbf{v} \times \hat{\mathbf{b}})] : \nabla \hat{\mathbf{b}}, \tag{3.29}$$

where $\widetilde{\Phi}$ is defined in (3.18). With this correction, $d\varphi/dt$ is gyrophase independent to order $O(v_i/L)$, that is,

$$\frac{d\varphi}{dt} = \left\langle \frac{d\varphi}{dt} \right\rangle + O(\delta_i^2 \Omega_i) = -\overline{\Omega}_i + O(\delta_i^2 \Omega_i), \tag{3.30}$$

where

$$\overline{\Omega}_i = \Omega_i(\mathbf{R}) + \frac{v_\parallel}{2}\hat{\mathbf{b}} \cdot \nabla \times \hat{\mathbf{b}} - v_\parallel \hat{\mathbf{b}} \cdot \nabla \hat{\mathbf{e}}_2 \cdot \hat{\mathbf{e}}_1 + \frac{Z^2 e^2}{M^2 c}\frac{\partial \langle \phi \rangle}{\partial \mu}. \tag{3.31}$$

The function $\overline{\Omega}_i$ is equal to Ω_i to lowest order. It might be surprising that the gyrophase φ_0 needs to be corrected by φ_1. The gyrokinetic gyrophase φ is in reality the fast time variation $\tau = -\varphi/\overline{\Omega}_i$. The correction φ_1 is necessary because $d\varphi_0/dt$ changes in the fast gyromotion time scale, making the dependence of φ_0 on τ non-trivial. There are two effects that contribute to the fast variation of $d\varphi_0/dt$, namely, the electromagnetic fields and geometric corrections. The electromagnetic fields contribute through the magnetic field strength $B(\mathbf{r})$ in $\Omega_i(\mathbf{r})$, giving the correction $-\Omega_i^{-1}\mathbf{v}_\perp \cdot \nabla \ln B$ in (3.29), and through the short wavelength turbulent structures

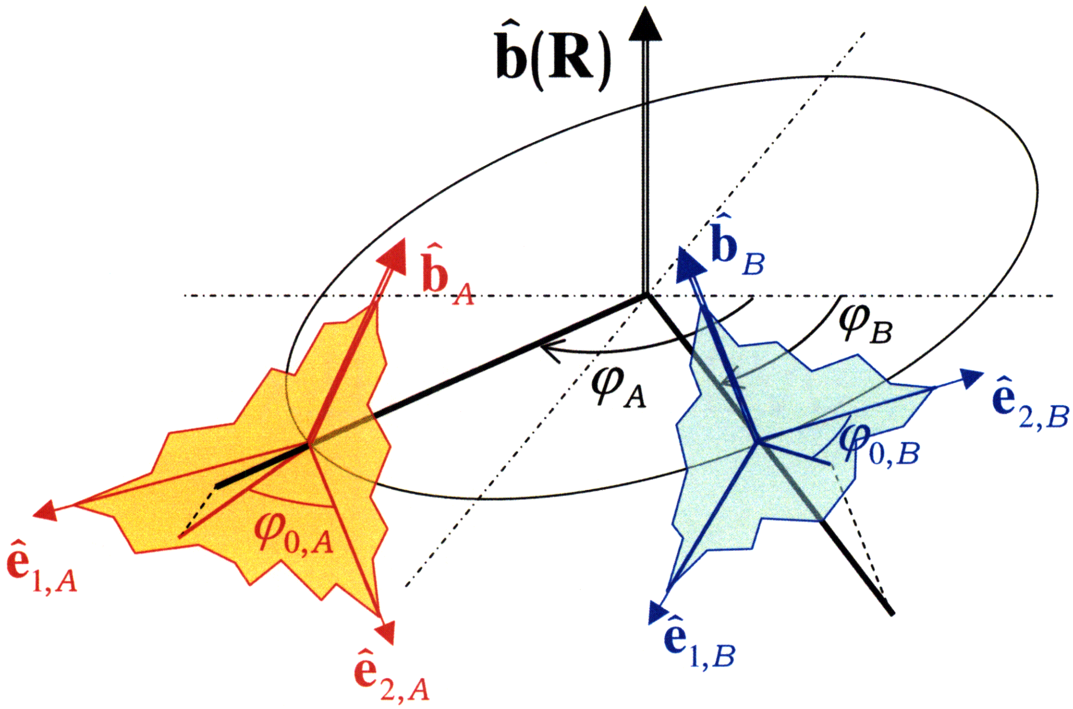

Figure 3-3: Geometric effects on the gyrokinetic gyrophase. The circle represents the gyromotion of a given particle. The variations of $\hat{\mathbf{e}}_1$, $\hat{\mathbf{e}}_2$ and $\hat{\mathbf{b}}$ are exaggerated.

of the potential. The particle feels a rapidly changing potential along its gyromotion. The perpendicular velocity then has a variation similar to the kinetic energy [recall figure 3-2], and the gyromotion accelerates and decelerates accordingly, giving the correction $-(Ze/MB)(\partial\widetilde{\Phi}/\partial\mu)$ in (3.29). The other contributions to the fast variation of $d\varphi_0/dt$ are geometrical. The gyrophase φ_0 is defined with respect to the vectors $\hat{\mathbf{e}}_1(\mathbf{r})$, $\hat{\mathbf{e}}_2(\mathbf{r})$ and $\hat{\mathbf{b}}(\mathbf{r})$ at the position $\mathbf{r}$ of the particle, not the gyrocenter position $\mathbf{R}$. Figure 3-3 shows that to find the gyrophase $\varphi_{0,A}$ that corresponds to the particle position A, the position vector $\mathbf{r} - \mathbf{R}$ must be projected onto the plane defined by the local vectors $\hat{\mathbf{e}}_{1,A}$ and $\hat{\mathbf{e}}_{2,A}$; $\varphi_{0,A}$ is the angle of the projection with respect to $\hat{\mathbf{e}}_{2,A}$ [recall from (2.2) that φ_0 is the angle of $\mathbf{v}_\perp$ with respect to the vector $\hat{\mathbf{e}}_1$ and hence it is the angle of $\mathbf{r} - \mathbf{R} = -\Omega_i^{-1}\mathbf{v}\times\hat{\mathbf{b}}$ with respect to $\hat{\mathbf{e}}_2$]. In figure 3-3, the geometric construction to determine φ_0 at point A is sketched in red, and the corresponding construction at another point B is given in blue, making explicit the distinction between the gyrokinetic gyrophases φ_A and φ_B, and the "local" gyrophases $\varphi_{0,A}$ and $\varphi_{0,B}$. Notice that both variation in $\hat{\mathbf{b}}$, that determines the "local" plane of $\hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_2$, and rotation of $\hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_2$ within that plane change the value of φ_0. For this reason, both $\nabla\hat{\mathbf{b}}$ and $\nabla\hat{\mathbf{e}}_2 \cdot \hat{\mathbf{e}}_1$ enter in φ_1 and $d\varphi/dt$.

Finally, the gyrokinetic magnetic moment variable is dealt with somewhat differently since we want to construct it to remain an adiabatic invariant order by order. The condition for the magnetic moment is not only that its derivative must be gyrophase independent, but that $\langle d\mu/dt \rangle$ must vanish order by order. In this thesis, I will define the magnetic moment μ such that $d\mu/dt$ has a gyrophase dependent component of order $\delta_i v_i^3/BL$ but satisfies

$$\left\langle \frac{d\mu}{dt} \right\rangle = O\left(\delta_i^2 \frac{v_i^3}{BL}\right) \simeq 0, \tag{3.32}$$

for μ to remain an adiabatic invariant. This variable μ is

$$\mu = \mu_0 + \mu_1, \tag{3.33}$$

where $\mu_0 = v_\perp^2/2B$ is the usual lowest order result, and $\mu_1 = O(\delta_i \mu_0)$. The correction $\mu_2 = O(\delta_i^2 \mu_0)$ is not necessary because the distribution function is assumed to be a stationary Maxwellian to zeroth order, making the dependence on μ weak. For μ to remain an adiabatic invariant, μ_1 must contain gyrophase independent contributions such that $\langle d\mu/dt \rangle = 0$ to the requisite order, given in (3.32). Solving for μ_1 as outlined in section A.1 of Appendix A gives

$$\mu_1 = \frac{Ze\widetilde{\phi}}{MB(\mathbf{R})} - \frac{1}{B}\mathbf{v}_\perp \cdot \mathbf{v}_M - \frac{v_{||}}{4B\Omega_i}[\mathbf{v}_\perp(\mathbf{v}\times\hat{\mathbf{b}}) + (\mathbf{v}\times\hat{\mathbf{b}})\mathbf{v}_\perp] : \nabla\hat{\mathbf{b}} - \frac{v_{||}v_\perp^2}{2B\Omega_i}\hat{\mathbf{b}}\cdot\nabla\times\hat{\mathbf{b}}. \tag{3.34}$$

To keep μ an adiabatic invariant, $\langle \mu_1 \rangle = -(v_{||}v_\perp^2/2B\Omega_i)(\hat{\mathbf{b}} \cdot \nabla \times \hat{\mathbf{b}}) \neq 0$. In subsection A.2.4 of Appendix A, $\langle \mu_1 \rangle$ is proven to make μ an adiabatic invariant to next order.

The procedure presented in this section is compared to the Lie transform techniques [25] in Appendix C. Both methods yield the same results, although the final equations look somewhat different. These apparent discrepancies are shown to be due to subtleties in the definitions of the functions $\langle \phi \rangle$, $\widetilde{\phi}$ and $\widetilde{\Phi}$. Finally, I also compare this derivation with drift kinetics. In particular, in Appendix E, I show that with the higher order corrections $\mathbf{R}_2$ and E_2 it is possible to recover the drift kinetic gyrophase dependent portion of f_i up to order $\delta_i^2 f_{Mi}$.

3.3 Gyrokinetic Fokker-Planck equation

The Fokker-Planck equation (3.3) becomes

$$\frac{\partial f_i}{\partial t} + \dot{\mathbf{R}} \cdot \nabla_{\mathbf{R}} f_i + \dot{E}\,\frac{\partial f_i}{\partial E} + \dot{\mu}\,\frac{\partial f_i}{\partial \mu} + \dot{\varphi}\,\frac{\partial f_i}{\partial \varphi} = C\{f_i\} \tag{3.35}$$

when written in gyrokinetic variables, where $\dot{Q} \equiv dQ/dt$, and Q is any of the gyrokinetic variables. The gyroaverage of this equation is

$$\frac{\partial \langle f_i \rangle}{\partial t} + \dot{\mathbf{R}} \cdot \nabla_{\mathbf{R}} \langle f_i \rangle + \dot{E}\,\frac{\partial \langle f_i \rangle}{\partial E} = \langle C\{f_i\} \rangle, \tag{3.36}$$

where $\langle f_i \rangle \equiv \langle f_i \rangle(\mathbf{R}, E, \mu, t)$ is the gyroaveraged ion distribution function. Here, I have used that E and $\mathbf{R}$ are defined such that their time derivatives are gyrophase independent to the orders given by (3.19) and (3.27). The term $\dot{\mu}(\partial f_i/\partial \mu)$ is neglected because the magnetic moment μ is defined such that $d\mu/dt = O(\delta_i v_i^3/BL)$ and the zeroth order distribution function is assumed to be a stationary Maxwellian, making $\partial f_i/\partial \mu = O(\delta_i f_{Mi} B/v_i^2)$. Therefore, in (3.36) I have neglected pieces that are $O(f_{Mi}\delta_i^2 v_i/L)$. I have also neglected the term $\langle \dot{\varphi}\,\partial f_i/\partial \varphi \rangle = O(\tilde{f}_i \delta_i v_i/L)$, where $\tilde{f}_i = f_i - \langle f_i \rangle$ is the gyrophase dependent piece of the distribution function. I will prove in the next paragraph that $\tilde{f}_i$ is $O(f_{Mi}\delta_i \nu_{ii}/\Omega_i)$, making all the neglected terms comparable to or smaller than $f_{Mi}\delta_i^2 v_i/L$, and the distribution function gyrophase independent to first order, $f_i \simeq \langle f_i \rangle$. Notice that, due to the missing pieces, I can only obtain contributions to the distribution function that are $O(\delta_i f_{Mi})$, as well as all terms with $k_\perp \rho_i \sim 1$.

The explicit equation for the gyrophase dependent part of the distribution function is obtained by subtracting from the full Fokker-Planck equation (3.35) its gyroaverage, giving to lowest order

$$-\Omega_i \frac{\partial \widetilde{f}_i}{\partial \varphi} = C\{f_i\} - \langle C\{f_i\}\rangle. \tag{3.37}$$

Therefore, the collisional term is the one that sets the size of $\widetilde{f}_i$. Since the distribution function is a Maxwellian to zeroth order, the collision operator vanishes to zeroth order, $C\{f_i\} = O(\delta_i \nu_{ii} f_{Mi})$, giving $C\{f_i\} - \langle C\{f_i\}\rangle = O(\delta_i \nu_{ii} f_{Mi})$. As a result, $\widetilde{f}_i$ is

$$\widetilde{f}_i \simeq -\frac{1}{\Omega_i} \int^{\varphi} d\varphi' \left(C\{f_i\} - \langle C\{f_i\}\rangle\right) = O\left(\frac{\delta_i \nu_{ii}}{\Omega_i} f_{Mi}\right), \tag{3.38}$$

where $\nu_{ii}/\Omega_i \ll 1$.

Using the values of $d\mathbf{R}/dt$ from (3.19) and dE/dt from (3.27), and using $\langle f_i\rangle \simeq f_i$, the equation for f_i in gyrokinetic variables is

$$\frac{\partial f_i}{\partial t} + [u\hat{\mathbf{b}}(\mathbf{R}) + \mathbf{v}_d] \cdot \left(\nabla_{\mathbf{R}} f_i - \frac{Ze}{M} \nabla_{\mathbf{R}} \langle \phi \rangle \frac{\partial f_i}{\partial E}\right) = \langle C\{f_i\}\rangle, \tag{3.39}$$

where $\langle \phi \rangle$ is defined in (3.16), and $f_i \equiv f_i(\mathbf{R}, E, \mu, t)$ is gyrophase independent.

The gyrokinetic equation can be also written in conservative form. To do so, the Jacobian of the gyrokinetic transformation is needed. Conservation of particles in phase space requires the Jacobian of the transformation, $J = \partial(\mathbf{r}, \mathbf{v})/\partial(\mathbf{R}, E, \mu, \varphi)$, to satisfy

$$\frac{\partial J}{\partial t} + \nabla_{\mathbf{R}} \cdot (\dot{\mathbf{R}} J) + \frac{\partial}{\partial E}(\dot{E} J) + \frac{\partial}{\partial \mu}(\dot{\mu} J) + \frac{\partial}{\partial \varphi}(\dot{\varphi} J) = 0. \tag{3.40}$$

[This is the equality $\nabla \cdot \dot{\mathbf{r}} + \nabla_v \cdot \dot{\mathbf{v}} = 0$ written in gyrokinetic variables]. Employing this property, equation (3.35) can be written in conservative form by multiplying it by J to obtain

$$\frac{\partial}{\partial t}(J f_i) + \nabla_{\mathbf{R}} \cdot \left(\dot{\mathbf{R}} J f_i\right) + \frac{\partial}{\partial E}\left(\dot{E} J f_i\right) + \frac{\partial}{\partial \mu}\left(\dot{\mu} J f_i\right) + \frac{\partial}{\partial \varphi}\left(\dot{\varphi} J f_i\right) = J C\{f_i\}. \tag{3.41}$$

The gyroaverage of this equation is

$$\frac{\partial}{\partial t}(J f_i) + \nabla_{\mathbf{R}} \cdot (\dot{\mathbf{R}} J f_i) + \frac{\partial}{\partial E}(\dot{E} J f_i) = J\langle C\{f_i\}\rangle. \tag{3.42}$$

Here, I have taken into account that the Jacobian J is independent of φ to the order of interest, as can be seen by using (3.40). The equation for the gyrophase dependent part of the Jacobian is obtained by subtracting from (3.40) its gyroaverage. Notice that $J - \langle J \rangle$ depends on the differences $\dot{\mathbf{R}} - \langle \dot{\mathbf{R}} \rangle$, $\dot{E} - \langle \dot{E} \rangle$..., and those differences are small by definition of the gyrokinetic variables. The gyrophase-dependent part of the Jacobian is estimated to be $J - \langle J \rangle = O(\delta_i^2 B/v_i)$. Finally, I substitute $d\mathbf{R}/dt$ and dE/dt in (3.42) to get

$$\frac{\partial}{\partial t}(Jf_i) + \nabla_{\mathbf{R}} \cdot \{Jf_i[u\hat{\mathbf{b}}(\mathbf{R}) + \mathbf{v}_d]\} - \frac{\partial}{\partial E}\left\{Jf_i \frac{Ze}{M}[u\hat{\mathbf{b}}(\mathbf{R}) + \mathbf{v}_M] \cdot \nabla_{\mathbf{R}} \langle \phi \rangle \right\} = J\langle C\{f_i\}\rangle. \tag{3.43}$$

The calculation of the Jacobian is described in section A.3 of Appendix A. The final result is

$$J = \frac{\partial(\mathbf{r}, \mathbf{v})}{\partial(\mathbf{R}, E, \mu, \varphi)} = \frac{B(\mathbf{R})}{u} + \frac{Mc}{Ze}\hat{\mathbf{b}}(\mathbf{R}) \cdot \nabla_{\mathbf{R}} \times \hat{\mathbf{b}}(\mathbf{R}). \tag{3.44}$$

In section A.3, it is also proven that J satisfies the gyroaverage of (3.40).

Similar gyrokinetic equations to (3.39) and (3.43) can be found for the gyrokinetic variables $\mathbf{R}$, u and μ, where u is defined by (3.23). Combining equations (3.19), (3.23), (3.27) and $d\mu/dt \simeq 0$ to obtain

$$\dot{u} = -\left[\hat{\mathbf{b}}(\mathbf{R}) + \frac{u}{\Omega_i}\hat{\mathbf{b}}(\mathbf{R}) \times \boldsymbol{\kappa}(\mathbf{R})\right] \cdot \nabla_{\mathbf{R}} \left[\mu B(\mathbf{R}) + \frac{Ze\langle \phi \rangle}{M}\right] \tag{3.45}$$

gives the gyrokinetic equation

$$\frac{\partial f_i}{\partial t} + [u\hat{\mathbf{b}}(\mathbf{R}) + \mathbf{v}_d] \cdot \nabla_{\mathbf{R}} f_i + \dot{u}\frac{\partial f_i}{\partial u} = \langle C\{f_i\}\rangle. \tag{3.46}$$

This gyrokinetic equation can be written in conservative form by noticing that the new Jacobian is given by

$$J_u = \frac{\partial(\mathbf{r}, \mathbf{v})}{\partial(\mathbf{R}, u, \mu, \varphi)} = \frac{\partial(\mathbf{r}, \mathbf{v})}{\partial(\mathbf{R}, E, \mu, \varphi)}\frac{\partial E}{\partial u} = B(\mathbf{R}) + \frac{Mcu}{Ze}\hat{\mathbf{b}}(\mathbf{R}) \cdot \nabla_{\mathbf{R}} \times \hat{\mathbf{b}}(\mathbf{R}). \tag{3.47}$$

Using the new Jacobian, the gyrokinetic equation may be written as

$$\frac{\partial}{\partial t}(J_u f_i) + \nabla_{\mathbf{R}} \cdot \{J_u f_i[u\hat{\mathbf{b}}(\mathbf{R}) + \mathbf{v}_d]\} + \frac{\partial}{\partial u}(J_u f_i \dot{u}) = J_u \langle C\{f_i\}\rangle. \quad (3.48)$$

The ion gyrokinetic Fokker-Planck equations (3.39), (3.43), (3.46) and (3.48) have their counterpart for electrons. However, since in this thesis the wavelengths shorter than the ion gyroradius are not considered, the electron gyrokinetic equation reduces to the drift kinetic equation

$$\frac{\partial \overline{f}_e}{\partial t} + (v_{||}\hat{\mathbf{b}} + \mathbf{v}_{de}) \cdot \left(\overline{\nabla}\,\overline{f}_e + \frac{e}{m}\nabla\phi\frac{\partial \overline{f}_e}{\partial E_0}\right) = C\{\overline{f}_e\}, \quad (3.49)$$

with $\overline{f}_e = \overline{f}_e(\mathbf{r}, E_0, \mu_0, t)$ the gyrophase independent piece of the distribution function, $\overline{\nabla}$ the gradient holding E_0, μ_0, φ_0 and t fixed, and

$$\mathbf{v}_{de} = -\frac{\mu_0}{2\Omega_e}\hat{\mathbf{b}} \times \nabla B - \frac{v_{||}^2}{\Omega_e}\hat{\mathbf{b}} \times \boldsymbol{\kappa} - \frac{c}{B}\nabla\phi \times \hat{\mathbf{b}} \quad (3.50)$$

the electron drifts. Here, $\Omega_e = eB/mc$ is the electron gyrofrequency. The total distribution function for electrons contains the gyrophase independent piece $\overline{f}_e$ and the gyrophase dependent piece

$$f_e - \overline{f}_e = -\frac{f_{Me}}{\Omega_e}(\mathbf{v} \times \hat{\mathbf{b}}) \cdot \left[\frac{\nabla n_e}{n_e} - \frac{e}{T_e}\nabla\phi + \left(\frac{mE_0}{T_e} - \frac{3}{2}\right)\frac{\nabla T_e}{T_e}\right]. \quad (3.51)$$

In the rest of this thesis, the gyrokinetics variables to be used are $\mathbf{R}$, E and μ for ions and $\mathbf{r}$, E_0 and μ_0 for electrons. Then, the relevant equations are (3.39) for ions, and (3.49) for electrons.

3.4 Gyrokinetic quasineutrality equation

Modern gyrokinetics employs a low order quasineutrality condition to calculate the electrostatic potential. The ion and electron distribution functions f_i and f_e are calculated using the lower order equations (3.39) and (3.49). These distribution functions are integrated over velocity space to obtain the densities n_i and n_e that are substituted into the quasineutrality condition $Zn_i = n_e$, from which the potential is solved. Since equations (3.39) and (3.49) give distribution functions good to order $\delta_i f_{Mi}$ and $\delta_e f_{Me}$, respectively, the quasineutrality equation is correct only to order $\delta_i n_e$. In chapter 4, I will show that this is not enough to solve for the long wavelength axisymmetric radial electric field, and employing such a low order quasineutrality equation can lead to unphysical results. In this section, I will derive the modern gyrokinetic quasineutrality equation to the order that is usually implemented, so I can demonstrate later, in chapter 4, that it cannot provide the correct long wavelength axisymmetric radial electric field.

I will begin with Poisson's equation to explicitly show that the quasineutral approximation is valid for the range of wavelengths of interest. The distribution function f_i in Poisson's equation,

$$-\nabla^2 \phi = 4\pi e \left[Z \int d^3v \; f_i(\mathbf{R}, E, \mu, t) - n_e(\mathbf{r}, t) \right], \tag{3.52}$$

is obtained from the Fokker-Planck equation (3.39). Therefore, it is known as a function of the gyrokinetic variables. The distribution function can be rewritten more conveniently as a function of $\mathbf{r} + \Omega_i^{-1}\mathbf{v} \times \hat{\mathbf{b}}$, E_0 and μ_0 by Taylor expanding. However, it is important to remember that there are missing pieces of order $\delta_i^2 f_{Mi}$ in the distribution function since terms of this order must be neglected to derive (3.39). Thus, the expansion can only be carried out to the order where the distribution function is totally known, resulting in

$$f_i(\mathbf{R}, E, \mu, t) = f_i(\mathbf{R}_g, E_0, \mu_0, t) + E_1 \frac{\partial f_i}{\partial E_0} + \mu_1 \frac{\partial f_i}{\partial \mu_0} + O(\delta_i^2 f_{Mi}). \tag{3.53}$$

Notice that $f_i(\mathbf{R}_g, E_0, \mu_0, t)$, with $\mathbf{R}_g \equiv \mathbf{r} + \Omega_i^{-1}\mathbf{v} \times \hat{\mathbf{b}}$, cannot be Taylor expanded around $\mathbf{r}$ because $k_\perp \rho_i \sim 1$. In the higher order terms proportional to E_1 and μ_1, the function f_i is valid only to lowest order, i.e., $f_i \simeq f_{Mi}$, $\partial f_i / \partial E_0 \simeq (-M/T_i) f_{Mi}$ and $\partial f_i / \partial \mu_0 \simeq 0$. Moreover, according to the ordering in (3.1), the corrections arising from using $\mathbf{r}$ instead of $\mathbf{R}_g$ are small by δ_i because, even though small wavelengths are allowed, the amplitude of the fluctuations with small wavelengths is assumed to be of the next order. Therefore, in the higher order integrals, only the long wavelength distribution function f_{Mi} depending on $\mathbf{R}_g \simeq \mathbf{r}$ need be retained.

Since the turbulent wavelengths are much larger than the Debye length, the term in the left side of Poisson's equation (3.52) may be neglected. The resulting quasineutrality equation reduces to

$$Zn_{ip}\{\phi\} \simeq n_e(\mathbf{r}, t) - Z\hat{N}_i(\mathbf{r}, t), \tag{3.54}$$

with $n_e = \int d^3v\, \overline{f}_e$ the electron density,

$$n_{ip}\{\phi\} = - \int d^3v\, \frac{Ze\widetilde{\phi}}{T_i} f_{Mi} \tag{3.55}$$

the ion polarization density that depends explicitly in the potential, and

$$\hat{N}_i(\mathbf{r}, t) = \int d^3v\, f_i(\mathbf{R}_g, E_0, \mu_0, t) \tag{3.56}$$

the ion guiding center density. In equation (3.54), terms of order $\delta_i^2 n_e$ and $n_e k_\perp \lambda_D^2/L \ll \delta_i k_\perp \rho_i n_e$ have been neglected, with $\lambda_D = \sqrt{T_e/4\pi e^2 n_e}$ the Debye length.

In the quasineutrality equation (3.54), the ion density is composed of two terms, the guiding center density $\hat{N}_i$ and the polarization density $n_{ip}\{\phi\}$. Both terms have a clear physical meaning. In figure 3-4, the ion density calculation is sketched. In gyrokinetics, ions are substituted by rings of charge with radius the ion gyroradius. The ion density at a point is then calculated by counting the rings that pass through that point. In figure 3-4, the ion density in the black square is computed by summing all the gyrocenters whose rings of charge cross the square. In the figure, two of them, $\mathbf{R}_A$ and $\mathbf{R}_B$, are shown. Importantly, the charge density along a ring is not constant, but depends on the potential. In figure 3-1 we saw that the potential changes rapidly along the gyromotion. There is a Maxwell-Boltzmann response to this variation, $-Ze\widetilde{\phi}/T_e$, that gives a varying ion charge density along the ring, as indicated in

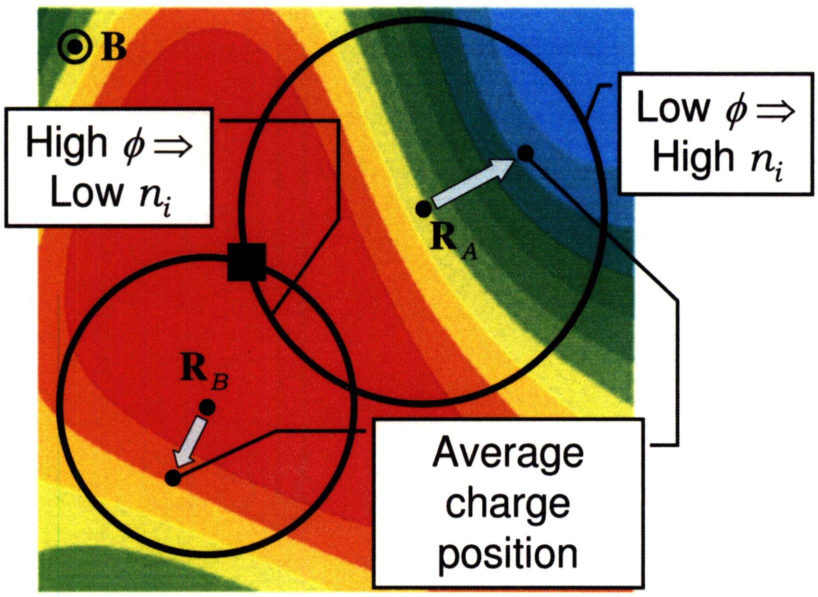

Figure 3-4: Ion density in the black square calculated using a gyrokinetic distribution function. The electrostatic potential is the background contour plot.

figure 3-4, where the left region of the ring $\mathbf{R}_A$ has lower density than the right side. In the ring, therefore, the average charge position is displaced towards the lower potential values, i.e., the charge moves in the same direction as the local electric field, hence the name polarization. For this reason, the ion density is separated into the guiding center density $\hat{N}_i$, due to the average charge in the ring, and the polarization density n_{ip} that originates in the non-uniform density along the ring that is induced by the short wavelength pieces of the potential.

Equation (3.54) is used to calculate ϕ for wavelengths of the order of the gyroradius, including zonal flow, in δf turbulence codes such as GS2 [1], GENE [2] or GYRO [3]. In most cases, the distribution function is obtained from the gyrokinetic equation (3.39) written for $f_i = f_{Mi} + h_i$, with $|h_i(\mathbf{R}, E, \mu, t)| \ll f_{Mi}$ and f_{Mi} only depending on ψ. The resulting equation is

$$\frac{\partial h_i}{\partial t} + [u\hat{\mathbf{b}}(\mathbf{R}) + \mathbf{v}_d] \cdot \nabla_{\mathbf{R}} h_i - \left\langle C^{(\ell)} \left\{ h_i - \frac{Ze\widetilde{\phi}}{T_i} f_{Mi} \right\} \right\rangle = \frac{Ze}{T_i} f_{Mi} \left\{ i\omega_*^{n,T} \langle \phi \rangle - [u\hat{\mathbf{b}}(\mathbf{R}) + \mathbf{v}_M] \cdot \nabla_{\mathbf{R}} \langle \phi \rangle \right\}, \qquad (3.57)$$

with $i = \sqrt{-1}$ and

$$\omega_*^{n,T} \equiv -i\frac{cT_i}{ZeB}(\hat{\mathbf{b}} \times \nabla\psi) \cdot \nabla_{\mathbf{R}} \ln\langle\phi\rangle \left[\frac{1}{n_i}\frac{dn_i}{d\psi} + \left(\frac{ME}{T_i} - \frac{3}{2}\right)\frac{1}{T_i}\frac{dT_i}{d\psi}\right] \tag{3.58}$$

the drift wave frequency. In equation (3.57), ϕ appears nonlinearly in $\mathbf{v}_d \cdot \nabla_{\mathbf{R}} h_i$ and linearly on the right side of the equation. The linear terms are usually solved implicitly. Then, h_i has a linear dependence on ϕ that will appear as a linear dependence in $\hat{N}_i$, and can be used to solve for ϕ in equation (3.54). The ion polarization density (3.55) also depends linearly on ϕ. However, at long wavelengths n_{ip} becomes too small to be important. For $\delta_i \ll k_\perp \rho_i \ll 1$, the polarization density is

$$n_{ip} \simeq \nabla \cdot \left(\frac{cn_i}{B\Omega_i}\nabla_\perp \phi\right) = O(\delta_i k_\perp \rho_i n_e). \tag{3.59}$$

The details of this calculation are given in section D.1 of Appendix D. To estimate the size of n_{ip}, I have used (3.1) to order $\nabla_\perp \phi \sim T_e/eL$ and $\nabla_\perp \nabla_\perp \phi \sim k_\perp T_e/eL$. With this estimate, for $k_\perp L \sim 1$ equation (3.54) becomes

$$Z\hat{N}_i(\mathbf{r}, t) = n_e(\mathbf{r}, t), \tag{3.60}$$

where terms of order $\delta_i^2 n_e$ have been neglected.

It is possible to obtain a higher order long wavelength quasineutrality equation for a non-turbulent plasma if the ion distribution function is assumed to be known to high enough order. The resulting equation is

$$\nabla \cdot \left(\frac{Zcn_i}{B\Omega_i}\nabla_\perp \phi\right) - \frac{ZMc^2 n_i}{2T_i B^2}|\nabla_\perp \phi|^2 = n_e - Z\hat{N}_i, \tag{3.61}$$

where $\hat{N}_i$ must be defined to higher order,

$$\hat{N}_i(\mathbf{r}, t) = \int d^3v\, f_i(\mathbf{r}, E_0, \mu_0, t)\left(1 + \frac{v_{||}}{\Omega_i}\hat{\mathbf{b}} \cdot \nabla \times \hat{\mathbf{b}}\right) + (\overleftrightarrow{\mathbf{I}} - \hat{\mathbf{b}}\hat{\mathbf{b}}) : \frac{\nabla\nabla p_i}{2M\Omega_i^2}. \tag{3.62}$$

The derivation of (3.61) is shown in section D.2 of Appendix D. Even though equation (3.61) is correct, it is only useful if we are able to evaluate the missing $O(\delta_i^2 f_{Mi})$ pieces in f_i that are of the same order as the left side in (3.61). Equation (3.39) misses these pieces. Equation (3.61) will only serve to demonstrate the problems that arise from the use of equation (3.54).

In chapter 4, I will prove that neither equation (3.60) or its higher order version (3.61) are able to provide the self-consistent, long wavelength radial electric field. I will even give an example in section 4.5 in which the higher order equation (3.61) leaves the radial electric field undetermined.

Chapter 4

Gyrokinetic vorticity equation

In this chapter, I rewrite the vorticity equation (2.9) in a convenient form for gyrokinetics. The gyrokinetic change of variables found in chapter 3 is especially well suited for simulation of drift wave turbulence. However, the gyrokinetic quasineutrality equation, traditionally used to calculate the electrostatic potential, has problems. By writing the vorticity equation (2.9) in gyrokinetic form, I can study the behavior of the quasineutrality equation at different time scales and wavelengths. In particular, I am able to prove that the radial current vanishes to a very high order for any radial electric field, i.e., the radial drift of ions and electrons is intrinsically ambipolar.

In section 4.1, I explain the notation employed in this chapter and I list the assumptions. These assumptions restrict the treatment to turbulence that has reached a statistical equilibrium and, therefore, only has small variations along magnetic flux surfaces. Under the assumptions of sections 3.1 and 4.1, I evaluate the size of the terms in the vorticity equation (2.9) and in the toroidal angular momentum conservation equation (2.21). The size of the different contributions to the vorticity equation depends on the perpendicular wavelength of interest, and this dependence makes some terms important for structures of the size of the ion gyroradius and negligible at wavelengths on the order of the minor radius of the device. In section 4.2, it will become clear that direct evaluation of the terms in equation (2.9) is too difficult to be of interest. Then, in the rest of the chapter, a different approach is taken. The equations for particle and momentum conservation are derived from the gyrokinetic equation in section 4.3. These conservation equations are then combined in section 4.4 to derive two different vorticity equations equivalent to (2.9) to lowest order. Importantly, these equations are easier to study and to evaluate numerically. With them, I show that the dependence of quasineutrality on the long wavelength radial electric field is not meaningful for gyrokinetic codes to retain since gyrokinetics will be shown to be intrinsically ambipolar as already stressed in section 2.3. The problems that arise from quasineutrality are exposed in a simplified example in section 4.5. I finish this chapter with a discussion in section 4.6. All the details of the calculation are relegated to Appendices F-K.

4.1 Notation and assumptions

In this chapter, I work in both the gyrokinetic phase space $\{\mathbf{R}, E, \mu, \varphi\}$ and the "physical" phase space $\{\mathbf{r}, \mathbf{v}\}$. I refer to it as physical phase space because spatial and velocity coordinates do not get mixed as they do in gyrokinetic phase space. I use the variables $\mathbf{r}$, E_0, μ_0 and φ_0 to describe this physical phase space. Whenever I write $\partial/\partial E_0$, it is implied that $\mathbf{r}$, μ_0, φ_0 and t are held fixed, and similarly for $\partial/\partial\mu_0$ and $\partial/\partial\varphi_0$. The gradient holding E_0, μ_0, φ_0 and t fixed will be written as $\overline{\nabla}$. In addition, any derivative with respect to a gyrokinetic variable is performed holding the other gyrokinetic variables constant. The partial derivative with respect to the time variable t deserves a special mention since it is necessary to indicate which variables are kept fixed. In this formulation, the time derivative holding $\mathbf{r}$ and $\mathbf{v}$ fixed is equivalent to holding $\mathbf{r}$, E_0, μ_0 and φ_0 fixed because the magnetic field is constant in time. Also, a gyroaverage holding $\mathbf{r}$, E_0, μ_0 and t fixed is denoted as $\overline{(\ldots)}$, as opposed to the gyrokinetic gyroaverage $\langle \ldots \rangle$ performed holding $\mathbf{R}$, E, μ and t fixed.

All the assumptions in section 3.1 are applicable here. Then, the zeroth order ion and electron distribution functions are assumed to be stationary Maxwellians, f_{Mi} and f_{Me}. In this chapter, the only spatial dependence allowed for these zeroth order solutions is in the radial variable ψ. Therefore, $\hat{\mathbf{b}} \cdot \nabla f_{Mi} = \hat{\mathbf{b}} \cdot \nabla f_{Me} = 0$. I assume that the radial gradients of f_{Mi} and f_{Me} are $O(1/L)$, with L of the order of the minor radius of the tokamak. The zeroth order potential ϕ works in a similar fashion, depending only on ψ and with a radial gradient on the longer scale L.

As in section 3.1, I allow wavelengths perpendicular to the magnetic field that are on the order of the ion gyroradius, $k_\perp \rho_i \sim 1$. The pieces of the potential and the distribution function with short perpendicular wavelengths are small in size, following the ordering in (3.1). Except for initial transients, I assume that the variation along the magnetic field of f_i, f_e and ϕ is slow, i.e., in general $\hat{\mathbf{b}} \cdot \nabla f_i \sim \delta_i f_{Mi}/L$, $\hat{\mathbf{b}} \cdot \nabla f_e \sim \delta_i f_{Me}/L$ and $\hat{\mathbf{b}} \cdot \nabla \phi \sim \delta_i T_e/eL$.

The ion distribution function $f_i(\mathbf{R}, E, \mu, t)$ is found employing the gyrokinetic equation (3.39) [the gyrophase dependent piece is $O(\delta_i f_{Mi} \nu_{ii}/\Omega_i)$ and given by (3.38)]. After the initial transient, equation (3.39) becomes $u\hat{\mathbf{b}} \cdot \nabla_\mathbf{R} f_i = \langle C\{f_i\}\rangle$ to zeroth order. At long wavelengths, this requires that f_i approach a Maxwellian f_{Mi} with $\hat{\mathbf{b}} \cdot \nabla f_{Mi} = 0$, giving $f_{Mi} \equiv f_{Mi}(\psi, E)$. The assumed long wavelength piece of the distribution function satisfies this condition. Importantly, $\hat{\mathbf{b}} \cdot \nabla f_{Mi} = 0$ does not impose any condition on the radial dependence of f_{Mi}. Consequently, the density and temperature in f_{Mi} may have short wavelength components as long as they satisfy the orderings in (3.1), i.e., $\nabla_\perp n_i \sim k_\perp n_{i,k} \sim n_{i,k=0}/L$, $\nabla_\perp T_i \sim k_\perp T_{i,k} \sim T_{i,k=0}/L$ and $\nabla_\perp \nabla_\perp f_{Mi,k} \sim k_\perp f_{Mi,k=0}/L$. Solving for the next order correction $f_i - f_{Mi}$ in equation (3.39) gives $f_i - f_{Mi} \sim \delta_i f_{Mi}$. Then, the average velocity $\mathbf{V}_i = n_i^{-1} \int d^3v \mathbf{v} f_i$ is of order $\delta_i v_i$. Furthermore, any variation of the distribution function within a flux surface is due to $f_i - f_{Mi}$, and thus small by δ_i as compared to the long wavelength piece of f_{Mi}. This means that when we consider average velocities or the gradients $\hat{\mathbf{b}} \cdot \nabla_\mathbf{R} f_i$ and $\hat{\boldsymbol{\zeta}} \cdot \nabla_\mathbf{R} f_i$, it will be useful to think about the distribution function as it is done in δf codes where $f_i = f_{Mi} + h_i$, with $h_i \sim \delta_i f_{Mi} \ll f_{Mi}$. Comparing the estimate for $f_i - f_{Mi}$ with the orderings in (3.1), I find that the gradients of f_i and ϕ parallel to the flux surfaces are smaller than the maximum allowed in gyrokinetics, i.e., $\hat{\mathbf{b}} \cdot \nabla_\mathbf{R} f_i \sim \delta_i f_{Mi}/L \lesssim \hat{\boldsymbol{\zeta}} \cdot \nabla_\mathbf{R} f_i \sim k_\perp (f_i - f_{Mi}) \sim k_\perp \rho_i f_{Mi}/L \lesssim f_{Mi}/L$ and $\hat{\mathbf{b}} \cdot \nabla\phi \sim \delta_i T_e/eL \lesssim \hat{\boldsymbol{\zeta}} \cdot \nabla\phi \sim k_\perp \rho_i T_e/eL \lesssim T_e/eL$. These estimates may fail for the initial transient, but I am interested in the electric field evolution at long times, when the transient has died away.

Interestingly, these assumptions imply that the long wavelength axisymmetric flows are neoclassical. At long wavelengths, the ion distribution function $f_i(\mathbf{R}, E, \mu, t)$ can be Taylor expanded around $\mathbf{r}$, E_0 and μ_0. Then, the gyroaveraged piece $\overline{f}_i$ is approximately $f_i(\mathbf{r}, E_0, \mu_0, t)$, where $\mathbf{R}$, E and μ have been replaced by $\mathbf{r}$, E_0 and μ_0. This gyroaveraged piece of the ion distribution function $\overline{f}_i$ satisfies the ion drift kinetic equation

$$\frac{\partial \overline{f}_i}{\partial t} + (v_{||}\hat{\mathbf{b}} + \mathbf{v}_M) \cdot \left(\nabla \overline{f}_i - \frac{Ze}{M} \nabla \phi \frac{\partial \overline{f}_i}{\partial E_0} \right) + \mathbf{v}_E \cdot \nabla_{\mathbf{R}} f_i = C\{\overline{f}_i\}. \tag{4.1}$$

This equation is obtained from (3.39) by realizing that the functional dependence of f_i on the gyrokinetic variables $\mathbf{R}$, E and μ is the same as the dependence of $\overline{f}_i$ on $\mathbf{r}$, E_0 and μ_0. Then, equation (4.1) is derived from (3.39) by replacing $\mathbf{R}$, E and μ by $\mathbf{r}$, E_0 and μ_0, and employing that $\langle \phi \rangle \simeq \phi$ for long wavelengths. The difference between the long wavelength ion equation (4.1) and a drift kinetic equation [see, for example, the electron equation (3.49)] is in the nonlinear term $\mathbf{v}_E \cdot \nabla_{\mathbf{R}} f_i$. In this term, the short wavelength components of f_i and ϕ beat together to give a long wavelength contribution. Due to the presence of these short wavelength pieces, f_i cannot be Taylor expanded and $\langle \phi \rangle \neq \phi$. Importantly, this term gives a negligible contribution to the axisymmetric piece of $\overline{f}_i$. For $\mathbf{v}_E \cdot \nabla_{\mathbf{R}} f_i$ to have an $n = 0$ toroidal mode number, the beating components $f_{i,n}$ and $\nabla_{\mathbf{R}} \langle \phi \rangle_{-n}$ must have toroidal mode numbers of the same magnitude and opposite sign. Moreover, these components must have $n \neq 0$ because otherwise $\nabla_{\mathbf{R}} \langle \phi \rangle$ is parallel to $\nabla_{\mathbf{R}} f_i$ and $\mathbf{v}_E \cdot \nabla_{\mathbf{R}} f_i$ vanishes exactly. Thus, only the non-axisymmetric components of $\langle \phi \rangle$ and f_i contribute to $\mathbf{v}_E \cdot \nabla_{\mathbf{R}} f_i$ and these are of order $\delta_i T_e/e$ and $\delta_i f_{Mi}$, respectively. Writing $\mathbf{v}_E \cdot \nabla_{\mathbf{R}} f_i = -(c/B)\hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \times (f_i \nabla_{\mathbf{R}} \langle \phi \rangle)$, it is easy to see that $\mathbf{v}_E \cdot \nabla_{\mathbf{R}} f_i \sim (c/B) k_\perp f_{i,n} |\nabla_{\mathbf{R}} \langle \phi \rangle_{-n}|$, with $k_\perp$ the radial wavenumber of the axisymmetric piece of $\overline{f}_i$. Since $f_{i,n} \sim \delta_i f_{Mi}$ and $(c/B)|\nabla_{\mathbf{R}} \langle \phi \rangle_{-n}| \sim \delta_i v_i$, the largest possible size for $\mathbf{v}_E \cdot \nabla_{\mathbf{R}} f_i$ is $\delta_i k_\perp \rho_i f_{Mi} v_i / L$. Consequently, at long wavelengths, $\mathbf{v}_E \cdot \nabla_{\mathbf{R}} f_i$ is negligibly small compared with the other terms in (4.1), and the equation for the axisymmetric component of $\overline{f}_i$ is the neoclassical drift kinetic equation. For this reason, the long wavelength axisymmetric flows must be neoclassical, i.e., they are given by

$$n_i \mathbf{V}_i = -\frac{cR}{Ze}\left(\frac{\partial p_i}{\partial \psi} + Zen_i \frac{\partial \phi}{\partial \psi}\right) + U(\psi)\mathbf{B}, \tag{4.2}$$

where $U(\psi)$ is proportional to $\partial T_i/\partial \psi$ in neoclassical theory [34, 35]. For $k_\perp \rho_i \sim \delta_i$, the turbulent term $\mathbf{v}_E \cdot \nabla_\mathbf{R} f_i$ is of order $\delta_i^2 f_{Mi} v_i / L$. By comparing its size with the smaller term in the drift kinetic equation, usually the collisional operator $C\{f_i\} \simeq C^{(\ell)}\{f_i - f_{Mi}\} \sim \delta_i f_{Mi} \nu_{ii}$, I obtain that the turbulent correction to the neoclassical flow $U(\psi)$ is of order $\delta_i v_i / L\nu_{ii}$.

To finish this section, I will present the different contributions to the distribution function. To order $\delta_i f_{Mi}$, the gyrokinetic distribution function can be written as

$$f_i \equiv f_i(\mathbf{R}, E, \mu, t) \simeq f_{ig} - \frac{Ze\tilde{\phi}}{T_i} f_{Mi}, \tag{4.3}$$

where

$$f_{ig} \equiv f_i(\mathbf{R}_g, E_0, \mu_0, t) \tag{4.4}$$

and

$$\mathbf{R}_g = \mathbf{r} + \frac{1}{\Omega_i}\mathbf{v} \times \hat{\mathbf{b}}. \tag{4.5}$$

In equation (4.3), I have Taylor expanded $f_i(\mathbf{R}, E, \mu, t)$ around E_0 and μ_0, and I have used the zeroth order Maxwellian f_{Mi} in the higher order terms. In the function $\tilde{\phi}$ in (4.3), it is enough to use the lowest order variables $\mathbf{R}_g$, μ_0 and φ_0 instead of $\mathbf{R}$, μ and φ (the dependence of $\tilde{\phi}$ on E is weak). The piece f_{ig} of the distribution function will be useful in section 4.3 to obtain moment equations from the gyrokinetic equation (3.39). However, the pieces of the distribution function given in (4.3) are not useful to evaluate terms in physical phase space since the variable $\mathbf{R}_g$ still mixes spatial and velocity space variables. In physical phase space, it is useful to distinguish between the gyrophase independent piece of the distribution function $\overline{f}_i$ and the gyrophase dependent piece. At long wavelengths, $f_{ig} \equiv f_i(\mathbf{R}_g, E_0, \mu_0, t)$ can be Taylor expanded

Term	Order of magnitude
ϖ	$\delta_i k_\perp \rho_i e n_e$
$\nabla \cdot (J_{\|\|}\hat{\mathbf{b}} + \mathbf{J}_d)$	$\delta_i e n_e v_i / L$
$\nabla \cdot [(c/B)\hat{\mathbf{b}} \times (\nabla \cdot \overleftrightarrow{\pi}_i)]$	$\delta_i (k_\perp \rho_i)^2 e n_e v_i / L$

Table 4.1: Order of magnitude estimates for vorticity equation (2.9).

around $\mathbf{r}$, and I can employ the lowest order result $\widetilde{\phi} \simeq \Omega_i^{-1}(\mathbf{v} \times \hat{\mathbf{b}}) \cdot \nabla\phi$, giving

$$\overline{f}_i = f_{i0} + O(\delta_i k_\perp \rho_i f_{Mi}) \tag{4.6}$$

and

$$f_i - \overline{f}_i = \frac{1}{\Omega_i}(\mathbf{v} \times \hat{\mathbf{b}}) \cdot \left[\frac{\nabla p_i}{p_i} + \frac{Ze\nabla\phi}{T_i} + \left(\frac{Mv^2}{2T_i} - \frac{5}{2} \right) \frac{\nabla T_i}{T_i} \right] f_{Mi} + O(\delta_i k_\perp \rho_i f_{Mi}), \tag{4.7}$$

with

$$f_{i0} \equiv f_i(\mathbf{r}, E_0, \mu_0, t). \tag{4.8}$$

Notice that f_{i0} differs from f_{ig} in that the gyrocenter position $\mathbf{R}$ in $f_i(\mathbf{R}, E, \mu, t)$ is replaced by the particle position $\mathbf{r}$ and not the intermediate variable $\mathbf{R}_g$.

4.2 General vorticity equation in gyrokinetics

In this section, I evaluate the size of the different terms in the vorticity equation (2.9) and the toroidal momentum equation (2.21). For these estimates I will use the assumptions in sections 3.1 and 4.1. The estimates are summarized in table 4.1. The final result will be that the vorticity equation (2.9) is not the most convenient to evaluate the electric field and a new vorticity equation is needed.

In equation (2.9), $\varpi \sim \delta_i k_\perp \rho_i e n_e$ since $\mathbf{V}_i \sim \delta_i v_i$ and $\nabla_\perp \sim k_\perp$. Both currents $\mathbf{J}_d$ and $J_{||}$ are of order $\delta_i e n_e v_i$. The divergence of $\mathbf{J}_d$ is of order $\delta_i e n_e v_i / L$ because, according to the orderings in (3.1), $\nabla p_i \sim k_\perp p_{i,k} \sim p_{i,k=0}/L$. The divergence of $J_{||}\hat{\mathbf{b}}$ is also of order $\delta_i e n_e v_i / L$, but in this case it is due to the small parallel gradients.

With these estimates, all the terms $\partial\varpi/\partial t$, $\nabla \cdot \mathbf{J}_d$ and $\nabla \cdot (J_{||}\hat{\mathbf{b}})$ compete with each other to determine the electric field. The remaining term, $\nabla \cdot [(c/B)\hat{\mathbf{b}} \times (\nabla \cdot \overleftrightarrow{\boldsymbol{\pi}}_i)]$, is more difficult to evaluate.

The ion viscosity $\overleftrightarrow{\boldsymbol{\pi}}_i$, given by (2.6), is of order $O(\delta_i k_\perp \rho_i p_i)$. It only depends on the gyrophase dependent part of the distribution function, and at long wavelengths, the gyrophase dependent piece is given by (4.7), making $\overleftrightarrow{\boldsymbol{\pi}}_i \sim \delta_i k_\perp \rho_i p_i$ because the lowest order gyrophase dependent piece vanishes. The estimate is also valid for electrons, for which it is assumed that the shortest wavelength is of the order of the ion gyroradius, giving an electron viscosity m/M times smaller than the ion viscosity, thereby justifying its neglect.

With this estimate for $\overleftrightarrow{\boldsymbol{\pi}}_i$, the term $\nabla \cdot [(c/B)\hat{\mathbf{b}} \times (\nabla \cdot \overleftrightarrow{\boldsymbol{\pi}}_i)]$ in (2.9) is formally of order $(k_\perp \rho_i)^3 e n_e v_i/L$, while the rest of the terms are of order $\delta_i e n_e v_i/L$. However, I will prove in section 4.4 that the formal estimate is too high, and in reality

$$\nabla \cdot \left[\frac{c}{B}\hat{\mathbf{b}} \times (\nabla \cdot \overleftrightarrow{\boldsymbol{\pi}}_i)\right] \sim \delta_i (k_\perp \rho_i)^2 e n_e v_i/L. \tag{4.9}$$

This term is the only one that enters in the equation for the radial electric field, as proven by (2.21). Since it becomes small for long wavelengths, the vorticity equation and hence its time integral, quasineutrality, are almost independent of the long wavelength radial electric field, i.e., the tokamak is intrinsically ambipolar even in the presence of turbulence. The order of magnitude estimate in equation (4.9) is the proof of intrinsic ambipolarity.

Except for the flux surface averaged vorticity equation, dominated by the term $\nabla \cdot [(c/B)\hat{\mathbf{b}} \times (\nabla \cdot \overleftrightarrow{\boldsymbol{\pi}}_i)] \sim \delta_i (k_\perp \rho_i)^2 e n_e v_i/L$, the lower order terms in the vorticity equation are always of order $\delta_i e n_e v_i/L$. Then, since the vorticity ϖ is $O(\delta_i k_\perp \rho_i e n_e)$, becoming small for the longer wavelengths, the time evolution of equation (2.9) requires a decreasing time step as $k_\perp \rho_i \to 0$, or an implicit numerical method that will ensure that the right side of (2.9) vanishes for long wavelengths. Solving implicitly for the potential is routinely done in gyrokinetic simulations [3, 62].

The flux surface averaged vorticity equation gives toroidal angular momentum conservation equation (2.21). According to the estimates in previous paragraphs, the radial toroidal viscosity would be $\langle R\hat{\boldsymbol{\zeta}}\cdot \overleftrightarrow{\boldsymbol{\pi}}_i \cdot\nabla\psi\rangle_\psi \sim \delta_i k_\perp \rho_i p_i R|\nabla\psi|$. Then, the characteristic time derivative for the toroidal velocity is $\partial/\partial t \sim k_\perp \rho_i v_i/L$. This estimate is in contradiction with neoclassical calculations [39, 40, 42, 43, 44] and the random walk estimate at the end of section 2.3, where the long wavelength toroidal-radial component of viscosity is found to be of order $\delta_i^3 p_i R|\nabla\psi|$. This discrepancy probably only occurs transiently for short periods of time. It is to be expected that for longer times, the transport of toroidal angular momentum $\langle R\hat{\boldsymbol{\zeta}}\cdot \overleftrightarrow{\boldsymbol{\pi}}_i \cdot\nabla\psi\rangle_\psi \sim \delta_i k_\perp \rho_i p_i R|\nabla\psi|$ gives a net zero contribution, and the time averaged toroidal-radial component of viscosity is actually of order $\delta_i^3 p_i R|\nabla\psi|$. The size of the time averaged $\langle R\hat{\boldsymbol{\zeta}}\cdot \overleftrightarrow{\boldsymbol{\pi}}_i \cdot\nabla\psi\rangle_\psi$ is discussed in chapter 5.

To summarize, the vorticity equation (2.9) has the right physics, and makes explicit the different times scales (from the fast turbulence times to the slow radial transport time). However, I show in section 4.4 that the divergence of $(c/B)\hat{\mathbf{b}} \times (\nabla\cdot \overleftrightarrow{\boldsymbol{\pi}}_i)$ is an order smaller than its formal estimate suggests for $k_\perp \rho_i \ll 1$, going from $(k_\perp \rho_i)^3 e n_i v_i/L$ to $\delta_i (k_\perp \rho_i)^2 e n_i v_i/L$. In other words, $(c/B)\hat{\mathbf{b}} \times (\nabla\cdot \overleftrightarrow{\boldsymbol{\pi}}_i)$ has a large divergence free piece. This difference between the real size and the formal ordering makes theoretical studies cumbersome and it may lead to numerical problems upon implementation. The rest of this chapter is devoted to finding a more convenient vorticity equation. In section 4.3, I will derive the particle and momentum conservation equations from the gyrokinetic equation (3.39). In the same way that particle and momentum conservation equations were used in section 2.2, I will employ the gyrokinetic conservation equations to find two gyrokinetic vorticity equations in section 4.4.

4.3 Transport in gyrokinetics

It is necessary to understand the transport of particles and momentum at wavelengths that are of the order of the ion gyroradius. It is at those wavelengths that the divergence of the viscosity becomes as important as the gradient of pressure, and we need to determine which one dominates in the vorticity equation. The gyrokinetic

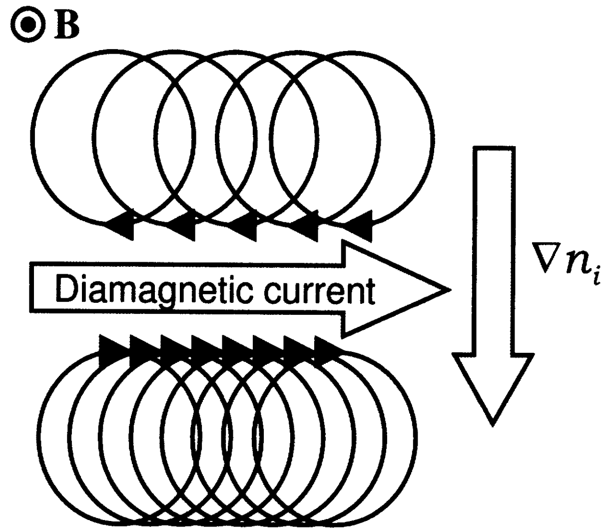

Figure 4-1: Ion diamagnetic flow due to a gradient in ion density.

equation (3.39) is especially well suited to this task. In this section, I derive moment equations from the gyrokinetic Fokker-Planck equation, in particular, conservation equations for particles and momentum. They will provide powerful insights, but we have to remember that the gyrokinetic equation is correct only to $O(\delta_i f_{Mi} v_i/L)$. Then, the conservation equations for particles and momentum are missing terms of order $\delta_i^2 n_e v_i/L$ and $\delta_i^2 p_i/L$.

With the conservation equations for particles and momentum derived in this section, I will obtain two gyrokinetic vorticity equations in section 4.4. These new vorticity equations will be equivalent to equation (2.9) up to, but not including, $O(\delta_i^2 e n_e v_i/L)$. They will have the advantage of explicitly cancelling the problematic divergence free component of the current density $(c/B)\hat{\mathbf{b}} \times (\nabla \cdot \overleftrightarrow{\pi}_i)$ discussed in section 4.2. The simplification originates in the fact that the ion distribution function is gyrophase independent in gyrokinetic variables. Physically, ions are replaced by rings of charge, thereby eliminating divergence free terms due to the particle gyromotion. A simple example is the diamagnetic current $-\nabla \times (cp_{i\perp}\hat{\mathbf{b}}/B)$. The physical mechanism responsible for the ion diamagnetic flow in the presence of a gradient of density is sketched in figure 4-1. In the figure, the ion density is higher in the bottom half. Then, due to the gyration, in the middle of the figure there are more ions moving towards the right than towards the left, giving a divergence free ion flow. Its divergence vanishes because the ions just gyrate around the fixed guiding centers and there is no net ion motion. In gyrokinetics, the gyromotion velocities are not considered because the gyromotion is replaced by rings of charge, removing the divergence free terms automatically and leaving only the net gyrocenter drifts.

The conservation equations for particles and momentum are of order $\delta_i n_e v_i/L$ and $\delta_i p_i/L$, respectively, and they miss terms of order $\delta_i^2 n_e v_i/L$ and $\delta_i^2 p_i/L$. In these equations, it is possible to study what happens for wavelengths longer than the ion gyroradius, $k_\perp \rho_i \ll 1$. Different terms will have different scalings in $k_\perp \rho_i$, and these scalings will define which terms dominate at longer wavelengths. For this reason, I will determine the scalings along with the conservation equations. It is important to keep in mind that there are missing terms of order $\delta_i^2 n_e v_i/L$ and $\delta_i^2 p_i/L$, and any terms from a subsidiary expansion in $k_\perp \rho_i$ are not meaningful in this limit. In particular, I will show in section 4.4 that the terms that determine the axisymmetric radial electric field are too small at long wavelengths to be determined by this subsidiary expansion.

In this section, I present the general method to obtain conservation equations from gyrokinetics. In subsection 4.3.1, I derive the gyrokinetic equation in the physical phase space variables $\mathbf{r}$, E_0, μ_0 and φ_0, and I write it in a conservative form that is convenient for deriving moment equations. The details of the calculation are contained in Appendix F. In subsection 4.3.2, I derive the general moment equation for a quantity $G(\mathbf{r}, \mathbf{v}, t)$. I will apply this general equation to obtain particle and momentum transport in subsections 4.3.3 and 4.3.4, respectively. The details of the calculations are in Appendix G. In Appendix H, I show how to treat the effect of the finite gyroradius on collisions.

4.3.1 Gyrokinetic equation in physical phase space

The distribution function shows a simpler structure when written in gyrokinetic variables, namely, it is independent of the gyrophase except for the piece in (3.38) responsible for classical collisional transport. The goal of this subsection is writing the Fokker-Planck equation, $df_i/dt = C\{f_i\}$, in the physical phase space variables $\mathbf{r}$, E_0, μ_0 and φ_0, while preserving the simple form obtained by employing the gyrokinetic

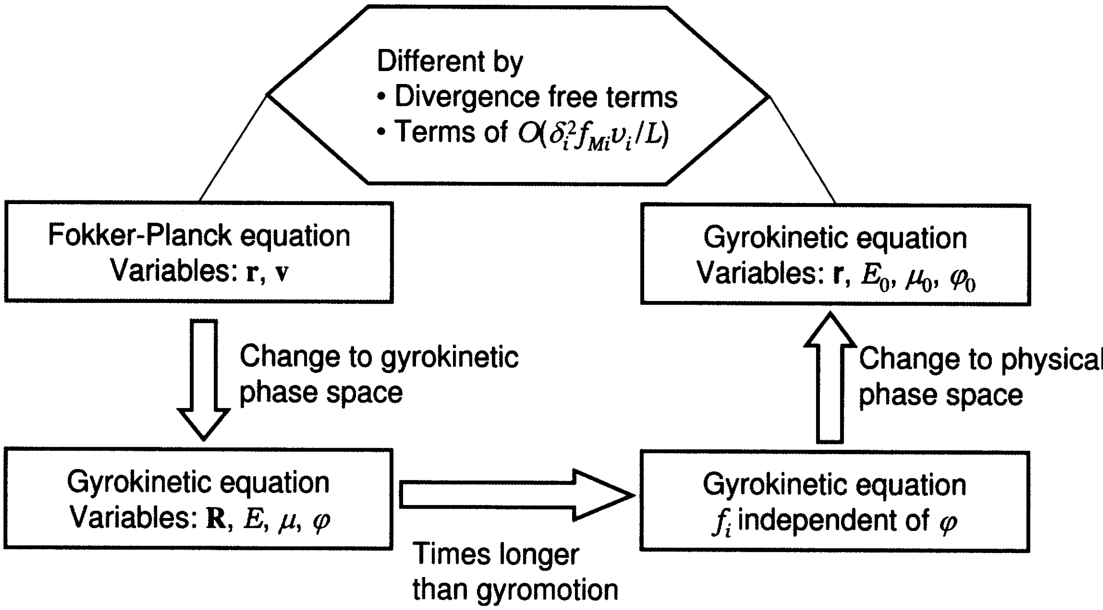

Figure 4-2: Gyrokinetic equation in physical phase space.

variables. The relation between the full Fokker-Planck equation and the gyrokinetic equation written in physical phase space variables is sketched in figure 4-2. They differ in two aspects. On the one hand, the gyrokinetic equation (3.39) misses terms of order $\delta_i^2 f_{Mi} v_i/L$. On the other hand, gyrokinetics is not only a change of variables, but it also implies a time scale separation between the gyromotion and the evolution of the slowly varying electrostatic potential. The ion distribution function is then gyrophase independent in the gyrokinetic phase space, i.e., the motion of the particles may be replaced by drifting rings of charge. The advantage of this distribution function is the lack of divergence free terms in the moment equations constructed from it, as explained in the introduction to this section.

I write the Fokker-Planck equation to order $\delta_i f_{Mi} v_i/L$, the order to which the gyrokinetic equation is deduced, by starting with

$$\frac{df_i}{dt} \equiv \left.\frac{\partial f_i}{\partial t}\right|_{\mathbf{r},\mathbf{v}} + \mathbf{v}\cdot\nabla f_i + \mathbf{a}\cdot\nabla_v f_i \simeq \left.\frac{\partial f_i}{\partial t}\right|_{\mathbf{R},E,\mu,\varphi} + \dot{\mathbf{R}}\cdot\nabla_{\mathbf{R}} f_i + \dot{E}\frac{\partial f_i}{\partial E} + \dot{\varphi}\frac{\partial f_i}{\partial \varphi}, \quad (4.10)$$

where $\mathbf{a} = -Ze\nabla\phi/M + \Omega_i(\mathbf{v}\times\hat{\mathbf{b}})$ is the acceleration of particles and I have written the Vlasov operator d/dt in both $\mathbf{r}$, $\mathbf{v}$ and gyrokinetic variables. The term $\dot{\mu}(\partial f_i/\partial\mu)$ does not appear in equation (4.10) because I assume that f_i is a stationary Maxwellian to zeroth order and $\dot{\mu}$ is small by definition of μ. The derivative respect to the gyrokinetic gyrophase φ is small and related to the collision operator by (3.38),

$$\dot{\varphi}\frac{\partial f_i}{\partial \varphi} = C\{f_i\} - \langle C\{f_i\}\rangle. \tag{4.11}$$

The difference between time derivatives of f_i can be written as

$$\left.\frac{\partial f_i}{\partial t}\right|_{\mathbf{R},E,\mu,\varphi} - \left.\frac{\partial f_i}{\partial t}\right|_{\mathbf{r},\mathbf{v}} \simeq -\frac{Ze}{M}\left.\frac{\partial \tilde{\phi}}{\partial t}\right|_{\mathbf{r},\mathbf{v}}\frac{\partial f_i}{\partial E}, \tag{4.12}$$

where I have employed (3.25) and that $\mathbf{B}$ is independent of time. Combining equations (4.10), (4.11) and (4.12), the Fokker-Planck equation $df_i/dt = C\{f_i\}$ becomes, to $O(\delta_i f_{Mi} v_i/L)$,

$$\left.\frac{\partial f_i}{\partial t}\right|_{\mathbf{r},\mathbf{v}} + \left.\frac{\partial}{\partial t}\left(\frac{Ze\tilde{\phi}}{T_i}f_{Mi}\right)\right|_{\mathbf{r},\mathbf{v}} + \dot{\mathbf{R}}\cdot\nabla_{\mathbf{R}} f_i + \dot{E}\frac{\partial f_i}{\partial E} = \langle C\{f_i\}\rangle. \tag{4.13}$$

Here, I have used that the time evolution of $\partial f_i/\partial E \simeq (-M/T_i) f_{Mi}$ is slow.

It is necessary to rewrite equation (4.13) in the variables $\mathbf{r}$, E_0, μ_0 and φ_0. Using equation (4.3) and considering that both the zeroth order distribution function and the zeroth order potential are almost constant along magnetic field lines, I can rewrite part of equation (4.13) in terms of the variables $\mathbf{r}$, E_0, μ_0 and φ_0. The details are in section F.1 of Appendix F, and the final result is

$$\dot{\mathbf{R}}\cdot\nabla_{\mathbf{R}} f_i + \dot{E}\frac{\partial f_i}{\partial E} \simeq \dot{\mathbf{R}}\cdot\nabla_{\mathbf{R}_g}\mathbf{r}\cdot\left(\overline{\nabla} f_{ig} - \frac{Ze}{M}\overline{\nabla}\langle\phi\rangle\frac{\partial f_{Mi}}{\partial E_0}\right), \tag{4.14}$$

where f_{ig} is missing the piece proportional to $\tilde{\phi}$ [see equation (4.3)]. The gradient $\nabla_{\mathbf{R}_g}$ is taken with respect to $\mathbf{R}_g$ holding E_0, μ_0, φ_0 and t fixed, and $\overline{\nabla}$ is the gradient with respect to $\mathbf{r}$ holding E_0, μ_0, φ_0 and t fixed. The quantity $\dot{\mathbf{R}}\cdot\nabla_{\mathbf{R}_g}\mathbf{r}$ is given by

$$\dot{\mathbf{R}}\cdot\nabla_{\mathbf{R}_g}\mathbf{r} = v_{||0}\hat{\mathbf{b}} + \mathbf{v}_{M0} + \mathbf{v}_{E0} + \tilde{\mathbf{v}}_1, \tag{4.15}$$

with

$$v_{||0} = v_{||} + \frac{v_\perp^2}{2\Omega_i}\hat{\mathbf{b}}\cdot\nabla\times\hat{\mathbf{b}} + \frac{2v_{||}}{\Omega_i}\boldsymbol{\kappa}\cdot(\mathbf{v}\times\hat{\mathbf{b}}) - \frac{v_{||}}{2B\Omega_i}(\mathbf{v}\times\hat{\mathbf{b}})\cdot\nabla B$$
$$+\frac{v_{||}}{\Omega_i}\mathbf{v}_\perp\cdot\nabla\hat{\mathbf{e}}_2\cdot\hat{\mathbf{e}}_1 + \frac{1}{4\Omega_i}[\mathbf{v}_\perp(\mathbf{v}\times\hat{\mathbf{b}}) + (\mathbf{v}\times\hat{\mathbf{b}})\mathbf{v}_\perp] : \nabla\hat{\mathbf{b}}, \qquad (4.16)$$

$$\mathbf{v}_{M0} = \frac{\mu_0}{\Omega_i}\hat{\mathbf{b}}\times\nabla B + \frac{v_{||}^2}{\Omega_i}\hat{\mathbf{b}}\times\boldsymbol{\kappa}, \qquad (4.17)$$

$$\mathbf{v}_{E0} = -\frac{c}{B}\overline{\nabla}\langle\phi\rangle\times\hat{\mathbf{b}} \qquad (4.18)$$

and using equation (F.6)

$$\tilde{\mathbf{v}}_1 = \frac{v_{||}}{\Omega_i}\overline{\nabla}\times\mathbf{v}_\perp. \qquad (4.19)$$

In equation (4.14), it is important to be aware of higher order terms (like $\mathbf{v}_{M0}\cdot\overline{\nabla}f_{ig}$), in which the full distribution function, not just f_{Mi}, must be retained. In these terms, the steep perpendicular gradients make the higher order pieces of the distribution function important [recall the orderings in (3.1)].

Equation (4.14) can be written in conservative form, more convenient for transport calculations. The details of this calculation are in section F.2 of Appendix F, and the result is

$$\dot{\mathbf{R}}\cdot\nabla_{\mathbf{R}}f_i + \dot{E}\frac{\partial f_i}{\partial E} \simeq \frac{v_{||}}{B}\left[\overline{\nabla}\cdot\left(\frac{B}{v_{||}}f_{ig}\dot{\mathbf{R}}\cdot\nabla_{\mathbf{R}_g}\mathbf{r}\right) - \frac{\partial}{\partial\mu_0}\left(f_{Mi}\mathbf{B}\cdot\overline{\nabla}\mu_{10}\right)\right.$$
$$\left. -\frac{\partial}{\partial\varphi_0}\left(f_{Mi}\mathbf{B}\cdot\overline{\nabla}\varphi_{10}\right) - \frac{\partial}{\partial E_0}\left(\frac{B}{v_{||}}f_{Mi}\frac{Ze}{M}\dot{\mathbf{R}}\cdot\nabla_{\mathbf{R}_g}\mathbf{r}\cdot\overline{\nabla}\langle\phi\rangle\right)\right], \qquad (4.20)$$

where $B/v_{||}$ is the Jacobian $\partial(\mathbf{v})/\partial(E_0, \mu, \varphi_0)$, and the quantities μ_{10} and φ_{10} are the pieces of the first order corrections μ_1 and φ_1 that do not depend on the potential. They are given by

$$\mu_{10} = \mu_1 - \frac{Ze\tilde{\phi}}{MB} \qquad (4.21)$$

and

$$\varphi_{10} = \varphi_1 + \frac{Ze}{MB}\frac{\partial\tilde{\Phi}}{\partial\mu}. \qquad (4.22)$$

The definitions of φ_1 and μ_1 are in equations (3.29) and (3.34), respectively.

Finally, substituting equation (4.20) into equation (4.13), I find

$$\left.\frac{\partial f_{ig}}{\partial t}\right|_{\mathbf{r},\mathbf{v}} + \frac{v_{||}}{B}\left[\overline{\nabla}\cdot\left(\frac{B}{v_{||}} f_{ig}\dot{\mathbf{R}}\cdot\nabla_{\mathbf{R}_g}\mathbf{r}\right) - \frac{\partial}{\partial \mu_0}\left(f_{Mi}\mathbf{B}\cdot\overline{\nabla}\mu_{10}\right)\right.$$
$$\left. - \frac{\partial}{\partial \varphi_0}\left(f_{Mi}\mathbf{B}\cdot\overline{\nabla}\varphi_{10}\right) - \frac{\partial}{\partial E_0}\left(\frac{B}{v_{||}} f_{Mi}\frac{Ze}{M}\dot{\mathbf{R}}\cdot\nabla_{\mathbf{R}_g}\mathbf{r}\cdot\overline{\nabla}\langle\phi\rangle\right)\right] = \langle C\{f_i\}\rangle. \quad (4.23)$$

Here, for $\langle\phi\rangle$ and $\widetilde{\phi}$, it is enough to consider the dependence on the lowest order variables, i.e., $\mathbf{R}_g$, μ_0 and φ_0 (the dependence of $\langle\phi\rangle$ and $\widetilde{\phi}$ on E is weak).

4.3.2 Transport of a general function $G(\mathbf{r},\mathbf{v},t)$ at $k_\perp\rho_i \sim 1$

Multiplying equation (4.23) by a function $G(\mathbf{r},\mathbf{v},t)$ and integrating over velocity space, I find the conservation equation for that function G to be

$$\frac{\partial}{\partial t}\left(\int d^3v\, G f_{ig}\right) + \nabla\cdot\left[\int d^3v\, f_{ig}\left(\dot{\mathbf{R}}\cdot\nabla_{\mathbf{R}_g}\mathbf{r}\right)G\right] = \int d^3v\, f_{ig}K\{G\}$$
$$+ \int d^3v\, G\langle C\{f_i\}\rangle, \quad (4.24)$$

with

$$K\{G\} = \left.\frac{\partial G}{\partial t}\right|_{\mathbf{r},\mathbf{v}} + \dot{\mathbf{R}}\cdot\nabla_{\mathbf{R}_g}\mathbf{r}\cdot\left(\overline{\nabla}G - \frac{Ze}{M}\overline{\nabla}\langle\phi\rangle\frac{\partial G}{\partial E_0}\right)$$
$$-v_{||}\hat{\mathbf{b}}\cdot\left(\overline{\nabla}\mu_{10}\frac{\partial G}{\partial \mu_0} + \overline{\nabla}\varphi_{10}\frac{\partial G}{\partial \varphi_0}\right). \quad (4.25)$$

In the next two subsections, I will use this formalism to study the transport of particles and momentum at short wavelengths.

4.3.3 Transport of particles at $k_\perp\rho_i \sim 1$

Particle transport for electrons is easy to obtain since I only need to consider $k_\perp\rho_e \ll 1$. In this limit, drift kinetics is valid giving

$$\frac{\partial n_e}{\partial t} + \nabla\cdot\left(n_e V_{e||}\hat{\mathbf{b}} + n_e\mathbf{V}_{ed} - \frac{cn_e}{B}\nabla\phi\times\hat{\mathbf{b}}\right) = 0, \quad (4.26)$$

with $n_e \mathbf{V}_{e||} = \int d^3v\, f_e v_{||}$ and

$$n_e \mathbf{V}_{ed} = -\frac{cp_e}{eB}\hat{\mathbf{b}}\hat{\mathbf{b}} \cdot \nabla \times \hat{\mathbf{b}} - \frac{cp_e}{eB^2}\hat{\mathbf{b}} \times \nabla B - \frac{cp_e}{eB}\hat{\mathbf{b}} \times \boldsymbol{\kappa}. \tag{4.27}$$

The same result may be deduced from equation (4.24) by neglecting electron pressure anisotropy terms that are small by a factor $\sqrt{m/M}$ compared to the ion pressure anisotropy. Obviously, the ion particle transport must be exactly the same as for the electrons due to quasineutrality. Nonetheless, we still must obtain the particle transport equation for ions to be able to calculate the electric field by requiring that both ions and electrons have the same density.

The conservation equation for ion particle number is given by equation (4.24) with $G = 1$. Employing section G.1 of Appendix G, it can be written as

$$\frac{\partial}{\partial t}(n_i - n_{ip}) + \nabla \cdot \left(n_i V_{i||}\hat{\mathbf{b}} + n_i \mathbf{V}_{igd} + n_i \mathbf{V}_{iE} + n_i \tilde{\mathbf{V}}_i + n_i \mathbf{V}_{iC} \right) = 0, \tag{4.28}$$

where n_{ip} is the polarization density, defined in (3.55), the parallel flow is

$$n_i V_{i||} = \int d^3v\, f_i v_{||} = \int d^3v\, f_{ig} v_{||}, \tag{4.29}$$

the term $n_i \tilde{\mathbf{V}}_i$ is a perpendicular flow that originates in finite gyroradius effects, given by

$$n_i \tilde{\mathbf{V}}_i = \int d^3v\, f_{ig} \tilde{\mathbf{v}}_1 = \int d^3v\, f_{ig} \frac{v_{||}}{\Omega_i} \overline{\nabla} \times \mathbf{v}_\perp, \tag{4.30}$$

and the flows due to the $E \times B$ and magnetic drifts are

$$n_i \mathbf{V}_{iE} = \int d^3v\, f_{ig} \mathbf{v}_{E0} = -\frac{c}{B} \int d^3v\, f_{ig} \overline{\nabla} \langle \phi \rangle \times \hat{\mathbf{b}} \tag{4.31}$$

and

$$n_i \mathbf{V}_{igd} = \frac{cp_{ig\perp}}{ZeB}\hat{\mathbf{b}}\hat{\mathbf{b}} \cdot \nabla \times \hat{\mathbf{b}} + \frac{cp_{ig\perp}}{ZeB^2}\hat{\mathbf{b}} \times \nabla B + \frac{cp_{ig||}}{ZeB}\hat{\mathbf{b}} \times \boldsymbol{\kappa}, \tag{4.32}$$

with $p_{ig||} = \int d^3v\, f_{ig} M v_{||}^2$ and $p_{ig\perp} = \int d^3v\, f_{ig} M v_\perp^2/2$. The collisional flow $n_i \mathbf{V}_{iC}$ is evaluated in Appendix H and is caused by ion-ion collisions due to finite gyroradius

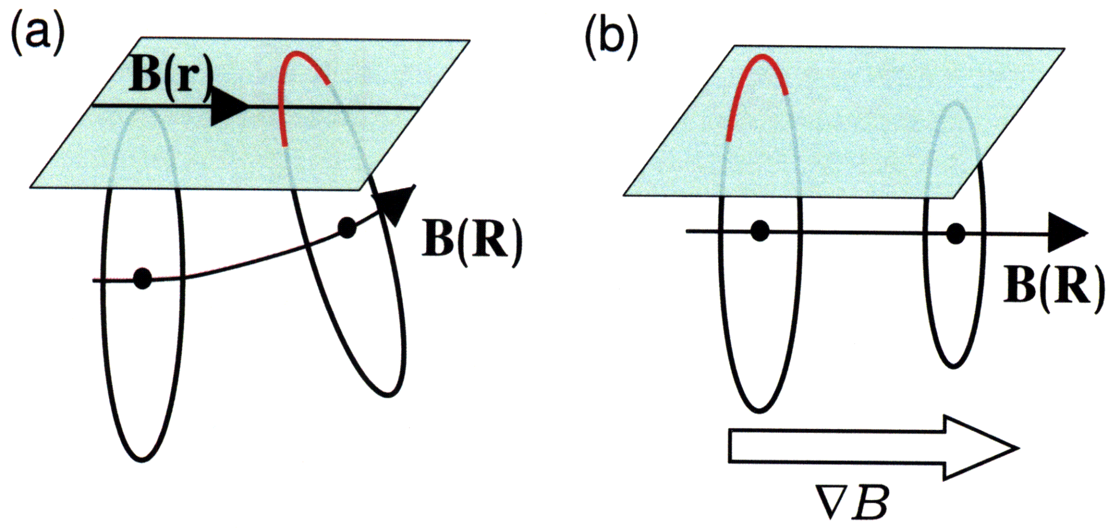

Figure 4-3: Finite gyroradius effects in $n_i \tilde{\mathbf{V}}_i$. (a) The magnetic field line along which the gyrocenter lies is the line that guides the parallel motion, making it possible for parallel motion to transport particles across magnetic field lines. (b) As the size of the gyromotion changes, a particle that spent time on both sides of the blue plane is now only on one of its sides, leading to an effective particle transport.

effects. It is given by

$$n_i \mathbf{V}_{iC} = -\frac{\gamma}{\Omega_i} \int d^3v \left(\langle \mathbf{\Gamma} \rangle \times \hat{\mathbf{b}} - \frac{1}{v_\perp^2} \langle \mathbf{\Gamma} \cdot \mathbf{v}_\perp \rangle \mathbf{v} \times \hat{\mathbf{b}} \right), \tag{4.33}$$

with $\gamma = 2\pi Z^4 e^4 \ln \Lambda / M^2$ and

$$\mathbf{\Gamma} = \int d^3v' \, f_{Mi} f'_{Mi} \nabla_g \nabla_g g \cdot \left[\nabla_v \left(\frac{f_i}{f_{Mi}} \right) - \nabla_{v'} \left(\frac{f'_i}{f'_{Mi}} \right) \right]. \tag{4.34}$$

Here, $f = f(\mathbf{v})$, $f' = f(\mathbf{v}')$, $\mathbf{g} = \mathbf{v} - \mathbf{v}'$, $g = |\mathbf{g}|$ and $\nabla_g \nabla_g g = (g^2 \overleftrightarrow{\mathbf{I}} - \mathbf{gg})/g^3$.

In the presence of potential structure on the order of the ion gyroradius, the contributions to $n_i \tilde{\mathbf{V}}_i$ no longer average to zero in a gyration since they can add coherently. In the integration $n_i \tilde{\mathbf{V}}_i = \int d^3v \, f_{ig} (v_{||}/\Omega_i) \overline{\nabla} \times \mathbf{v}_\perp$ only two terms contribute to its divergence so that $\nabla \cdot (n_i \tilde{\mathbf{V}}_i) = \nabla \cdot (n_i \tilde{\mathbf{V}}_{i0})$ with

$$n_i \tilde{\mathbf{V}}_{i0} = \int d^3v \, f_{ig} \left[\frac{v_{||}}{\Omega_i} (\mathbf{v} \times \hat{\mathbf{b}}) \cdot \nabla \hat{\mathbf{b}} + \frac{v_{||} \hat{\mathbf{b}} \cdot \nabla B}{2B\Omega_i} \mathbf{v} \times \hat{\mathbf{b}} \right]. \tag{4.35}$$

In section G.1 of Appendix G, I prove that all the other terms in $\tilde{\mathbf{v}}_1$ can be neglected.

Term	Order of magnitude
n_{ip}	$\delta_i k_\perp \rho_i n_e$
$\nabla \cdot (n_i V_{i\|\|} \hat{\mathbf{b}} + n_i \mathbf{V}_{igd} + n_i \mathbf{V}_{iE})$	$\delta_i n_e v_i / L$
$\nabla \cdot (n_i \tilde{\mathbf{V}}_i)$	$\delta_i (k_\perp \rho_i)^2 n_e v_i / L$
$\nabla \cdot (n_i \mathbf{V}_{iC})$	$\delta_i (k_\perp \rho_i)^2 n_e \nu_{ii}$

Table 4.2: Order of magnitude estimates for ion particle conservation equation (4.28).

The physical origin of $n_i \tilde{\mathbf{V}}_{i0}$ is sketched in figure 4-3. The drift $(v_{||}/\Omega_i)(\mathbf{v} \times \hat{\mathbf{b}}) \cdot \nabla \hat{\mathbf{b}}$ is presented in figure 4-3(a), where there is difference between the direction of the magnetic field at the gyrocenter $\hat{\mathbf{b}}(\mathbf{R})$, and the direction of the magnetic field at the real position of the particle $\hat{\mathbf{b}}(\mathbf{r})$. Due to this difference, the parallel motion of the gyrocenter drives part of the gyromotion, plotted in red in the figure, across the blue plane. Notice that the magnetic field line $\mathbf{B}(\mathbf{r})$ lies in the plane, leading to the "paradoxical" parallel motion across magnetic field lines. The other term in $n_i \tilde{\mathbf{V}}_{i0}$, $(v_{||} \hat{\mathbf{b}} \cdot \nabla B / 2B\Omega_i) \mathbf{v} \times \hat{\mathbf{b}}$, is explained in figure 4-3(b). Here, the change in the size of the gyroradius due to the change in magnitude of the magnetic field, $dB/dt = v_{||} \hat{\mathbf{b}} \cdot \nabla B$, drives part of the gyromotion, plotted in red, across the blue plane.

In section 4.4, I will obtain a vorticity equation for ϕ by imposing $Zn_i = n_e$. Equations (4.26) and (4.28) will provide the time evolution of $Zn_i - n_e$. It will be useful to know the size of the different terms in equation (4.28). I summarize the estimates of order of magnitude in table 4.2. These estimates include the scaling with $k_\perp \rho_i$ for long wavelengths. The wavenumber $\mathbf{k}_\perp$ is the overall perpendicular wavenumber, i.e., given $\mathbf{k}_\perp$, the corresponding Fourier component for nonlinear terms like $A(\mathbf{r}) \times B(\mathbf{r})$ is $\int d^2k'_\perp \tilde{A}(\mathbf{k}'_\perp) \times \tilde{B}(\mathbf{k}_\perp - \mathbf{k}'_\perp)$, with $\tilde{A}$ and $\tilde{B}$ the Fourier transforms of functions A and B. The divergence of the drift flow, $\nabla \cdot (n_i \mathbf{V}_{igd} + n_i \mathbf{V}_{iE})$, is of order $\delta_i n_e v_i / L$ since $\overline{\nabla} f_{Mi} \sim \overline{\nabla} f_{ik}$ [recall (3.1)]. The flow $n_i \tilde{\mathbf{V}}_i$ is of order $\delta_i^2 k_\perp \rho_i n_e v_i$ because, for $k_\perp \rho_i \ll 1$, the gyrophase dependent piece of f_{ig}, similar to (4.7), is even in $v_{||}$ to zeroth order, making the integral vanish. Its divergence, $\nabla \cdot (n_i \tilde{\mathbf{V}}_i)$, is then of order $\delta_i (k_\perp \rho_i)^2 n_e v_i / L$. The divergence of the collisional flow, $\nabla \cdot (n_i \mathbf{V}_{iC})$, is of order $\delta_i (k_\perp \rho_i)^2 \nu_{ii} n_e$, as proven in Appendix H. The polarization density n_{ip} is of order $\delta_i k_\perp \rho_i n_e$, as shown in (3.59). This result means that for long wavelengths, the polarization term becomes unimportant. Therefore, at long wavelengths only the balance between the time evolution of the density, the parallel flow, and the magnetic and $E \times B$ drifts matter.

4.3.4 Transport of momentum at $k_\perp \rho_i \sim 1$

From electron momentum conservation, I will only need the parallel component, given by

$$\hat{\mathbf{b}} \cdot \nabla p_{e||} + (p_{e||} - p_{e\perp}) \nabla \cdot \hat{\mathbf{b}} = e n_e \hat{\mathbf{b}} \cdot \nabla \phi + F_{ei||}, \tag{4.36}$$

where $F_{ei||} = m \int d^3v \, v_{||} C_{ei}\{f_e\}$ is the collisional parallel momentum exchange. The time derivative of $n_e m \mathbf{V}_e$ and the viscosity have been neglected because they are small by a factor of m/M. In this equation, there are terms of two different orders of magnitude. The dominant terms are $\hat{\mathbf{b}} \cdot \nabla p_{e||} \simeq \hat{\mathbf{b}} \cdot \nabla p_e$ and $e n_e \hat{\mathbf{b}} \cdot \nabla \phi$, both of order $O(\delta_i p_e / L)$ for turbulent fluctuations at $k_\perp \rho_i \sim 1$. The friction force $F_{ei||}$ and the terms that contain the pressure anisotropy $p_{e||} - p_{e\perp} \sim \delta_e p_e$ are an order $\sqrt{m/M}$ smaller than the dominant terms. However, these smaller terms are crucial because they provide the non-adiabatic behavior and hence allow radial transport of particles. In the vorticity equation (2.9), the non-adiabatic electron response is kept in the integral $\int d^3v \, f_e v_{||}$ in $J_{||}$. Thus, for most purposes, equation (4.36) can be simplified to

$$\hat{\mathbf{b}} \cdot \nabla p_e = e n_e \hat{\mathbf{b}} \cdot \nabla \phi. \tag{4.37}$$

Equation (4.36) can also be recovered by using equation (4.24) and neglecting terms small by m/M.

For the ions, using equation (4.24) with $G = \mathbf{v}$ and employing section G.2 of Appendix G, I find the momentum conservation equation

$$\frac{\partial}{\partial t} (n_i M \mathbf{V}_{ig}) + \hat{\mathbf{b}} [\hat{\mathbf{b}} \cdot \nabla p_{ig||} + (p_{ig||} - p_{ig\perp}) \nabla \cdot \hat{\mathbf{b}} + \nabla \cdot \boldsymbol{\pi}_{ig||}] + \nabla \cdot \overleftrightarrow{\boldsymbol{\pi}}_{ig\times} = -Z e n_i \hat{\mathbf{b}} \hat{\mathbf{b}} \cdot \nabla \phi + \tilde{F}_{iE} \hat{\mathbf{b}} + \mathbf{F}_{iB} + \mathbf{F}_{iC}, \tag{4.38}$$

where $n_i \mathbf{V}_{ig} = \int d^3v \, f_{ig} \mathbf{v}$ is the average gyrocenter velocity; the vector $\boldsymbol{\pi}_{ig||}$ is the parallel momentum transported by the drifts and is given by

$$\boldsymbol{\pi}_{ig||} = \int d^3v \, f_{ig}(\mathbf{v}_{M0} + \mathbf{v}_{E0} + \tilde{\mathbf{v}}_1) M v_{||}; \tag{4.39}$$

the tensor $\overleftrightarrow{\boldsymbol{\pi}}_{ig\times}$ gives the transport of perpendicular momentum by the parallel velocity and the drifts,

$$\overleftrightarrow{\boldsymbol{\pi}}_{ig\times} = \int d^3v \, f_{ig}(v_{||}\hat{\mathbf{b}} + \mathbf{v}_{M0} + \mathbf{v}_{E0} + \tilde{\mathbf{v}}_1) M \mathbf{v}_\perp; \tag{4.40}$$

the vector $\tilde{F}_{iE}\hat{\mathbf{b}}$ is a correction due to the short wavelengths of the electric field with

$$\tilde{F}_{iE} = Ze \int d^3v \, f_{Mi}(\hat{\mathbf{b}} + \Omega_i^{-1} \nabla \times \mathbf{v}_\perp) \cdot \overline{\nabla \tilde{\phi}}; \tag{4.41}$$

the vector $\mathbf{F}_{iB}$ contains the effect on the gyromotion of the variation in the magnetic field and is given by

$$\mathbf{F}_{iB} = \int d^3v \, M f_{ig} v_{||} \hat{\mathbf{b}} \cdot \overline{\nabla \mathbf{v}_\perp} + \int d^3v \, \frac{Mc}{B} f_{Mi} (\overline{\nabla \tilde{\phi} \times \hat{\mathbf{b}}) \cdot \nabla \mathbf{v}_\perp}; \tag{4.42}$$

and the finite gyroradius effects on collisions are included in

$$\begin{aligned} \mathbf{F}_{iC} \equiv M \int d^3v \mathbf{v} \langle C\{f_i\} \rangle = -M\gamma \int d^3v \left(\langle \boldsymbol{\Gamma} \rangle \cdot \hat{\mathbf{b}}\hat{\mathbf{b}} + \frac{1}{v_\perp^2} \langle \boldsymbol{\Gamma} \cdot \mathbf{v}_\perp \rangle \mathbf{v}_\perp \right) \\ + \nabla \cdot \left[\frac{M\gamma}{\Omega_i} \int d^3v \left(\langle \boldsymbol{\Gamma} \rangle \times \hat{\mathbf{b}} - \frac{1}{v_\perp^2} \langle \boldsymbol{\Gamma} \cdot \mathbf{v}_\perp \rangle \mathbf{v} \times \hat{\mathbf{b}} \right) \mathbf{v} \right]. \end{aligned} \tag{4.43}$$

The force $\tilde{F}_{iE}$ originates in the change in the parallel velocity magnitude due to potential structures on the size of the ion gyroradius. The force $\mathbf{F}_{iB}$ accounts for the change in perpendicular velocity due to variation in the magnetic field that the particle feels during its motion. Interestingly, the parallel component of equation (4.38) is simply

$$\frac{\partial}{\partial t}(n_i M V_{i||}) + \hat{\mathbf{b}} \cdot \nabla p_{ig||} + (p_{ig||} - p_{ig\perp}) \nabla \cdot \hat{\mathbf{b}} + \nabla \cdot \boldsymbol{\pi}_{ig||} = -Zen_i \hat{\mathbf{b}} \cdot \nabla \phi$$

Term	Order of magnitude
$n_i M \mathbf{V}_{ig}$	$\delta_i n_e M v_i$
$\hat{\mathbf{b}} \cdot \nabla p_{ig\|\|}$, $(p_{ig\|\|} - p_{ig\perp}) \nabla \cdot \hat{\mathbf{b}}$, $Zen_i \hat{\mathbf{b}} \cdot \nabla \phi$	$\delta_i p_i / L$
$\nabla \cdot \boldsymbol{\pi}_{ig\|\|}$, $\nabla \cdot \overleftrightarrow{\boldsymbol{\pi}}_{ig\times}$, $\tilde{F}_{iE}$, $\mathbf{F}_{iB}$	$\delta_i k_\perp \rho_i p_i / L$
$\mathbf{F}_{iC}$	$\delta_i k_\perp \rho_i \nu_{ii} n_e M v_i$

Table 4.3: Order of magnitude estimates for ion momentum conservation equation (4.38).

$$+\tilde{F}_{iE} + \mathbf{F}_{iC} \cdot \hat{\mathbf{b}}. \quad (4.44)$$

The parallel components of $\nabla \cdot \overleftrightarrow{\boldsymbol{\pi}}_{ig\times}$ and $\mathbf{F}_{iB}$ cancel each other, as proven in section G.2 of Appendix G.

Equation (4.38) will be used in section 4.4 to get one of the forms of the vorticity equation. Thus, it is useful to estimate the size of the different terms in it. The estimates of order of magnitude are summarized in table 4.3. The pressure terms, $\hat{\mathbf{b}} \cdot \nabla p_{ig||} + (p_{ig||} - p_{ig\perp}) \nabla \cdot \hat{\mathbf{b}}$, and the electric field term, $Zen_i \hat{\mathbf{b}} \cdot \nabla \phi$, are $O(\delta_i p_i / L)$. The terms $\tilde{F}_{iE}$ and $\mathbf{F}_{iB}$ are of order $\delta_i k_\perp \rho_i p_i / L$. These estimates are obvious for the integrals $(Ze/\Omega_i) \int d^3v\, f_{Mi} (\overline{\nabla} \times \mathbf{v}_\perp) \cdot \overline{\nabla} \tilde{\phi} \sim (Mc/B) \int d^3v\, f_{Mi} (\overline{\nabla} \tilde{\phi} \times \hat{\mathbf{b}}) \cdot \overline{\nabla} \mathbf{v}_\perp$ since $\overline{\nabla} \tilde{\phi} \sim k_\perp \rho_i T_e / eL$ due to $e\tilde{\phi}/T_e \sim \delta_i$. The integral $Ze \int d^3v\, f_{Mi} \hat{\mathbf{b}} \cdot \overline{\nabla} \tilde{\phi}$ would seem to be of order $\delta_i p_i / L$ since $\hat{\mathbf{b}} \cdot \overline{\nabla} \tilde{\phi} \sim \delta_i T_e / eL$ but it is an order $k_\perp \rho_i$ smaller because the integral of $\tilde{\phi}$ in the gyrophase φ_0 vanishes to zeroth order. The integral $M \int d^3v f_{ig} v_{||} \hat{\mathbf{b}} \cdot \overline{\nabla} \mathbf{v}_\perp$ is of order $\delta_i k_\perp \rho_i p_i / L$ because the leading order gyrophase dependent piece of f_{ig} is even in $v_{||}$ [recall (4.7)]. The collisional force, $\mathbf{F}_{iC}$ is order $\delta_i k_\perp \rho_i \nu_{ii} n_e M v_i$, as proven in Appendix H. The vector $\boldsymbol{\pi}_{ig||}$ is $O(\delta_i^2 p_i)$ because f_{ig} is even in $v_{||}$ and $\mathbf{v}_\perp$ up to order $\delta_i f_{Mi}$. The matrix $\overleftrightarrow{\boldsymbol{\pi}}_{ig\times}$ has three different pieces: the integral $\int d^3v\, f_{ig} (\mathbf{v}_{M0} + \tilde{\mathbf{v}}_1) M \mathbf{v}_\perp$ also of order $\delta_i^2 p_i$, the integral $\int d^3v\, f_{ig} M v_{||} \hat{\mathbf{b}} \mathbf{v}_\perp$ of order $\delta_i k_\perp \rho_i p_i$, and the integral $\int d^3v\, f_{ig} M \mathbf{v}_{E0} \mathbf{v}_\perp$ of order $\delta_i p_i$. The integral $\int d^3v\, f_{ig} M v_{||} \hat{\mathbf{b}} \mathbf{v}_\perp$ is of order $\delta_i k_\perp \rho_i p_i$ because the leading order gyrophase dependent piece of f_{ig} is even in $v_{||}$ as in (4.7). On the other hand, the size of the integral $\int d^3v\, f_{ig} M \mathbf{v}_{E0} \mathbf{v}_\perp$ is estimated by employing $\mathbf{v}_{E0} = -(c/B) \overline{\nabla} \langle \phi \rangle \times \hat{\mathbf{b}} =$

$-(c/B)\nabla\phi \times \hat{\mathbf{b}} + (c/B)\overline{\nabla}\widetilde{\phi} \times \hat{\mathbf{b}}$. Then, I find that $(Mc/B)\int d^3v\, f_{ig}(\nabla\phi \times \hat{\mathbf{b}})\mathbf{v}_\perp \sim \delta_i^2 p_i$ since f_{ig} is gyrophase dependent at $O(\delta_i f_{Mi})$, and $(Mc/B)\int d^3v\, f_{ig}(\overline{\nabla}\widetilde{\phi} \times \hat{\mathbf{b}})\mathbf{v}_\perp \lesssim \delta_i p_i$. It is difficult to refine this last estimate because it is a nonlinear term and short wavelength pieces of f_{ig} and $\widetilde{\phi}$ can beat to give a long wavelength result. The divergences of $\boldsymbol{\pi}_{ig||}$ and $\overleftrightarrow{\boldsymbol{\pi}}_{ig\times}$ are both $O(\delta_i k_\perp \rho_i p_i/L)$. Importantly, the divergence of $\overleftrightarrow{\boldsymbol{\pi}}_{ig\times} \lesssim \delta_i p_i$ is not of order $k_\perp \rho_i p_i/L$ but of order $\delta_i k_\perp \rho_i p_i/L$. The divergence of $\int d^3v\, f_{ig} M v_{||} \hat{\mathbf{b}}\mathbf{v}_\perp$ is of order $\delta_i k_\perp \rho_i p_i/L$ because it only contains a parallel gradient, and $\nabla \cdot [(Mc/B)\int d^3v\, f_{ig}(\overline{\nabla}\widetilde{\phi} \times \hat{\mathbf{b}})\mathbf{v}_\perp] = -\nabla \cdot [Mc \int dE_0\, d\mu_0\, d\varphi_0\, \widetilde{\phi}\, \overline{\nabla} \times (\hat{\mathbf{b}} f_{ig}\mathbf{v}_\perp/v_{||})] \sim \delta_i k_\perp \rho_i p_i/L$, where I have used $d^3v = (B/v_{||}) dE_0\, d\mu_0\, d\varphi_0$ and $\overline{\nabla} \cdot [\overline{\nabla} \times (\ldots)] = 0$ [since $(f_{ig}/v_{||})(\overline{\nabla}\widetilde{\phi} \times \hat{\mathbf{b}})\mathbf{v}_\perp = \overline{\nabla} \times (\hat{\mathbf{b}}\widetilde{\phi} f_{ig}\mathbf{v}_\perp/v_{||}) - \widetilde{\phi}\,\overline{\nabla} \times (\hat{\mathbf{b}} f_{ig}\mathbf{v}_\perp/v_{||})$].

4.4 Vorticity equation for gyrokinetics

In section 4.2, I showed that the term that contains the ion viscosity in the vorticity equation (2.9) seems to dominate at short wavelengths. However, in reality it is smaller than $\nabla \cdot \mathbf{J}_d$, as I demonstrate in this section. As a result, the viscosity must be evaluated carefully; otherwise, spurious terms may appear in numerical simulations. Here, I propose two different vorticity equations that avoid this numerical problem and are valid for short wavelengths. Long wavelength, transport time scale phenomena, like the self-consistent calculation of the radial electric field, can be included, but this will be the subject of chapter 5.

The vorticity equation (2.9) provides a way to temporally evolve the electric field perpendicular to the magnetic field. However, the parallel electric field strongly depends on the parallel electron dynamics, hidden in the parallel current $J_{||}$ in equation (2.9). Fortunately, it is enough to use the integral $J_{||} = Ze\int d^3v\, v_{||} f_{ig} - e\int d^3v\, v_{||} f_e$ for the parallel current since $J_{||}$ does not alter the higher order calculation of the radial electric field determined by equation (2.21). In several codes [3, 62], the electron and ion distribution functions are solved implicitly in the potential. These implicit solutions are then substituted in $J_{||}$ to find the potential in the next time step from the vorticity equation.

In subsection 4.4.1, a vorticity equation is derived directly from gyrokinetic quasineutrality. The advantage of this form is its close relation to previous algorithms, but it differs greatly from the general vorticity equation (2.9). In subsection 4.4.2, I present a modified vorticity equation that has more similarities with equation (2.9). I ensure that both forms are equivalent and satisfy the desired condition at long wavelengths, namely, that they provide a fixed toroidal velocity.

4.4.1 Vorticity from quasineutrality

The first version of the vorticity equation is obtained by taking the time derivative of the gyrokinetic quasineutrality ($Zn_i = n_e$). In other words, I find the time evolution of ion and electron density and ensure that its difference is constant in time. This is equivalent to subtracting equation (4.26) from Z times (4.28) to obtain

$$\frac{\partial}{\partial t}(Zen_{ip}) = \nabla \cdot \left(J_{||}\hat{\mathbf{b}} + \mathbf{J}_{gd} + \tilde{\mathbf{J}}_i + Zen_i\tilde{\mathbf{V}}_i + Zen_i\mathbf{V}_{iC} \right), \tag{4.45}$$

where $\tilde{\mathbf{J}}_i$ is the polarization current

$$\tilde{\mathbf{J}}_i \equiv \frac{Zec}{B}\left(\nabla\phi \times \hat{\mathbf{b}} \int d^3v\, f_i - \int d^3v\, f_{ig}\overline{\nabla}\langle\phi\rangle \times \hat{\mathbf{b}} \right) = \frac{Zec}{B}\left(\int d^3v\, f_{ig}\overline{\nabla}\tilde{\phi} \times \hat{\mathbf{b}} - \nabla\phi \times \hat{\mathbf{b}} \int d^3v\, \frac{Ze\tilde{\phi}}{T_i} f_{Mi} \right) \tag{4.46}$$

and $\mathbf{J}_{gd}$ is the drift current

$$\mathbf{J}_{gd} \equiv Zen_i\mathbf{V}_{igd} - en_e\mathbf{V}_{ed} = \frac{cp_{g||}}{B}\hat{\mathbf{b}}\hat{\mathbf{b}} \cdot \nabla \times \hat{\mathbf{b}} + \frac{cp_{g\perp}}{B^2}\hat{\mathbf{b}} \times \nabla B + \frac{cp_{g||}}{B}\hat{\mathbf{b}} \times \boldsymbol{\kappa}, \tag{4.47}$$

with $p_{g||} = p_{ig||} + p_e$ and $p_{g\perp} = p_{ig\perp} + p_e$. Here, to write the second form of $\tilde{\mathbf{J}}_i$, I use $f_i - f_{ig} = -(Ze\tilde{\phi}/T_i)f_{Mi}$, given in (4.3). In equation (4.45), the ion polarization density, $n_{ip} = -\int d^3v\,(Ze\tilde{\phi}/T_i)f_{Mi}$ is advanced in time, and the electric field is solved from n_{ip}.

It is necessary to check if equation (4.45) satisfies the right conditions at long wavelengths. In the present form, though, it is a tedious task. To perform this check,

Term	Order of magnitude
Zen_{ip}	$\delta_i k_\perp \rho_i e n_e$
$\nabla \cdot (J_{\|\|}\hat{\mathbf{b}} + \mathbf{J}_{gd})$	$\delta_i e n_e v_i / L$
$\nabla \cdot (\tilde{\mathbf{J}}_i + Zen_i\tilde{\mathbf{V}}_i)$	$\delta_i (k_\perp \rho_i)^2 e n_e v_i / L$
$\nabla \cdot (Zen_i \mathbf{V}_{iC})$	$\delta_i (k_\perp \rho_i)^2 e n_e \nu_{ii}$

Table 4.4: Order of magnitude estimates for vorticity equation (4.45).

I will use the much more convenient form in subsection 4.4.2, that I will prove is equivalent.

Finally, I give estimates for the size of all the terms in (4.45) in table 4.4. These estimates will be useful in subsection 4.4.2 to study the behavior of the toroidal angular momentum for $k_\perp \rho_i \ll 1$. The size of most of the terms in equation (4.45) can be obtained from the estimates given in table 4.2, giving $\nabla \cdot \mathbf{J}_{gd} \sim \delta_i e n_e v_i / L$, $\nabla \cdot (Zen_i \tilde{\mathbf{V}}_i) \sim \delta_i (k_\perp \rho_i)^2 e n_e v_i / L$ and $\nabla \cdot (Zen_i \mathbf{V}_{iC}) \sim \delta_i (k_\perp \rho_i)^2 \nu_{ii} e n_e$. The size of $\nabla \cdot \tilde{\mathbf{J}}_i$ requires more work. Even though a cancellation between the drift kinetic $E \times B$ flow, $(cn_i/B)\nabla\phi \times \hat{\mathbf{b}}$, and the corresponding gyrokinetic flow, $(c/B)\int d^3v\, f_{ig} \overline{\nabla}\langle\phi\rangle \times \hat{\mathbf{b}}$, is expected, due to the nonlinear character of these terms, where the short wavelength components of f_{ig} and ϕ can beat to give a long wavelength term, I can only give a bound for the size of $\tilde{\mathbf{J}}_i$. Given the definition of $\tilde{\mathbf{J}}_i$ in equation (4.46), its size is bounded by $|\tilde{\mathbf{J}}_i| \lesssim \delta_i e n_e v_i$. Then, it would seem that its divergence must be $|\nabla \cdot \tilde{\mathbf{J}}_i| \lesssim k_\perp \rho_i e n_e v_i / L$, but the lowest order terms contain $\nabla \times \nabla\phi = 0$ and $\overline{\nabla} \times \overline{\nabla}\langle\phi\rangle = 0$, leading to $|\nabla \cdot \tilde{\mathbf{J}}_i| \lesssim \delta_i e n_e v_i / L$. To refine this bound, I use that $\nabla \cdot \tilde{\mathbf{J}}_i = \int dE_0\, d\mu_0\, d\varphi_0\, \overline{\nabla} \cdot \tilde{\mathbf{j}}_i$, where $\overline{\nabla} \cdot \tilde{\mathbf{j}}_i$ is found from equation (4.46) to be

$$\overline{\nabla} \cdot \tilde{\mathbf{j}}_i \equiv \overline{\nabla} \cdot \left[\frac{Zec}{v_{\|\|}} \left(f_i \nabla\phi \times \hat{\mathbf{b}} - f_{ig} \overline{\nabla}\langle\phi\rangle \times \hat{\mathbf{b}} \right) \right] =$$

$$-\overline{\nabla} \cdot \left[\tilde{\phi}\, \overline{\nabla} \times \left(\hat{\mathbf{b}} \frac{Zec}{v_{\|\|}} f_{ig} \right) + \frac{Zec}{v_{\|\|}} \frac{Ze\tilde{\phi}}{T_i} f_{Mi} \nabla\phi \times \hat{\mathbf{b}} \right] \sim \delta_i k_\perp \rho_i e f_{Mi} B / L. \qquad (4.48)$$

I have used $d^3v = (B/v_{\|\|}) dE_0\, d\mu_0\, d\varphi_0$ and $\overline{\nabla} \cdot (v_{\|\|}^{-1} f_{ig} \overline{\nabla}\tilde{\phi} \times \hat{\mathbf{b}}) = -\overline{\nabla} \cdot [\tilde{\phi}\, \overline{\nabla} \times (\hat{\mathbf{b}} v_{\|\|}^{-1} f_{ig})]$ to find this result. The second form of (4.48) is useful to estimate the size of $\overline{\nabla} \cdot \tilde{\mathbf{j}}_i$

because I can use $e\tilde{\phi}/T_e \sim \delta_i$ and $\overline{\nabla} f_{ig} \sim f_{Mi}/L$ to find $\overline{\nabla} \cdot \tilde{\mathbf{j}}_i \sim \delta_i k_\perp \rho_i e f_{Mi} B/L$. In $\overline{\nabla} \cdot \tilde{\mathbf{j}}_i$, given in (4.48), there are short wavelength components of f_{ig} and ϕ that beat together to give a long wavelength component. These short wavelength components of $f_{ig} \equiv f_i(\mathbf{R}_g, E_0, \mu_0, t)$ and $\tilde{\phi}(\mathbf{R}_g, \mu_0, \varphi_0, t)$ cannot be expanded around $\mathbf{r}$, but the long wavelength component of $\overline{\nabla} \cdot \tilde{\mathbf{j}}_i$ can be expanded as a whole to find

$$\overline{\nabla} \cdot \tilde{\mathbf{j}}_i(\mathbf{R}_g, E_0, \mu_0, \varphi_0) \simeq \overline{\nabla} \cdot \tilde{\mathbf{j}}_i(\mathbf{r}, E_0, \mu_0, \varphi_0) + \frac{1}{\Omega_i}(\mathbf{v} \times \hat{\mathbf{b}}) \cdot \overline{\nabla}(\overline{\nabla} \cdot \tilde{\mathbf{j}}_i). \tag{4.49}$$

The difference between $\overline{\nabla} \cdot \tilde{\mathbf{j}}_i(\mathbf{r}, E_0, \mu_0, \varphi_0)$ and $\overline{\nabla} \cdot \tilde{\mathbf{j}}_i(\mathbf{R}_g, E_0, \mu_0, \varphi_0)$ is negligible in the higher order term. The interesting property of equation (4.49) is that the velocity integral of the zeroth order term $\overline{\nabla} \cdot \tilde{\mathbf{j}}_i(\mathbf{r}, E_0, \mu_0, \varphi_0)$ can be done because the gyrophase dependence in $\mathbf{R}_g$ has disappeared. Employing the second form of equation (4.48) for $\overline{\nabla} \cdot \tilde{\mathbf{j}}_i(\mathbf{r}, E_0, \mu_0, \varphi_0)$, I find $\int dE_0\, d\mu_0\, d\varphi_0\, \overline{\nabla} \cdot \tilde{\mathbf{j}}_i(\mathbf{r}, E_0, \mu_0, \varphi_0) = -\nabla \cdot [(Zec/B) \int d^3v\, (Ze\tilde{\phi}/T_i) f_{Mi} \overline{\nabla}\tilde{\phi} \times \hat{\mathbf{b}}] \sim \delta_i^2 k_\perp \rho_i e n_e v_i/L$ since $\tilde{\phi}\overline{\nabla}\tilde{\phi} = \overline{\nabla}(\tilde{\phi}^2/2)$. This result is negligible compared with the term $\Omega_i^{-1}(\mathbf{v} \times \hat{\mathbf{b}}) \cdot \overline{\nabla}(\overline{\nabla} \cdot \tilde{\mathbf{j}}_i)$ in (4.49), which gives $\nabla \cdot \tilde{\mathbf{J}}_i \sim \delta_i (k_\perp \rho_i)^2 e n_e v_i/L$ since $\overline{\nabla} \cdot \tilde{\mathbf{j}}_i \sim \delta_i k_\perp \rho_i e f_{Mi} B/L$. Using this result, I find that the $k_\perp \rho_i \ll 1$ limit of $\nabla \cdot \tilde{\mathbf{J}}_i$ is

$$\nabla \cdot \tilde{\mathbf{J}}_i \simeq \nabla \cdot \left[\int dE_0\, d\mu_0\, d\varphi_0 \frac{1}{\Omega_i}(\mathbf{v} \times \hat{\mathbf{b}}) \overline{\nabla} \cdot \tilde{\mathbf{j}}_i \right], \tag{4.50}$$

where I use that $\Omega_i^{-1}(\mathbf{v} \times \hat{\mathbf{b}}) \cdot \overline{\nabla}(\overline{\nabla} \cdot \tilde{\mathbf{j}}_i) \simeq \overline{\nabla} \cdot [\Omega_i^{-1}(\mathbf{v} \times \hat{\mathbf{b}})(\overline{\nabla} \cdot \tilde{\mathbf{j}}_i)]$ because the gradient of $\Omega_i^{-1}\mathbf{v} \times \hat{\mathbf{b}}$ is of order $1/L$ and the gradient of $\overline{\nabla} \cdot \tilde{\mathbf{j}}_i$ is of order $k_\perp$.

4.4.2 Vorticity from moment description

Equation (4.45) has the advantage of having a direct relation with the gyrokinetic quasineutrality. However, its relation with the full vorticity equation (2.9) and the evolution of toroidal angular momentum is not explicit. For those reasons, I next derive an alternative vorticity equation.

I define a new useful function

$$\varpi_G = \nabla \cdot \left(\frac{Ze}{\Omega_i} \int d^3v \, f_{ig} \mathbf{v} \times \hat{\mathbf{b}} \right) - \int d^3v \, \frac{Z^2 e^2 \widetilde{\phi}}{T_i} f_{Mi} \tag{4.51}$$

that I will call gyrokinetic "vorticity" because, for $k_\perp \rho_i \ll 1$, it tends to $\varpi = \nabla \cdot [(Ze/\Omega_i)\mathbf{V}_i \times \hat{\mathbf{b}}]$. To see this, I use $-\int d^3v \, (Ze\widetilde{\phi}/T_i) f_{Mi} \to \nabla \cdot [(cn_i/B\Omega_i)\nabla_\perp \phi]$ and $\int d^3v \, f_{ig} \mathbf{v} \times \hat{\mathbf{b}} \to (c/ZeB)\nabla_\perp p_i$ to obtain

$$\varpi_G \simeq \nabla \cdot \left(\frac{Zecn_i}{B\Omega_i} \nabla_\perp \phi + \frac{c}{B\Omega_i} \nabla_\perp p_i \right), \tag{4.52}$$

as given by ϖ in equation (2.12) to first order. The new version of the vorticity equation will evolve in time the gyrokinetic "vorticity" (4.51). The advantage of this new equation is that it will tend to a form similar to the moment vorticity equation (2.9) for $k_\perp \rho_i \ll 1$. It is important to point out that the new version of the vorticity equation, to be given in (4.53), and equations (2.9) and (4.45) are totally equivalent up to $O(\delta_i^2 e n_e v_i / L)$. The advantage of the new version is a similarity with the full vorticity equation (2.9). This similarity helps to study the size of the term $\nabla \cdot [(c/B)\hat{\mathbf{b}} \times (\nabla \cdot \overleftrightarrow{\boldsymbol{\pi}}_i)]$ and the behavior of the transport of toroidal angular momentum for $k_\perp \rho_i \ll 1$. I will prove that the toroidal velocity varies slowly, as expected. Additionally, since the new vorticity equation is derived from quasineutrality and the gyrokinetic equation, it is equivalent to the gyrokinetic quasineutrality equation and provides a way to study its limitations.

The new version of the vorticity equation is obtained by adding equations (4.45) and $\nabla \cdot \{(c/B)[\text{equation (4.38)}] \times \hat{\mathbf{b}}\}$ to obtain

$$\frac{\partial \varpi_G}{\partial t} = \nabla \cdot \left[J_{||} \hat{\mathbf{b}} + \mathbf{J}_{gd} + \tilde{\mathbf{J}}_{i\phi} + \frac{c}{B} \hat{\mathbf{b}} \times (\nabla \cdot \overleftrightarrow{\boldsymbol{\pi}}_{iG}) + Zen_i \mathbf{V}_{iC} - \frac{c}{B} \hat{\mathbf{b}} \times \mathbf{F}_{iC} \right]. \tag{4.53}$$

The terms $\tilde{\mathbf{J}}_i$, $Zen_i\tilde{\mathbf{V}}_i$, $(c/B)\hat{\mathbf{b}} \times (\nabla \cdot \overleftrightarrow{\boldsymbol{\pi}}_{ig\times})$ and $(c/B)\hat{\mathbf{b}} \times \mathbf{F}_{iB}$ are recombined to give $\tilde{\mathbf{J}}_{i\phi}$, $(c/B)\hat{\mathbf{b}} \times (\nabla \cdot \overleftrightarrow{\boldsymbol{\pi}}_{iG})$ and some other terms that have vanished because they are divergence free. The details on how to obtain equation (4.53) are in Appendix I. Here

Term	Order of magnitude
ϖ_G	$\delta_i k_\perp \rho_i e n_e$
$\nabla \cdot (J_{\|\|} \hat{\mathbf{b}} + \mathbf{J}_{gd})$	$\delta_i e n_e v_i / L$
$\nabla \cdot [\tilde{\mathbf{J}}_{i\phi} + (c/B)\hat{\mathbf{b}} \times (\nabla \cdot \overleftrightarrow{\boldsymbol{\pi}}_{iG})]$	$\delta_i (k_\perp \rho_i)^2 e n_e v_i / L$
$\nabla \cdot [Z e n_i \mathbf{V}_{iC} - (c/B)\hat{\mathbf{b}} \times \mathbf{F}_{iC}]$	$\delta_i (k_\perp \rho_i)^2 e n_e \nu_{ii}$

Table 4.5: Order of magnitude estimates for vorticity equation (4.53).

I have defined the new viscosity tensor

$$\overleftrightarrow{\boldsymbol{\pi}}_{iG} \equiv M \int d^3v\, f_{ig} \mathbf{v}_\perp v_{\|} \hat{\mathbf{b}} + \overleftrightarrow{\boldsymbol{\pi}}_{ig\times} =$$
$$\int d^3v\, f_{ig} M \left[v_{\|} (\mathbf{v}_\perp \hat{\mathbf{b}} + \hat{\mathbf{b}} \mathbf{v}_\perp) + (\mathbf{v}_{M0} + \mathbf{v}_{E0} + \tilde{\mathbf{v}}_1)\, \mathbf{v}_\perp \right] \tag{4.54}$$

and the new polarization current

$$\tilde{\mathbf{J}}_{i\phi} = \tilde{\mathbf{J}}_i - \frac{Zec}{B\Omega_i} \hat{\mathbf{b}} \times \int d^3v\, f_{Mi} (\overline{\nabla} \tilde{\phi} \times \hat{\mathbf{b}}) \cdot \overline{\nabla} \mathbf{v}_\perp, \tag{4.55}$$

with $\tilde{\mathbf{J}}_i$ given in (4.46). The derivation of equation (4.53) implies that both equations (4.45) and (4.53) are equivalent as long as the perpendicular component of equation (4.38) is satisfied, and any property proved for one of them is valid for the other.

Equation (4.53) gives the evolution of ϖ_G and the potential is then found by solving equation (4.51). Equation (4.53) does not contain terms that are almost divergence free, as was the case of $(c/B)\hat{\mathbf{b}} \times (\nabla \cdot \overleftrightarrow{\boldsymbol{\pi}}_i)$ in equation (2.9). This will ease implementation in existing simulations.

It is important to know the size of the different terms in (4.53) for implementation purposes. The order of magnitude of the different terms is summarized in table 4.5. In subsection 4.4.1, in table 4.4, I showed $\nabla \cdot \mathbf{J}_{gd} \sim \delta_i e n_e v_i / L$ and $\nabla \cdot (Zen_i \mathbf{V}_{iC}) \sim \delta_i (k_\perp \rho_i)^2 \nu_{ii} e n_e$. The term $\nabla \cdot [(c/B)\hat{\mathbf{b}} \times \mathbf{F}_{iC}]$ is of order $\delta_i (k_\perp \rho_i)^2 \nu_{ii} e n_e$ according to the results in Appendix H. For the flow $(c/B)\hat{\mathbf{b}} \times (\nabla \cdot \overleftrightarrow{\boldsymbol{\pi}}_{iG})$, there are two different pieces in $\overleftrightarrow{\boldsymbol{\pi}}_{iG}$, given in (4.54), namely, $M \int d^3v\, f_{ig} \mathbf{v}_\perp v_{\|} \hat{\mathbf{b}}$ and $\overleftrightarrow{\boldsymbol{\pi}}_{ig\times}$ as defined in (4.40). The first component gives $(Mc/B)\hat{\mathbf{b}} \times [\nabla \cdot (\int d^3v f_{ig} \mathbf{v}_\perp v_{\|} \hat{\mathbf{b}})] = (Ze/\Omega_i) \hat{\mathbf{b}} \times (\int d^3v\, f_{ig} v_{\|} \mathbf{v}_\perp \cdot \nabla \hat{\mathbf{b}}) \sim$

$\delta_i^2 k_\perp \rho_i e n_e v_i$, where I have used that the lowest order gyrophase dependent piece of f_{ig} is even in $v_{||}$ [recall (4.7)]. The divergence $\nabla\cdot \overleftrightarrow{\boldsymbol{\pi}}_{ig\times}$ is of order $\delta_i k_\perp \rho_i p_i/L$ as proven in subsection 4.3.4, giving $\nabla\cdot[(c/B)\hat{\mathbf{b}}\times(\nabla\cdot \overleftrightarrow{\boldsymbol{\pi}}_{iG})] \sim \delta_i (k_\perp\rho_i)^2 e n_e v_i/L$. Finally, $\tilde{\mathbf{J}}_{i\phi}$ is also composed of two pieces shown in (4.55). The divergence of $\nabla\cdot\tilde{\mathbf{J}}_i$ was already found in (4.50), and it is of order $\delta_i (k_\perp\rho_i)^2 e n_e v_i/L$. The second term in (4.55) is of order $\delta_i^2 k_\perp \rho_i e n_e v_i$, giving $\nabla\cdot\tilde{\mathbf{J}}_{i\phi} \sim \delta_i (k_\perp\rho_i)^2 e n_e v_i/L$. Interestingly, employing equations (4.48) and (4.50), and the definition of $\tilde{\mathbf{J}}_{i\phi}$ in (4.55), I find that for $k_\perp\rho_i \ll 1$, $\nabla\cdot\tilde{\mathbf{J}}_{i\phi}$ tends to

$$\nabla\cdot\tilde{\mathbf{J}}_{i\phi} = \nabla\cdot\left\{\frac{c}{B}\hat{\mathbf{b}}\times\left[\nabla\cdot\left\{\int d^3v \left(f_{ig}\frac{c}{B}\overline{\nabla}\langle\phi\rangle\times\hat{\mathbf{b}} - f_i\frac{c}{B}\nabla\phi\times\hat{\mathbf{b}}\right) M\mathbf{v}_\perp\right\}\right]\right\}. \quad (4.56)$$

To obtain this expression I neglect

$$\nabla\cdot\left\{\frac{Ze}{\Omega_i}\hat{\mathbf{b}}\times\left[\int d^3v\,(f_i - f_{ig})\frac{c}{B}(\nabla\phi\times\hat{\mathbf{b}})\cdot\overline{\nabla}\mathbf{v}_\perp\right]\right\} \sim \delta_i^2 k_\perp\rho_i e n_e v_i/L. \qquad (4.57)$$

Equation (4.56) will be useful in the $k_\perp\rho_i \ll 1$ limit worked out in Appendix J.

The estimates in table 4.5 are useful to determine the size of the problematic term $\nabla\cdot[(c/B)\hat{\mathbf{b}}\times(\nabla\cdot \overleftrightarrow{\boldsymbol{\pi}}_i)]$. Subtracting equation (2.9) from (4.53) gives

$$\nabla\cdot\left[\frac{c}{B}\hat{\mathbf{b}}\times(\nabla\cdot \overleftrightarrow{\boldsymbol{\pi}}_i)\right] = \frac{\partial}{\partial t}(\varpi - \varpi_G) + \nabla\cdot\left[\mathbf{J}_{gd} - \mathbf{J}_d + \tilde{\mathbf{J}}_{i\phi} + \frac{c}{B}\hat{\mathbf{b}}\times(\nabla\cdot \overleftrightarrow{\boldsymbol{\pi}}_{iG}) + Zen_i\mathbf{V}_{iC} - \frac{c}{B}\hat{\mathbf{b}}\times\mathbf{F}_{iC}\right] \sim \delta_i(k_\perp\rho_i)^2 e n_e v_i/L. \quad (4.58)$$

This result was anticipated in equation (4.9) and proves that turbulent tokamaks are intrinsically ambipolar!

Most of the estimates for the terms in equation (4.58) are obtained from table 4.5. Only $\varpi - \varpi_G$ and $\nabla\cdot(\mathbf{J}_{gd} - \mathbf{J}_d)$ need clarification. The long wavelength limit of $\varpi_G \sim \delta_i k_\perp\rho_i e n_e$, given in (4.52), is the same as the long wavelength limit of ϖ in (2.12). Thus, they can only differ in the next order in $k_\perp\rho_i$, giving $\varpi - \varpi_G \sim \delta_i (k_\perp\rho_i)^2 e n_e$.

The difference $\mathbf{J}_{gd} - \mathbf{J}_d$ can be rewritten using $f_i - f_{ig} = -(Ze\tilde{\phi}/T_i)f_{Mi}$ to obtain

$$\mathbf{J}_{gd} - \mathbf{J}_d = \int d^3v \, \frac{Z^2e^2\tilde{\phi}}{T_i} f_{Mi} \left(\frac{v_\perp^2}{2\Omega_i} \hat{\mathbf{b}} \cdot \nabla \times \hat{\mathbf{b}} + \frac{v_\perp^2}{2B\Omega_i} \hat{\mathbf{b}} \times \nabla B + \frac{v_\parallel^2}{\Omega_i} \hat{\mathbf{b}} \times \boldsymbol{\kappa} \right). \quad (4.59)$$

The size of $\mathbf{J}_{gd} - \mathbf{J}_d$ is $\delta_i^2 k_\perp \rho_i e n_e v_i$ since the integral of $\tilde{\phi}$ in the gyrophase vanishes to zeroth order. Then, $\nabla \cdot (\mathbf{J}_{gd} - \mathbf{J}_d) \sim \delta_i (k_\perp \rho_i)^2 e n_e v_i / L$. Employing these estimates, I find that $\nabla \cdot [(c/B)\hat{\mathbf{b}} \times (\nabla \cdot \overleftrightarrow{\boldsymbol{\pi}}_i)]$ is of order $\delta_i (k_\perp \rho_i)^2 e n_e v_i / L$, as given in (4.58) and asserted in equation (4.9) in section 4.2. Therefore, there is a piece of the viscosity of order $\delta_i k_\perp \rho_i p_i$ that vanishes to zeroth order in $\nabla \cdot [(c/B)\hat{\mathbf{b}} \times (\nabla \cdot \overleftrightarrow{\boldsymbol{\pi}}_i)]$. In equations (4.45) and (4.53) this piece has already been cancelled.

Finally, I study the evolution of the toroidal velocity implicit in equation (4.53). To do so, I flux surface average equation (4.53) as I did in section 2.3 for equation (2.9). The result is

$$\frac{\partial}{\partial t} \langle \varpi_G \rangle_\psi = \frac{1}{V'} \frac{\partial}{\partial \psi} V' \Big\langle \mathbf{J}_{gd} \cdot \nabla \psi + \tilde{\mathbf{J}}_{i\phi} \cdot \nabla \psi + Zen_i \mathbf{V}_{iC} \cdot \nabla \psi - \frac{c}{B} (\nabla \cdot \overleftrightarrow{\boldsymbol{\pi}}_{iG} - \mathbf{F}_{iC}) \cdot (\hat{\mathbf{b}} \times \nabla \psi) \Big\rangle_\psi . \quad (4.60)$$

The term $\mathbf{J}_{gd} \cdot \nabla \psi$ can be manipulated in the same way as the term $\mathbf{J}_d$ in equation (2.14) to give $\langle \mathbf{J}_{gd} \cdot \nabla \psi \rangle_\psi = -\langle (cI/B)[\hat{\mathbf{b}} \cdot \nabla p_{g\parallel} + (p_{g\parallel} - p_{g\perp}) \nabla \cdot \hat{\mathbf{b}}] \rangle_\psi$. Employing the parallel momentum equation for ions, given by (4.44), and electrons, given by (4.37), to write

$$\hat{\mathbf{b}} \cdot \nabla p_{g\parallel} + (p_{g\parallel} - p_{g\perp}) \nabla \cdot \hat{\mathbf{b}} = -\frac{\partial}{\partial t}(n_i M V_{i\parallel}) - \nabla \cdot \boldsymbol{\pi}_{ig\parallel} + \tilde{F}_{iE} + \mathbf{F}_{iC} \cdot \hat{\mathbf{b}}, \quad (4.61)$$

I find

$$\frac{\partial}{\partial t} \left(\langle \varpi_G \rangle_\psi - \frac{1}{V'} \frac{\partial}{\partial \psi} V' \left\langle \frac{ZeI}{\Omega_i} n_i V_{i\parallel} \right\rangle_\psi \right) = \frac{1}{V'} \frac{\partial}{\partial \psi} V' \Big\langle \frac{cI}{B} (\nabla \cdot \boldsymbol{\pi}_{ig\parallel} - \tilde{F}_{iE}) + \tilde{\mathbf{J}}_{i\phi} \cdot \nabla \psi - \frac{c}{B} (\nabla \cdot \overleftrightarrow{\boldsymbol{\pi}}_{iG}) \cdot (\hat{\mathbf{b}} \times \nabla \psi) + Zen_i \mathbf{V}_{iC} \cdot \nabla \psi - cR \mathbf{F}_{iC} \cdot \hat{\boldsymbol{\zeta}} \Big\rangle_\psi , \quad (4.62)$$

where I have employed $(I/B)\mathbf{F}_{iC} \cdot \hat{\mathbf{b}} - B^{-1} \mathbf{F}_{iC} \cdot (\hat{\mathbf{b}} \times \nabla \psi) = R \mathbf{F}_{iC} \cdot \hat{\boldsymbol{\zeta}}$ [recall (2.16)].

Taking the limit $k_\perp \rho_i \ll 1$, for which $\varpi_G \to \nabla \cdot [(Zen_i/\Omega_i)\mathbf{V}_i \times \hat{\mathbf{b}}]$, equation (4.62) can be shown to give

$$\frac{\partial}{\partial t}\langle R n_i M \mathbf{V}_i \cdot \hat{\boldsymbol{\zeta}} \rangle_\psi = -\frac{1}{V'}\frac{\partial}{\partial \psi} V' \langle R\hat{\boldsymbol{\zeta}} \cdot \overleftrightarrow{\boldsymbol{\pi}}_i^{(0)} \cdot \nabla\psi \rangle_\psi, \tag{4.63}$$

where I have integrated once in ψ. The details of the calculation are in Appendix J. The zeroth order off-diagonal viscosity is given by

$$\langle R\hat{\boldsymbol{\zeta}} \cdot \overleftrightarrow{\boldsymbol{\pi}}_i^{(0)} \cdot \nabla\psi \rangle_\psi = \left\langle \int d^3v\, f_i R M (\mathbf{v} \cdot \hat{\boldsymbol{\zeta}}) \left(\mathbf{v}_{M0} + \tilde{\mathbf{v}}_1 - \frac{c}{B}\nabla\phi \times \hat{\mathbf{b}} \right) \cdot \nabla\psi \right\rangle_\psi . \tag{4.64}$$

The distribution function $f_i = f_{ig} - (Ze\tilde{\phi}/T_i) f_{Mi}$ has both the adiabatic and the non-adiabatic pieces. The viscosity in (4.64) includes the nonlinear Reynolds stress, describing the $E \times B$ transport of toroidal angular momentum, and the transport due to the magnetic drifts $\mathbf{v}_{M0}$ and finite gyroradius effects $\tilde{\mathbf{v}}_1$. In the absence of collisions, only the Reynolds stress gives a non-vanishing contribution as the other terms correspond to the gyroviscosity. In section K.1 of Appendix K, I prove that

$$\begin{aligned}\left\langle \int d^3v\, f_i R M (\mathbf{v} \cdot \hat{\boldsymbol{\zeta}})(\mathbf{v}_{M0} + \tilde{\mathbf{v}}_1) \cdot \nabla\psi \right\rangle_\psi &= \frac{\partial}{\partial t} \left\langle \frac{1}{2B\Omega_i} (|\nabla\psi|^2 p_{i\perp} + I^2 p_{i||}) \right\rangle_\psi \\ &- \left\langle \frac{M}{2B\Omega_i} \int d^3v\, C\{f_i\} \left(|\nabla\psi|^2 \frac{v_\perp^2}{2} + I^2 v_{||}^2 \right) \right\rangle_\psi . \end{aligned} \tag{4.65}$$

It becomes apparent that when statistical equilibrium is reached and the net radial transport of energy is slow so that $\partial/\partial t \simeq 0$, the magnetic drifts only provide momentum transport proportional to the collision frequency. Since collisions are usually weak, this term will tend to be small. Moreover, in section K.2 of Appendix K, I show that this collisional piece vanishes exactly in up-down symmetric tokamaks, leaving only the Reynolds stress,

$$\langle R\hat{\boldsymbol{\zeta}} \cdot \overleftrightarrow{\boldsymbol{\pi}}_i^{(0)} \cdot \nabla\psi \rangle_\psi \simeq - \left\langle \int d^3v\, f_i R M (\mathbf{v} \cdot \hat{\boldsymbol{\zeta}}) \frac{c}{B} (\nabla\phi \times \hat{\mathbf{b}}) \cdot \nabla\psi \right\rangle_\psi . \tag{4.66}$$

In any case, the zeroth order viscosity is of order $\delta_i^2 p_i$, and the corresponding piece in the vorticity equation (4.53) is of order $\delta_i (k_\perp \rho_i)^2 e n_e v_i / L$, which becomes of order

$\delta_i^3 en_e v_i/L$ as $k_\perp \to 1/L$.

An important conclusion that can be derived from equation (4.63) is that the rate of change of the toroidal velocity is $\partial/\partial t \sim k_\perp \rho_i v_i/L$, becoming slower and slower as we reach longer wavelengths. This behavior must be reproduced by any equation used to calculate the radial electric field. Equations (4.45) and (4.53) satisfy this condition, but in addition they have the advantage of showing this property explicitly. The terms that determine the radial electric field turn out to be $\tilde{\mathbf{J}}_i \cdot \nabla\psi$, $Zen_i\tilde{\mathbf{V}}_i \cdot \nabla\psi$ and $Zen_i\mathbf{V}_{iC} \cdot \nabla\psi$ in equation (4.45), and $\tilde{\mathbf{J}}_{i\phi} \cdot \nabla\psi$, $(c/B)[\hat{\mathbf{b}} \times (\nabla \cdot \overleftrightarrow{\boldsymbol{\pi}}_{iG})] \cdot \nabla\psi$, $Zen_i\mathbf{V}_{iC} \cdot \nabla\psi$ and $(c/B)(\hat{\mathbf{b}} \times \mathbf{F}_{iC}) \cdot \nabla\psi$ in equation (4.53). Additionally, it is necessary to keep the terms $\nabla \cdot \boldsymbol{\pi}_{ig||}$, $\tilde{F}_{iE}$ and $\mathbf{F}_{iC} \cdot \hat{\mathbf{b}}$ in the parallel momentum equation (4.44). Any simulation must make sure that these terms have the correct behavior at long wavelengths and give equation (4.63). In the traditional gyrokinetic approach, the terms $\tilde{\mathbf{J}}_i \cdot \nabla\psi$, $Zen_i\tilde{\mathbf{V}}_i \cdot \nabla\psi$ and $Zen_i\mathbf{V}_{iC} \cdot \nabla\psi$ of equation (4.45) can be tracked back to terms in the gyrokinetic Fokker-Planck equation. They correspond to the difference between the ion and electron gyroaveraged $E \times B$ flows, $(c/B)(f_i \nabla\phi \times \hat{\mathbf{b}} - f_{ig} \nabla_{\mathbf{R}} \langle\phi\rangle \times \hat{\mathbf{b}})$, and the finite gyroradius effects that make $\hat{\mathbf{b}}(\mathbf{R}_g) \neq \hat{\mathbf{b}}(\mathbf{r})$, $\nabla_{\mathbf{R}_g} \neq \overline{\nabla}$ and $\langle C\{f_i\}\rangle \neq C\{f_i\}$. This identification is the advantage of equation (4.45) since it allows easier analysis of existing simulations. Equation (4.45) can be used to check if the simulations reproduce the correct transport of toroidal angular momentum.

Since the vorticity equations (4.45) and (4.53) give equation (4.63) for $k_\perp \rho_i \ll 1$, it may seem that they provide the correct radial electric field at long wavelengths. Moreover, I have deduced these vorticity equations employing only the gyrokinetic Fokker-Planck equation and the corresponding quasineutrality, making it tempting to argue that the traditional gyrokinetic method is good enough to find the radial electric field. This argument is flawed because there are missing terms of order $\delta_i^2 en_e v_i/L$ in equations (4.45) and (4.53). Then, the transport of toroidal angular momentum (4.63), that corresponds to a term of order $\delta_i (k_\perp \rho_i)^2 en_e v_i/L$, will remain correct only if $(k_\perp \rho_i)^2 \gg \delta_i$. Consequently, the gyrokinetic quasineutrality should provide the correct radial electric field up to wavelengths of order $\sqrt{\rho_i L}$. For longer wavelengths, there will be missing terms. This estimate only considers the terms that the gyroki-

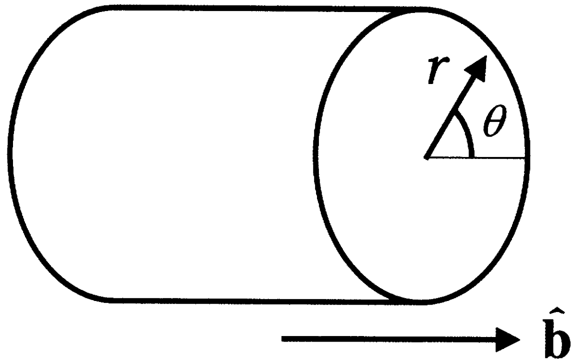

Figure 4-4: Geometry of the θ-pinch.

netic equation is missing and neglects possible numerical inaccuracies.

In the next section, I will show with a simplified example that gyrokinetic indeed has problems determining the radial electric field in axisymmetric configurations. This example is intended to illustrate the difficulties that arise from the use of gyrokinetic Fokker-Planck equation together with the gyrokinetic quasineutrality equation (3.54).

4.5 Example: quasineutrality in a θ-pinch

In this section, I try to find the solution to the non-turbulent, axisymmetric θ-pinch. Without turbulence, the perpendicular wavelengths are of the order of the characteristic size of the θ-pinch, $k_\perp L \sim 1$. This section is intended only as an example, and neglecting the turbulence greatly simplifies the calculation without fundamentally changing the properties of quasineutrality. In this simplified problem, I find that current gyrokinetic treatments, even if extended to a higher order in $\delta_i = \rho_i/L \ll 1$ than in chapter 3, do not yield a solution for the long wavelength radial electric field, leaving it as a free parameter. The gyrokinetic Fokker-Planck equation and the quasineutrality equation are intrinsically ambipolar and cannot determine the radial electric field. In chapter 5, I will show that the radial electric field is recovered if a different approach is employed.

In the θ-pinch, the magnetic field is given by $\mathbf{B} = B(r)\hat{\mathbf{b}}$, where here $\hat{\mathbf{b}}$ is a constant unit vector in the axial direction, and r is the radial coordinate in cylindrical geometry.

The geometry is sketched in figure 4-4. For long wavelengths, the gyrokinetic equation can be found to order $\delta_i^2 f_{Mi} v_i / L$. The simplified geometry of the magnetic field yields more manageable expressions for the gyrokinetic variables, i.e., μ_1 and $\mathbf{R}_2$ become

$$\mu_1 = \frac{Ze\tilde{\phi}}{MB} - \frac{v_\perp^2}{2B^2\Omega_i}(\mathbf{v} \times \hat{\mathbf{b}}) \cdot \nabla B \tag{4.67}$$

and

$$\mathbf{R}_2 = \frac{1}{4B\Omega_i^2}[\mathbf{v}_\perp \mathbf{v}_\perp - (\mathbf{v} \times \hat{\mathbf{b}})(\mathbf{v} \times \hat{\mathbf{b}})] \cdot \nabla B, \tag{4.68}$$

where the term $(c/B\Omega_i)\nabla_\mathbf{R}\tilde{\Phi} \times \hat{\mathbf{b}} \sim \delta_i^2 k_\perp \rho_i L$ has been neglected because I assume that $k_\perp L \sim 1$. Using $\mathbf{R}_2$, the gyroaverage of $\dot{\mathbf{R}}$ is calculated to be

$$\langle \dot{\mathbf{R}} \rangle \simeq \langle v_{||} \rangle \hat{\mathbf{b}} + \left\langle \frac{\mu_0}{\Omega_i} \hat{\mathbf{b}} \times \nabla B - \frac{c}{B} \nabla \phi \times \hat{\mathbf{b}} + \mathbf{v}_\perp \cdot \nabla \mathbf{R}_2 - \frac{Ze}{M} \nabla \phi \cdot \nabla_v \mathbf{R}_2 \right\rangle, \tag{4.69}$$

where I have used that in a θ-pinch $\hat{\mathbf{b}} \cdot \nabla B = 0$ to write $\mathbf{v} \cdot \nabla \mathbf{R}_2 = \mathbf{v}_\perp \cdot \nabla \mathbf{R}_2$. The gyroaverages are performed by employing the long wavelength approximation $\nabla \phi \simeq \nabla_\mathbf{R} \phi - \Omega_i^{-1} (\mathbf{v} \times \hat{\mathbf{b}}) \cdot \nabla_\mathbf{R} \phi$ and the relation $\langle \mathbf{v}_\perp \mathbf{v}_\perp \mathbf{v}_\perp \rangle = 0$ to get

$$\langle \dot{\mathbf{R}} \rangle \simeq \langle v_{||} \rangle \hat{\mathbf{b}} + \frac{\mu}{\Omega_i(\mathbf{R})} \hat{\mathbf{b}} \times \nabla_\mathbf{R} B - \frac{c}{B(\mathbf{R})} \nabla_\mathbf{R} \phi \times \hat{\mathbf{b}}. \tag{4.70}$$

The gyroaverage of $\dot{E}$ is found by using (A.17) to write

$$\dot{E} = -\frac{Ze}{M} \left(\mathbf{v} \cdot \nabla \phi - \frac{d\tilde{\phi}}{dt} \right) \simeq -\frac{Ze}{M} \left(\dot{\mathbf{R}} \cdot \nabla_\mathbf{R} \langle \phi \rangle + \dot{\mu} \frac{\partial \langle \phi \rangle}{\partial \mu} - \frac{\partial \tilde{\phi}}{\partial t} \right), \tag{4.71}$$

where I employ that $\partial/\partial t \sim \delta_i v_i / L$ for long wavelengths and $\partial \phi / \partial E = O(\delta_i^3 M/e)$ in the θ-pinch. Considering that $\langle \dot{\mu} \rangle = O(\delta_i^2 v_i^3 / BL)$ and $\partial \langle \phi \rangle / \partial \mu = O(\delta_i MB/e)$, the gyroaverage of (4.71) is calculated to be

$$\langle \dot{E} \rangle \simeq -\frac{Ze}{M} \langle \dot{\mathbf{R}} \rangle \cdot \nabla_\mathbf{R} \langle \phi \rangle. \tag{4.72}$$

Thus, the gyrokinetic equation to order $O(\delta_i^2 f_{Mi} v_i/L)$ is

$$\frac{\partial f_i}{\partial t} + \langle \dot{\mathbf{R}} \rangle \cdot \left(\nabla_{\mathbf{R}} f_i - \frac{Ze}{M} \nabla_{\mathbf{R}} \langle \phi \rangle \frac{\partial f_i}{\partial E} \right) = \langle C\{f_i\} \rangle. \tag{4.73}$$

I have neglected the derivative $\partial f_i/\partial \mu$ because the distribution function is Maxwellian to zeroth order and $\langle \dot{\mu} \rangle$ is already small by definition of μ. For an axisymmetric steady state solution, the terms on the left side of (4.73) vanish, the second term because the gyrocenter parallel and perpendicular drifts, $\langle \dot{\mathbf{R}} \rangle$, remain in surfaces of constant f_i and ϕ (for this reason, $\langle v_{||} \rangle$ need not be evaluated to second order). Thus, equation (4.73) becomes $\langle C\{f_i\} \rangle = 0$. Such an equation can be solved for a simplified collision operator. I use a Krook operator, $C\{f_i\} = -\nu(f_i - f_M)$, with constant collision frequency ν and a shifted Maxwellian,

$$f_M = n_i \left(\frac{M}{2\pi T_i} \right)^{3/2} \exp \left[-\frac{M(\mathbf{v} - \mathbf{V}_i)^2}{2T_i} \right], \tag{4.74}$$

where n_i, T_i and $\mathbf{V}_i$ are functions of the position $\mathbf{r}$. I assume that the parallel average velocity, $V_{i||} = \hat{\mathbf{b}} \cdot \mathbf{V}_i$, is zero and I order $\mathbf{V}_i$ as $O(\delta_i v_i)$ to obtain

$$f_M \simeq f_{M0} \left[1 + \frac{M\mathbf{v}_\perp \cdot \mathbf{V}_i}{T_i} + \frac{M^2(\mathbf{v}_\perp \cdot \mathbf{V}_i)^2}{2T_i^2} - \frac{MV_i^2}{2T_i} \right], \tag{4.75}$$

with

$$f_{M0} = n_i \left(\frac{M}{2\pi T_i} \right)^{3/2} \exp \left(-\frac{Mv^2}{2T_i} \right). \tag{4.76}$$

With the Krook operator, the gyrokinetic solution is

$$f_i = \langle f_M \rangle = \langle f_{M0} \rangle + \left\langle \frac{M\mathbf{v}_\perp \cdot \mathbf{V}_i}{T_i} f_{M0} \right\rangle + \frac{M^2 v_\perp^2 V_i^2}{4T_i^2} F_M - \frac{MV_i^2}{2T_i} F_M, \tag{4.77}$$

where I have used that in the higher order terms, $f_{M0} \simeq F_M$, with

$$F_M = n_i(\mathbf{R}) \left[\frac{M}{2\pi T_i(\mathbf{R})} \right]^{3/2} \exp \left[-\frac{ME}{T_i(\mathbf{R})} \right]. \tag{4.78}$$

For the first two terms in equation (4.77), it is necessary to Taylor expand $f_{M0}(\mathbf{r}, E_0)$,

$\mathbf{V}_i(\mathbf{r})$ and $\mathbf{v}_\perp = \sqrt{2\mu_0 B(\mathbf{r})}[\hat{\mathbf{e}}_1(\mathbf{r})\cos\varphi_0 + \hat{\mathbf{e}}_2(\mathbf{r})\sin\varphi_0]$ around $\mathbf{R}$, E, μ and φ. The final result is

$$f_i = \langle f_M \rangle = F_M \Bigg[1 - \frac{x_\perp^2}{n_i} \nabla \cdot \left(\frac{cn_i}{B\Omega_i} \nabla_\perp \phi \right) + \frac{Mc^2}{2T_i B^2}(2 - x_\perp^2)|\nabla_\perp \phi|^2 + \frac{MV_i^2}{2T}(x_\perp^2 - 1) + \frac{x_\perp^2}{2M\Omega_i^2}\left(x^2 - \frac{5}{2} \right) \nabla_\perp^2 T_i - \frac{x_\perp^2}{2n_i M\Omega_i^2} \nabla_\perp^2 p_i + \frac{x_\perp^2}{2T_i M\Omega_i^2}\left(\frac{35}{4} - 7x^2 + x^4 \right) |\nabla_\perp T_i|^2 + \frac{2x_\perp^2}{MB\Omega_i^2}\left(\frac{5}{2} - x^2 \right) \nabla_\perp B \cdot \nabla_\perp T_i + \frac{c}{T_i B\Omega_i} \nabla_\perp \phi \cdot \nabla_\perp T_i \left(\frac{5}{2} - x_\perp^2 - x^2 \right) \Bigg], \tag{4.79}$$

where $x^2 = Mv^2/2T_i \simeq ME/T_i$, $x_\perp^2 = Mv_\perp^2/2T_i \simeq M\mu B/T_i$ and I have employed

$$\mathbf{V}_i = \frac{1}{n_i M\Omega_i} \hat{\mathbf{b}} \times \nabla p_i - \frac{c}{B} \nabla\phi \times \hat{\mathbf{b}}. \tag{4.80}$$

The distribution function in (4.79) has been calculated by using a gyrokinetic equation that is correct to order $\delta_i^2 f_{Mi} v_i / L$ for both the Vlasov operator and the gyroaveraged collision operator. Using the definitions $\mathbf{R} = \mathbf{r} + \mathbf{R}_1 + \mathbf{R}_2$, $E = E_0 + E_1 + E_2$ and $\mu = \mu_0 + \mu_1$, and the gyrophase dependent collisional piece $\tilde{f}_i$ given in (3.38), I can find the distribution function f_i in $\mathbf{r}$, $\mathbf{v}$ variables to order $O(\delta_i^2 f_{Mi})$. As a check, the same solution has been also obtained without resorting to gyrokinetics to order $O(\delta_i^2 f_{Mi})$. This check is omitted here.

If I had gyroaveraged $C\{f_i\}$ only to order δ_i, as most gyrokinetic models do, the solution would have been simply

$$f_i \simeq F_M. \tag{4.81}$$

Substituting this solution into the higher order gyrokinetic quasineutrality equation (3.61), I find the inconsistent result

$$\nabla \cdot \left(\frac{Zcn_i}{B\Omega_i} \nabla_\perp \phi \right) - \frac{ZMc^2 n_i}{2T_i B^2} |\nabla_\perp \phi|^2 = n_e - Zn_i - \frac{Z}{2M\Omega_i^2} \nabla_\perp^2 p_i. \tag{4.82}$$

However, this quasineutrality equation is very different from the one we obtain by using the full $O(\delta_i^2 f_{Mi})$ solution in (4.79), which simply gives

$$Zn_i = n_e. \tag{4.83}$$

Therefore, the gyrokinetic quasineutrality equation reduces to the quasineutrality condition when the exact $O(\delta_i^2 f_{Mi})$ distribution function of (4.79) is employed. Equation (4.82) is wrong because the $O(\delta_i f_{Mi})$ result of (4.81) is either inducing an $O(\delta_i^2 e n_e)$ charge difference or imposing the non-physical condition

$$\nabla \cdot \left(\frac{Zcn_i}{B\Omega_i} \nabla_\perp \phi \right) - \frac{ZMc^2 n_i}{2T_i B^2} |\nabla_\perp \phi|^2 = -\frac{Z}{2M\Omega_i^2} \nabla_\perp^2 p_i. \tag{4.84}$$

The difference between (4.82) and (4.83), given by (4.84), originates in $O(\delta_i^2 n_e)$ terms that should have been cancelled by pieces of the distribution function of the same order.

The θ-pinch example illustrates the problem of using a lower order gyrokinetic equation than needed, but it also highlights another issue. The potential does not appear in the quasineutrality equation (4.83), and, therefore, it cannot be found using it. In a computer simulation, the potential is obtained from the gyrokinetic quasineutrality equation (3.61), and the distribution function is evolved employing the gyrokinetic equation (4.73). A possible initial condition $f_{i,t=0}(\mathbf{R}, E, \mu)$ is the stationary solution (4.79), where the potential $\phi(r)$ is a free function that is set to be $\phi_{t=0}(r)$. Equation (4.83) proves that the solution to the gyrokinetic quasineutrality equation (3.61) at $t = 0$ must be $\phi(r, t = 0) = \phi_{t=0}(r)$, where $\phi_{t=0}(r)$ is the free function that I chose for the initial condition. Since the initial condition $f_{i,t=0}(\mathbf{R}, E, \mu)$ is a stationary solution, I find that the solution for all times is $f_i = f_{i,t=0}$ and $\phi = \phi_{t=0}$, and the radial electric field is solely determined by the initial condition. If there were any numerical errors that made the solution invalid to order $\delta_i^2 f_{Mi}$, the radial electric field would suffer a non-physical evolution. In modern gyrokinetics, the radial electric field is then determined by the initial condition, in particular, by a piece of order $\delta_i^2 f_{Mi}$ of the initial condition. This result is not surprising since in this chapter I have proved that, in axisymmetric configurations, the axisymmetric piece of the vorticity equation, or time derivative of $Zn_i - n_e$, is of order $\delta_i^3 n_e v_i / L$ at the most, as given by (4.9). The gyrokinetic equation is only calculated to order $\delta_i^2 f_{Mi} v_i / L$ in this section (and only to order $\delta_i f_{Mi} v_i / L$ in codes), leading to an effectively constant $Zn_i - n_e$ and hence a radial electric field dependent only on the initial condition. In section 5.2, I will show that in reality the time derivative of $Zn_i - n_e$ is even smaller than $\delta_i^3 n_e v_i / L$ by a factor of ν / Ω_i.

4.6 Discussion

In this chapter, I have shown how the vorticity equation recovers the physics of quasineutrality and at the same time retains the effect of the transport of toroidal angular momentum in the radial electric field. I have proposed two possible vorticity equations, (4.45) and (4.53). With these two equations, I estimate the size of the term that determines the radial electric field, given by equation (4.9). In this manner, I prove that, with the usual gyrokinetic equation, setting the radial current to zero, $\langle \mathbf{J} \cdot \nabla \psi \rangle_\psi = 0$, cannot determine the long wavelength axisymmetric radial electric field. Therefore, modern gyrokinetic formulations are intrinsically ambipolar, and thereby unable to determine the long wavelength axisymmetric radial electric field.

I illustrated the problems that arise from a failure to satisfy intrinsic ambipolarity with a simplified problem in section 4.5. In the example, the long wavelength, axisymmetric radial electric field was left undetermined by the gyrokinetic quasineutrality equation even if the distribution was calculated to an order higher than current codes can achieve. More importantly, if there is an error in the density as small as $\delta_i^2 n_e$, the gyrokinetic quasineutrality equation yields an erroneous long wavelength radial electric field. This feature places a strong requirement on the accuracy of any code that calculates the radial electric field. The vorticity equation, on the other hand, makes the dependence of the radial electric field on the toroidal transport of angular momentum explicit. Both vorticity equations (4.45) and (4.53) would yield a long wavelength radial electric field constant for the short turbulence saturation time.

Between the two vorticity equations, equation (4.45) is closer to the gyrokinetic quasineutrality and is probably the best candidate to implement and compare with existing results. In fact, a similar, but less complete, vorticity equation has already been implemented in the PIC gyrokinetic code GEM [63]. On the other hand, equation (4.53) is similar to the traditional vorticity equation (2.9), making the study of conservation of toroidal angular momentum straightforward.

These vorticity equations are valid for short wavelengths on the order of the ion gyroradius. They must be supplemented with long wavelength physics to be extended to wavelengths longer than $\sqrt{\rho_i L}$. Only then will the transport of toroidal angular momentum be correctly described. The extension to longer wavelengths is treated in chapter 5.

Finally, any numerical implementation of either of the vorticity equations needs to make sure that the properties derived and discussed are satisfied, namely, the scaling of the different terms with $k_\perp \rho_i$ should be ensured, and the cancellations that take place due to the flux surface average should also be maintained in codes. It is for this reason that I give all the details of the analytical calculations including detailed appendices.

Chapter 5

Solving for the radial electric field

In this chapter, I describe the method that I propose to calculate the radial electric field. As I showed in section 2.3, the flux surface averaged vorticity equation reduces to the transport of toroidal angular momentum. Then, to obtain the radial electric field, it is necessary to solve the conservation equation for the transport of toroidal angular momentum.

In section 5.1, I obtain an equation for the toroidal-radial component of the ion viscosity to order $\delta_i^3 p_i$ that only requires the ion distribution function to order $\delta_i^2 f_{Mi}$. I already argued in section 2.3 that $\delta_i^3 p_i$ is the order to which the ion viscosity should be found to recover gyroBohm transport of angular momentum. In this chapter, I give some arguments that suggest that the radial transport of angular momentum is indeed of order $\delta_i^3 p_i$. Interestingly, the transport of toroidal angular momentum found in (4.64), of order $\delta_i k_\perp \rho_i p_i$, should then vanish for long wavelengths. In reality, the transport probably is of order $\delta_i^2 p_i$ at each time step yet its time average vanish to that order. In other words, there might be fast local exchange of toroidal angular momentum, leading to zonal flow structure, but the irreversible transport of angular momentum from the edge to the core is much slower. In section 5.2, I apply the equation for the ion viscosity obtained in section 5.1 to solve for the radial electric field in the example presented in section 4.5.

Even with the convenient equation that gives the radial transport of toroidal angular momentum to order $\delta_i^3 p_i$ with only an $O(\delta_i^2 f_{Mi})$ distribution function, this still makes it necessary to find the ion distribution function and the potential to an order higher than the order to which they are usually calculated. At the end of this chapter, in section 5.3, I prove that under certain assumptions, the lowest order full f gyrokinetic equation (3.39) is valid even to calculate the second order distribution function, and the vorticity equations (4.45) and (4.53) are easily extended to yield a higher order potential. Finally, I discuss all the results of this chapter in section 5.4.

5.1 Ion viscosity and the axisymmetric potential

The evolution of the toroidal velocity, given in (2.21), is determined by the flux surface averaged toroidal-radial component of the ion viscosity $\langle R\hat{\boldsymbol{\zeta}} \cdot \overleftrightarrow{\boldsymbol{\pi}}_i \cdot \nabla\psi\rangle_\psi$. According to the gyroBohm estimates at the end of section 2.3, the ion viscosity has to be obtained to order $\delta_i^3 p_i$. If the ion viscosity is to be determined directly from its definition in (2.6), the ion distribution function must be calculated to order $\delta_i^3 f_{Mi}$, too high of an order to be practical or implementable.

To avoid direct evaluation of the ion viscosity, I propose using moments of the Fokker-Planck equation. This is the approach followed in drift kinetics [64] and to formulate a hybrid gyrokinetic-fluid description [36]. The ion viscosity can be solved from the $M\mathbf{vv}$ moment of the Fokker-Planck equation, given by

$$\Omega_i(\overleftrightarrow{\boldsymbol{\pi}}_i \times\hat{\mathbf{b}} - \hat{\mathbf{b}}\times \overleftrightarrow{\boldsymbol{\pi}}_i) = \overleftrightarrow{\mathbf{K}}, \tag{5.1}$$

with

$$\overleftrightarrow{\mathbf{K}} = \frac{\partial \overleftrightarrow{\mathbf{P}}_i}{\partial t} + \nabla\cdot\left(M\int d^3v\, f_i\mathbf{vvv}\right) + Zen_i(\nabla\phi\mathbf{V}_i + \mathbf{V}_i\nabla\phi) - M\int d^3v\, C\{f_i\}\mathbf{vv}. \tag{5.2}$$

Here, $\overleftrightarrow{\mathbf{P}}_i = M\int d^3v\, f_i\mathbf{vv}$. From the moment equation (5.1), the off diagonal elements of $\overleftrightarrow{\boldsymbol{\pi}}_i$ can be solved for as a function of $\overleftrightarrow{\mathbf{K}}$. Additionally, equation (5.1) contains the energy conservation equation, $\mathrm{Trace}(\overleftrightarrow{\mathbf{K}}) = 0$, and the parallel pressure equation, $\hat{\mathbf{b}}\cdot \overleftrightarrow{\mathbf{K}} \cdot\hat{\mathbf{b}} = 0$.

To solve for the toroidal-radial component $R\hat{\boldsymbol{\zeta}}\cdot \overleftrightarrow{\boldsymbol{\pi}}_i \cdot\nabla\psi$, I pre-multiply and postmultiply equation (5.1) by $R\hat{\boldsymbol{\zeta}}$, giving

$$R\hat{\boldsymbol{\zeta}}\cdot \overleftrightarrow{\boldsymbol{\pi}}_i \cdot\nabla\psi = \frac{Mc}{2Ze}\frac{\partial}{\partial t}(R^2\hat{\boldsymbol{\zeta}}\cdot \overleftrightarrow{\mathbf{P}}_i \cdot\hat{\boldsymbol{\zeta}}) + \frac{M^2c}{2Ze}\nabla\cdot\left[\int d^3v\, \mathbf{v} f_i R^2(\mathbf{v}\cdot\hat{\boldsymbol{\zeta}})^2\right]$$
$$+Mcn_i R^2(\hat{\boldsymbol{\zeta}}\cdot\nabla\phi)(\mathbf{V}_i\cdot\hat{\boldsymbol{\zeta}}) - \frac{M^2c}{2Ze}\int d^3v\, C\{f_i\}R^2(\mathbf{v}\cdot\hat{\boldsymbol{\zeta}})^2, \qquad (5.3)$$

where I use $R(\hat{\mathbf{b}}\times\hat{\boldsymbol{\zeta}}) = \nabla\psi/B$ and $\nabla(R\hat{\boldsymbol{\zeta}}) = (\nabla R)\hat{\boldsymbol{\zeta}} - \hat{\boldsymbol{\zeta}}(\nabla R)$. Flux surface averaging this expression gives

$$\langle R\hat{\boldsymbol{\zeta}}\cdot \overleftrightarrow{\boldsymbol{\pi}}_i \cdot\nabla\psi\rangle_\psi = \frac{M^2c}{2Ze}\frac{1}{V'}\frac{\partial}{\partial\psi}V'\left\langle\int d^3v\,(f_i - \overline{f}_i)(\mathbf{v}\cdot\nabla\psi)R^2(\mathbf{v}\cdot\hat{\boldsymbol{\zeta}})^2\right\rangle_\psi$$
$$+\langle Mcn_i R^2(\hat{\boldsymbol{\zeta}}\cdot\nabla\phi)(\mathbf{V}_i\cdot\hat{\boldsymbol{\zeta}})\rangle_\psi + \frac{Mc}{2Ze}\frac{\partial}{\partial t}\langle R^2\hat{\boldsymbol{\zeta}}\cdot \overleftrightarrow{\mathbf{P}}_i \cdot\hat{\boldsymbol{\zeta}}\rangle_\psi$$
$$-\frac{M^2c}{2Ze}\left\langle\int d^3v\, C\{f_i\}R^2(\mathbf{v}\cdot\hat{\boldsymbol{\zeta}})^2\right\rangle_\psi. \qquad (5.4)$$

In the first term of the right side, I use that $\overline{(\mathbf{v}\cdot\nabla\psi)(\mathbf{v}\cdot\hat{\boldsymbol{\zeta}})^2} = 0$ to write the integral only as a function of the gyrophase dependent piece of the distribution function. Equation (5.4) has the advantage that a distribution function correct to order $\delta_i^2 f_{Mi}$ gives a viscosity good to order $\delta_i^3 p_i$! The method by which we have gained an order in δ_i is similar to the calculation of the perpendicular ion flow employing the momentum equation. To evaluate the perpendicular ion flow $n_i\mathbf{V}_{i\perp} = \int d^3v\, f_i\mathbf{v}_\perp$ to order $\delta_i n_e v_i$ by direct integration over velocity space, the distribution function f_i must be correct to order $\delta_i f_{Mi}$. Instead, it is possible to use the ion perpendicular momentum equation to order p_i/L, where the Lorentz force $(Ze/c)n_i\mathbf{V}_i\times\mathbf{B}$ balances the perpendicular pressure gradient $\nabla_\perp p_i$ and the perpendicular electric field $Zen_i\nabla_\perp\phi$, giving the ion flow $n_i\mathbf{V}_{i\perp} = (c/ZeB)\hat{\mathbf{b}}\times\nabla p_i - (cn_i/B)\nabla\phi\times\hat{\mathbf{b}}$. Notice that only the lowest order Maxwellian f_{Mi} has been used to find p_i. We have gained an order in δ_i.

If the gyroBohm estimates done at the end of section 2.3 are correct, the ion viscosity must identically vanish to order $\delta_i^2 p_i$ without determining the evolution of the long wavelength axisymmetric radial electric field on transport time scales. To obtain the toroidal-radial component of $\overleftrightarrow{\boldsymbol{\pi}}_i$ to this order, it is enough to use a distribution function good to order $\delta_i f_{Mi}$ in (5.4). For long wavelengths and to the order of interest, the gyrophase dependent piece of the distribution function is given by (4.7), i.e., it is proportional to $(\mathbf{v}\times\hat{\mathbf{b}})\cdot\nabla\psi$. Then, the first integral in (5.4) vanishes. Additionally, I find that for this gyrophase dependence, $\overleftrightarrow{\mathbf{P}}_i \simeq p_{i\perp}(\overleftrightarrow{\mathbf{I}} - \hat{\mathbf{b}}\hat{\mathbf{b}}) + p_{i||}\hat{\mathbf{b}}\hat{\mathbf{b}}$ and $\int d^3v\, \mathbf{v}\mathbf{v}C\{f_i\} \simeq \int d^3v\, [(v_\perp^2/2)(\overleftrightarrow{\mathbf{I}} - \hat{\mathbf{b}}\hat{\mathbf{b}}) + v_{||}^2\hat{\mathbf{b}}\hat{\mathbf{b}}]C\{f_i\}$. With all these simplifications, $\langle R\hat{\boldsymbol{\zeta}}\cdot \overleftrightarrow{\boldsymbol{\pi}}_i \cdot\nabla\psi\rangle_\psi$ becomes to $O(\delta_i^2 p_i R|\nabla\psi|)$

$$\langle R\hat{\boldsymbol{\zeta}}\cdot \overleftrightarrow{\boldsymbol{\pi}}_i \cdot\nabla\psi\rangle_\psi \simeq -\left\langle n_i RM(\mathbf{V}_i\cdot\hat{\boldsymbol{\zeta}})\frac{c}{B}(\nabla\phi\times\hat{\mathbf{b}})\cdot\nabla\psi\right\rangle_\psi$$
$$+\frac{\partial}{\partial t}\left\langle\frac{1}{2B\Omega_i}(p_{i\perp}|\nabla\psi|^2 + p_{i||}I^2)\right\rangle_\psi - \left\langle\frac{M}{2B\Omega_i}\int d^3v\, C\{f_i\}\left(\frac{v_\perp^2}{2}|\nabla\psi|^2 + v_{||}^2 I^2\right)\right\rangle_\psi, \quad (5.5)$$

where I have used $\hat{\mathbf{b}}\cdot\nabla\phi \simeq 0$ and $R\hat{\boldsymbol{\zeta}} = I\hat{\mathbf{b}}/B - (\hat{\mathbf{b}}\times\nabla\psi)/B$ to write $cR\hat{\boldsymbol{\zeta}}\cdot\nabla\phi \simeq -(c/B)(\nabla\phi\times\hat{\mathbf{b}})\cdot\nabla\phi$. Equation (5.5) is exactly the transport of momentum found from the gyrokinetic vorticity equation, given in (4.64) and (4.65). For turbulence in statistical equilibrium, the time derivative term can be neglected. If in addition the tokamak is up-down symmetric, the collisional term vanishes as proven in section K.2 of Appendix K, leaving only the Reynolds stress

$$\langle R\hat{\boldsymbol{\zeta}}\cdot \overleftrightarrow{\boldsymbol{\pi}}_i \cdot\nabla\psi\rangle_\psi = -\left\langle n_i RM(\mathbf{V}_i\cdot\hat{\boldsymbol{\zeta}})\frac{c}{B}(\nabla\phi\times\hat{\mathbf{b}})\cdot\nabla\psi\right\rangle_\psi =$$
$$-\left\langle\int d^3v\, f_i RM(\mathbf{v}\cdot\hat{\boldsymbol{\zeta}})\frac{c}{B}(\nabla\phi\times\hat{\mathbf{b}})\cdot\nabla\psi\right\rangle_\psi. \quad (5.6)$$

This Reynolds stress is formally of order $\delta_i^2 p_i R|\nabla\psi|$. If the Reynolds stress were this big, the transport of toroidal angular momentum would be much larger than the gyroBohm estimate. It is more plausible that the Reynolds stress averaged over time is almost zero. Therefore, the Reynolds stress to order $\delta_i^2 p_i$ does not determine the evolution of the long wavelength axisymmetric radial electric field on transport time scales. This possibility does not conflict with possible fast growth and evolution of zonal flow structure, that happens in relatively short times, but does not transport angular momentum through large distances.

It is difficult to prove unarguably that the Reynolds stress (5.6) must vanish to order $\delta_i^2 p_i$. In δf flux tube codes like GS2 [1] and GENE [2], only the gradients of density and temperature enter the equation for the correction to the Maxwellian [recall (3.57)]. The gradient of the velocity and hence of the long wavelength axisymmetric radial electric field is ordered out because the average velocity in the plasma is assumed to be small by δ_i. Then, the system does not have a preferred direction and it is unlikely that there is any transport of angular momentum. Quasilinear calculations suggest that in up-down symmetric tokamaks, δf flux tube formulations must give zero transport [65]. If the average velocity is ordered as large as the thermal velocity, the symmetry in the flux tube is broken and there is a net radial momentum transport [66], but such a description is not relevant in many tokamaks.

It seems more reasonable to assume that, at least in a time averaged sense, the Reynolds stress (5.6) becomes of order $\delta_i^3 p_i$. Therefore, from now on, I consider the fast time average of equation (5.4) to filter the fluctuations in the transport of toroidal angular momentum. Fast time here is an intermediate time between the transit time of the particle motion around the tokamak, L/v_i, and the much slower transport time scale, $\delta_i^{-2} L/v_i$. Since this time average should make the Reynolds stress of order $\delta_i^3 p_i$, the rest of the terms in equation (5.4) must be evaluated to order $\delta_i^3 p_i$. To that end, the ion distribution function and the potential must be known to order $\delta_i^2 f_{Mi}$ and order $\delta_i^2 T_e/e$; an order higher than solved for in gyrokinetic codes. In section 5.3, I will prove that this problem can be circumvented under some simplifying assumptions. The rest of this section is on how to evaluate the first term in (5.4) in a convenient way.

The first term in equation (5.4) only depends on the gyrophase dependent piece of the ion distribution function. For this reason, it can be solved by employing the moment $\mathbf{vvv}$ of the Fokker-Planck equation, given by

$$\Omega_i \int d^3v\, f_i M[(\mathbf{v} \times \hat{\mathbf{b}})\mathbf{vv} + \mathbf{v}(\mathbf{v} \times \hat{\mathbf{b}})\mathbf{v} + \mathbf{vv}(\mathbf{v} \times \hat{\mathbf{b}})] = \frac{\partial}{\partial t} \left(\int d^3v\, f_i M \mathbf{vvv} \right)$$
$$+ \nabla \cdot \left(\int d^3v\, f_i M \mathbf{vvvv} \right) + Ze \int d^3v\, f_i (\nabla\phi \mathbf{vv} + \mathbf{v}\nabla\phi\mathbf{v} + \mathbf{vv}\nabla\phi)$$
$$- \int d^3v\, C\{f_i\} M \mathbf{vvv}. \quad (5.7)$$

Multiplying every index in this tensor by $R\hat{\boldsymbol{\zeta}}$, employing $R(\hat{\mathbf{b}} \times \hat{\boldsymbol{\zeta}}) = \nabla\psi/B$ and flux surface averaging gives

$$\left\langle M \int d^3v\,(f_i - \overline{f}_i)(\mathbf{v}\cdot\nabla\psi)R^2(\mathbf{v}\cdot\hat{\boldsymbol{\zeta}})^2 \right\rangle_\psi = \frac{M^2 c}{3Ze}\frac{\partial}{\partial t}\left\langle \int d^3v\, f_i R^3(\mathbf{v}\cdot\hat{\boldsymbol{\zeta}})^3 \right\rangle_\psi$$
$$+\frac{M^2c}{3Ze}\frac{1}{V'}\frac{\partial}{\partial\psi}V'\left\langle \int d^3v\, f_i(\mathbf{v}\cdot\nabla\psi)R^3(\mathbf{v}\cdot\hat{\boldsymbol{\zeta}})^3 \right\rangle_\psi + c\langle R^3(\hat{\boldsymbol{\zeta}}\cdot\nabla\phi)(\hat{\boldsymbol{\zeta}}\cdot \overleftrightarrow{\mathbf{P}}_i \cdot\hat{\boldsymbol{\zeta}})\rangle_\psi$$
$$-\frac{M^2c}{3Ze}\left\langle \int d^3v\, C\{f_i\}R^3(\mathbf{v}\cdot\hat{\boldsymbol{\zeta}})^3 \right\rangle_\psi . \quad (5.8)$$

This equation has to be evaluated to order $\delta_i^2 p_i v_i R^2|\nabla\psi|$ to give terms of order $\delta_i^3 p_i R|\nabla\psi|$ in (5.4). The first term in the right side is then negligible because it has a time derivative. With turbulence that has reached statistical equilibrium and after fast time averaging, the time derivative becomes of the order of the transport time scale at long wavelengths, i.e., $\partial/\partial t \sim D_{gB}/L^2 \sim \delta_i^2 v_i/L$. The contribution of such a time derivative is negligible since it gives a term of order $\delta_i^3 p_i v_i R^2|\nabla\psi|$. The second term in equation (5.8) is also negligible since $\overline{(\mathbf{v}\cdot\nabla\psi)(\mathbf{v}\cdot\hat{\boldsymbol{\zeta}})^3} = 0$ means that only the gyrophase dependent piece of the distribution function contributes. To the order of interest, the gyrophase dependent piece is given by (4.7), and its contribution vanishes. Then, the only terms left are

$$\left\langle M \int d^3v\,(f_i - \overline{f}_i)(\mathbf{v}\cdot\nabla\psi)R^2(\mathbf{v}\cdot\hat{\boldsymbol{\zeta}})^2 \right\rangle_\psi = c\langle R^3(\hat{\boldsymbol{\zeta}}\cdot\nabla\phi)(\hat{\boldsymbol{\zeta}}\cdot \overleftrightarrow{\mathbf{P}}_i \cdot\hat{\boldsymbol{\zeta}})\rangle_\psi$$
$$-\frac{M^2c}{3Ze}\left\langle \int d^3v\, C\{f_i\}R^3(\mathbf{v}\cdot\hat{\boldsymbol{\zeta}})^3 \right\rangle_\psi . \quad (5.9)$$

Substituting this relation into equation (5.4) gives

$$\langle R\hat{\boldsymbol{\zeta}}\cdot \overleftrightarrow{\boldsymbol{\pi}}_i \cdot\nabla\psi\rangle_\psi = \frac{Mc}{2Ze}\langle R^2\rangle_\psi \frac{\partial p_i}{\partial t} + \langle Mcn_iR^2(\hat{\boldsymbol{\zeta}}\cdot\nabla\phi)(\mathbf{V}_i\cdot\hat{\boldsymbol{\zeta}})\rangle_\psi$$
$$-\frac{M^2c}{2Ze}\left\langle \int d^3v\, C\{f_i\}R^2(\mathbf{v}\cdot\hat{\boldsymbol{\zeta}})^2 \right\rangle_\psi + \frac{Mc^2}{2Ze}\frac{1}{V'}\frac{\partial}{\partial\psi}V'\langle R^3(\hat{\boldsymbol{\zeta}}\cdot\nabla\phi)(\hat{\boldsymbol{\zeta}}\cdot \overleftrightarrow{\mathbf{P}}_i \cdot\hat{\boldsymbol{\zeta}})\rangle_\psi$$
$$-\frac{M^3c^2}{6Z^2e^2}\frac{1}{V'}\frac{\partial}{\partial\psi}V'\left\langle \int d^3v\, C\{f_i\}R^3(\mathbf{v}\cdot\hat{\boldsymbol{\zeta}})^3 \right\rangle_\psi . \quad (5.10)$$

In the time derivative term, I used that for statistical equilibrium and after fast time averaging, only the slow transport time scales are left. Then, the dominant term in $\partial(\hat{\boldsymbol{\zeta}}\cdot \overleftrightarrow{\mathbf{P}}_i \cdot\hat{\boldsymbol{\zeta}})/\partial t$ is $\partial p_i/\partial t$. This term is determined by the turbulent heat transport and heating in the plasma. Of the rest of the terms in equation (5.10), the second and the third term require a distribution function good to order $\delta_i^2 f_{Mi}$, and a potential good to $\delta_i^2 T_e/e$. The last two terms only need the distribution function to order $\delta_i f_{Mi}$ and the potential to order $\delta_i T_e/e$.

In the next section, I apply the methodology suggested in this section to solve for the radial electric field in the θ-pinch problem presented in section 4.5. In section 5.3, I will explain how the distribution function can be found to second order in tokamak geometry. Notice that this piece of the distribution function is only needed for the irreversible transport of angular momentum since the transport of angular momentum to order $\delta_i^2 p_i$ is enough to capture the fast evolution of zonal flow.

5.2 Example: the solution of a θ-pinch

In section 4.5, I showed that, in the θ-pinch, the quasineutrality condition applied to the second order solution (4.79), valid to $O(\delta_i^2 f_{Mi})$, does not determine the radial electric field. However, I will show in this section that the electrostatic potential can be obtained from the conservation of azimuthal angular momentum, equivalent to the conservation of toroidal angular momentum in tokamaks since both momentums are in the direction of symmetry. The momentum equation has the advantage of showing how quasineutrality depends on the long wavelength axisymmetric potential without having to calculate the distribution function to higher order than $O(\delta_i^2 f_{Mi})$. The methodology I use here is presented for screw pinches and dipolar configurations in [67].

For a steady state solution, the azimuthal angular momentum must be conserved, giving

$$\frac{1}{r}\frac{\partial}{\partial r}(r^2 \hat{\mathbf{r}} \cdot \overleftrightarrow{\boldsymbol{\pi}}_i \cdot \hat{\boldsymbol{\theta}}) = 0, \tag{5.11}$$

where $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ are the unit vectors in the radial and azimuthal directions, with $\hat{\mathbf{r}} \times \hat{\boldsymbol{\theta}} = \hat{\mathbf{b}}$ [recall figure 4-4], and $\overleftrightarrow{\boldsymbol{\pi}}_i$ is the ion viscosity, given by (2.6). In a case without sources or sinks of momentum, the final equation for the potential is $r^2 \hat{\mathbf{r}} \cdot \overleftrightarrow{\boldsymbol{\pi}}_i \cdot \hat{\boldsymbol{\theta}} = 0$. Finding $r^2 \hat{\mathbf{r}} \cdot \overleftrightarrow{\boldsymbol{\pi}}_i \cdot \hat{\boldsymbol{\theta}}$ directly from the distribution function requires a higher order solution than the one provided by the $O(\delta_i^2 f_{Mi} v_i/L)$ gyrokinetic equation (4.73) used so far. However, this problem can be circumvented by using the equivalent to equation (5.10) for θ-pinches, given by

$$r^2 \hat{\mathbf{r}} \cdot \overleftrightarrow{\boldsymbol{\pi}}_i \cdot \hat{\boldsymbol{\theta}} = \frac{Mr^2}{2\Omega_i} \int d^3v\, C\{f_i\} (\mathbf{v} \cdot \hat{\boldsymbol{\theta}})^2 - \frac{M}{6r\Omega_i} \frac{\partial}{\partial r} \left[\frac{r^3}{\Omega_i} \int d^3v\, C\{f_i\} (\mathbf{v} \cdot \hat{\boldsymbol{\theta}})^3 \right]. \quad (5.12)$$

To obtain this equation, I have used that $\hat{\boldsymbol{\theta}} \cdot \nabla\phi = 0$ due to axisymmetry. In this particular case, this expression can be reduced to integrals of the gyrophase dependent piece of the distribution function, terms much simpler to obtain to order $\delta_i^2 f_{Mi}$. To see this, I use $\overline{(\mathbf{v} \cdot \hat{\boldsymbol{\theta}})^2} = v_\perp^2/2$, $(\mathbf{v} \cdot \hat{\boldsymbol{\theta}})^2 - \overline{(\mathbf{v} \cdot \hat{\boldsymbol{\theta}})^2} = (1/2)\hat{\boldsymbol{\theta}} \cdot [\mathbf{v}_\perp \mathbf{v}_\perp - (\mathbf{v} \times \hat{\mathbf{b}})(\mathbf{v} \times \hat{\mathbf{b}})] \cdot \hat{\boldsymbol{\theta}} = (1/2)[(\mathbf{v} \cdot \hat{\boldsymbol{\theta}})^2 - (\mathbf{v} \cdot \hat{\mathbf{r}})^2]$ and $\overline{(\mathbf{v} \cdot \hat{\boldsymbol{\theta}})^3} = 0$ to write

$$r^2 \hat{\mathbf{r}} \cdot \overleftrightarrow{\boldsymbol{\pi}}_i \cdot \hat{\boldsymbol{\theta}} = -\frac{Mr^2}{4\Omega_i} \int d^3v \left(C\{f_i\} - \overline{C\{f_i\}} \right) [(\mathbf{v} \cdot \hat{\mathbf{r}})^2 - (\mathbf{v} \cdot \hat{\boldsymbol{\theta}})^2]$$
$$+ \frac{Mr^2}{2\Omega_i} \int d^3v\, C\{f_i\} \frac{v_\perp^2}{2} - \frac{M}{6r\Omega_i} \frac{\partial}{\partial r} \left[\frac{r^3}{\Omega_i} \int d^3v \left(C\{f_i\} - \overline{C\{f_i\}} \right) (\mathbf{v} \cdot \hat{\boldsymbol{\theta}})^3 \right]. \quad (5.13)$$

For the Krook operator $C\{f_i\} = -\nu(f_i - f_M)$, with f_M given in (4.75),

$$C\{f_i\} - \overline{C\{f_i\}} = -\nu \left\{ f_i - \overline{f}_i - \frac{M\mathbf{v}_\perp \cdot \mathbf{V}_i}{T_i} f_{M0} \right.$$
$$\left. - \frac{M^2}{2T_i^2} (\mathbf{v}_\perp \mathbf{v}_\perp - \overline{\mathbf{v}_\perp \mathbf{v}_\perp}) : (\mathbf{V}_i \mathbf{V}_i) f_{M0} \right\}, \quad (5.14)$$

where, according to (4.80),

$$\mathbf{V}_i = \frac{c}{B} \hat{\boldsymbol{\theta}} \left(\frac{1}{Zen_i} \frac{\partial p_i}{\partial r} + \frac{\partial \phi}{\partial r} \right). \quad (5.15)$$

The term $\int d^3v\, C\{f_i\}(Mv_\perp^2/2)$ in (5.13) can be found from the equation for the perpendicular pressure, given by

$$\frac{\partial p_{i\perp}}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} \left[rM \int d^3v\, f_i (\mathbf{v} \cdot \hat{\mathbf{r}}) \frac{v_\perp^2}{2} \right] = -Zen_i \mathbf{V}_{i\perp} \cdot \nabla\phi + M \int d^3v\, C\{f_i\} \frac{v_\perp^2}{2}. \quad (5.16)$$

Since in this case, $\partial/\partial t = 0$ and $\mathbf{V}_{i\perp} \cdot \nabla\phi = (\mathbf{V}_{i\perp} \cdot \hat{\mathbf{r}})(\partial\phi/\partial r) = 0$ [recall (5.15)], equation (5.13) finally becomes

$$\begin{aligned} r^2\hat{\mathbf{r}}\cdot \overleftrightarrow{\boldsymbol{\pi}}_i \cdot\hat{\boldsymbol{\theta}} = -\frac{Mr^2}{4\Omega_i}\int d^3v \left(C\{f_i\} - \overline{C\{f_i\}}\right)[(\mathbf{v}\cdot\hat{\mathbf{r}})^2 - (\mathbf{v}\cdot\hat{\boldsymbol{\theta}})^2] \\ +\frac{Mr}{2\Omega_i}\frac{\partial}{\partial r}\left[r\int d^3v\,(f_i - \overline{f}_i)(\mathbf{v}\cdot\hat{\mathbf{r}})\frac{v_\perp^2}{2}\right] \\ -\frac{M}{6r\Omega_i}\frac{\partial}{\partial r}\left[\frac{r^3}{\Omega_i}\int d^3v \left(C\{f_i\} - \overline{C\{f_i\}}\right)(\mathbf{v}\cdot\hat{\boldsymbol{\theta}})^3\right]. \end{aligned} \tag{5.17}$$

In this equation, only the gyrophase dependent piece of the distribution function enters in the integrals. The corrections $\mathbf{R}_1$, $\mathbf{R}_2$, E_1, E_2 and μ_1 depend on the gyrophase. Then $f_i(\mathbf{R}, E, \mu, t)$ must be Taylor expanded around $\mathbf{r}$, E_0 and μ_0 to get the second order gyrophase dependent piece. The calculation is done in section E.2 of Appendix E, and the final result is

$$\begin{aligned} (f_i - \overline{f}_i)_g = \frac{1}{\Omega_i}(\mathbf{v}\cdot\hat{\boldsymbol{\theta}})\left(\frac{\partial f_{M0}}{\partial r} + \frac{Ze}{T_i}\frac{\partial\phi}{\partial r}f_{M0}\right) \\ -\frac{r}{4\Omega_i}[(\mathbf{v}\cdot\hat{\mathbf{r}})^2 - (\mathbf{v}\cdot\hat{\boldsymbol{\theta}})^2]\frac{\partial}{\partial r}\left[\frac{1}{r\Omega_i}\left(\frac{\partial f_{M0}}{\partial r} + \frac{Ze}{T_i}\frac{\partial\phi}{\partial r}f_{M0}\right)\right] \\ -\frac{Mc}{4B\Omega_i}[(\mathbf{v}\cdot\hat{\mathbf{r}})^2 - (\mathbf{v}\cdot\hat{\boldsymbol{\theta}})^2]\frac{\partial\phi}{\partial r}\left[\frac{\partial}{\partial r}\left(\frac{f_{M0}}{T_i}\right) + \frac{Ze}{T_i^2}\frac{\partial\phi}{\partial r}f_{M0}\right], \end{aligned} \tag{5.18}$$

where f_{M0} is given in (4.76), and the subindex g indicates the non-collisional origin of this gyrophase dependence. The gyrophase dependent piece given by (3.38) is also necessary. For the Krook operator it becomes

$$(f_i - \overline{f}_i)_c = -\frac{\nu}{\Omega_i^2}f_{M0}\left(\frac{Mv^2}{2T_i} - \frac{5}{2}\right)\mathbf{v}_\perp\cdot\nabla\ln T_i. \tag{5.19}$$

Employing equations (5.14), (5.15), (5.18) and (5.19), I find

$$\begin{aligned} \frac{Mr^2}{4\Omega_i}\int d^3v\,(C\{f_i\} - \overline{C\{f_i\}})[(\mathbf{v}\cdot\hat{\mathbf{r}})^2 - (\mathbf{v}\cdot\hat{\boldsymbol{\theta}})^2] = \\ \frac{\nu r^3 p_i}{4\Omega_i^2}\frac{\partial}{\partial r}\left[\frac{c}{rB}\left(\frac{\partial\phi}{\partial r} + \frac{1}{Zen_i}\frac{\partial p_i}{\partial r}\right)\right] + \frac{\nu r^3}{4\Omega_i^2}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{p_i}{M\Omega_i}\frac{\partial T_i}{\partial r}\right), \end{aligned} \tag{5.20}$$

$$\frac{Mr}{2\Omega_i}\frac{\partial}{\partial r}\left[r\int d^3v\,(f_i - \overline{f}_i)(\mathbf{v}\cdot\hat{\mathbf{r}})\frac{v_\perp^2}{2}\right] = -\frac{r}{\Omega_i}\frac{\partial}{\partial r}\left(\frac{\nu p_i r}{M\Omega_i^2}\frac{\partial T_i}{\partial r}\right) \tag{5.21}$$

and

$$\frac{M}{6r\Omega_i}\frac{\partial}{\partial r}\left[\frac{r^3}{\Omega_i}\int d^3v\,(C\{f_i\} - \overline{C\{f_i\}})(\mathbf{v}\cdot\hat{\boldsymbol{\theta}})^3\right] = -\frac{1}{2r\Omega_i}\frac{\partial}{\partial r}\left(\frac{\nu p_i r^3}{M\Omega_i^2}\frac{\partial T_i}{\partial r}\right). \tag{5.22}$$

Substituting these results into (5.17), $r^2\hat{\mathbf{r}}\cdot\overleftrightarrow{\boldsymbol{\pi}}_i\cdot\hat{\boldsymbol{\theta}} = 0$ gives

$$c\left(\frac{\partial\phi}{\partial r} + \frac{1}{Zen_i}\frac{\partial p_i}{\partial r}\right) = rB(r)\int_0^r dr'\,\frac{U(r')}{r'}\left[\frac{\partial}{\partial r'}\ln B(r') - \frac{3}{2}\frac{\partial}{\partial r'}\ln\left(\frac{p_i(r')U(r')}{r'}\right)\right], \tag{5.23}$$

where $U = (2/M\Omega_i)(\partial T_i/\partial r)$. Notice the difference between this equation and (4.84). In particular, notice that for an isothermal f_{M0}, $\partial T_i/\partial r = 0$, a radial Maxwell-Boltzmann response is recovered from (5.23) as expected, but this is not a feature of the non-physical forms (4.82) and (4.84).

Finally, I remark that equation (5.17) gives a radial transport of toroidal angular momentum $\hat{\mathbf{r}}\cdot\overleftrightarrow{\boldsymbol{\pi}}_i\cdot\hat{\boldsymbol{\theta}} \sim \delta_i^2(\nu/\Omega_i)p_i$, corresponding to the term in the vorticity equation

$$\nabla\cdot\left[\frac{c}{B}\hat{\mathbf{b}}\times(\nabla\cdot\overleftrightarrow{\boldsymbol{\pi}}_i)\right] = -\frac{1}{r}\frac{\partial}{\partial r}\left[\frac{c}{rB}\frac{\partial}{\partial r}(r^2\hat{\mathbf{r}}\cdot\overleftrightarrow{\boldsymbol{\pi}}_i\cdot\hat{\boldsymbol{\theta}})\right] \sim \frac{\nu\delta_i^3}{\Omega_i}en_e v_i/L. \tag{5.24}$$

The radial current density represented by this term is too small to be recovered with a gyrokinetic equation good only to order $\delta_i^2 f_{Mi}v_i/L$, as already shown in section 4.5.

5.3 Distribution function and potential to second order

To evaluate (5.10), the ion distribution function and the potential have to be found to order $\delta_i^2 f_{Mi}$ and $\delta_i^2 T_e/e$, respectively. In this section, I show how both the distribution function and the potential can be calculated to higher order without the full second order Fokker-Planck and vorticity equations. I take advantage of the usually small ratio $1/q \sim B_p/B \ll 1$, where $q(\psi) = (2\pi)^{-1}\oint d\theta(\mathbf{B}\cdot\nabla\zeta/\mathbf{B}\cdot\nabla\theta)$ is the safety factor, and $B_p = |\nabla\psi|/R$ is the poloidal component of the magnetic field. Expanding in $q \gg 1$, I will find the ion distribution function to order $q\delta_i^2 f_{Mi}$, neglecting terms of order $\delta_i^2 f_{Mi}$. The potential is calculated consistently with this higher order solution for f_i by employing a higher order vorticity equation. Importantly, the vorticity equations obtained in this section only give to higher order the short wavelength, non-axisymmetric part of the potential. The axisymmetric piece of the potential must be calculated employing another equation. The axisymmetric component is composed of the flux surface averaged piece, given by the conservation of toroidal angular momentum equation (2.21) and the higher order viscosity (5.10), and a poloidally varying modification. The poloidal variation is the Geodesic Acoustic Mode (GAM) response [68, 69]; the initial transient of an axisymmetric perturbation in the potential. The perturbation initially induces poloidal density variations that rapidly Landau damp towards a constant zonal flow known as the Hinton-Rosenbluth residual [10, 11]. This initial decay or GAM is axisymmetric and thus does not drive radial transport. It can, however, shear the turbulence. The lower order vorticity equations (4.45) and (4.53) reproduce the lower order GAM response. The new higher order vorticity equations derived in this subsection will not have, however, higher order corrections to GAMs. It is relatively straightforward to calculate the higher order corrections analytically, but they complicate the vorticity equations unnecessarily since in reality the GAM response is believed to be less important than the Hinton-Rosenbluth residual, that is adequately kept by the gyrokinetic ion Fokker-Planck equation (3.39) and the toroidal angular momentum conservation equation (2.21). For this reason, I will drop the higher order axisymmetric corrections to the gyrokinetic vorticity equations (4.45) and (4.53).

In subsection 5.3.1, I show that equation (3.39) is enough to calculate the distribution function to order $q\delta_i^2 f_{Mi}$. I also argue that the second order corrections $\mathbf{R}_2$ and E_2 are not needed since they only provide corrections of order $\delta_i^2 f_{Mi}$. Employing these two results in subsection 5.3.2, I extend the gyrokinetic vorticity equations (4.45) and (4.53) to give the electrostatic potential consistent with the higher order f_i. To do so, I develop an extended gyrokinetic equation in the physical phase space, as I did in subsection 4.3.1, but now to order $q\delta_i^2 f_{Mi} v_i / L$. Taking moments of this equation, I obtain the new extended vorticity equations that retain the short wavelength,

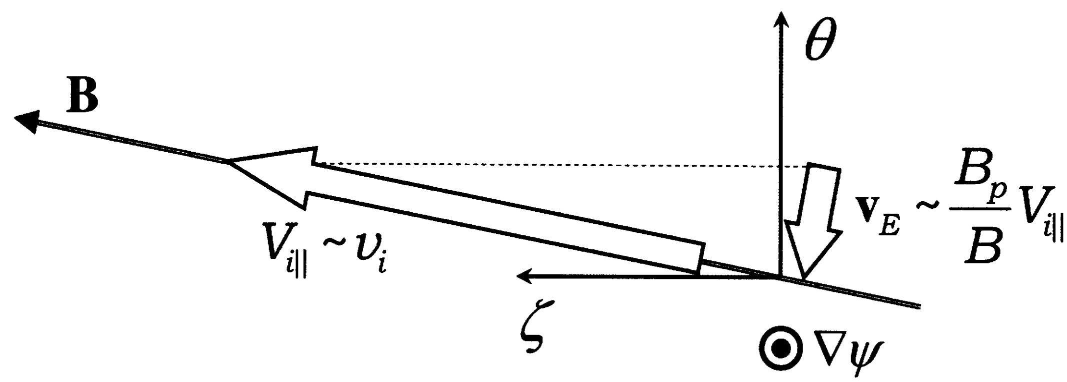

Figure 5-1: High flow ordering for the ion parallel velocity with $B_p/B \ll 1$. Notice that the poloidal projection of the $E \times B$ drift must be comparable to the poloidal projection of the ion parallel velocity, giving $\mathbf{v}_E \sim (B_p/B)V_{i\|} \sim (B_p/B)v_i \ll v_i$.

non-axisymmetric pieces of the potential to higher order.

5.3.1 Higher order ion distribution function

To find $f_i(\mathbf{R}, E, \mu, t)$ to order $\delta_i^2 f_{Mi}$, it is necessary to solve a higher order gyrokinetic Fokker-Planck equation. Similarly, for the higher order potential, it is necessary to find a higher order gyrokinetic vorticity equation. In this section, I show that under certain assumptions, the second order gyrokinetic Fokker-Planck and vorticity equations can be easily deduced from their first order versions.

There has already been some work in transport of toroidal angular momentum in gyrokinetics. For these studies, it was necessary to realize that the Reynolds stress tends to vanish to order $\delta_i^2 p_i$ in the low flow limit, becoming of order $\delta_i^3 p_i$. In references [65, 66, 70] the revised approach is ordering the parallel velocity as comparable to the ion thermal speed. Since for sonic velocities, the plasma can only rotate toroidally [71, 72], a sonic parallel velocity requires, in general, a sonic $E \times B$ drift to cancel its poloidal component. However, sonic $E \times B$ drifts invalidate the gyrokinetic derivation of chapter 3. To avoid this problem, references [65, 66, 70] do not reach sonic $E \times B$ velocities because they take advantage of the expansion parameter $B_p/B \ll 1$. The perpendicular $E \times B$ drift is small compared to the thermal speed by B_p/B, making the traditional gyrokinetic formulation based on subsonic $E \times B$ motion still valid [see figure 5-1]. Under these assumptions, the toroidal velocity is given by $-cR(\partial\phi/\partial\psi)$, and the radial transport of toroidal angular momentum is of order $\delta_i^2 p_i$, as can be found from equation (5.5). The term with the time derivative in equation (5.5) is still negligible if the turbulence reaches its statistical equilibrium, but the collisional term, proven to vanish for up-down symmetric tokamaks in section K.2 of Appendix K, contributes to order $\delta_i^2 p_i$ because the sonic parallel velocity breaks the up-down symmetry by introducing a preferred direction in the Maxwellian. Similarly, the Reynolds stress (5.6) in this case is of order $\delta_i^2 p_i$. In reference [66], the Reynolds stress is calculated employing the δf code GYRO with sonic parallel velocities, and it does not vanish to order $\delta_i^2 p_i$, as expected.

This approach has the disadvantage of making the toroidal velocity only depend on the radial electric field $\partial\phi/\partial\psi$. Density and temperature gradients cannot compete with the radial electric field, and therefore it is not possible to recover naturally the isothermal radial Maxwell-Boltzmann solution, or the dependence of the velocity on the temperature gradient. I propose an alternative approach with subsonic velocities that at the same time avoids solving a full second order gyrokinetic equation. It exploits the extra expansion parameter $B_p/B \ll 1$ in a different manner.

In the new ordering with $B_p/B \ll 1$, the parallel gradient is of order $1/qR$, with R the major radius and $q \sim B/B_p \gg 1$ the safety factor. As in section 4.1, the ion and electron zeroth order distribution functions are assumed to be stationary Maxwellians with only radial dependence, i.e., $f_i \simeq f_{Mi}(\psi)$ and $f_e \simeq f_{Me}(\psi)$. With the new orderings, the size of the first order correction to the Maxwellian h_{i1} changes depending on the nature of the correction, i.e., depending on whether it is turbulent, due to non-axisymmetric potential fluctuations, or neoclassical, given by the long wavelength, axisymmetric pieces. For the neoclassical banana regime pieces, the term $\mathbf{v}_E \cdot \nabla_\mathbf{R} f_i$ is negligible [recall the discussion in section 4.1], and the neoclassical correction h_{i1}^{nc} is determined by a balance between the parallel streaming term $u\hat{\mathbf{b}}\cdot\nabla_\mathbf{R} h_{i1}^{\mathrm{nc}} \sim (v_i/qR)h_{i1}^{\mathrm{nc}}$ and the magnetic drift term $\mathbf{v}_M \cdot \nabla_\mathbf{R} f_{Mi} \sim (\rho_i/R) v_i f_{Mi}/a$, giving a neoclassical piece of order $h_{i1}^{\mathrm{nc}} \sim q\delta_i f_{Mi}$ when the transit average collisional constraint is satisfied. On the other hand, in tokamaks, turbulence is driven by toroidal drift wave modes in which the parallel streaming term $u\hat{\mathbf{b}} \cdot \nabla_\mathbf{R} f_i$ is of secondary importance. The turbulent contributions $h_{i1}^{\rm tb}$ are determined by the competition between the magnetic drift term $\mathbf{v}_M \cdot \nabla_{\mathbf{R}} h_{i1}^{\rm tb}$ and the $E \times B$ drift term $\mathbf{v}_E^{\rm tb} \cdot \nabla_{\mathbf{R}} h_{i1}^{\rm tb}$. The orderings of (3.1) still hold, giving that for $k_\perp \rho_i \sim 1$, $h_{i1}^{\rm tb}/f_{Mi} \sim e\phi^{\rm tb}/T_e \sim \delta_i$. The turbulent contribution $h_{i1}^{\rm tb}$ is then smaller than the neoclassical piece $h_{i1}^{\rm nc}$ by a factor of $1/q$.

To calculate the axisymmetric radial electric field in a completely general manner, the gyrokinetic treatment needs to be extended to provide the pieces $h_{i2} \sim \delta_i^2 f_{Mi}$ of the ion distribution function. This would require calculating the gyrokinetic Fokker-Planck equation (3.39) to higher order, i.e., obtaining the time derivatives of the gyrokinetic variables $\mathbf{R}$, E and μ to an order higher in δ_i. However, as just noted, there are terms that are larger by $q \gg 1$. Instead of calculating the complete $O(\delta_i^2 f_{Mi} v_i/L)$ gyrokinetic Fokker-Planck equation, I will only keep the terms that are larger by q. To identify these terms, I let $f_i = f_{Mi} + h_{i1} + h_{i2} + \ldots$ and then write the gyrokinetic equation for the second order perturbation as

$$\frac{\partial h_{i2}}{\partial t} + [u\hat{\mathbf{b}}(\mathbf{R}) + \mathbf{v}_d] \cdot \nabla_{\mathbf{R}} h_{i2} - \langle C\{f_i\}\rangle^{(2)} = -\mathbf{v}_d \cdot \nabla_{\mathbf{R}} h_{i1}^{\rm nc}$$
$$-\dot{\mathbf{R}}^{(2)} \cdot \nabla_{\mathbf{R}} (f_{Mi} + h_{i1}^{\rm tb}) + \frac{Ze}{M}[u\hat{\mathbf{b}}(\mathbf{R}) + \mathbf{v}_M] \cdot \nabla_{\mathbf{R}} \langle\phi\rangle \frac{\partial h_{i1}}{\partial E} + \dot{E}^{(2)} \frac{M f_{Mi}}{T_i}, \quad (5.25)$$

with $\langle C\{f_i\}\rangle^{(2)} = \langle C\{f_i\}\rangle - \langle C^{(\ell)}\{f_{Mi} + h_{i1}\}\rangle$, $\dot{\mathbf{R}}^{(2)} = \langle\dot{\mathbf{R}}\rangle - [u\hat{\mathbf{b}}(\mathbf{R}) + \mathbf{v}_d]$ and $\dot{E}^{(2)} = \langle\dot{E}\rangle + (Ze/M)[u\hat{\mathbf{b}}(\mathbf{R}) + \mathbf{v}_M] \cdot \nabla_{\mathbf{R}} \langle\phi\rangle$. Here, $\langle C\{f_i\}\rangle$, $\langle\dot{E}\rangle$ and $\langle\dot{\mathbf{R}}\rangle$ are calculated to order $\delta_i^2 \nu_{ii} f_{Mi}$, $\delta_i^2 v_i^3/L$ and $\delta_i^2 v_i$, respectively; an order higher than in equation (3.39). Notice that the first order correction h_{i1} enters differently depending on its nature. The turbulent short wavelength piece $h_{i1}^{\rm tb}$ has large gradients and it is multiplied by the small quantity $\dot{\mathbf{R}}^{(2)}$, while the gradient of the neoclassical piece $h_{i1}^{\rm nc}$ is small but is multiplied by the lowest order term $\mathbf{v}_d \gg \dot{\mathbf{R}}^{(2)}$.

On the right side of equation (5.25), the dominant terms are $-\mathbf{v}_d \cdot \nabla_{\mathbf{R}} h_{i1}^{\rm nc}$ and $(Ze/M)[u\hat{\mathbf{b}}(\mathbf{R}) + \mathbf{v}_M] \cdot \nabla_{\mathbf{R}} \langle\phi\rangle (\partial h_{i1}^{\rm nc}/\partial E)$ because $h_{i1}^{\rm nc}$ is larger than all other terms by a factor of q. The higher order corrections $\dot{\mathbf{R}}^{(2)}$ and $\dot{E}^{(2)}$ are finite gyroradius correction that do not contain any q factors. Since $h_{i1}^{\rm nc}$ determines the parallel velocity and the parallel heat flow, the term $\mathbf{v}_d \cdot \nabla_{\mathbf{R}} h_{i1}^{\rm nc}$ represents the effect of the gradient of the parallel velocity and parallel heat flow on turbulence. This term is not kept in δf flux tube codes because only the short wavelength pieces of the first order correction to the distribution function are calculated. We see from (5.25) that it is possible to simply add this neoclassical term to the δf gyrokinetic equation along with $(Ze/M)[u\hat{\mathbf{b}}(\mathbf{R}) + \mathbf{v}_M] \cdot \nabla_\mathbf{R}\langle\phi\rangle(\partial h_{i1}^{\mathrm{nc}}/\partial E)$. In full f codes, the distribution function is solved from the lowest order equation (3.39). In this equation, the terms $-\mathbf{v}_d \cdot \nabla_\mathbf{R} h_{i1}^{\mathrm{nc}}$ and $(Ze/M)[u\hat{\mathbf{b}}(\mathbf{R}) + \mathbf{v}_M] \cdot \nabla_\mathbf{R}\langle\phi\rangle(\partial h_{i1}^{\mathrm{nc}}/\partial E)$ are naturally included, so it is not necessary to write the gyrokinetic equation to higher order than (3.39).

As for h_{i1}, the function h_{i2} has a turbulent piece h_{i2}^{tb}, and a neoclassical piece h_{i2}^{nc}. The turbulent piece is given by the balance between the drifts $\mathbf{v}_d \cdot \nabla_\mathbf{R} h_{i2}^{\mathrm{tb}} \sim \delta_i v_i k_\perp h_{i2}^{\mathrm{tb}}$ and the driving term $\mathbf{v}_d \cdot \nabla_\mathbf{R} h_{i1}^{\mathrm{nc}} \sim q\delta_i^2 v_i f_{Mi}/a$, giving $h_{i2}^{\mathrm{tb}} \sim q\delta_i^2 f_{Mi}$ for $k_\perp \rho_i \sim 1$. The neoclassical piece is a result of a balance between the parallel streaming term $u\hat{\mathbf{b}} \cdot \nabla_\mathbf{R} h_{i2}^{\mathrm{nc}} \sim (v_i/qR) h_{i2}^{\mathrm{nc}}$ and the magnetic drift term $\mathbf{v}_M \cdot \nabla_\mathbf{R} h_{i1}^{\mathrm{nc}} \sim (\rho_i/R) v_i q \delta_i f_{Mi}/a$, leading to $h_{i2}^{\mathrm{nc}} \sim q^2\delta_i^2 f_{Mi}$.

Since h_{i2}^{tb} is larger than $\delta_i^2 f_{Mi}$ by a factor of q, the second order corrections $\mathbf{R}_2$ and E_2 give negligible contributions to the second order piece of the distribution function. To see this, Taylor expand $f_i(\mathbf{R}, E, \mu, t)$ around $\mathbf{R}_g = \mathbf{r} + \Omega_i^{-1}\mathbf{v} \times \hat{\mathbf{b}}$, E_0 and μ_0. Then, the terms $\mathbf{R}_2 \cdot \nabla_{\mathbf{R}_g} f_{ig}$ and $E_2(\partial f_{Mi}/\partial E_0)$, of order $\delta_i^2 f_{Mi}$, are negligible. This fact simplifies the integration in velocity space in (5.10), since it is enough to keep only the first order corrections $\mathbf{R}_1$, E_1 and μ_1. Finally, the gyrophase dependent piece $\tilde{f}_i$, defined in (3.38), vanishes $[\tilde{f}_i \simeq -\Omega_i^{-1} \int^\varphi d\varphi' (C^{(\ell)}\{h_{i1}^{\mathrm{nc}}\} - \langle C^{(\ell)}\{h_{i1}^{\mathrm{nc}}\}\rangle) = 0$ since $\partial h_{i1}^{\mathrm{nc}}/\partial \varphi_0 = 0]$.

5.3.2 Higher order electrostatic potential

In this subsection, I find the gyrokinetic vorticity equations (4.45) and (4.53) to higher order. To simplify the derivation, I limit myself to the short wavelength, non-axisymmetric contributions to the vorticity equation – the ones responsible for the turbulence. The flux surface averaged component of the potential will be given by the conservation of toroidal angular momentum. For the usually unimportant GAM response [68, 69], it is enough to retain the first order terms, already in the lower order gyrokinetic vorticity equations (4.45) and (4.53).

Finding the higher order vorticity equation becomes a simple task because the higher order corrections to the gyrokinetic variables $\mathbf{R}_2$ and E_2 are negligible. Quasineutrality must be enforced for the gyrokinetic Fokker-Planck equation with the new terms $\mathbf{v}_d \cdot \nabla_\mathbf{R} h_{i1}^{\rm nc}$ and $(Ze/M)[u\hat{\mathbf{b}}(\mathbf{R}) + \mathbf{v}_M] \cdot \nabla_\mathbf{R} \langle \phi \rangle (\partial h_{i1}^{\rm nc}/\partial E)$, and a $O(q\delta_i^2 f_{Mi})$ distribution function given by

$$f_i(\mathbf{R}, E, \mu, t) \simeq f_{iG} + \frac{Ze\tilde{\phi}}{M}\left[\frac{\partial}{\partial E_0}(f_{Mi} + h_{i1}^{\rm nc}) + \frac{1}{B}\frac{\partial h_{i1}^{\rm nc}}{\partial \mu_0}\right], \tag{5.26}$$

with $f_{iG} \equiv f_i(\mathbf{R}_g, E_0, \mu_g, t)$, $\mathbf{R}_g = \mathbf{r} + \Omega_i^{-1}\mathbf{v} \times \hat{\mathbf{b}}$ and $\mu_g = \mu - Ze\tilde{\phi}/MB \neq \mu_0$. In expression (5.26), $h_{i1}^{\rm nc}(\mathbf{R}, E, \mu, t) \simeq h_{i1}^{\rm nc}(\mathbf{r}, E_0, \mu_0, t)$. It is convenient to extract the $Ze\tilde{\phi}/MB$ portion of μ by introducing μ_g so we can take advantage of previous results.

As I did in section 4.3, I will find the vorticity equations by taking moments of the gyrokinetic Fokker-Planck equation (3.39); in this case to order $q\delta_i^2 f_{Mi} v_i/L$. To write (3.39) in physical phase space, I will first find the evolution equation for f_{iG}. Equation (3.39) gives the evolution of f_{iG} if $\mathbf{R}$ is replaced by $\mathbf{R}_g$, E by E_0 and μ by μ_g, giving to order $q\delta_i^2 f_{Mi} v_i/L$

$$\left.\frac{\partial f_{iG}}{\partial t}\right|_{\mathbf{r},\mathbf{v}} + [u_g\hat{\mathbf{b}}(\mathbf{R}_g) + \mathbf{v}_{dg}] \cdot \left[\nabla_{\mathbf{R}_g} f_{iG} - \frac{Ze}{M}\nabla_{\mathbf{R}_g}\langle\phi\rangle \frac{\partial}{\partial E_0}(f_{Mi} + h_{i1}^{\rm nc})\right] = \langle C\{f_i\}\rangle|_{\mathbf{R}\to\mathbf{R}_g, E\to E_0, \mu\to\mu_g}, \tag{5.27}$$

with

$$\mathbf{v}_{dg} \equiv \frac{\mu_g}{\Omega_i(\mathbf{R}_g)}\hat{\mathbf{b}}(\mathbf{R}_g) \times \nabla_{\mathbf{R}_g} B(\mathbf{R}_g) + \frac{u_g^2}{\Omega_i(\mathbf{R}_g)}\hat{\mathbf{b}}(\mathbf{R}_g) \times \boldsymbol{\kappa}(\mathbf{R}_g) - \frac{c}{B(\mathbf{R}_g)}\nabla_{\mathbf{R}_g}\langle\phi\rangle(\mathbf{R}_g, \mu_g, t) \times \hat{\mathbf{b}}(\mathbf{R}_g) \tag{5.28}$$

and

$$u_g \equiv \sqrt{2[E_0 - \mu_g B(\mathbf{R}_g)]} = \sqrt{2[E - \mu B(\mathbf{R})]} = u, \tag{5.29}$$

where $E_1 = Ze\tilde{\phi}/M$ and $\mu - \mu_g = Ze\tilde{\phi}/MB(\mathbf{R}_g)$ cancel exactly to give the second equality. In equation (5.27), I have neglected terms of order $\delta_i^2 f_{Mi}$ by taking the approximation $\partial f_{iG}/\partial E_0 \simeq \partial(f_{Mi} + h_{i1}^{\rm nc})/\partial E_0$.

Equation (5.27) needs to be written in the physical phase space variables. To the order of interest, $\nabla_{\mathbf{R}_g} f_{iG} = \nabla_{\mathbf{R}_g}\mathbf{r} \cdot \overline{\nabla} f_{iG} + \nabla_{\mathbf{R}_g}\mu_0 (\partial h_{i1}^{\rm nc}/\partial \mu_0)$ and $\nabla_{\mathbf{R}_g}\langle\phi\rangle \simeq \nabla_{\mathbf{R}_g}\mathbf{r} \cdot \overline{\nabla}\langle\phi\rangle$, with $\nabla_{\mathbf{R}_g}\mu_0 = -\nabla_{\mathbf{R}_g}\mu_{10} \simeq -\overline{\nabla}\mu_{10}$ and $\mu_g - \mu_0 = \mu_{10} = \mu_1 - Ze\widetilde{\phi}/MB$. Then, equation (5.27) becomes

$$\left.\frac{\partial f_{iG}}{\partial t}\right|_{\mathbf{r},\mathbf{v}} + [u\hat{\mathbf{b}}(\mathbf{R}_g) + \mathbf{v}_{M0} + \mathbf{v}_{E0}] \cdot \nabla_{\mathbf{R}_g}\mathbf{r} \cdot \left[\overline{\nabla} f_{iG} - \frac{Ze}{M}\overline{\nabla}\langle\phi\rangle \frac{\partial}{\partial E_0}(f_{Mi} + h_{i1}^{\rm nc})\right] - v_{||}\hat{\mathbf{b}} \cdot \overline{\nabla}\mu_{10}\frac{\partial h_{i1}^{\rm nc}}{\partial \mu_0} = \langle C\{f_i\}\rangle. \quad (5.30)$$

Here, the term $(\mathbf{v}_{dg} - \mathbf{v}_{M0} - \mathbf{v}_{E0}) \cdot \nabla_{\mathbf{R}} f_{iG} \sim \delta_i^2 f_{Mi} v_i/L$ has been neglected. In the collisional term $\langle C\{f_i\}\rangle$ in (5.30), it is necessary to consider pieces of order $q\delta_i^2 \nu_{ii} f_{Mi}$ that come from the gyroaverage of $C^{(\ell)}\{h_{i1}^{\rm nc}\}$ performed holding the higher order gyrokinetic variables $\mathbf{R}_g$, E and μ fixed, with $C^{(\ell)}$ the linearized collision operator. These terms are not considered in Appendix H, where the gyrokinetic variables are approximated by $\mathbf{R}_g$, E_0 and μ_0. Fortunately, the collision frequency is usually small, making these terms negligible. Ignoring these terms and assuming that the collision operator can be treated as in Appendix H is reasonable and simplifies the rest of the derivation. Finally, employing the results in section F.2 of Appendix F, equation (5.30) gives

$$\left.\frac{\partial f_{iG}}{\partial t}\right|_{\mathbf{r},\mathbf{v}} + \frac{v_{||}}{B}\Bigg\{\overline{\nabla} \cdot \left[\frac{B}{v_{||}} f_{iG}\left(\dot{\mathbf{R}} \cdot \nabla_{\mathbf{R}_g}\mathbf{r}\right)\right] - \frac{\partial}{\partial \mu_0}\left[(f_{Mi} + h_{i1}^{\rm nc})\mathbf{B} \cdot \overline{\nabla}\mu_{10}\right] - \frac{\partial}{\partial \varphi_0}\left[(f_{Mi} + h_{i1}^{\rm nc})\mathbf{B} \cdot \overline{\nabla}\varphi_{10}\right] - \frac{\partial}{\partial E_0}\left[\frac{B}{v_{||}}(f_{Mi} + h_{i1}^{\rm nc})\frac{Ze}{M}\left(\dot{\mathbf{R}} \cdot \nabla_{\mathbf{R}_g}\mathbf{r}\right) \cdot \overline{\nabla}\langle\phi\rangle\right]\Bigg\} = \langle C\{f_i\}\rangle, \quad (5.31)$$

with $\dot{\mathbf{R}} \cdot \nabla_{\mathbf{R}_g}\mathbf{r}$ from (4.15). Notice that equation (5.31) is equivalent to equation (4.23) except for the changes $f_{Mi} \rightarrow f_{Mi} + h_{i1}^{\rm nc}$ and $f_{ig} \rightarrow f_{iG}$.

The same moment equations that were obtained with (4.23) can be found for equation (5.31), but now with f_{iG} instead of f_{ig} and $f_{Mi} + h_{i1}^{\rm nc}$ instead of f_{Mi}. The differences between equations (4.23) and (5.31) are enough to invalidate some of the cancellations that were found in Appendix G. For example, in the momentum conservation equation for ions (4.38), the piece of the viscosity $M\int d^3v\, f_{ig} v_{||0} \mathbf{v}_\perp$, with $v_{||0}$ given in (4.16), reduced to $M\int d^3v\, f_{ig} v_{||} \mathbf{v}_\perp$ because the small correction $M(v_{||0} - v_{||})\mathbf{v}_\perp$, composed of terms that are either odd in $v_{||}$ or in $\mathbf{v}_\perp$, vanished when integrated over the lowest order Maxwellian. In the new vorticity equation, terms of order $q\delta_i^2 p_i/L$ must be retained in the momentum conservation equation, leading to a new non-vanishing term $M\int d^3v\, h_{i1}^{\mathrm{nc}}(v_{||0} - v_{||})\mathbf{v}_\perp$. Fortunately, most terms like this one are axisymmetric and only enter in the calculation of the flux surface averaged radial electric field and the higher order GAM response. To calculate the flux surface averaged radial electric field, the toroidal angular momentum conservation equation is to be used, and to simplify the equations I ignore the higher order corrections to the GAM response that is expected to be unimportant, as already discussed. Therefore, the vorticity equation is only employed to solve for the non-axisymmetric, turbulent fluctuations in the potential, and many of the cancellations employed in Appendix G are recovered. Additionally, the difference $f_{iG} - f_{ig} \simeq (\mu_g - \mu_0)(\partial h_{i1}^{\mathrm{nc}}/\partial \mu_0) \sim q\delta_i^2 f_{Mi}$ is also a long wavelength, axisymmetric piece, and it will not enter in the equations for the non-axisymmetric electric field. Then, f_{iG} can be approximated by the simpler distribution function $f_{ig} \equiv f_i(\mathbf{R}_g, E_0, \mu_0, t)$. The generalized particle conservation equation can be found following section G.1 of Appendix G by ignoring the purely axisymmetric contributions of h_{i1}^{nc}. Then, the non-axisymmetric component of particle conservation is

$$\frac{\partial}{\partial t}\left(n_i - n_{ip}^{(2)}\right) + \nabla \cdot \left(n_i V_{ig||}^{(2)} \hat{\mathbf{b}} + n_i \mathbf{V}_{igd} + n_i \mathbf{V}_{iE} + n_i \tilde{\mathbf{V}}_i + n_i \mathbf{V}_{iC}\right) = 0, \tag{5.32}$$

with

$$n_{ip}^{(2)} = -\int d^3v\, \frac{Ze\tilde{\phi}}{T_i} f_{Mi} + \int d^3v\, \frac{Ze\tilde{\phi}}{M}\left(\frac{\partial h_{i1}^{\mathrm{nc}}}{\partial E_0} + \frac{1}{B}\frac{\partial h_{i1}^{\mathrm{nc}}}{\partial \mu_0}\right) \tag{5.33}$$

and

$$n_i V_{ig||}^{(2)} = \int d^3v\, f_{ig} v_{||} \neq n_i V_{i||}. \tag{5.34}$$

The rest of the terms in (5.32) are as defined in subsection 4.3.3, although now the turbulent second order contribution h_{i2}^{tb} implicitly enters in the integrals via the solution

f_i to the full gyrokinetic equation (3.39). The non-axisymmetric piece of the momentum conservation equation can be obtained following section G.2 of Appendix G, finally giving

$$\frac{\partial}{\partial t}(n_i M \mathbf{V}_{ig}) + \hat{\mathbf{b}}[\hat{\mathbf{b}} \cdot \nabla p_{ig||} + (p_{ig||} - p_{ig\perp})\nabla \cdot \hat{\mathbf{b}} + \nabla \cdot \boldsymbol{\pi}_{ig||}] + \nabla \cdot \overleftrightarrow{\boldsymbol{\pi}}_{ig\times} = \\ -Zen_i \hat{\mathbf{b}} \left(\hat{\mathbf{b}} + \frac{V_{i||}^{\mathrm{nc}}}{\Omega_i} \hat{\mathbf{b}} \times \boldsymbol{\kappa} \right) \cdot \nabla \phi + \tilde{F}_{iE}^{(2)} \hat{\mathbf{b}} + \mathbf{F}_{iB}^{(2)} + \mathbf{F}_{iC}, \quad (5.35)$$

with $n_i V_{i||}^{\mathrm{nc}} = \int d^3v\, v_{||} h_{i1}^{\mathrm{nc}}$,

$$\tilde{F}_{iE}^{(2)} = Ze \int d^3v\, (f_{Mi} + h_{i1}^{\mathrm{nc}}) \left(\hat{\mathbf{b}} + \frac{v_{||}}{\Omega_i} \hat{\mathbf{b}} \times \boldsymbol{\kappa} + \frac{1}{\Omega_i} \overline{\nabla} \times \mathbf{v}_\perp \right) \cdot \overline{\nabla} \tilde{\phi} \quad (5.36)$$

and

$$\mathbf{F}_{iB}^{(2)} = \int d^3v\, M f_{ig} v_{||} \hat{\mathbf{b}} \cdot \overline{\nabla} \mathbf{v}_\perp + \int d^3v\, \frac{Mc}{B} (f_{Mi} + h_{i1}^{\mathrm{nc}}) (\overline{\nabla} \tilde{\phi} \times \hat{\mathbf{b}}) \cdot \overline{\nabla} \mathbf{v}_\perp. \quad (5.37)$$

Again, the rest of the terms are as defined in subsection 4.3.4, but with the higher order piece h_{i2}^{tb} implicitly included.

The moment equations (5.32) and (5.35) can be used to extend the gyrokinetic vorticity equations (4.45) and (4.53) to order $q\delta_i^2 en_e v_i/L$. Combining equation (5.32) with the electron number conservation equation (4.26) gives the vorticity equation

$$\frac{\partial}{\partial t}(Zen_{ip}^{(2)}) = \nabla \cdot \left(J_{g||}^{(2)} \hat{\mathbf{b}} + \mathbf{J}_{gd} + \tilde{\mathbf{J}}_i + Zen_i \tilde{\mathbf{V}}_i + Zen_i \mathbf{V}_{iC} \right), \quad (5.38)$$

with

$$J_{g||}^{(2)} = Zen_i V_{ig||}^{(2)} - en_e V_{e||}, \quad (5.39)$$

and the rest of the terms as defined in subsection 4.4.1 with h_{i2}^{tb} implicit. Finally, combining (5.38) with (5.35) gives

$$\frac{\partial \varpi_G^{(2)}}{\partial t} = \nabla \cdot \left[J_{g||}^{(2)} \hat{\mathbf{b}} + \mathbf{J}_{gd} + \tilde{\mathbf{J}}_{i\phi}^{(2)} + \frac{c}{B} \hat{\mathbf{b}} \times (\nabla \cdot \overleftrightarrow{\boldsymbol{\pi}}_{iG}) + Zen_i \mathbf{V}_{iC} - \frac{c}{B} \hat{\mathbf{b}} \times \mathbf{F}_{iC} \right], \quad (5.40)$$

with

$$\varpi_G^{(2)} = \nabla\cdot\left(\frac{Ze}{\Omega_i}\int d^3v\, f_{ig}\mathbf{v}\times\hat{\mathbf{b}}\right) - \int d^3v\,\frac{Z^2e^2\tilde{\phi}}{T_i}f_{Mi} + \int d^3v\,\frac{Z^2e^2\tilde{\phi}}{M}\left(\frac{\partial h_{i1}^{\mathrm{nc}}}{\partial E_0} + \frac{1}{B}\frac{\partial h_{i1}^{\mathrm{nc}}}{\partial \mu_0}\right), \tag{5.41}$$

$$\tilde{\mathbf{J}}_{i\phi}^{(2)} = \tilde{\mathbf{J}}_i - \frac{Zec}{B\Omega_i}\hat{\mathbf{b}}\times\int d^3v\,(f_{Mi} + h_{i1}^{\mathrm{nc}})(\nabla\tilde{\phi}\times\hat{\mathbf{b}})\cdot\nabla\mathbf{v}_\perp, \tag{5.42}$$

and the rest of the terms as defined in subsection 4.4.2. Both equations (5.38) and (5.40) can be used to find the short wavelength non-axisymmetric pieces of the potential consistent with the higher order f_i. The flux surface averaged component of the potential is given by the conservation equation of toroidal angular momentum. Finally, it is important to remember that dropping the higher order axisymmetric terms in equations (5.38) and (5.40) implies dropping the higher order corrections to the GAM response [68, 69]. I expect this response to be unimportant for core turbulence based on previous experience with tokamak core simulations.

5.4 Discussion

The fast time average of equation (5.10) provides the irreversible transport of toroidal angular momentum across the tokamak. This irreversible transport determines the toroidal rotation profile and hence the self-consistent radial electric field.

It is expected that the fast time average of equation (5.10) is of order $\delta_i^3 p_i$, requiring then a distribution function good to order $\delta_i^2 f_{Mi}$ and a potential consistent with this higher order distribution function. It is in principle possible (if not in practice) to obtain a gyrokinetic equation able to provide such accurate results, but in this thesis I propose an alternative approach. To simplify the problem, I take advantage of the expansion in $B_p/B \ll 1$ to prove that the gyrokinetic Fokker-Planck equation (3.39) is enough to obtain the distribution function up to order $q\delta_i^2 f_{Mi}$. In δf flux tube codes, the distribution function cannot be calculated to higher order than $\delta_i f_{Mi}$ because of the present implementation technique, but these codes can be adapted by adding terms that contain the first order neoclassical correction h_{i1}^{nc}. In full f codes, the gyrokinetic equation (3.39) is fully implemented. Once the turbulence has

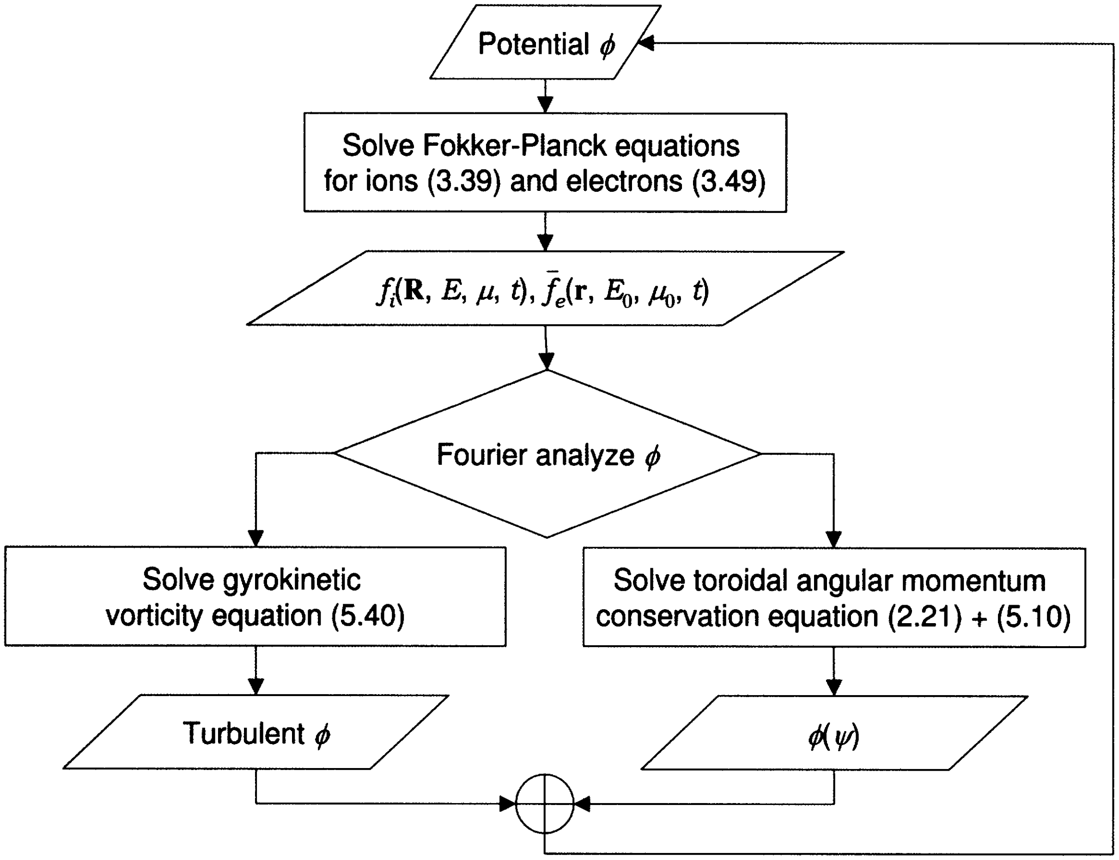

Figure 5-2: Procedure proposed to obtain the long wavelength axisymmetric radial electric field.

reached statistical equilibrium, the long wavelength axisymmetric flows must be close to the neoclassical solution, and the orderings described in this chapter hold, leading to solutions valid up to $q\delta_i^2 f_{Mi}$.

Finally, I have extended the gyrokinetic vorticity equations (4.45) and (4.53), and written them in (5.38) and (5.40). These vorticity equations have only been found for non-axisymmetric pieces of the potential, giving then the non-axisymmetric turbulent fluctuations consistent with the higher order f_i. The flux surface averaged component of the potential cannot be calculated from these vorticity equations, but it can be solved from the conservation of toroidal angular momentum (2.21).

The method proposed to self-consistently solve for the long wavelength axisymmetric radial electric field in the presence of drift wave turbulence is summarized in figure 5-2. To be specific I employ the higher order gyrokinetic vorticity equation

(5.40). The solution must be reached by time evolution of ϕ, f_i and $\overline{f}_e$. Employing the gyrokinetic Fokker-Planck equation (3.39) and the electron drift kinetic equation (3.49), the ion and electron distribution functions $f_i(\mathbf{R}, E, \mu, t)$ and $\overline{f}_e(\mathbf{r}, E_0, \mu_0, t)$ are evolved in time. To find the corresponding electrostatic potential, we first split it into the flux averaged component and the turbulent piece by, for example, Fourier analyzing it into toroidal and poloidal modes. For the turbulent pieces, we must employ the higher order gyrokinetic vorticity equation (5.40). The flux surface averaged component of the potential is obtained by evolving the axisymmetric toroidal rotation $\mathbf{V}_i \cdot \hat{\boldsymbol{\zeta}}$ with equation (2.21) and the viscosity given in (5.10). Once the toroidal rotation is found, the axisymmetric radial electric field is obtained using the lower order result (4.4) to write

$$Rn_i\mathbf{V}_i \cdot \hat{\boldsymbol{\zeta}} = \int d^3v\, f_{ig} R(\mathbf{v} \cdot \hat{\boldsymbol{\zeta}}) - \int d^3v\, \frac{Ze\widetilde{\phi}}{T_i} f_{Mi} R(\mathbf{v}_\perp \cdot \hat{\boldsymbol{\zeta}})$$
$$\xrightarrow[k_\perp \rho_i \to 0]{} IU(\psi) - \frac{cR^2}{Ze}\left(\frac{\partial p_i}{\partial \psi} + Zen_i \frac{\partial \phi}{\partial \psi}\right), \tag{5.43}$$

where to write this last expression I have used the neoclassical relation (4.2). Here, f_{ig} depends implicitly on $\phi(\psi)$. Finally, the turbulent and flux surface averaged components of the electrostatic potential are added to obtain the total electric field, and the distribution functions f_i and $\overline{f}_e$ may be evolved in time again. Notice that equation (5.10), employed here to find the radial electric field, requires that the radial wavelengths be longer than the ion gyroradius. This limitation is probably unimportant since the zonal flow is characterized by $k_\perp \rho_i \sim 0.1$ [12].

Chapter 6

Conclusions

In this thesis, I have proven that the current gyrokinetic treatments, composed of a gyrokinetic Fokker-Planck equation and a gyrokinetic quasineutrality equation, cannot provide the long wavelength, axisymmetric radial electric field. Employing the vorticity equation (2.9), I first showed that setting the radial current to zero to obtain the axisymmetric radial electric field is equivalent to solving the toroidal angular momentum conservation equation, given in (2.21).

In chapter 3, I present a new derivation of electrostatic gyrokinetics that generalizes the linear treatment of [58, 59]. This derivation is useful in chapter 4 to study the current conservation or vorticity equation in steady state turbulence. To simplify the problem, I assume that, in statistical equilibrium, the turbulent fluctuations within a flux surface must be small by $\delta_i = \rho_i/L$ because of the fast transport along magnetic field lines. Then, the long wavelength axisymmetric flows must remain neoclassical, and the tokamak is intrinsically ambipolar even in the presence of turbulence, i.e., $\langle \mathbf{J} \cdot \nabla \psi \rangle_\psi \simeq 0$ for any long wavelength axisymmetric radial electric field. According to the estimate in (4.9), the radial current density associated with transport of toroidal angular momentum is so small that modern gyrokinetic treatments are unable to self-consistently calculate the long wavelength radial electric field. For most codes, long wavelengths are those above $\sqrt{\rho_i L}$.

To solve this issue, I propose to solve a vorticity equation instead of the gyrokinetic quasineutrality equation. The vorticity equation has the advantage of showing explicitly the dependence on the transport of toroidal angular momentum. I have derived two approximate gyrokinetic vorticity equations. Vorticity equation (4.45) is similar to the gyrokinetic quasineutrality equation, and vorticity equation (4.53), on the other hand, is constructed to resemble the full vorticity equation (2.9). The two gyrokinetic vorticity equations (4.45) and (4.53) are equivalent to the full vorticity equation (2.9) within terms of order δ_i. The long wavelength radial electric field cannot be found from these gyrokinetic vorticity equations because they are missing crucial terms. However, they satisfy a very desirable property explicitly, namely, the long wavelength toroidal velocity tends to be constant for the short turbulence saturation time scales.

To complement the gyrokinetic vorticity equation, I propose using the conservation equation for the toroidal angular momentum (2.21), where the toroidal-radial component of the ion viscosity $\langle R\hat{\boldsymbol{\zeta}} \cdot \overleftrightarrow{\boldsymbol{\pi}}_i \cdot \nabla\psi \rangle_\psi$ is given to order $\delta_i^3 p_i$ in (5.10). Unfortunately, expression (5.10) requires a distribution function and a potential of order higher than calculated in gyrokinetic codes. In section 5.3, I show that, for $q \gg 1$, the ion distribution function can be found to high enough order by employing the full f gyrokinetic equation (3.39). The gyrokinetic vorticity equations (4.45) and (4.53), however, have to be extended to determine the higher order potential. Equation (5.38) and (5.40) are the higher order versions of (4.45) and (4.53).

To summarize, to obtain the self-consistent electric field, it is necessary to refrain from using the lower order gyrokinetic quasineutrality equation. Instead, the electric field has to be found by employing a higher order formulation like the proposed vorticity equation (2.9). Moreover, since the axisymmetric contributions to the vorticity equation are equivalent to the conservation of toroidal angular momentum, only the non-axisymmetric pieces of the potential must be found by employing higher order gyrokinetic vorticity equations like equation (5.38) or equation (5.40). The axisymmetric electric field should be found from (2.21) employing the toroidal-radial component of the ion viscosity in (5.10). At the same time, the ion distribution function evolves according to (3.39). In order to implement this methodology, I foresee several steps. First, the lowest order gyrokinetic vorticity equations (4.45) and (4.53) should be implemented in δf flux tube codes. These codes are well understood and easy to study. Equation (4.45) is appealing since it is very similar to the gyrokinetic quasineutrality equation. If the vorticity equations show good numerical behavior in δf flux tube codes, they should be then implemented in full f codes. For runs that stay below transport time scales, these vorticity equations are still valid. Finally, the transport of toroidal angular momentum given in (5.10) must be studied. To do so, it is necessary to calculate the potential and distribution function to higher order, requiring then the higher order versions of the gyrokinetic vorticity equations (5.38) and (5.40). Again, it will probably be easier to study these equations and the transport of toroidal angular momentum in δf flux tube codes first, and then, finally, implement this method in full f models.

Appendix A

Derivation of the gyrokinetic variables

In this Appendix the detailed calculation of the gyrokinetic variables is carried out. In section A.1, the gyrokinetic variables are computed to first order in δ_i. In section A.2, the gyrokinetic variables $\mathbf{R}$ and E are extended to second order, and the gyrokinetic magnetic moment μ is proven to be an adiabatic invariant to higher order. Finally, in section A.3, the Jacobian of the transformation from the variables $\mathbf{r}$, $\mathbf{v}$ to the gyrokinetic variables is calculated. The Jacobian is employed to write the gyrokinetic equation in conservative form.

A.1 First order gyrokinetic variables

It is convenient to express any term that contains the electrostatic potential ϕ in gyrokinetic variables, mainly because the electrostatic potential components with $k_\perp \rho_i \sim 1$ cannot be Taylor expanded. In order to do so, I will develop some useful relations involving the potential ϕ in subsection A.1.1. With these relations, the first order corrections, $\mathbf{R}_1$, E_1, φ_1 and μ_1, are derived.

A.1.1 Useful relations for ϕ

I first derive all possible gyrokinetic partial derivatives of ϕ and their relation to one another. To do so, only $\mathbf{R} = \mathbf{r} + \Omega_i^{-1}\mathbf{v} \times \hat{\mathbf{b}} + O(\delta_i^2 L)$ is needed.

The derivative respect to the gyrocenter position is

$$\nabla_{\mathbf{R}}\phi(\mathbf{r}) = \nabla\phi + \nabla_{\mathbf{R}}(\mathbf{r} - \mathbf{R}) \cdot \nabla\phi = \nabla\phi + O(\delta_i T_e/eL) \simeq \nabla\phi. \tag{A.1}$$

The derivative respect to the energy is

$$\frac{\partial\phi}{\partial E} = \frac{\partial}{\partial E}(\mathbf{r} - \mathbf{R}) \cdot \nabla\phi = O(\delta_i^2 M/e) \simeq 0, \tag{A.2}$$

since $\mathbf{r} - \mathbf{R}$ only depends on E at $O(\delta_i^2 L)$.

Using $\mathbf{r} - \mathbf{R} \propto \sqrt{\mu}(\hat{\mathbf{e}}_1 \sin\varphi - \hat{\mathbf{e}}_2 \cos\varphi)$, the derivatives with respect to μ and φ are calculated to be

$$\frac{\partial\phi}{\partial\mu} = \frac{\partial}{\partial\mu}(\mathbf{r} - \mathbf{R}) \cdot \nabla\phi \simeq -\frac{Mc}{Zev_\perp^2}(\mathbf{v} \times \hat{\mathbf{b}}) \cdot \nabla\phi \tag{A.3}$$

and

$$\frac{\partial\phi}{\partial\varphi} = \frac{\partial}{\partial\varphi}(\mathbf{r} - \mathbf{R}) \cdot \nabla\phi \simeq -\frac{1}{\Omega_i}\mathbf{v}_\perp \cdot \nabla\phi. \tag{A.4}$$

I will need more accurate relationship than (A.1) and (A.4) for the second order corrections. They will be developed in subsection A.2.1.

A.1.2 Calculation of $\mathbf{R}_1$

The first order correction $\mathbf{R}_1$ is given by (3.9), where in this case, $Q_0 = \mathbf{R}_0 = \mathbf{r}$. The total derivative of $\mathbf{R}_0$ is $d\mathbf{R}_0/dt = \mathbf{v} = v_{||}\hat{\mathbf{b}} + \mathbf{v}_\perp$, and its gyroaverage gives $\langle d\mathbf{R}_0/dt \rangle = v_{||}\hat{\mathbf{b}} + O(\delta_i v_i)$. By employing $\mathbf{v}_\perp = \partial(\mathbf{v} \times \hat{\mathbf{b}})/\partial\varphi_0$, equation (3.9) gives (3.14).

A.1.3 Calculation of E_1

The first order correction E_1 is given by (3.9), where $Q_0 = E_0 = v^2/2$ and $dQ_0/dt = dE_0/dt = -(Ze/M)\mathbf{v} \cdot \nabla\phi$. It is convenient to write E_1 as a function of $\mathbf{R}$, E, μ and φ. To do so, I use (A.1) and (A.4) to find

$$-\mathbf{v} \cdot \nabla\phi = -v_{||}\hat{\mathbf{b}} \cdot \nabla\phi - \mathbf{v}_\perp \cdot \nabla\phi \simeq -v_{||}\hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}}\phi + \Omega_i \frac{\partial\phi}{\partial\varphi}. \tag{A.5}$$

Notice that $\hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}}\tilde{\phi} \ll \hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}}\langle\phi\rangle$ because $\tilde{\phi}$ is smaller than $\langle\phi\rangle$. As a result, $dE_0/dt \simeq -(Ze/M)v_{||}\hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}}\langle\phi\rangle + (Ze\Omega_i/M)\partial\phi/\partial\varphi$ and $\langle dE_0/dt\rangle = -(Ze/M)v_{||}\hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}}\langle\phi\rangle + O(\delta_i v_i^3/L)$. Then, equation (3.9) gives (3.25).

A.1.4 Calculation of φ_1

The first order correction φ_1 is given by (3.9), where $Q_0 = \varphi_0$. The zeroth order gyrophase φ_0 is defined by equation (2.2). According to this definition, upon using $\nabla_v\varphi_0 = -v_\perp^{-2}\mathbf{v} \times \hat{\mathbf{b}}$ and $\nabla\varphi_0 = (v_{||}/v_\perp^2)\nabla\hat{\mathbf{b}} \cdot (\mathbf{v} \times \hat{\mathbf{b}}) + \nabla\hat{\mathbf{e}}_2 \cdot \hat{\mathbf{e}}_1$, the total derivative of φ_0 is

$$\frac{d\varphi_0}{dt} = -\Omega_i - \frac{Z^2e^2}{M^2c}\frac{\partial\tilde{\phi}}{\partial\mu} + (\mathbf{v} \times \hat{\mathbf{b}}) \cdot \left[\nabla \ln \Omega_i + \frac{v_{||}^2}{v_\perp^2}\hat{\mathbf{b}} \cdot \nabla\hat{\mathbf{b}} - \hat{\mathbf{b}} \times \nabla\hat{\mathbf{e}}_2 \cdot \hat{\mathbf{e}}_1\right] + \frac{v_{||}}{2v_\perp^2}[\mathbf{v}_\perp(\mathbf{v} \times \hat{\mathbf{b}}) + (\mathbf{v} \times \hat{\mathbf{b}})\mathbf{v}_\perp] : \nabla\hat{\mathbf{b}}, \tag{A.6}$$

where the potential $\phi(\mathbf{r}, t)$ and the gyrofrequency $\Omega_i(\mathbf{r})$ have been written as functions of the gyrokinetic variables by using (A.3) and $\Omega_i(\mathbf{r}) \simeq \Omega_i(\mathbf{R}) + (\mathbf{r} - \mathbf{R}) \cdot \nabla\Omega_i$, respectively, and I have used the relations $\langle\mathbf{v}_\perp\mathbf{v}_\perp\rangle \simeq \overline{\mathbf{v}_\perp\mathbf{v}_\perp} = (v_\perp^2/2)(\overleftrightarrow{\mathbf{I}} - \hat{\mathbf{b}}\hat{\mathbf{b}})$ and

$$\mathbf{v}_\perp\mathbf{v}_\perp - \langle\mathbf{v}_\perp\mathbf{v}_\perp\rangle = \frac{1}{2}[\mathbf{v}_\perp\mathbf{v}_\perp - (\mathbf{v} \times \hat{\mathbf{b}})(\mathbf{v} \times \hat{\mathbf{b}})]. \tag{A.7}$$

Here, $\overline{(\ldots)}$ is the gyroaverage holding $\mathbf{r}$, $v_{||}$, $v_\perp$ and t fixed, and $\langle\ldots\rangle$ is the gyroaverage holding $\mathbf{R}$, E, μ and t fixed. These two gyroaverages are equivalent in this case because the functions involved do not have short wavelengths. A detailed derivation of (A.7) and other velocity relations can be found in Appendix B.

In equation (A.6), the function $\overline{\Omega}_i$ is given by equation (3.31). Upon gyroaveraging (A.6), I obtain $\langle d\varphi_0/dt\rangle = -\overline{\Omega}_i + O(\delta_i^2\Omega_i)$. Finally, φ_1 is obtained from (3.9) by employing $\mathbf{v}\times\hat{\mathbf{b}} = -\partial\mathbf{v}_\perp/\partial\varphi_0$ and

$$\mathbf{v}_\perp\mathbf{v}_\perp - (\mathbf{v}\times\hat{\mathbf{b}})(\mathbf{v}\times\hat{\mathbf{b}}) = \frac{1}{2}\frac{\partial}{\partial\varphi_0}[\mathbf{v}_\perp(\mathbf{v}\times\hat{\mathbf{b}}) + (\mathbf{v}\times\hat{\mathbf{b}})\mathbf{v}_\perp], \tag{A.8}$$

giving equation (3.29). Relation (A.8) is proven in Appendix B.

A.1.5 Calculation of μ_1

Calculating μ_1 requires more work than calculating any of the other first order corrections since we want μ to be an adiabatic invariant to all orders of interest. This requirement imposes two conditions to μ_1. One of them is similar to the requirements already imposed to $\mathbf{R}_1$, E_1 and φ_1, $d\mu_0/dt - \Omega_i(\partial\mu_1/\partial\varphi) = \langle d\mu_0/dt\rangle = 0$, but there is an additional condition making $\mu_0+\mu_1$ an adiabatic invariant to first order,

$$\left\langle \frac{d}{dt}(\mu_0+\mu_1)\right\rangle = O\left(\delta_i^2\frac{v_i^3}{BL}\right). \tag{A.9}$$

The solution to both conditions is given by

$$\mu_1 = \frac{1}{\Omega_i}\int^{\varphi} d\varphi' \left(\frac{d\mu_0}{dt} - \left\langle\frac{d\mu_0}{dt}\right\rangle\right) + \langle\mu_1\rangle. \tag{A.10}$$

Notice that the only difference with the result in (3.9) is that the gyrophase independent term, $\langle\mu_1\rangle$, must be retained, making it possible to satisfy condition (A.9).

Employing $\nabla_v\mu_0 = \mathbf{v}_\perp/B$ and $\nabla\mu_0 = -(v_\perp^2/2B^2)\nabla B - (v_{||}/B)\nabla\hat{\mathbf{b}}\cdot\mathbf{v}$, I find that the total derivative for $\mu_0 = v_\perp^2/2B$ is

$$\begin{aligned}\frac{d\mu_0}{dt} = -\frac{Ze}{MB}\mathbf{v}_\perp\cdot\nabla\phi - \frac{v_\perp^2}{2B^2}\mathbf{v}_\perp\cdot\nabla B - \frac{v_{||}^2}{B}\hat{\mathbf{b}}\cdot\nabla\hat{\mathbf{b}}\cdot\mathbf{v}_\perp \\ -\frac{v_{||}}{2B}[\mathbf{v}_\perp\mathbf{v}_\perp - (\mathbf{v}\times\hat{\mathbf{b}})(\mathbf{v}\times\hat{\mathbf{b}})] : \nabla\hat{\mathbf{b}},\end{aligned} \tag{A.11}$$

where I have used the relations $\langle\mathbf{v}_\perp\mathbf{v}_\perp\rangle \simeq \overline{\mathbf{v}_\perp\mathbf{v}_\perp} = (v_\perp^2/2)(\overleftrightarrow{\mathbf{I}} - \hat{\mathbf{b}}\hat{\mathbf{b}})$ and (A.7).

Notice that the gyrophase independent terms in (A.11) cancel exactly due to

$\hat{\mathbf{b}} \cdot \nabla \ln B + \nabla \cdot \hat{\mathbf{b}} = 0$, making μ_0 an adiabatic invariant to zeroth order. The term that contains ϕ in (A.11) is rewritten as a function of the gyrokinetic variables by using (A.4), to give $-(Ze/MB)\mathbf{v}_\perp \cdot \nabla\phi = (Ze\Omega_i/MB)\partial\phi/\partial\varphi$.

Applying (A.10), μ_1 is found to be given by (3.34). To get this result, I have employed $\mathbf{v}_\perp = \partial(\mathbf{v} \times \hat{\mathbf{b}})/\partial\varphi_0$ and (A.8). The average value $\langle\mu_1\rangle = -(v_{||}v_\perp^2/2B\Omega_i)(\hat{\mathbf{b}} \cdot \nabla \times \hat{\mathbf{b}})$ was chosen to ensure that condition (A.9) is satisfied. In previous works [19, 73], it has been noticed that solving (A.9) may be avoided and replaced by imposing the relation $E = [d\mathbf{R}/dt \cdot \hat{\mathbf{b}}(\mathbf{R})]^2/2 + \mu B(\mathbf{R})$ on the gyrokinetic variables. This procedure works in this case, and allows me to find $\langle\mu_1\rangle$. I will prove that the chosen $\langle\mu_1\rangle$ satisfies condition (A.9) in subsection A.2.4.

A.2 Second order gyrokinetic variables

To construct the gyrokinetic variables to second order, higher order relations than the ones developed in subsection A.1.1 are needed to express ϕ as a function of the gyrokinetic variables. These extended relations are deduced in subsection A.2.1. Using them, the second order corrections $\mathbf{R}_2$ and E_2 and the gyrophase independent piece of the first order correction $\langle\mu_1\rangle$ are calculated. The magnetic moment and the gyrophase are not required to higher order.

A.2.1 More useful relations for ϕ

To calculate the second order correction E_2 and the gyrophase independent piece $\langle\mu_1\rangle$, the expressions $\hat{\mathbf{b}} \cdot (\nabla\phi - \nabla_{\mathbf{R}}\phi)$ and $\mathbf{v} \cdot \nabla\phi$ must be given in gyrokinetic variables to order $\delta_i T_e/eL$ and $\delta_i T_e v_i/eL$, respectively.

For $\hat{\mathbf{b}} \cdot (\nabla\phi - \nabla_{\mathbf{R}}\phi)$, I use $\nabla\phi = \nabla\mathbf{R} \cdot \nabla_{\mathbf{R}}\phi + \nabla E(\partial\phi/\partial E) + \nabla\mu(\partial\phi/\partial\mu) + \nabla\varphi(\partial\phi/\partial\varphi)$ to write

$$\hat{\mathbf{b}} \cdot (\nabla\phi - \nabla_{\mathbf{R}}\phi) \simeq \hat{\mathbf{b}} \cdot \nabla\mathbf{R}_1 \cdot \nabla_{\mathbf{R}}\phi + \hat{\mathbf{b}} \cdot \nabla\mu_0 \frac{\partial\phi}{\partial\mu} + \hat{\mathbf{b}} \cdot \nabla\varphi_0 \frac{\partial\phi}{\partial\varphi}, \qquad \text{(A.12)}$$

where I have neglected higher order terms. Here, it is important that the gradient is parallel to the magnetic field, since $\mathbf{R}_2$, μ_1 and φ_1 have pieces with short perpendicular wavelengths that are important for the perpendicular component of the gradient. Employing $\nabla\mathbf{R}_1 = -\Omega_i^{-1}[(\nabla \ln B)(\mathbf{v}\times\hat{\mathbf{b}})+\nabla\hat{\mathbf{b}}\times\mathbf{v}]$, $\nabla\mu_0 = -(v_\perp^2/2B^2)\nabla B-(v_{||}/B)\nabla\hat{\mathbf{b}}\cdot\mathbf{v}$, $\nabla\varphi_0 = (v_{||}/v_\perp^2)\nabla\hat{\mathbf{b}}\cdot(\mathbf{v}\times\hat{\mathbf{b}}) + \nabla\hat{\mathbf{e}}_2\cdot\hat{\mathbf{e}}_1$ and the lowest order relations (A.3) and (A.4) for $\partial\phi/\partial\mu$ and $\partial\phi/\partial\varphi$, I obtain

$$\hat{\mathbf{b}}\cdot(\nabla\phi - \nabla_{\mathbf{R}}\phi) = -\frac{1}{2B\Omega_i}(\hat{\mathbf{b}}\cdot\nabla B)(\mathbf{v}\times\hat{\mathbf{b}})\cdot\nabla\phi$$
$$-\frac{1}{\Omega_i}\hat{\mathbf{b}}\cdot\nabla\hat{\mathbf{b}}\cdot(\mathbf{v}\times\hat{\mathbf{b}})(\hat{\mathbf{b}}\cdot\nabla\phi) + \hat{\mathbf{b}}\cdot\nabla\hat{\mathbf{e}}_2\cdot\hat{\mathbf{e}}_1\frac{\partial\phi}{\partial\varphi}. \tag{A.13}$$

To find this result, I have used

$$-\frac{v_{||}}{B}\hat{\mathbf{b}}\cdot\nabla\hat{\mathbf{b}}\cdot\mathbf{v}_\perp\frac{\partial\phi}{\partial\mu} + \frac{v_{||}}{v_\perp^2}\hat{\mathbf{b}}\cdot\nabla\hat{\mathbf{b}}\cdot(\mathbf{v}\times\hat{\mathbf{b}})\frac{\partial\phi}{\partial\varphi} = \frac{v_{||}}{\Omega_i}\hat{\mathbf{b}}\cdot\nabla\hat{\mathbf{b}}\cdot(\hat{\mathbf{b}}\times\nabla\phi), \tag{A.14}$$

where I employ (A.3) and (A.4) for $\partial\phi/\partial\mu$ and $\partial\phi/\partial\varphi$, and the relation $\mathbf{v}_\perp(\mathbf{v}\times\hat{\mathbf{b}}) - (\mathbf{v}\times\hat{\mathbf{b}})\mathbf{v}_\perp = v_\perp^2(\overleftrightarrow{\mathbf{I}}\times\hat{\mathbf{b}})$. This relation is obtained from the fact that $\mathbf{v}_\perp$ and $\mathbf{v}\times\hat{\mathbf{b}}$ expand the vector space perpendicular to the magnetic field, giving $\mathbf{v}_\perp\mathbf{v}_\perp + (\mathbf{v}\times\hat{\mathbf{b}})(\mathbf{v}\times\hat{\mathbf{b}}) = v_\perp^2(\overleftrightarrow{\mathbf{I}} - \hat{\mathbf{b}}\hat{\mathbf{b}})$.

To calculate $\mathbf{v}\cdot\nabla\phi$, I use that the total time derivative for ϕ in $\mathbf{r}$, $\mathbf{v}$ variables is

$$\frac{d\phi}{dt} = \left.\frac{\partial\phi}{\partial t}\right|_{\mathbf{r}} + \mathbf{v}\cdot\nabla\phi, \tag{A.15}$$

while as a function of the new gyrokinetic variables it becomes

$$\frac{d\phi}{dt} = \left.\frac{\partial\phi}{\partial t}\right|_{\mathbf{R},\,E,\,\mu,\,\varphi} + \dot{\mathbf{R}}\cdot\nabla_{\mathbf{R}}\phi + \dot{E}\frac{\partial\phi}{\partial E} + \dot{\varphi}\frac{\partial\phi}{\partial\varphi}. \tag{A.16}$$

Combining these equations gives an equation for $\mathbf{v}\cdot\nabla\phi$,

$$-\mathbf{v}\cdot\nabla\phi = \left(\left.\frac{\partial\phi}{\partial t}\right|_{\mathbf{r}} - \left.\frac{\partial\phi}{\partial t}\right|_{\mathbf{R},\,E,\,\mu,\,\varphi}\right) - \dot{E}\frac{\partial\phi}{\partial E} - \dot{\mathbf{R}}\cdot\nabla_{\mathbf{R}}\phi - \dot{\varphi}\frac{\partial\phi}{\partial\varphi}, \tag{A.17}$$

where the left side of the equation is of order $O(T_e v_i/eL)$. I analyze the right side term by term, keeping terms up to order $\delta_i T_e v_i/eL$. Noticing that $\phi(\mathbf{r},t) = \phi(\mathbf{R}+(\mathbf{r}-\mathbf{R}),t)$, the partial derivatives with respect to time give the negligible contribution $(\partial\phi/\partial t|_{\mathbf{r}} - \partial\phi/\partial t|_{\mathbf{R},\,E,\,\mu,\,\varphi}) = -\partial(\mathbf{r}-\mathbf{R})/\partial t \cdot \nabla\phi = O(\delta_i^2 T_e v_i/eL)$, since the time derivative of $\mathbf{r}-\mathbf{R}$ can only be of order $\delta_i^2 v_i$ for a static magnetic field. The partial derivative with respect to E is estimated in (A.2), giving that $\dot{E}\,\partial\phi/\partial E = O(\delta_i^2 T_e v_i/eL)$ is negligible. The total derivative $\dot{\mathbf{R}}$ has two different components, which I will calculate in detail in subsection A.2.2. These components are the parallel velocity of the gyrocenter, $u\hat{\mathbf{b}}(\mathbf{R})$, of order v_i, and the drift velocity, $\mathbf{v}_d$, of order $\delta_i v_i$. Using this information, I find $u\hat{\mathbf{b}}(\mathbf{R})\cdot\nabla_{\mathbf{R}}\phi = O(T_e v_i/eL)$ and $\mathbf{v}_d\cdot\nabla_{\mathbf{R}}\phi = O(\delta_i T_e v_i/eL)$. Finally, the last term in the right side of (A.17) is $\dot{\varphi}(\partial\phi/\partial\varphi) = O(T_e v_i/eL)$, since $\dot{\varphi}\sim\Omega_i$ and $\partial\phi/\partial\varphi = \partial\tilde{\phi}/\partial\varphi \sim \delta_i T_e/e$ according to (3.17). Neglecting all the terms smaller than $\delta_i T_e v_i/eL$, equation (A.17) becomes

$$-\mathbf{v}\cdot\nabla\phi = -u\hat{\mathbf{b}}(\mathbf{R})\cdot\nabla_{\mathbf{R}}\phi - \mathbf{v}_d\cdot\nabla_{\mathbf{R}}\phi + \overline{\Omega}_i\frac{\partial\tilde{\phi}}{\partial\varphi}. \qquad \text{(A.18)}$$

A.2.2 Calculation of $\mathbf{R}_2$

The second order correction $\mathbf{R}_2$ is given by (3.10), where $Q_0 = \mathbf{R}_0 = \mathbf{r}$ and $Q_1 = \mathbf{R}_1 = \Omega_i^{-1}\mathbf{v}\times\hat{\mathbf{b}}$. The total time derivative of $\mathbf{R}_0 + \mathbf{R}_1$ is

$$\frac{d}{dt}(\mathbf{R}_0+\mathbf{R}_1) = v_{||}\hat{\mathbf{b}} - \mathbf{v}\cdot\nabla\left(\frac{\hat{\mathbf{b}}}{\Omega_i}\right)\times\mathbf{v} - \frac{c}{B}\nabla\phi\times\hat{\mathbf{b}}, \qquad \text{(A.19)}$$

and its gyroaverage may be written as $\langle d(\mathbf{R}_0+\mathbf{R}_1)/dt\rangle = u\hat{\mathbf{b}}(\mathbf{R}) + \mathbf{v}_d$, where $u = \langle v_{||}\rangle + (v_\perp^2/2\Omega_i)(\hat{\mathbf{b}}\cdot\nabla\times\hat{\mathbf{b}})$, and $\mathbf{v}_d$ has been already defined in (3.20). The function u can be written as a function of the gyrokinetic variables. I express $v_{||}$ as a function of $\mathbf{r}$, E_0 and μ_0, expand around $\mathbf{R}$, E and μ, and insert $\mathbf{R}_1$, μ_1 and E_1 to obtain

$$v_{||} = \sqrt{2(E_0-\mu_0 B(\mathbf{r}))} \simeq \sqrt{2(E-\mu B(\mathbf{R}))} - \frac{v_\perp^2}{2\Omega_i}\hat{\mathbf{b}}\cdot\nabla\times\hat{\mathbf{b}}$$
$$-\frac{v_{||}}{\Omega_i}\hat{\mathbf{b}}\cdot\nabla\hat{\mathbf{b}}\cdot(\mathbf{v}\times\hat{\mathbf{b}}) - \frac{1}{4\Omega_i}[\mathbf{v}_\perp(\mathbf{v}\times\hat{\mathbf{b}}) + (\mathbf{v}\times\hat{\mathbf{b}})\mathbf{v}_\perp] : \nabla\hat{\mathbf{b}}. \qquad \text{(A.20)}$$

Finally, gyroaveraging and using $\langle\mathbf{v}_\perp(\mathbf{v}\times\hat{\mathbf{b}}) + (\mathbf{v}\times\hat{\mathbf{b}})\mathbf{v}_\perp\rangle = 0$ [a result that is deduced from (A.7)] give $u = \sqrt{2[E-\mu B(\mathbf{R})]}$, which can be rewritten as (3.23).

Using (A.19) and (A.20), Taylor expanding $\hat{\mathbf{b}}(\mathbf{r})$ about $\mathbf{R}$ and inserting the result into (3.10) gives (3.15) and (3.19). To integrate over gyrophase, $\mathbf{v}\times\hat{\mathbf{b}} = -\partial\mathbf{v}_\perp/\partial\varphi_0$ and (A.8) have been used.

A.2.3 Calculation of E_2

Equation (3.10) gives E_2, where $Q_0 = E_0 = v^2/2$ and $Q_1 = E_1 = Ze\tilde{\phi}/M$. The total derivative of $E_0 = v^2/2$ can be expressed as a function of the new gyrokinetic variables to the requisite order by using (A.18) to obtain

$$\frac{dE_0}{dt} = -\frac{Ze}{M}\mathbf{v}\cdot\nabla\phi \simeq \frac{Ze}{M}\left\{\overline{\Omega}_i\frac{\partial\tilde{\phi}}{\partial\varphi} - [u\hat{\mathbf{b}}(\mathbf{R}) + \mathbf{v}_d]\cdot\nabla_{\mathbf{R}}\phi\right\}. \tag{A.21}$$

From the definition of $E_1 = Ze\tilde{\phi}/M$, use of gyrokinetic variables yields

$$\frac{dE_1}{dt} = \frac{Ze}{M}\left\{\frac{\partial\tilde{\phi}}{\partial t} + [u\hat{\mathbf{b}}(\mathbf{R}) + \mathbf{v}_d]\cdot\nabla_{\mathbf{R}}\tilde{\phi} - \overline{\Omega}_i\frac{\partial\tilde{\phi}}{\partial\varphi}\right\}. \tag{A.22}$$

Adding both contributions together leaves

$$\frac{d}{dt}(E_0 + E_1) = -\frac{Ze}{M}[u\hat{\mathbf{b}}(\mathbf{R}) + \mathbf{v}_d]\cdot\nabla_{\mathbf{R}}\langle\phi\rangle + \frac{Ze}{M}\frac{\partial\tilde{\phi}}{\partial t}. \tag{A.23}$$

As a result, E_2 is as shown in (3.26), and to this order, dE/dt is given by (3.27).

A.2.4 Calculation of $\langle\mu_1\rangle$

In this subsection, I will check that $\langle\mu_1\rangle = -(v_{||}v_\perp^2/2B\Omega_i)\hat{\mathbf{b}}\cdot\nabla\times\hat{\mathbf{b}}$ satisfies the condition in (A.9). To do so, it is going to be useful to distinguish between the part of $d(\mu_0+\mu_1)/dt$ that depends on ϕ and the part that does not depend on ϕ at all since these pieces will vanish independently of each other. I will write the piece of $d(\mu_0+\mu_1)/dt$ that depends on ϕ as a function of the gyrokinetic variables, finding

$$\left.\frac{d}{dt}(\mu_0+\mu_1)\right|_\phi = -\frac{Ze}{M}\nabla\phi\cdot\nabla_v\mu_0 - \frac{Ze}{M}\nabla\phi\cdot\nabla_v\mu_1|_{\mathbf{r},\mathbf{v}} + \frac{d}{dt}\mu_1|_\phi, \tag{A.24}$$

with $\mu_1|_\phi = Ze\tilde{\phi}/MB$ and $\mu_1|_{\mathbf{r},\mathbf{v}} = \mu_1 - Ze\tilde{\phi}/MB$. The piece of $d(\mu_0+\mu_1)/dt$ that does not depend on ϕ is

$$\frac{d}{dt}(\mu_0+\mu_1)\bigg|_{\mathbf{r},\mathbf{v}} = \mathbf{v}\cdot\nabla\mu_0 - \Omega_i\frac{\partial}{\partial\varphi_0}\mu_1|_{\mathbf{r},\mathbf{v}} + \mathbf{v}\cdot\nabla\mu_1|_{\mathbf{r},\mathbf{v}}. \tag{A.25}$$

In this equation, the two first terms cancel by definition of $\mu_1|_{\mathbf{r},\mathbf{v}}$, leaving

$$\frac{d}{dt}(\mu_0+\mu_1)\bigg|_{\mathbf{r},\mathbf{v}} = \mathbf{v}\cdot\nabla\mu_1|_{\mathbf{r},\mathbf{v}}. \tag{A.26}$$

I will first prove that the gyroaverage of (A.26), $\langle\mathbf{v}\cdot\nabla\mu_1|_{\mathbf{r},\mathbf{v}}\rangle$, vanishes to $O(\delta_i v_i^3/BL)$ due to the choice of $\langle\mu_1\rangle$. Afterwards, I will prove that the gyroaverage of (A.24) vanishes to the same order, demonstrating then that $\mu_0+\mu_1$ satisfies condition (A.9).

To prove that $\langle\mathbf{v}\cdot\nabla\mu_1|_{\mathbf{r},\mathbf{v}}\rangle = 0$, the function $\mu_1|_{\mathbf{r},\mathbf{v}}$, is conveniently rewritten as

$$\begin{aligned}\mu_1|_{\mathbf{r},\mathbf{v}} = {} & -\frac{v_\perp^2}{2B^2\Omega_i}(\mathbf{v}\times\hat{\mathbf{b}})\cdot\nabla B - \frac{v_{||}^2}{B\Omega_i}\mathbf{v}_\perp\cdot\nabla\times\hat{\mathbf{b}} \\ & -\frac{v_{||}}{2B\Omega_i}(\mathbf{v}\times\hat{\mathbf{b}})\cdot\nabla\hat{\mathbf{b}}\cdot\mathbf{v} - \frac{v_{||}v_\perp^2}{4B\Omega_i}\hat{\mathbf{b}}\cdot\nabla\times\hat{\mathbf{b}},\end{aligned} \tag{A.27}$$

where I use $\hat{\mathbf{b}}\cdot\nabla\hat{\mathbf{b}} = \boldsymbol{\kappa}$ and (2.8) to write $\hat{\mathbf{b}}\cdot\nabla\hat{\mathbf{b}}\cdot(\mathbf{v}\times\hat{\mathbf{b}}) = \mathbf{v}_\perp\cdot\nabla\times\hat{\mathbf{b}}$, and I employ equation (A.7) and $\langle\mathbf{v}_\perp\mathbf{v}_\perp\rangle = (v_\perp^2/2)(\overleftrightarrow{\mathbf{I}} - \hat{\mathbf{b}}\hat{\mathbf{b}})$ to find

$$\frac{1}{2}[\mathbf{v}_\perp(\mathbf{v}\times\hat{\mathbf{b}}) + (\mathbf{v}\times\hat{\mathbf{b}})\mathbf{v}_\perp] : \nabla\hat{\mathbf{b}} = (\mathbf{v}\times\hat{\mathbf{b}})\cdot\nabla\hat{\mathbf{b}}\cdot\mathbf{v}_\perp - \frac{v_\perp^2}{2}\hat{\mathbf{b}}\cdot\nabla\times\hat{\mathbf{b}}. \tag{A.28}$$

I will examine term by term the gyroaverage of $\mathbf{v}\cdot\nabla$[expression (A.27)]. Employing $\mathbf{v}\cdot\nabla(v_\perp^2/2) = -\mathbf{v}\cdot\nabla(v_{||}^2/2) = -v_{||}\mathbf{v}\cdot\nabla v_{||}$, $\mathbf{v}\cdot\nabla v_{||} = \mathbf{v}\cdot\nabla\hat{\mathbf{b}}\cdot\mathbf{v}$ and $\langle\mathbf{v}\mathbf{v}\rangle = (v_\perp^2/2)\overleftrightarrow{\mathbf{I}} + [v_{||}^2 - (v_\perp^2/2)]\hat{\mathbf{b}}\hat{\mathbf{b}}$, the first, second and fourth terms in (A.27) give

$$\begin{aligned}\left\langle\mathbf{v}\cdot\nabla\left[\frac{v_\perp^2}{2B^2\Omega_i}(\mathbf{v}\times\hat{\mathbf{b}})\cdot\nabla B\right]\right\rangle = {} & -\frac{v_{||}^2v_\perp^2}{2B^2\Omega_i}\hat{\mathbf{b}}\cdot\nabla\hat{\mathbf{b}}\cdot(\hat{\mathbf{b}}\times\nabla B) \\ + \frac{v_\perp^4}{4}\nabla\cdot\left(\frac{\hat{\mathbf{b}}\times\nabla B}{B^2\Omega_i}\right) & + \frac{v_\perp^2}{2}\left(v_{||}^2 - \frac{v_\perp^2}{2}\right)\hat{\mathbf{b}}\cdot\nabla\left(\frac{\hat{\mathbf{b}}\times\nabla B}{B^2\Omega_i}\right)\cdot\hat{\mathbf{b}},\end{aligned} \tag{A.29}$$

$$\left\langle \mathbf{v}\cdot\nabla\left(\frac{v_{\|}^2}{B\Omega_i}\mathbf{v}_\perp\cdot\nabla\times\hat{\mathbf{b}}\right)\right\rangle = \frac{v_{\|}^2 v_\perp^2}{B\Omega_i}\hat{\mathbf{b}}\cdot\nabla\hat{\mathbf{b}}\cdot(\nabla\times\hat{\mathbf{b}})$$
$$+\frac{v_{\|}^2 v_\perp^2}{2}\nabla\cdot\left[\frac{(\nabla\times\hat{\mathbf{b}})_\perp}{B\Omega_i}\right] + v_{\|}^2\left(v_{\|}^2 - \frac{v_\perp^2}{2}\right)\hat{\mathbf{b}}\cdot\nabla\left[\frac{(\nabla\times\hat{\mathbf{b}})_\perp}{B\Omega_i}\right]\cdot\hat{\mathbf{b}} \qquad \text{(A.30)}$$

and

$$\left\langle \mathbf{v}\cdot\nabla\left(\frac{v_{\|}v_\perp^2}{4B\Omega_i}\hat{\mathbf{b}}\cdot\nabla\times\hat{\mathbf{b}}\right)\right\rangle = \frac{v_\perp^4}{8B\Omega_i}(\nabla\cdot\hat{\mathbf{b}})\hat{\mathbf{b}}\cdot\nabla\times\hat{\mathbf{b}}$$
$$-\frac{v_{\|}^2 v_\perp^2}{4B\Omega_i}(\nabla\cdot\hat{\mathbf{b}})\hat{\mathbf{b}}\cdot\nabla\times\hat{\mathbf{b}} + \frac{v_{\|}^2 v_\perp^2}{4}\hat{\mathbf{b}}\cdot\nabla\left(\frac{\hat{\mathbf{b}}\cdot\nabla\times\hat{\mathbf{b}}}{B\Omega_i}\right). \qquad \text{(A.31)}$$

The contribution to $\langle \mathbf{v}\cdot\nabla\mu_1|_{\mathbf{r},\mathbf{v}}\rangle$ of the third term in (A.27) is calculated by using $\mathbf{v}\cdot\nabla v_{\|} = \mathbf{v}\cdot\nabla\hat{\mathbf{b}}\cdot\mathbf{v}$, $\langle\mathbf{v}\mathbf{v}\rangle = (v_\perp^2/2)\overleftrightarrow{\mathbf{I}} + [v_{\|}^2 - (v_\perp^2/2)]\hat{\mathbf{b}}\hat{\mathbf{b}}$ and

$$\langle v_{\perp,j}v_{\perp,k}v_{\perp,l}v_{\perp,m}\rangle = \frac{v_\perp^4}{8}[(\delta_{jk} - \hat{b}_j\hat{b}_k)(\delta_{lm} - \hat{b}_l\hat{b}_m) + (\delta_{jl} - \hat{b}_j\hat{b}_l)(\delta_{km} - \hat{b}_k\hat{b}_m)$$
$$+(\delta_{jm} - \hat{b}_j\hat{b}_m)(\delta_{kl} - \hat{b}_k\hat{b}_l)]. \qquad \text{(A.32)}$$

This result is proven in Appendix B. With these relations, the third term in (A.27) gives

$$\left\langle \mathbf{v}\cdot\nabla\left[\frac{v_{\|}}{2B\Omega_i}(\mathbf{v}\times\hat{\mathbf{b}})\cdot\nabla\hat{\mathbf{b}}\cdot\mathbf{v}\right]\right\rangle = \frac{v_\perp^4}{16B\Omega_i}(\nabla\cdot\hat{\mathbf{b}})\hat{\mathbf{b}}\cdot\nabla\times\hat{\mathbf{b}}$$
$$+\frac{v_\perp^4}{16B\Omega_i}\overleftrightarrow{\mathbf{I}}\colon[(\hat{\mathbf{b}}\times\nabla\hat{\mathbf{b}})\cdot\nabla\hat{\mathbf{b}}] + \frac{v_\perp^4}{16B\Omega_i}\overleftrightarrow{\mathbf{I}}\colon[(\hat{\mathbf{b}}\times\nabla\hat{\mathbf{b}})\cdot(\nabla\hat{\mathbf{b}})^T]$$
$$+\frac{v_{\|}^2}{2}\left(v_{\|}^2 - \frac{3}{2}v_\perp^2\right)\hat{\mathbf{b}}\cdot\left[\hat{\mathbf{b}}\cdot\nabla\left(\frac{\hat{\mathbf{b}}\times\nabla\hat{\mathbf{b}}}{B\Omega_i}\right)\cdot\hat{\mathbf{b}}\right] + \frac{v_{\|}^2 v_\perp^2}{4}\hat{\mathbf{b}}\cdot\nabla\left(\frac{\hat{\mathbf{b}}\cdot\nabla\times\hat{\mathbf{b}}}{B\Omega_i}\right)$$
$$+\frac{v_{\|}^2 v_\perp^2}{4}\nabla\cdot\left(\frac{\hat{\mathbf{b}}\times\nabla\hat{\mathbf{b}}}{B\Omega_i}\right)\cdot\hat{\mathbf{b}} + \frac{v_{\|}^2 v_\perp^2}{4}\nabla\cdot\left[\frac{(\hat{\mathbf{b}}\times\nabla\hat{\mathbf{b}})^T}{B\Omega_i}\right]\cdot\hat{\mathbf{b}}, \qquad \text{(A.33)}$$

with $(\nabla\hat{\mathbf{b}})^T$ and $(\hat{\mathbf{b}}\times\nabla\hat{\mathbf{b}})^T$ the transposes of $\nabla\hat{\mathbf{b}}$ and $\hat{\mathbf{b}}\times\nabla\hat{\mathbf{b}}$.

Several terms in equations (A.29), (A.30), (A.31) and (A.33) simplify because they vanish. In particular, equation (2.8), with $\hat{\mathbf{b}}\cdot\nabla\hat{\mathbf{b}} = \boldsymbol{\kappa}$, leads to $\hat{\mathbf{b}}\cdot\nabla\hat{\mathbf{b}}\cdot(\nabla\times\hat{\mathbf{b}}) = 0$ and $\hat{\mathbf{b}}\cdot\nabla[(\nabla\times\hat{\mathbf{b}})_\perp/B\Omega_i]\cdot\hat{\mathbf{b}} = -(B\Omega_i)^{-1}\hat{\mathbf{b}}\cdot\nabla\hat{\mathbf{b}}\cdot(\nabla\times\hat{\mathbf{b}}) = 0$. Also, $\hat{\mathbf{b}}\cdot\{\hat{\mathbf{b}}\cdot\nabla[(\hat{\mathbf{b}}\times\nabla\hat{\mathbf{b}})/B\Omega_i]\cdot\hat{\mathbf{b}}\} = 0$, $\overleftrightarrow{\mathbf{I}}\colon[(\hat{\mathbf{b}}\times\nabla\hat{\mathbf{b}})\cdot(\nabla\hat{\mathbf{b}})^T] = -\nabla\cdot(\hat{\mathbf{b}}\times\nabla\hat{\mathbf{b}})\cdot\hat{\mathbf{b}} = 0$ and $\nabla\cdot[(\hat{\mathbf{b}}\times\nabla\hat{\mathbf{b}})/B\Omega_i]\cdot\hat{\mathbf{b}} = 0$.

With these cancellations, I find that equations (A.30) and (A.33) reduce to

$$\left\langle \mathbf{v}\cdot\nabla\left(\frac{v_{\|}^2}{B\Omega_i}\mathbf{v}_\perp\cdot\nabla\times\hat{\mathbf{b}}\right)\right\rangle = -\frac{v_{\|}^2 v_\perp^2}{2B\Omega_i}\nabla\cdot(\hat{\mathbf{b}}\hat{\mathbf{b}}\cdot\nabla\times\hat{\mathbf{b}}) - \frac{v_{\|}^2 v_\perp^2}{B^2\Omega_i}(\nabla\times\hat{\mathbf{b}})_\perp\cdot\nabla B, \tag{A.34}$$

where I have employed $\nabla\cdot[(\nabla\times\hat{\mathbf{b}})_\perp] = -\nabla\cdot(\hat{\mathbf{b}}\hat{\mathbf{b}}\cdot\nabla\times\hat{\mathbf{b}})$, and

$$\left\langle \mathbf{v}\cdot\nabla\left[\frac{v_{\|}}{2B\Omega_i}(\mathbf{v}\times\hat{\mathbf{b}})\cdot\nabla\hat{\mathbf{b}}\cdot\mathbf{v}\right]\right\rangle = \frac{v_\perp^4}{8B\Omega_i}(\nabla\cdot\hat{\mathbf{b}})\hat{\mathbf{b}}\cdot\nabla\times\hat{\mathbf{b}} + \frac{v_{\|}^2 v_\perp^2}{4B\Omega_i}\nabla\cdot(\hat{\mathbf{b}}\hat{\mathbf{b}}\cdot\nabla\times\hat{\mathbf{b}}), \tag{A.35}$$

where I have used $B\Omega_i\hat{\mathbf{b}}\cdot\nabla[(\hat{\mathbf{b}}\cdot\nabla\times\hat{\mathbf{b}})/B\Omega_i] = \nabla\cdot(\hat{\mathbf{b}}\hat{\mathbf{b}}\cdot\nabla\times\hat{\mathbf{b}}) - (\nabla\cdot\hat{\mathbf{b}} + 2\hat{\mathbf{b}}\cdot\nabla\ln B)\hat{\mathbf{b}}\cdot\nabla\times\hat{\mathbf{b}} = \nabla\cdot(\hat{\mathbf{b}}\hat{\mathbf{b}}\cdot\nabla\times\hat{\mathbf{b}}) + (\nabla\cdot\hat{\mathbf{b}})\hat{\mathbf{b}}\cdot\nabla\times\hat{\mathbf{b}}$, $B\Omega_i\nabla\cdot[(\hat{\mathbf{b}}\times\nabla\hat{\mathbf{b}})^T/B\Omega_i]\cdot\hat{\mathbf{b}} = -\overleftrightarrow{\mathbf{I}}\colon[(\hat{\mathbf{b}}\times\nabla\hat{\mathbf{b}})\cdot\nabla\hat{\mathbf{b}}] = -(\nabla\cdot\hat{\mathbf{b}})\hat{\mathbf{b}}\cdot\nabla\times\hat{\mathbf{b}}$ and

$$\overleftrightarrow{\mathbf{I}}\colon[(\hat{\mathbf{b}}\times\nabla\hat{\mathbf{b}})\cdot\nabla\hat{\mathbf{b}}] = (\nabla\cdot\hat{\mathbf{b}})\hat{\mathbf{b}}\cdot\nabla\times\hat{\mathbf{b}}. \tag{A.36}$$

To prove this last expression, I write $\overleftrightarrow{\mathbf{I}}\colon[(\hat{\mathbf{b}}\times\nabla\hat{\mathbf{b}})\cdot\nabla\hat{\mathbf{b}}]$ as a divergence, giving

$$\overleftrightarrow{\mathbf{I}}\colon[(\hat{\mathbf{b}}\times\nabla\hat{\mathbf{b}})\cdot\nabla\hat{\mathbf{b}}] = \nabla\cdot[(\hat{\mathbf{b}}\cdot\nabla\hat{\mathbf{b}})\times\hat{\mathbf{b}}] - \overleftrightarrow{\mathbf{I}}\colon(\hat{\mathbf{b}}\cdot\nabla\nabla\hat{\mathbf{b}}\times\hat{\mathbf{b}}) + \hat{\mathbf{b}}\cdot\nabla\hat{\mathbf{b}}\cdot(\nabla\times\hat{\mathbf{b}}). \tag{A.37}$$

Then, relation (A.36) is recovered by using $\overleftrightarrow{\mathbf{I}}\colon(\hat{\mathbf{b}}\cdot\nabla\nabla\hat{\mathbf{b}}\times\hat{\mathbf{b}}) = \hat{\mathbf{b}}\cdot\nabla(\nabla\times\hat{\mathbf{b}})\cdot\hat{\mathbf{b}} = \nabla\cdot(\hat{\mathbf{b}}\hat{\mathbf{b}}\cdot\nabla\times\hat{\mathbf{b}}) - (\nabla\cdot\hat{\mathbf{b}})\hat{\mathbf{b}}\cdot\nabla\times\hat{\mathbf{b}} - \hat{\mathbf{b}}\cdot\nabla\hat{\mathbf{b}}\cdot(\nabla\times\hat{\mathbf{b}})$ and equation (2.8) to write $(\hat{\mathbf{b}}\cdot\nabla\hat{\mathbf{b}})\times\hat{\mathbf{b}} = -(\nabla\times\hat{\mathbf{b}})_\perp$ and $\hat{\mathbf{b}}\cdot\nabla\hat{\mathbf{b}}\cdot(\nabla\times\hat{\mathbf{b}}) = 0$.

In addition to equations (A.34) and (A.35), equation (A.29) can be written as

$$\left\langle \mathbf{v}\cdot\nabla\left[\frac{v_\perp^2}{2B^2\Omega_i}(\mathbf{v}\times\hat{\mathbf{b}})\cdot\nabla B\right]\right\rangle = \frac{v_{\|}^2 v_\perp^2}{B^2\Omega_i}(\nabla\times\hat{\mathbf{b}})_\perp\cdot\nabla B - \frac{v_\perp^4}{4B\Omega_i}(\nabla\cdot\hat{\mathbf{b}})\hat{\mathbf{b}}\cdot\nabla\times\hat{\mathbf{b}}, \tag{A.38}$$

where I use equation (2.8) to write $\hat{\mathbf{b}}\cdot\nabla\hat{\mathbf{b}}\cdot(\hat{\mathbf{b}}\times\nabla B) = -(\nabla\times\hat{\mathbf{b}})_\perp\cdot\nabla B$, and I employ $B^2\Omega_i\nabla\cdot[(\hat{\mathbf{b}}\times\nabla B)/B^2\Omega_i] = (\nabla\times\hat{\mathbf{b}})\cdot\nabla B$, $B^2\Omega_i\hat{\mathbf{b}}\cdot\nabla[(\hat{\mathbf{b}}\times\nabla B)/B^2\Omega_i]\cdot\hat{\mathbf{b}} =$

$-\hat{\mathbf{b}}\cdot\nabla\hat{\mathbf{b}}\cdot(\hat{\mathbf{b}}\times\nabla B) = (\nabla\times\hat{\mathbf{b}})_\perp\cdot\nabla B$ and $\hat{\mathbf{b}}\cdot\nabla B = -B(\nabla\cdot\hat{\mathbf{b}})$. Finally, I add equations (A.31), (A.34), (A.35) and (A.38) to obtain

$$\langle \mathbf{v}\cdot\nabla\mu_1|_{\mathbf{r},\mathbf{v}}\rangle = \frac{v_{||}^2 v_\perp^2}{4B\Omega_i}\left[\nabla\cdot(\hat{\mathbf{b}}\hat{\mathbf{b}}\cdot\nabla\times\hat{\mathbf{b}}) + (\nabla\cdot\hat{\mathbf{b}})\hat{\mathbf{b}}\cdot\nabla\times\hat{\mathbf{b}} - B\Omega_i\hat{\mathbf{b}}\cdot\nabla\left(\frac{\hat{\mathbf{b}}\cdot\nabla\times\hat{\mathbf{b}}}{B\Omega_i}\right)\right] = 0, \tag{A.39}$$

the property I was trying to prove. Notice that the time derivative of $\langle\mu_1\rangle = -(v_{||}v_\perp^2/2B\Omega_i)\hat{\mathbf{b}}\cdot\nabla\times\hat{\mathbf{b}}$ was necessary to obtain this result. The derivative $d\langle\mu_1\rangle/dt$ is in part responsible for the term (A.31).

There is still the piece of $d(\mu_0+\mu_1)/dt$ that depends explicitly on the potential, given in (A.24). I will next prove that it also gyroaverages to zero, the desired result. The terms in (A.24) must be written as a function of the gyrokinetic variables in order to make the gyroaverage easier. I will do so for each term to the required order. The first term in (A.24) is $-(Ze/M)\nabla\phi\cdot\nabla_v\mu_0 = -(Ze/MB)\mathbf{v}_\perp\cdot\nabla\phi$. Using $\mathbf{v}_\perp\cdot\nabla\phi = \mathbf{v}\cdot\nabla\phi - v_{||}\hat{\mathbf{b}}\cdot\nabla\phi$ and relation (A.18), I find to order $\delta_i v_i^3/BL$

$$-\frac{Ze}{M}\nabla\phi\cdot\nabla_v\mu_0 = \frac{Ze}{MB}[v_{||}\hat{\mathbf{b}}(\mathbf{r})\cdot\nabla\phi - u\hat{\mathbf{b}}(\mathbf{R})\cdot\nabla_{\mathbf{R}}\phi] - \frac{Ze}{MB}\mathbf{v}_d\cdot\nabla_{\mathbf{R}}\phi + \frac{Ze\overline{\Omega}_i}{MB(\mathbf{r})}\frac{\partial\phi}{\partial\varphi}. \tag{A.40}$$

In the lower order term $[Ze\overline{\Omega}_i/MB(\mathbf{r})](\partial\phi/\partial\varphi)$, the difference $B(\mathbf{r})-B(\mathbf{R}) \simeq -\Omega_i^{-1}(\mathbf{v}\times\hat{\mathbf{b}})\cdot\nabla B$ is important, giving

$$\frac{Ze\overline{\Omega}_i}{MB(\mathbf{r})}\frac{\partial\phi}{\partial\varphi} \simeq \frac{Ze\overline{\Omega}_i}{MB(\mathbf{R})}\frac{\partial\phi}{\partial\varphi} - \frac{c}{B^3}(\mathbf{v}_\perp\cdot\nabla\phi)[(\mathbf{v}\times\hat{\mathbf{b}})\cdot\nabla B], \tag{A.41}$$

where I employ the lowest order result $\partial\phi/\partial\varphi \simeq -\Omega_i^{-1}\mathbf{v}_\perp\cdot\nabla\phi$. The term $v_{||}\hat{\mathbf{b}}(\mathbf{r})\cdot\nabla\phi - u\hat{\mathbf{b}}(\mathbf{R})\cdot\nabla_{\mathbf{R}}\phi$ in (A.40) is to the order of interest

$$v_{||}\hat{\mathbf{b}}(\mathbf{r})\cdot\nabla\phi - u\hat{\mathbf{b}}(\mathbf{R})\cdot\nabla_{\mathbf{R}}\phi \simeq (v_{||}-u)\hat{\mathbf{b}}(\mathbf{R})\cdot\nabla_{\mathbf{R}}\phi + u[\hat{\mathbf{b}}(\mathbf{r})-\hat{\mathbf{b}}(\mathbf{R})]\cdot\nabla_{\mathbf{R}}\phi + u\hat{\mathbf{b}}(\mathbf{R})\cdot(\nabla\phi-\nabla_{\mathbf{R}}\phi), \tag{A.42}$$

where $\hat{\mathbf{b}}(\mathbf{r}) - \hat{\mathbf{b}}(\mathbf{R}) \simeq -\Omega_i^{-1}(\mathbf{v} \times \hat{\mathbf{b}}) \cdot \nabla\hat{\mathbf{b}}$ and $\hat{\mathbf{b}}(\mathbf{R}) \cdot (\nabla\phi - \nabla_{\mathbf{R}}\phi)$ is given in (A.13). The difference $v_{||} - u$ can be obtained from (A.20), giving

$$v_{||} - u = -\frac{v_{||}}{\Omega_i}\hat{\mathbf{b}} \cdot \nabla\hat{\mathbf{b}} \cdot (\mathbf{v} \times \hat{\mathbf{b}}) - \frac{1}{4\Omega_i}[\mathbf{v}_\perp(\mathbf{v} \times \hat{\mathbf{b}}) + (\mathbf{v} \times \hat{\mathbf{b}})\mathbf{v}_\perp] : \nabla\hat{\mathbf{b}}$$
$$-\frac{v_\perp^2}{2\Omega_i}\hat{\mathbf{b}} \cdot \nabla \times \hat{\mathbf{b}} = -\frac{v_{||}}{\Omega_i}\hat{\mathbf{b}} \cdot \nabla\hat{\mathbf{b}} \cdot (\mathbf{v} \times \hat{\mathbf{b}}) - \frac{1}{2\Omega_i}(\mathbf{v} \times \hat{\mathbf{b}}) \cdot \nabla\hat{\mathbf{b}} \cdot \mathbf{v}_\perp$$
$$-\frac{v_\perp^2}{4\Omega_i}\hat{\mathbf{b}} \cdot \nabla \times \hat{\mathbf{b}}, \qquad \text{(A.43)}$$

where I use (A.28) to obtain the second equality. Employing (A.13), (A.41), (A.42) and the definition of $\overline{\Omega}_i$ in (3.31), equation (A.40) becomes

$$-\frac{Ze}{M}\nabla\phi \cdot \nabla_v\mu_0 = \frac{Ze}{MB}(\hat{\mathbf{b}} \cdot \nabla\phi)\left[(v_{||} - u) - \frac{v_{||}}{\Omega_i}\hat{\mathbf{b}} \cdot \nabla\hat{\mathbf{b}} \cdot (\mathbf{v} \times \hat{\mathbf{b}})\right]$$
$$-\frac{cv_{||}}{B^2}(\mathbf{v} \times \hat{\mathbf{b}}) \cdot \nabla\hat{\mathbf{b}} \cdot \nabla\phi - \frac{cv_{||}}{2B^3}[(\mathbf{v} \times \hat{\mathbf{b}}) \cdot \nabla\phi](\hat{\mathbf{b}} \cdot \nabla B)$$
$$-\frac{c}{B^3}(\mathbf{v}_\perp \cdot \nabla\phi)[(\mathbf{v} \times \hat{\mathbf{b}}) \cdot \nabla B] - \frac{cv_{||}}{2B^2}(\mathbf{v}_\perp \cdot \nabla\phi)(\hat{\mathbf{b}} \cdot \nabla \times \hat{\mathbf{b}})$$
$$-\frac{Ze}{MB}\mathbf{v}_d \cdot \nabla_{\mathbf{R}}\phi + \left(\frac{Z^2e^2}{M^2c} + \frac{Z^3e^3}{M^3cB}\frac{\partial\langle\phi\rangle}{\partial\mu}\right)\frac{\partial\phi}{\partial\varphi}. \qquad \text{(A.44)}$$

To obtain this form, I used the lowest order result $\partial\phi/\partial\varphi \simeq -\Omega_i^{-1}\mathbf{v}_\perp \cdot \nabla\phi$ in the term $(v_{||}/2)(\hat{\mathbf{b}} \cdot \nabla \times \hat{\mathbf{b}})(\partial\phi/\partial\varphi)$.

The second term in (A.24) is calculated employing (A.27), giving

$$-\frac{Ze}{M}\nabla\phi \cdot \nabla_v\mu_1|_{\mathbf{r},\mathbf{v}} = \frac{Ze}{MB}(\hat{\mathbf{b}} \cdot \nabla\phi)\left[\frac{v_{||}}{\Omega_i}\hat{\mathbf{b}} \cdot \nabla\hat{\mathbf{b}} \cdot (\mathbf{v} \times \hat{\mathbf{b}}) - (v_{||} - u)\right]$$
$$+\frac{c}{B^3}(\mathbf{v}_\perp \cdot \nabla\phi)[(\mathbf{v} \times \hat{\mathbf{b}}) \cdot \nabla B] + \frac{Ze}{MB}\mathbf{v}_M \cdot \nabla\phi$$
$$+\frac{cv_{||}}{2B^2}(\nabla\phi \times \hat{\mathbf{b}}) \cdot \nabla\hat{\mathbf{b}} \cdot \mathbf{v}_\perp + \frac{cv_{||}}{2B^2}(\mathbf{v} \times \hat{\mathbf{b}}) \cdot \nabla\hat{\mathbf{b}} \cdot \nabla\phi$$
$$+\frac{cv_{||}}{2B^2}(\mathbf{v}_\perp \cdot \nabla\phi)(\hat{\mathbf{b}} \cdot \nabla \times \hat{\mathbf{b}}), \qquad \text{(A.45)}$$

with $v_{||} - u$ from (A.43). Substituting equations (A.44) and (A.45) in equation (A.24), the piece of $d(\mu_0 + \mu_1)/dt$ that depends on ϕ is written as

$$\left.\frac{d}{dt}(\mu_0 + \mu_1)\right|_\phi = \left(\frac{Z^2e^2}{M^2c} + \frac{Z^3e^3}{M^3cB}\frac{\partial\langle\phi\rangle}{\partial\mu}\right)\frac{\partial\tilde{\phi}}{\partial\varphi}$$

$$+\frac{Zec}{MB^2}(\nabla_{\mathbf{R}}\langle\phi\rangle \times \hat{\mathbf{b}}) \cdot \nabla_{\mathbf{R}}\widetilde{\phi} + \frac{d}{dt}\mu_1|_\phi. \tag{A.46}$$

Here, I have used

$$(\nabla\phi \times \hat{\mathbf{b}}) \cdot \nabla\hat{\mathbf{b}} \cdot \mathbf{v}_\perp - (\mathbf{v} \times \hat{\mathbf{b}}) \cdot \nabla\hat{\mathbf{b}} \cdot \nabla\phi = -(\nabla \cdot \hat{\mathbf{b}})(\mathbf{v} \times \hat{\mathbf{b}}) \cdot \nabla\phi. \tag{A.47}$$

To prove this last expression, I employ $\nabla_\perp\phi = v_\perp^{-2}[\mathbf{v}_\perp\mathbf{v}_\perp \cdot \nabla\phi + (\mathbf{v} \times \hat{\mathbf{b}})(\mathbf{v} \times \hat{\mathbf{b}}) \cdot \nabla\phi]$ to find

$$\begin{aligned}(\nabla\phi \times \hat{\mathbf{b}}) \cdot \nabla\hat{\mathbf{b}} \cdot \mathbf{v}_\perp - (\mathbf{v} \times \hat{\mathbf{b}}) \cdot \nabla\hat{\mathbf{b}} \cdot \nabla\phi = \\ -\frac{1}{v_\perp^2}[(\mathbf{v} \times \hat{\mathbf{b}}) \cdot \nabla\phi][\mathbf{v}_\perp\mathbf{v}_\perp + (\mathbf{v} \times \hat{\mathbf{b}})(\mathbf{v} \times \hat{\mathbf{b}})] : \nabla\hat{\mathbf{b}}.\end{aligned} \tag{A.48}$$

Upon using $\overleftrightarrow{\mathbf{I}} - \hat{\mathbf{b}}\hat{\mathbf{b}} = v_\perp^{-2}[\mathbf{v}_\perp\mathbf{v}_\perp + (\mathbf{v} \times \hat{\mathbf{b}})(\mathbf{v} \times \hat{\mathbf{b}})]$, equation (A.47) is recovered.

Finally, the gyroaverage of equation (A.46) is zero. The term

$$\frac{d}{dt}\mu_1|_\phi = \frac{\partial}{\partial t}\mu_1|_\phi + \dot{\mathbf{R}} \cdot \nabla_{\mathbf{R}}\mu_1|_\phi + \dot{\varphi}\frac{\partial}{\partial\varphi}\mu_1|_\phi \tag{A.49}$$

vanishes when gyroaveraged because $\mu_1|_\phi = Ze\widetilde{\phi}/MB(\mathbf{R})$ gyroaverages to zero and the gyrokinetic variables are defined such that $\dot{\mathbf{R}}$ and $\dot{\varphi}$ are gyrophase independent. Notice that here it is important that $B(\mathbf{R})$ in $\mu_1|_\phi = Ze\widetilde{\phi}/MB(\mathbf{R})$ depends on $\mathbf{R}$ and not on $\mathbf{r}$.

A.3 Jacobian of the gyrokinetic transformation

In this section the Jacobian of the transformation from variables $\mathbf{r}$, $\mathbf{v}$ to variables $\mathbf{R}$, E, μ, φ is calculated, and the gyroaverage of condition (3.40) is checked.

The inverse of the Jacobian is

$$\frac{1}{J}=\left|\begin{array}{cc|c|c|c} \ddots & & \vdots & \vdots & \vdots \\ & \nabla\mathbf{R} & \nabla E & \nabla\mu & \nabla\varphi \\ & \ddots & \vdots & \vdots & \vdots \\ \hline \ddots & & \vdots & \vdots & \vdots \\ & \nabla_v\mathbf{R} & \nabla_v E & \nabla_v\mu & \nabla_v\varphi \\ & \ddots & \vdots & \vdots & \vdots \end{array}\right| = \left|\begin{array}{cc|c|c|c} \ddots & & \vdots & \vdots & \vdots \\ & \nabla\mathbf{R} & \nabla E & \nabla\mu & \nabla\varphi \\ & \ddots & \vdots & \vdots & \vdots \\ \hline \ddots & & \vdots & \vdots & \vdots \\ & \mathbf{0} & \partial E & \partial\mu & \partial\varphi \\ & \ddots & \vdots & \vdots & \vdots \end{array}\right|. \tag{A.50}$$

Employing that the terms in the left columns of the first form are to first approximation $\nabla\mathbf{R} \simeq \overleftrightarrow{\mathbf{I}}$ and $\nabla_v\mathbf{R} \simeq \Omega_i^{-1}\overleftrightarrow{\mathbf{I}}\times\hat{\mathbf{b}}$, the determinant is simplified by combining linearly the rows in the matrix to determine the second form, where

$$\partial(\ldots) = \nabla_v(\ldots) - \frac{1}{\Omega_i}\hat{\mathbf{b}}\times\nabla(\ldots). \tag{A.51}$$

The second form of (A.50) can be simplified by noticing that the lower left piece of the matrix is zero. Thus, the determinant may be written as

$$J^{-1} = \det(\nabla\mathbf{R})[\partial E\cdot(\partial\mu\times\partial\varphi)]. \tag{A.52}$$

I analyze the two determinants on the right side independently. The matrix $\nabla\mathbf{R}$ is $\overleftrightarrow{\mathbf{I}} + \nabla(\Omega_i^{-1}\mathbf{v}\times\hat{\mathbf{b}}+\mathbf{R}_2)$. Hence, $\det(\nabla\mathbf{R}) \simeq 1+\nabla\cdot(\Omega_i^{-1}\mathbf{v}\times\hat{\mathbf{b}}+\mathbf{R}_2)$. The Jacobian must be obtained to first order only. The only important term to that order in $\mathbf{R}_2$ is the term that contains the potential ϕ, since its gradient may be large, but $\nabla\cdot\mathbf{R}_2 \simeq -\nabla\cdot[(c/B\Omega_i)\nabla_{\mathbf{R}}\tilde{\Phi}\times\hat{\mathbf{b}}] \simeq 0$. Therefore, the determinant of $\nabla\mathbf{R}$ becomes

$$\det(\nabla\mathbf{R}) = 1 - \mathbf{v}\cdot\nabla\times\left(\frac{\hat{\mathbf{b}}}{\Omega_i}\right). \tag{A.53}$$

For the second determinant in (A.52), I evaluate the columns of the matrix ∂E,

$\partial\mu$ and $\partial\varphi$ to the order of interest, using

$$\partial E = \mathbf{v} + \nabla_v E_1 - \frac{1}{\Omega_i}\hat{\mathbf{b}} \times \nabla E, \tag{A.54}$$

$$\partial\mu = \frac{\mathbf{v}_\perp}{B} + \nabla_v \mu_1 - \frac{1}{\Omega_i}\hat{\mathbf{b}} \times \nabla\mu \tag{A.55}$$

and

$$\partial\varphi = -\frac{1}{v_\perp^2}\mathbf{v} \times \hat{\mathbf{b}} + \nabla_v \varphi_1 - \frac{1}{\Omega_i}\hat{\mathbf{b}} \times \nabla\varphi. \tag{A.56}$$

The determinant becomes

$$\partial E \cdot (\partial\mu \times \partial\varphi) \simeq \frac{v_{||}}{B(\mathbf{r})} + \frac{\hat{\mathbf{b}}}{B} \cdot \nabla_v E_1 + \left(\frac{v_{||}}{v_\perp^2}\mathbf{v}_\perp - \hat{\mathbf{b}}\right) \cdot \left(\nabla_v \mu_1 - \frac{1}{\Omega_i}\hat{\mathbf{b}} \times \nabla\mu\right) - \frac{v_{||}}{B}(\mathbf{v} \times \hat{\mathbf{b}}) \cdot \left(\nabla_v \varphi_1 - \frac{1}{\Omega_i}\hat{\mathbf{b}} \times \nabla\varphi\right). \tag{A.57}$$

In the lower order term $v_{||}/B(\mathbf{r})$, the difference $B(\mathbf{r}) - B(\mathbf{R}) \simeq -\Omega_i^{-1}(\mathbf{v}\times\hat{\mathbf{b}})\cdot\nabla B \sim \delta_i B$ is important. From the definitions of E_1, μ_1 and φ_1, I find their gradients in velocity space. I need the gradients in velocity space of $\tilde{\phi}$ and $\partial\tilde{\Phi}/\partial\mu$. The gradient $\nabla_v \tilde{\phi}$ is given by

$$\begin{aligned} \nabla_v \tilde{\phi} &= \nabla_v E \frac{\partial\tilde{\phi}}{\partial E} + \nabla_v \mu \frac{\partial\tilde{\phi}}{\partial\mu} + \nabla_v \varphi \frac{\partial\tilde{\phi}}{\partial\varphi} + \nabla_v \mathbf{R} \cdot \nabla_{\mathbf{R}} \tilde{\phi} \\ &= \frac{\mathbf{v}_\perp}{B}\frac{\partial\tilde{\phi}}{\partial\mu} - \frac{1}{v_\perp^2}\mathbf{v} \times \hat{\mathbf{b}}\frac{\partial\tilde{\phi}}{\partial\varphi} + \frac{1}{\Omega_i}\hat{\mathbf{b}} \times \nabla_{\mathbf{R}}\tilde{\phi}. \end{aligned} \tag{A.58}$$

The gradient $\nabla_v(\partial\tilde{\Phi}/\partial\mu)$ is found in a similar way. The gradients in real space are only to be obtained to zeroth order. However, some terms of the first order quantities that contain ϕ are important because they have steep gradients. Considering this, I find

$$\nabla E = \frac{Ze}{M}\nabla\tilde{\phi}, \tag{A.59}$$

$$\nabla\mu = -\frac{v_\perp^2}{2B^2}\nabla B - \frac{v_{||}}{B}\nabla\hat{\mathbf{b}} \cdot \mathbf{v}_\perp + \frac{Ze}{MB}\nabla\tilde{\phi} \tag{A.60}$$

and

$$\nabla\varphi = \frac{v_{||}}{v_\perp^2}\nabla\hat{\mathbf{b}}\cdot(\mathbf{v}\times\hat{\mathbf{b}}) + \nabla\hat{\mathbf{e}}_2\cdot\hat{\mathbf{e}}_1 - \frac{Ze}{MB}\nabla\left(\frac{\partial\tilde{\Phi}}{\partial\mu}\right). \tag{A.61}$$

Due to the preceding considerations, equation (A.57) becomes

$$\partial E\cdot(\partial\mu\times\partial\varphi) = \frac{u}{B(\mathbf{r})}\left[1 + \frac{1}{\Omega_i}\mathbf{v}\cdot(\hat{\mathbf{b}}\times\boldsymbol{\kappa})\right], \tag{A.62}$$

where I have employed (A.43) to express $v_{||}$ as a function of the gyrokinetic variables, and I have used $(\mathbf{v}\times\hat{\mathbf{b}})\cdot\nabla\hat{\mathbf{b}}\cdot\mathbf{v}_\perp - \mathbf{v}_\perp\cdot\nabla\hat{\mathbf{b}}\cdot(\mathbf{v}\times\hat{\mathbf{b}}) = v_\perp^2\hat{\mathbf{b}}\cdot\nabla\times\hat{\mathbf{b}}$. This last result is deduced from $v_\perp^2(\overleftrightarrow{\mathbf{I}} - \hat{\mathbf{b}}\hat{\mathbf{b}}) = \mathbf{v}_\perp\mathbf{v}_\perp + (\mathbf{v}\times\hat{\mathbf{b}})(\mathbf{v}\times\hat{\mathbf{b}})$. In equation (A.62), in the lower order term $u/B(\mathbf{r})$, the difference $B(\mathbf{r}) - B(\mathbf{R}) = -\Omega_i^{-1}(\mathbf{v}\times\hat{\mathbf{b}})\cdot\nabla B$ is important. Combining (A.53) and (A.62), and using (2.8) to write $\mathbf{v}\cdot(\hat{\mathbf{b}}\times\boldsymbol{\kappa}) = \mathbf{v}_\perp\cdot\nabla\times\hat{\mathbf{b}}$, the Jacobian of the transformation is found to be as given by (3.44). Notice that to this order $J = \langle J\rangle$ as required by (3.40).

Finally, I prove that J satisfies the gyroaverage of (3.40) to the required order, namely,

$$\frac{\partial J}{\partial t} + \nabla_{\mathbf{R}}\cdot\{J[u\hat{\mathbf{b}}(\mathbf{R}) + \mathbf{v}_d]\} - \frac{Ze}{M}\frac{\partial}{\partial E}\{J[u\hat{\mathbf{b}}(\mathbf{R}) + \mathbf{v}_d]\cdot\nabla_{\mathbf{R}}\langle\phi\rangle\} = 0, \tag{A.63}$$

where $\mathbf{v}_d$ is given by (3.20). To first order, I obtain

$$J[u\hat{\mathbf{b}}(\mathbf{R}) + \mathbf{v}_d] \simeq \mathbf{B}(\mathbf{R}) + \frac{Mc\mu}{Zeu}\hat{\mathbf{b}}\times\nabla_{\mathbf{R}}B + \frac{Mcu}{Ze}\nabla_{\mathbf{R}}\times\hat{\mathbf{b}} - \frac{c}{u}\nabla_{\mathbf{R}}\langle\phi\rangle\times\hat{\mathbf{b}}, \tag{A.64}$$

where I have employed (2.8). Inserting (A.64) into (A.63) and recalling that $u = \sqrt{2[E - \mu B(\mathbf{R})]}$ is enough to prove that (A.63), and thus (3.40) gyroaveraged, are satisfied by the Jacobian to first order.

Appendix B

Useful gyroaverages and gyrophase derivatives

The various derivations require various averages and integrals respect to the gyrophase φ_0. In particular, the integrals for terms that contain several $\mathbf{v}_\perp$ are recurrent. In this Appendix, I show how to work out these type of terms.

According to (2.2), the perpendicular velocity, $\mathbf{v}_\perp$, is $\mathbf{v}_\perp = v_\perp(\hat{\mathbf{e}}_1 \cos\varphi_0 + \hat{\mathbf{e}}_2 \sin\varphi_0)$. In order to find gyroaverages and integrals, it is useful to express the perpendicular velocity as

$$\mathbf{v}_\perp = \mathrm{Re}[v_\perp \exp(i\varphi_0)\mathbf{u}] = \frac{v_\perp}{2}[\exp(i\varphi_0)\mathbf{u} + \exp(-i\varphi_0)\mathbf{u}^*], \tag{B.1}$$

where $i = \sqrt{-1}$, $\mathbf{u} = \hat{\mathbf{e}}_1 - i\hat{\mathbf{e}}_2$, $\mathbf{u}^*$ is the conjugate of the complex vector $\mathbf{u}$ and $\mathrm{Re}(\mathbf{a})$ is the real part of the complex tensor $\mathbf{a}$. Notice also that

$$v_\perp \exp(i\varphi_0)\mathbf{u} = \mathbf{v}_\perp + i\mathbf{v} \times \hat{\mathbf{b}}. \tag{B.2}$$

It is very common to find tensors composed of tensor products of $\mathbf{v}_\perp$,

$$(\mathbf{v}_\perp)^n \equiv \underbrace{\mathbf{v}_\perp\mathbf{v}_\perp \ldots \mathbf{v}_\perp}_{n \text{ times}}. \tag{B.3}$$

It is possible to find a more convenient form for this tensor product by using equation (B.1). It is useful to distinguish between odd and even n. For even n, $n = 2m$,

$$(\mathbf{v}_\perp)^{2m} = \frac{v_\perp^{2m}}{2^{2m-1}}\mathrm{Re}\Big\{ \exp(i2m\varphi_0)\mathbf{W}^{(2m,0)} + \exp[i2(m-1)\varphi_0]\mathbf{W}^{(2m,1)} + \ldots + \exp(i2\varphi_0)\mathbf{W}^{(2m,m-1)}\Big\} + \frac{v_\perp^{2m}}{2^{2m}}\mathbf{W}^{(2m,m)}, \qquad \text{(B.4)}$$

and for odd n, $n = 2m+1$,

$$(\mathbf{v}_\perp)^{2m+1} = \frac{v_\perp^{2m+1}}{2^{2m}}\mathrm{Re}\Big\{ \exp[i(2m+1)\varphi_0]\mathbf{W}^{(2m+1,0)} + \exp[i(2m-1)\varphi_0]\mathbf{W}^{(2m+1,1)} + \ldots + \exp(i\varphi_0)\mathbf{W}^{(2m+1,m)}\Big\}, \qquad \text{(B.5)}$$

where the tensor $\mathbf{W}^{(n,p)}$ is the tensor formed by the addition of all the possible different tensor products between $(n-p)$ $\mathbf{u}$ vectors and p $\mathbf{u}^*$ vectors, i.e.,

$$\mathbf{W}^{(n,p)} \equiv \underbrace{\mathbf{u}\ldots\mathbf{u}}_{n-p}\underbrace{\mathbf{u}^*\mathbf{u}^*\ldots\mathbf{u}^*}_{p} + \underbrace{\mathbf{u}\ldots\mathbf{u}}_{n-p-1}\ \mathbf{u}^*\mathbf{u}\underbrace{\mathbf{u}^*\ldots\mathbf{u}^*}_{p-1} + \ldots + \underbrace{\mathbf{u}^*\ldots\mathbf{u}^*}_{p}\underbrace{\mathbf{u}\ldots\mathbf{u}}_{n-p}. \qquad \text{(B.6)}$$

There are $n!/[p!(n-p)!]$ different terms in the summation. For example, $\mathbf{W}^{(5,2)}$ has 10 summands, given by

$$\mathbf{W}^{(5,2)} \equiv \mathbf{uuuu^*u^*} + \mathbf{uuu^*uu^*} + \mathbf{uu^*uuu^*} + \mathbf{u^*uuuu^*} + \mathbf{uuu^*u^*u} + \mathbf{uu^*uu^*u} + \mathbf{u^*uuu^*u} + \mathbf{uu^*u^*uu} + \mathbf{u^*uu^*uu} + \mathbf{u^*u^*uuu}. \qquad \text{(B.7)}$$

The tensor $\mathbf{W}^{(n,p)}$ can be written in a form in which only $\mathbf{W}^{(n-2p,0)}$ and the matrix $\overleftrightarrow{\mathbf{I}} - \hat{\mathbf{b}}\hat{\mathbf{b}}$ appear. The tensor of the form $\mathbf{W}^{(m,0)}$ is more convenient because it is part of $\mathrm{Re}[v_\perp^m \exp(im\varphi_0)\mathbf{W}^{(m,0)}]$ and easy to write in a recognizable manner. For example, for $m = 2$, employing (B.2), I find

$$\mathrm{Re}[v_\perp^2 \exp(i2\varphi_0)\mathbf{W}^{(2,0)}] = \mathbf{v}_\perp\mathbf{v}_\perp - (\mathbf{v}\times\hat{\mathbf{b}})(\mathbf{v}\times\hat{\mathbf{b}}). \qquad \text{(B.8)}$$

For $m = 3$,

$$\mathrm{Re}[v_\perp^3 \exp(i3\varphi_0)\mathbf{W}^{(3,0)}] = \mathbf{v}_\perp\mathbf{v}_\perp\mathbf{v}_\perp - \mathbf{v}_\perp(\mathbf{v}\times\hat{\mathbf{b}})(\mathbf{v}\times\hat{\mathbf{b}}) - (\mathbf{v}\times\hat{\mathbf{b}})\mathbf{v}_\perp(\mathbf{v}\times\hat{\mathbf{b}}) - (\mathbf{v}\times\hat{\mathbf{b}})(\mathbf{v}\times\hat{\mathbf{b}})\mathbf{v}_\perp. \quad \text{(B.9)}$$

These tensors are not only easy to write but also easy to integrate in φ_0 because

$$\frac{\partial}{\partial\varphi_0}\mathrm{Im}[v_\perp^m \exp(im\varphi_0)\mathbf{W}^{(m,0)}] = m\mathrm{Re}[v_\perp^m \exp(im\varphi_0)\mathbf{W}^{(m,0)}], \quad \text{(B.10)}$$

with $\mathrm{Im}(\mathbf{a})$ the imaginary part of the complex tensor $\mathbf{a}$. Equation (B.10) for $m = 2$ is

$$\frac{\partial}{\partial\varphi_0}[\mathbf{v}_\perp(\mathbf{v}\times\hat{\mathbf{b}}) + (\mathbf{v}\times\hat{\mathbf{b}})\mathbf{v}_\perp] = 2[\mathbf{v}_\perp\mathbf{v}_\perp - (\mathbf{v}\times\hat{\mathbf{b}})(\mathbf{v}\times\hat{\mathbf{b}})], \quad \text{(B.11)}$$

and for $m = 3$ is

$$\frac{\partial}{\partial\varphi_0}[\mathbf{v}_\perp\mathbf{v}_\perp(\mathbf{v}\times\hat{\mathbf{b}}) + \mathbf{v}_\perp(\mathbf{v}\times\hat{\mathbf{b}})\mathbf{v}_\perp + (\mathbf{v}\times\hat{\mathbf{b}})\mathbf{v}_\perp\mathbf{v}_\perp - (\mathbf{v}\times\hat{\mathbf{b}})(\mathbf{v}\times\hat{\mathbf{b}})(\mathbf{v}\times\hat{\mathbf{b}})]$$
$$= 3[\mathbf{v}_\perp\mathbf{v}_\perp\mathbf{v}_\perp - \mathbf{v}_\perp(\mathbf{v}\times\hat{\mathbf{b}})(\mathbf{v}\times\hat{\mathbf{b}}) - (\mathbf{v}\times\hat{\mathbf{b}})\mathbf{v}_\perp(\mathbf{v}\times\hat{\mathbf{b}}) - (\mathbf{v}\times\hat{\mathbf{b}})(\mathbf{v}\times\hat{\mathbf{b}})\mathbf{v}_\perp]. \text{(B.12)}$$

Equation (B.11) is used in the derivation of the gyrokinetic variables.

The decomposition of $\mathbf{W}^{(n,p)}$ into $\mathbf{W}^{(n-2p,0)}$ and matrices $\overleftrightarrow{\mathbf{I}} - \hat{\mathbf{b}}\hat{\mathbf{b}}$ is also interesting for gyroaverages. According to equation (B.4), the tensor products of an even number of vectors $\mathbf{v}_\perp$ has the gyroaverage

$$\overline{(\mathbf{v}_\perp)^{2m}} = \frac{v_\perp^{2m}}{2^{2m}}\mathbf{W}^{(2m,m)}, \quad \text{(B.13)}$$

where $\mathbf{W}^{(2m,m)}$ is a summation of tensor products of p ($\overleftrightarrow{\mathbf{I}} - \hat{\mathbf{b}}\hat{\mathbf{b}}$) matrices. At the end of this appendix, I will give the result as a function of $\overleftrightarrow{\mathbf{I}} - \hat{\mathbf{b}}\hat{\mathbf{b}}$. The gyroaverage of an odd number of $\mathbf{v}_\perp$ is zero, as shown in (B.5).

I will prove now that $\mathbf{W}^{(n,p)}$ can be written such that it only contains $\mathbf{W}^{(n-2p,0)}$ and the matrix $\overleftrightarrow{\mathbf{I}} - \hat{\mathbf{b}}\hat{\mathbf{b}}$. To do so, I will first prove that $\mathbf{W}^{(n,p)}$ can be decomposed into a summation of tensors formed by the product of $\mathbf{W}^{(n-2p,0)}$ and $\mathbf{W}^{(2p,p)}$. After that, I will show that $\mathbf{W}^{(2p,p)}$ is formed by a summation of tensor products of p $(\overleftrightarrow{\mathbf{I}} - \hat{\mathbf{b}}\hat{\mathbf{b}})$ matrices.

By multiplying $\mathbf{W}^{(n,p)}$ by $(n-p)!/[p!(n-2p)!]$, $(n-2p)$ $\mathbf{u}$ vectors can be distinguished from the rest of $\mathbf{u}$ vectors. I will denote them as $\mathbf{u}^d$ (this is a trick to make the derivation easier). Then, a new tensor $\mathbf{T}^{(n,p,p)}$ is defined as the tensor formed by p $\mathbf{u}$ vectors, p $\mathbf{u}^*$ vectors and $(n-2p)$ $\mathbf{u}^d$ vectors, giving

$$\frac{(n-p)!}{p!(n-2p)!}\mathbf{W}^{(n,p)} = \mathbf{T}^{(n,p,p)} \equiv \underbrace{\mathbf{u}^d\ldots\mathbf{u}^d}_{n-2p}\underbrace{\mathbf{u}\ldots\mathbf{u}\mathbf{u}}_{p}\underbrace{\mathbf{u}^*\mathbf{u}^*\ldots\mathbf{u}^*}_{p}$$
$$+\ \underbrace{\mathbf{u}^d\ldots\mathbf{u}^d}_{n-2p-1}\ \mathbf{u}\mathbf{u}^d\underbrace{\mathbf{u}\ldots\mathbf{u}\mathbf{u}}_{p}\underbrace{\mathbf{u}^*\mathbf{u}^*\ldots\mathbf{u}^*}_{p-1}+\ldots+\underbrace{\mathbf{u}^*\ldots\mathbf{u}^*}_{p}\underbrace{\mathbf{u}\ldots\mathbf{u}}_{p}\underbrace{\mathbf{u}^d\ldots\mathbf{u}^d}_{n-2p}. \quad \text{(B.14)}$$

For instance, $3\mathbf{W}^{(5,2)} = \mathbf{T}^{(5,2,2)}$ can be deduced from (B.7) to give

$$\begin{aligned}\mathbf{T}^{(5,2,2)} \equiv\ & \mathbf{u}^d\mathbf{u}\mathbf{u}\mathbf{u}^*\mathbf{u}^* + \mathbf{u}^d\mathbf{u}\mathbf{u}^*\mathbf{u}\mathbf{u}^* + \mathbf{u}^d\mathbf{u}^*\mathbf{u}\mathbf{u}\mathbf{u}^* + \mathbf{u}^*\mathbf{u}^d\mathbf{u}\mathbf{u}\mathbf{u}^* + \mathbf{u}^d\mathbf{u}\mathbf{u}^*\mathbf{u}^*\mathbf{u}\\
&+\mathbf{u}^d\mathbf{u}^*\mathbf{u}\mathbf{u}^*\mathbf{u} + \mathbf{u}^*\mathbf{u}^d\mathbf{u}\mathbf{u}^*\mathbf{u} + \mathbf{u}^d\mathbf{u}^*\mathbf{u}^*\mathbf{u}\mathbf{u} + \mathbf{u}^*\mathbf{u}^d\mathbf{u}^*\mathbf{u}\mathbf{u} + \mathbf{u}^*\mathbf{u}^*\mathbf{u}^d\mathbf{u}\mathbf{u}\\
&+\mathbf{u}\mathbf{u}^d\mathbf{u}\mathbf{u}^*\mathbf{u}^* + \mathbf{u}\mathbf{u}^d\mathbf{u}^*\mathbf{u}\mathbf{u}^* + \mathbf{u}\mathbf{u}^*\mathbf{u}^d\mathbf{u}\mathbf{u}^* + \mathbf{u}^*\mathbf{u}\mathbf{u}^d\mathbf{u}\mathbf{u}^* + \mathbf{u}\mathbf{u}^d\mathbf{u}^*\mathbf{u}^*\mathbf{u}\\
&+\mathbf{u}\mathbf{u}^*\mathbf{u}^d\mathbf{u}^*\mathbf{u} + \mathbf{u}^*\mathbf{u}\mathbf{u}^d\mathbf{u}^*\mathbf{u} + \mathbf{u}\mathbf{u}^*\mathbf{u}^*\mathbf{u}^d\mathbf{u} + \mathbf{u}^*\mathbf{u}\mathbf{u}^*\mathbf{u}^d\mathbf{u} + \mathbf{u}^*\mathbf{u}^*\mathbf{u}\mathbf{u}^d\mathbf{u}\\
&+\mathbf{u}\mathbf{u}\mathbf{u}^d\mathbf{u}^*\mathbf{u}^* + \mathbf{u}\mathbf{u}\mathbf{u}^*\mathbf{u}^d\mathbf{u}^* + \mathbf{u}\mathbf{u}^*\mathbf{u}\mathbf{u}^d\mathbf{u}^* + \mathbf{u}^*\mathbf{u}\mathbf{u}\mathbf{u}^d\mathbf{u}^* + \mathbf{u}\mathbf{u}\mathbf{u}^*\mathbf{u}^*\mathbf{u}^d\\
&+\mathbf{u}\mathbf{u}^*\mathbf{u}\mathbf{u}^*\mathbf{u}^d + \mathbf{u}^*\mathbf{u}\mathbf{u}\mathbf{u}^*\mathbf{u}^d + \mathbf{u}\mathbf{u}^*\mathbf{u}^*\mathbf{u}\mathbf{u}^d + \mathbf{u}^*\mathbf{u}\mathbf{u}^*\mathbf{u}\mathbf{u}^d + \mathbf{u}^*\mathbf{u}^*\mathbf{u}\mathbf{u}\mathbf{u}^d. \quad \text{(B.15)}\end{aligned}$$

The tensor $\mathbf{T}^{(n,p,p)}$ can be written as a combination of a tensor $\mathbf{W}^{(n-2p,0)}$ formed by $(n-2p)$ $\mathbf{u}^d$ vectors and a tensor $\mathbf{W}^{(2p,p)}$ formed by p $\mathbf{u}$ vectors and p $\mathbf{u}^*$ vectors, leading to

$$\begin{aligned}T^{(n,p,p)}_{j_1j_2\cdots j_n} \equiv\ & W^{(2p,p)}_{j_1\cdots j_{2p}}W^{(n-2p,0)}_{j_{2p+1}\cdots j_n} + W^{(2p,p)}_{j_1\cdots j_{2p-1}j_{2p+1}}W^{(n-2p,0)}_{j_{2p}j_{2p+2}\cdots j_n} + \ldots\\
&+W^{(2p,p)}_{j_{n-2p+1}\cdots j_n}W^{(n-2p,0)}_{j_1\cdots j_{n-2p}} = \frac{n!}{(n-2p)!(2p)!}W^{(2p,p)}_{(j_1\cdots j_{2p}}W^{(n-2p,0)}_{j_{2p+1}\cdots j_n)}. \quad \text{(B.16)}\end{aligned}$$

Here, the number of summands is $n!/[(n-2p)!(2p)!]$ because the tensors $\mathbf{W}^{(m,0)}$ and $\mathbf{W}^{(2p,p)}$ are symmetric respect to all of their indexes. Additionally, the tensor $\mathbf{W}^{(2p,p)}$ is always real, since $[\mathbf{W}^{(2p,p)}]^* = \mathbf{W}^{(2p,p)}$ by definition of $\mathbf{W}^{(2p,p)}$. The last expression in (B.16) is based on the typical tensor notation, where the parenthesis around the indexes indicate symmetrization of the tensor. As an example of the equivalence in (B.16), the tensor $\mathbf{T}^{(5,2,2)}$ in (B.15) may be written as

$$T^{(5,2,2)}_{jklmn} = W^{(4,2)}_{jklm} W^{(1,0)}_{n} + W^{(4,2)}_{jkln} W^{(1,0)}_{m} + W^{(4,2)}_{jkmn} W^{(1,0)}_{l} + W^{(4,2)}_{jlmn} W^{(1,0)}_{k} + W^{(4,2)}_{klmn} W^{(1,0)}_{j}, \qquad \text{(B.17)}$$

with $\mathbf{W}^{(1,0)} \equiv \mathbf{u} \equiv \mathbf{u}^d$, and

$$\mathbf{W}^{(4,2)} \equiv \mathbf{uuu^*u^*} + \mathbf{uu^*uu^*} + \mathbf{u^*uuu^*} + \mathbf{uu^*u^*u} + \mathbf{u^*uu^*u} + \mathbf{u^*u^*uu}. \qquad \text{(B.18)}$$

The tensor $\mathbf{W}^{(2p,p)}$ can be rewritten employing yet a third tensor, $\mathbf{I}^{(2p)}$. This tensor is formed by adding all the possible different tensor products that are formed by p matrices $(\overleftrightarrow{\mathbf{I}} - \hat{\mathbf{b}}\hat{\mathbf{b}})_{jk} = \delta_{jk} - \hat{b}_j \hat{b}_k = \delta^{\perp}_{jk}$, i.e.,

$$I^{(2p)}_{j_1 j_2 \ldots j_{2p}} = \delta^{\perp}_{j_1 j_2} \delta^{\perp}_{j_3 j_4} \ldots \delta^{\perp}_{j_{2p-1} j_{2p}} + \delta^{\perp}_{j_1 j_3} \delta^{\perp}_{j_2 j_4} \ldots \delta^{\perp}_{j_{2p-1} j_{2p}} + \ldots = \frac{(2p)!}{2^p p!} \delta^{\perp}_{(j_1 j_2} \delta^{\perp}_{j_3 j_4} \ldots \delta^{\perp}_{j_{2p-1} j_{2p})}. \qquad \text{(B.19)}$$

This tensor is formed by $(2p)!/(2^p p!)$ summands. For example, $\mathbf{I}^{(4)}$ is

$$I^{(4)}_{jklm} = \delta^{\perp}_{jk} \delta^{\perp}_{lm} + \delta^{\perp}_{jl} \delta^{\perp}_{km} + \delta^{\perp}_{jm} \delta^{\perp}_{kl}. \qquad \text{(B.20)}$$

The tensor $\mathbf{I}^{(2p)}$ can be written in terms of $\mathbf{W}^{(2p,p)}$ because

$$2(\overleftrightarrow{\mathbf{I}} - \hat{\mathbf{b}}\hat{\mathbf{b}}) = 2(\hat{\mathbf{e}}_1 \hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2 \hat{\mathbf{e}}_2) = \mathbf{uu^*} + \mathbf{u^*u}. \qquad \text{(B.21)}$$

Using this relation, the tensor $2^p \mathbf{I}^{(2p)}$, formed by all possible tensor products of p $2(\overleftrightarrow{\mathbf{I}} - \hat{\mathbf{b}}\hat{\mathbf{b}})$ matrices, is written as a summation of tensor products of p $\mathbf{u}$ vectors and p $\mathbf{u}^*$ vectors by substituting equation (B.21) in it. Each summand $2^p \delta^{\perp}_{j_1 j_2} \delta^{\perp}_{j_3 j_4} \ldots \delta^{\perp}_{j_{2p-1} j_{2p}}$ of $2^p \mathbf{I}^{(2p)}$ [recall (B.19)] gives 2^p different tensors formed by p $\mathbf{u}$ vectors and p $\mathbf{u}^*$ vectors. Adding all the terms in $\mathbf{I}^{(2p)}$, there is a total of $(2p)!/p!$ terms formed by tensor products of p $\mathbf{u}$ vectors and p $\mathbf{u}^*$ vectors. In this summation, each summand of $\mathbf{W}^{(2p,p)}$ is present $p!$ times. According to (B.6), a summand of $\mathbf{W}^{(2p,p)}$ is uniquely determined by the positions that the p $\mathbf{u}$ vectors occupy, or alternatively, which p indexes of $j_1 j_2 \dots j_{2p}$ correspond to the p $\mathbf{u}$ vectors. Without loss of generality, these indexes can be chosen to be $j_1 j_2 \dots j_p$. A tensor of the form $u_{j_1} u_{j_2} \dots u_{j_p} u^*_{j_{p+1}} \dots u^*_{j_{2p}}$ is obtained from every term of $2^p \mathbf{I}^{(2p)}$ of the form $2^p \delta^{\perp}_{j_1 k_1} \delta^{\perp}_{j_2 k_2} \dots \delta^{\perp}_{j_p k_p}$, with $k_1 k_2 \dots k_p$ being any permutation of $j_{p+1} j_{p+2} \dots j_{2p}$. There are $p!$ of these terms, proving that

$$2^p \mathbf{I}^{(2p)} = p! \mathbf{W}^{(2p,p)}. \tag{B.22}$$

Considering the examples (B.18) and (B.20), equation (B.22) implies that their relation is $4\mathbf{I}^{(4)} = 2\mathbf{W}^{(4,2)}$. Combining equations (B.14), (B.16) and (B.22) gives

$$W^{(n,p)}_{j_1 j_2 \cdots j_n} = \frac{2^p (n-2p)!}{(n-p)!} \Big[I^{(2p)}_{j_1 \cdots j_{2p}} W^{(n-2p,0)}_{j_{2p+1} \cdots j_n} + I^{(2p)}_{j_1 \cdots j_{2p-1} j_{2p+1}} W^{(n-2p,0)}_{j_{2p} j_{2p+2} \cdots j_n} + \dots$$
$$+ I^{(2p)}_{j_{n-2p+1} \cdots j_n} W^{(n-2p,0)}_{j_1 \cdots j_{n-2p}} \Big] = \frac{n!}{p!(n-p)!} I^{(2p)}_{(j_1 \cdots j_{2p}} W^{(n-2p,0)}_{j_{2p+1} \cdots j_n)}. \tag{B.23}$$

As promised, $\mathbf{W}^{(n,p)}$ is written in terms of $\mathbf{W}^{(n-2p,0)}$ and $\overleftrightarrow{\mathbf{I}} - \hat{\mathbf{b}}\hat{\mathbf{b}}$ because $\mathbf{I}^{(2p)}$ is formed by addition of tensor products of $\overleftrightarrow{\mathbf{I}} - \hat{\mathbf{b}}\hat{\mathbf{b}}$. Continuing with the example in (B.7), $\mathbf{W}^{(5,2)}$ gives

$$W^{(5,2)}_{jklmn} = \frac{2}{3} \Big[I^{(4)}_{jklm} u_n + I^{(4)}_{jkln} u_m + I^{(4)}_{jkmn} u_l + I^{(4)}_{jlmn} u_k + I^{(4)}_{klmn} u_j \Big], \tag{B.24}$$

with $\mathbf{I}^{(4)}$ given in (B.20). This tensor appears in $(\mathbf{v}_\perp)^5 \equiv \mathbf{v}_\perp \mathbf{v}_\perp \mathbf{v}_\perp \mathbf{v}_\perp \mathbf{v}_\perp$ as [recall (B.5)]

$$\frac{1}{2^4} \mathrm{Re}[v_\perp^5 \exp(i\varphi_0) W^{(5,2)}_{jklmn}] = \frac{v_\perp^4}{24} \Big[I^{(4)}_{jklm} v_{\perp,n} + I^{(4)}_{jkln} v_{\perp,m} + I^{(4)}_{jkmn} v_{\perp,l}$$
$$+ I^{(4)}_{jlmn} v_{\perp,k} + I^{(4)}_{klmn} v_{\perp,j} \Big]. \tag{B.25}$$

To finish, the gyroaverage of an even number of vectors $\mathbf{v}_\perp$ is calculated. Combining equations (B.13) and (B.22) gives

$$\overline{(\mathbf{v}_\perp)^{2m}} = \frac{v_\perp^{2m}}{2^m m!}\mathbf{I}^{(2m)}, \tag{B.26}$$

with $\mathbf{I}^{(2m)}$ given in (B.19). For $m = 1$, the familiar result

$$\overline{\mathbf{v}_\perp \mathbf{v}_\perp} = \frac{v_\perp^2}{2}(\overleftrightarrow{\mathbf{I}} - \hat{\mathbf{b}}\hat{\mathbf{b}}) \tag{B.27}$$

is found. This result, combined with (B.4) for $m = 2$ and (B.8) gives (A.7). The gyroaverage for $m = 4$ gives (A.32).

Appendix C

Gyrokinetic equivalence

In this Appendix, a summary of [61], I prove that the gyrokinetic results in chapter 3 are completely consistent with the pioneering results by Dubin *et al* [21], obtained using a Hamiltonian formalism and Lie transforms [25]. This comparison confirms that both approaches, the recursive method in chapter 3 and the Lie transforms, yield the same final gyrokinetic formalism.

In chapter 3, the recursive approach developed in [58, 59] was generalized for nonlinear electrostatic gyrokinetics in a general magnetic field. In reference [21], the nonlinear electrostatic gyrokinetic equation was derived for a constant magnetic field and a collisionless plasma using a Hamiltonian formalism. The asymptotic expansion was carried out to higher order in [21] because the calculation is easier in a constant magnetic field. When the method proposed in chapter 3 is extended to next order, the results are different in appearance, but I will prove that these differences are due to subtleties in some definitions.

Both methods are asymptotic expansions in the small parameter $\delta = \rho/L \ll 1$. Here L is a characteristic macroscopic length in the problem and $\rho = v_{\mathrm{th}}/\Omega$ is the gyroradius, with $\Omega = ZeB/Mc$ the gyrofrequency, $v_{\mathrm{th}} = \sqrt{2T/M}$ the thermal velocity and T the temperature. In both methods, the phase space $\{\mathbf{r}, \mathbf{v}\}$, with $\mathbf{r}$ and $\mathbf{v}$ the position and velocity of the particles, is expressed in gyrokinetic variables, defined order by order in δ. In reference [21], the gyrokinetic variables are obtained by Lie transform and the gyrokinetic equation is found to second order in δ. In chapter 3, the gyrokinetic variables are found by imposing that their time derivative is gyrophase independent. Here $d/dt \equiv \partial/\partial t + \mathbf{v} \cdot \nabla + (-Ze\nabla\phi/M + \Omega\mathbf{v} \times \hat{\mathbf{b}}) \cdot \nabla_v$ is the Vlasov operator. In chapter 3, the gyrokinetic equation was only found to first order. In this Appendix, I will calculate the gyrokinetic equation and the gyrokinetic variables to higher order for a constant magnetic field, and I will compare the results with those in [21]. The orderings and assumptions are more general than in section 3.1. The pieces of the distribution function and the potential with short wavelengths also scale as

$$\frac{f_k}{f_s} \sim \frac{e\phi_k}{T} \sim \frac{1}{k_\perp L} \lesssim 1, \tag{C.1}$$

with $k_\perp \rho \lesssim 1$. Here f_s is the lowest order distribution function with a slow variation in both $\mathbf{r}$ and $\mathbf{v}$. In chapter 3, the lowest order distribution functions was assumed to be a Maxwellian. In this Appendix, to ease the comparison with reference [21], I relax that assumption. Finally, I order the time derivatives as $\partial/\partial t \sim v_{\text{th}}/L$.

C.1 Constant magnetic field results

The general gyrokinetic variables obtained in section 3.2 are $\mathbf{R} = \mathbf{r} + \mathbf{R}_1 + \mathbf{R}_2$, $E = E_0 + E_1 + E_2$, $\mu = \mu_0 + \mu_1 + \mu_2$ and $\varphi = \varphi_0 + \varphi_1 + \varphi_2$. Since the unit vector $\hat{\mathbf{b}}$ is assumed constant in space and time, I can define $\hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_2$ in (2.2) so that they are also constant, and I do so to simplify the comparison with [21]. The corrections found in section 3.2 specialized to constant magnetic field are, for the gyrocenter position $\mathbf{R}$, $\mathbf{R}_1 = \Omega^{-1}\mathbf{v} \times \hat{\mathbf{b}}$ and $\mathbf{R}_2 = -(c/B\Omega)\nabla_{\mathbf{R}}\tilde{\Phi} \times \hat{\mathbf{b}}$; for the kinetic energy E, $E_1 = Ze\tilde{\phi}/M$ and $E_2 = (c/B)(\partial\tilde{\Phi}/\partial t)$; and for the magnetic moment μ and the gyrophase φ, $\mu_1 = Ze\tilde{\phi}/MB$ and $\varphi_1 = -(Ze/MB)(\partial\tilde{\Phi}/\partial\mu)$. The corrections μ_2 and φ_2 were not calculated because they were not needed to obtain the gyrokinetic equation to first order in δ under the assumptions in chapter 3.

I require the gyroaverage of $d\mathbf{R}/dt$ and dE/dt to higher order than in section 3.2, and I need the second order correction μ_2. For constant magnetic fields, $\langle d\mathbf{R}/dt\rangle$, $\langle dE/dt\rangle$ and the correction μ_2 can be easily calculated by employing the methodology in chapter 3. I will define μ_2 so that the gyroaverage of $d\mu/dt$ is zero to order $\delta^2 v_{\text{th}}^3/BL$.

The second order correction φ_2 is needed to obtain $d\varphi/dt$ to higher order. However, f is gyroaveraged, making the dependence on φ weak. Thus, φ_2 never enters in the final results and will not be necessary for our purposes. Once I have $\mathbf{R}$, E, μ and their derivatives to higher order, I will compare these results to both the gyrokinetic Vlasov equation and the gyrokinetic Poisson's equation in [21].

C.2 Time derivative of R

Employing the definitions of $\mathbf{R}_1$ and $\mathbf{R}_2$, I find

$$\frac{d\mathbf{R}}{dt} = v_{||}\hat{\mathbf{b}} - \frac{c}{B}\nabla\phi \times \hat{\mathbf{b}} + \frac{d\mathbf{R}_2}{dt}. \tag{C.2}$$

The gyroaverage of this expression is performed holding the gyrokinetic variables $\mathbf{R}$, E, μ and t fixed to obtain

$$\left\langle \frac{d\mathbf{R}}{dt} \right\rangle = u\hat{\mathbf{b}} - \frac{c}{B}\langle \nabla\phi \rangle \times \hat{\mathbf{b}}, \tag{C.3}$$

where $u = \langle v_{||} \rangle$. My gyrokinetic variables are defined so that when the Vlasov operator is applied to a function with a vanishing gyroaverage, like $\mathbf{R}_2 = \mathbf{R}_2(\mathbf{R}, E, \mu, \varphi, t)$, the result also has a zero gyroaverage; namely $\langle d\mathbf{R}_2/dt \rangle = 0$.

The gradient $\nabla\phi$ is written in the gyrokinetic variables by using

$$\nabla\phi = \nabla\mathbf{R}\cdot\nabla_{\mathbf{R}}\phi + \frac{\partial\phi}{\partial\mu}\nabla\mu + \frac{\partial\phi}{\partial\varphi}\nabla\varphi \simeq \nabla_{\mathbf{R}}\phi + \nabla\mathbf{R}_2\cdot\nabla_{\mathbf{R}}\phi + \frac{\partial\phi}{\partial\mu}\nabla\mu_1 + \frac{\partial\phi}{\partial\varphi}\nabla\varphi_1. \tag{C.4}$$

Here, I neglect $\partial\phi/\partial E \simeq -(\partial\mathbf{R}_1/\partial E)\cdot\nabla\phi$ because the function $\mathbf{R}_1$ does not depend on E to order δL. To obtain the second equality, I use that $\nabla\mathbf{R}_1 = 0 = \nabla\mu_0 = \nabla\varphi_0$. The gyroaverage of equation (C.4), obtained employing the definitions of $\mathbf{R}_2$, μ_1 and φ_1, gives

$$\langle\nabla\phi\rangle \simeq \nabla_{\mathbf{R}}\langle\phi\rangle - \frac{c}{B\Omega}\langle\nabla_{\mathbf{R}}\nabla_{\mathbf{R}}\tilde{\Phi}\cdot(\hat{\mathbf{b}}\times\nabla_{\mathbf{R}}\phi)\rangle + \frac{Ze}{MB}\left\langle\frac{\partial\phi}{\partial\mu}\nabla_{\mathbf{R}}\tilde{\phi} - \frac{\partial\phi}{\partial\varphi}\nabla_{\mathbf{R}}\left(\frac{\partial\tilde{\Phi}}{\partial\mu}\right)\right\rangle. \tag{C.5}$$

This equation can be simplified by integrating by parts in φ to obtain

$$\left\langle \frac{\partial \phi}{\partial \mu} \nabla_{\mathbf{R}} \widetilde{\phi} - \frac{\partial \phi}{\partial \varphi} \nabla_{\mathbf{R}} \left(\frac{\partial \widetilde{\Phi}}{\partial \mu} \right) \right\rangle = \left\langle \frac{\partial \widetilde{\phi}}{\partial \mu} \nabla_{\mathbf{R}} \widetilde{\phi} + \widetilde{\phi} \nabla_{\mathbf{R}} \left(\frac{\partial \widetilde{\phi}}{\partial \mu} \right) \right\rangle = \frac{1}{2} \nabla_{\mathbf{R}} \left(\frac{\partial}{\partial \mu} \langle \widetilde{\phi}^2 \rangle \right). \quad \text{(C.6)}$$

I next demonstrate that

$$\langle \nabla_{\mathbf{R}} \nabla_{\mathbf{R}} \widetilde{\Phi} \cdot (\hat{\mathbf{b}} \times \nabla_{\mathbf{R}} \phi) \rangle = \frac{1}{2} \nabla_{\mathbf{R}} \langle \nabla_{\mathbf{R}} \widetilde{\Phi} \cdot (\hat{\mathbf{b}} \times \nabla_{\mathbf{R}} \widetilde{\phi}) \rangle \quad \text{(C.7)}$$

by first noticing that

$$\langle \nabla_{\mathbf{R}} \nabla_{\mathbf{R}} \widetilde{\Phi} \cdot (\hat{\mathbf{b}} \times \nabla_{\mathbf{R}} \phi) \rangle = \nabla_{\mathbf{R}} \langle \nabla_{\mathbf{R}} \widetilde{\Phi} \cdot (\hat{\mathbf{b}} \times \nabla_{\mathbf{R}} \widetilde{\phi}) \rangle + \langle \nabla_{\mathbf{R}} \nabla_{\mathbf{R}} \widetilde{\phi} \cdot (\hat{\mathbf{b}} \times \nabla_{\mathbf{R}} \widetilde{\Phi}) \rangle. \quad \text{(C.8)}$$

Integrating by parts in φ in the second term, I find $\langle \nabla_{\mathbf{R}} \nabla_{\mathbf{R}} \widetilde{\phi} \cdot (\hat{\mathbf{b}} \times \nabla_{\mathbf{R}} \widetilde{\Phi}) \rangle = -\langle \nabla_{\mathbf{R}} \nabla_{\mathbf{R}} \widetilde{\Phi} \cdot (\hat{\mathbf{b}} \times \nabla_{\mathbf{R}} \widetilde{\phi}) \rangle$, giving the result in (C.7).

Finally, substituting equations (C.6) and (C.7) into equation (C.5) and using the result in (C.3) gives

$$\left\langle \frac{d\mathbf{R}}{dt} \right\rangle = u \hat{\mathbf{b}} - \frac{c}{B} \nabla_{\mathbf{R}} \Psi \times \hat{\mathbf{b}}, \quad \text{(C.9)}$$

with

$$\Psi = \langle \phi \rangle + \frac{Ze}{2MB} \frac{\partial}{\partial \mu} \langle \widetilde{\phi}^2 \rangle + \frac{c}{2B\Omega} \langle \nabla_{\mathbf{R}} \widetilde{\phi} \cdot (\hat{\mathbf{b}} \times \nabla_{\mathbf{R}} \widetilde{\Phi}) \rangle. \quad \text{(C.10)}$$

To find u, I need $v_{||}$ as a function of the gyrokinetic variables. To do so, I use

$$\frac{v_{||}^2}{2} = E_0 - \mu_0 B = E - \mu B - (E_2 - \mu_2 B), \quad \text{(C.11)}$$

where I employ $E_1 - \mu_1 B = 0$. According to this result, the difference between $u = \langle v_{||} \rangle$ and $v_{||}$ is necessarily of order $\delta^2 v_{\text{th}}$. Once I calculate μ_2, I will be able to find u.

C.3 Time derivative of E

Employing the definitions of E_1 and E_2, and gyroaveraging, I find

$$\left\langle \frac{dE}{dt} \right\rangle = -\frac{Ze}{M} \langle \mathbf{v} \cdot \nabla\phi \rangle. \tag{C.12}$$

Here, I have used that $\langle dE_1/dt \rangle = 0 = \langle dE_2/dt \rangle$.

The term $\mathbf{v} \cdot \nabla\phi$ can be conveniently rewritten by employing

$$\frac{d\phi}{dt} = \left.\frac{\partial\phi}{\partial t}\right|_{\mathbf{r}} + \mathbf{v} \cdot \nabla\phi = \left.\frac{\partial\phi}{\partial t}\right|_{\mathbf{R},E,\mu,\varphi} + \frac{d\mathbf{R}}{dt} \cdot \nabla_{\mathbf{R}}\phi + \frac{d\mu}{dt}\frac{\partial\phi}{\partial\mu} + \frac{d\varphi}{dt}\frac{\partial\phi}{\partial\varphi}. \tag{C.13}$$

Here, I neglect $\partial\phi/\partial E$ again. Solving for $\mathbf{v} \cdot \nabla\phi$ and gyroaveraging, I find

$$\langle \mathbf{v} \cdot \nabla\phi \rangle = \left\langle \frac{d\mathbf{R}}{dt} \cdot \nabla_{\mathbf{R}}\phi \right\rangle + \left\langle \frac{d\mu}{dt}\frac{\partial\phi}{\partial\mu} \right\rangle + \left\langle \frac{d\varphi}{dt}\frac{\partial\phi}{\partial\varphi} \right\rangle - \left\langle \left.\frac{\partial\phi}{\partial t}\right|_{\mathbf{r}} - \left.\frac{\partial\phi}{\partial t}\right|_{\mathbf{R},E,\mu,\varphi} \right\rangle. \tag{C.14}$$

To simplify the calculation, I will assume that I know the corrections $\mathbf{R}_3$, $\mu_3 - \langle \mu_3 \rangle$, φ_2 and φ_3 (obtaining these corrections is straightforward following the procedure in section 3.2 but will be unnecessary). With these corrections, I find that to the order needed, $d\mathbf{R}/dt = \langle d\mathbf{R}/dt \rangle$, given in (C.9), $d\mu/dt = \langle d\mu/dt \rangle \simeq 0$ and $d\varphi/dt = \langle d\varphi/dt \rangle$. Then, equation (C.14) simplifies to

$$\langle \mathbf{v} \cdot \nabla\phi \rangle = \left(u\hat{\mathbf{b}} - \frac{c}{B}\nabla_{\mathbf{R}}\Psi \times \hat{\mathbf{b}} \right) \cdot \nabla_{\mathbf{R}}\langle\phi\rangle - \left\langle \left.\frac{\partial\phi}{\partial t}\right|_{\mathbf{r}} - \left.\frac{\partial\phi}{\partial t}\right|_{\mathbf{R},E,\mu,\varphi} \right\rangle, \tag{C.15}$$

where Ψ is given in equation (C.10) and $\langle \partial\phi/\partial\varphi \rangle = 0$. Notice that assuming that I already have $\mathbf{R}_3$, $\mu_3 - \langle \mu_3 \rangle$, φ_2 and φ_3 is only a shortcut to find the result in (C.15). To obtain $\langle\phi\rangle$ to the order required, these higher order corrections are not needed, neither are they necessary for the difference between time derivatives, as I will prove next. The difference between time derivatives is

$$\left.\frac{\partial\phi}{\partial t}\right|_{\mathbf{r}} - \left.\frac{\partial\phi}{\partial t}\right|_{\mathbf{R},E,\mu,\varphi} = \left.\frac{\partial\mathbf{R}}{\partial t}\right|_{\mathbf{r},\mathbf{v}} \cdot \nabla_{\mathbf{R}}\phi + \left.\frac{\partial\mu}{\partial t}\right|_{\mathbf{r},\mathbf{v}} \frac{\partial\phi}{\partial\mu} + \left.\frac{\partial\varphi}{\partial t}\right|_{\mathbf{r},\mathbf{v}} \frac{\partial\phi}{\partial\varphi}. \tag{C.16}$$

The procedure for rewriting (C.16) is analogous to that used on (C.4). Using the definitions of $\mathbf{R}_1$, $\mathbf{R}_2$, μ_1 and φ_1, I find that $\partial \mathbf{R}/\partial t|_{\mathbf{r},\mathbf{v}} \simeq \partial \mathbf{R}_2/\partial t|_{\mathbf{r},\mathbf{v}}$, $\partial \mu/\partial t|_{\mathbf{r},\mathbf{v}} \simeq \partial \mu_1/\partial t|_{\mathbf{r},\mathbf{v}}$ and $\partial \varphi/\partial t|_{\mathbf{r},\mathbf{v}} \simeq \partial \varphi_1/\partial t|_{\mathbf{r},\mathbf{v}}$, giving

$$\left\langle \left.\frac{\partial \phi}{\partial t}\right|_{\mathbf{r}} - \left.\frac{\partial \phi}{\partial t}\right|_{\mathbf{R},E,\mu,\varphi} \right\rangle = \frac{\partial}{\partial t}(\Psi - \langle \phi \rangle), \tag{C.17}$$

where I use the equivalent to equations (C.6) and (C.7) with $\partial/\partial t$ replacing $\nabla_{\mathbf{R}}$. The final result, obtained by combining equations (C.12), (C.15) and (C.17), is

$$\left\langle \frac{dE}{dt} \right\rangle = \frac{Ze}{M} \left[\frac{\partial}{\partial t}(\Psi - \langle \phi \rangle) - \left(u\hat{\mathbf{b}} - \frac{c}{B} \nabla_{\mathbf{R}} \Psi \times \hat{\mathbf{b}} \right) \cdot \nabla_{\mathbf{R}} \langle \phi \rangle \right]. \tag{C.18}$$

C.4 Second order correction μ_2

The correction μ_2, according to section 3.2, is given by

$$\mu_2 = \frac{1}{\Omega} \int^{\varphi} d\varphi' \left[\frac{d}{dt}(\mu_0 + \mu_1) - \left\langle \frac{d}{dt}(\mu_0 + \mu_1) \right\rangle \right] + \langle \mu_2 \rangle, \tag{C.19}$$

where $\langle \mu_2 \rangle$ is found by requiring that $\langle d\mu/dt \rangle = 0$ to order $\delta^2 v_{\mathrm{th}}^3/BL$.

The time derivative of $\mu_0 + \mu_1$ is given by

$$\frac{d}{dt}(\mu_0 + \mu_1) = \frac{Ze}{MB} \left(-\mathbf{v}_\perp \cdot \nabla \phi + \frac{d\tilde{\phi}}{dt} \right). \tag{C.20}$$

To rewrite $\mathbf{v}_\perp \cdot \nabla \phi$ as a function of the gyrokinetic variables, I employ $\mathbf{v}_\perp \cdot \nabla \phi = \mathbf{v} \cdot \nabla \phi - v_{||} \hat{\mathbf{b}} \cdot \nabla \phi$ and equation (C.13) to find

$$-\mathbf{v}_\perp \cdot \nabla \phi + \frac{d\tilde{\phi}}{dt} = -\frac{d\langle \phi \rangle}{dt} + \left.\frac{\partial \phi}{\partial t}\right|_{\mathbf{r}} + v_{||} \hat{\mathbf{b}} \cdot \nabla \phi. \tag{C.21}$$

To the order I am interested in, $d\mathbf{R}/dt \simeq u\hat{\mathbf{b}} - (c/B)\nabla_{\mathbf{R}} \langle \phi \rangle \times \hat{\mathbf{b}}$, giving

$$-\mathbf{v}_\perp \cdot \nabla \phi + \frac{d\tilde{\phi}}{dt} = - \left.\frac{\partial \langle \phi \rangle}{\partial t}\right|_{\mathbf{R},E,\mu,\varphi} - u\hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \langle \phi \rangle + \left.\frac{\partial \phi}{\partial t}\right|_{\mathbf{r}} + v_{||} \hat{\mathbf{b}} \cdot \nabla \phi. \tag{C.22}$$

According to equation (C.11), the difference between $u = \langle v_{||} \rangle$ and $v_{||}$ is higher order, and according to equation (C.16), the difference between $\partial\phi/\partial t|_{\mathbf{r}}$ and $\partial\phi/\partial t|_{\mathbf{R},E,\mu,\varphi}$ is negligible. Therefore, equations (C.20) and (C.22) give

$$\frac{d}{dt}(\mu_0+\mu_1) = \frac{Ze}{MB}\left(\frac{\partial\tilde{\phi}}{\partial t} + u\hat{\mathbf{b}}\cdot\nabla_{\mathbf{R}}\tilde{\phi}\right), \tag{C.23}$$

which in turn, using equation (C.19), yields

$$\mu_2 = \frac{c}{B^2}\left(\frac{\partial\tilde{\Phi}}{\partial t} + u\hat{\mathbf{b}}\cdot\nabla_{\mathbf{R}}\tilde{\Phi}\right) + \langle\mu_2\rangle. \tag{C.24}$$

To find $\langle\mu_2\rangle$ I require that $\langle d\mu/dt\rangle = 0$ to order $\delta^2 v_{\mathrm{th}}^3/BL$. The gyroaverage of $d\mu/dt$ is given by

$$\left\langle\frac{d\mu}{dt}\right\rangle = \left\langle\frac{d}{dt}(\mu_0+\mu_1+\mu_2)\right\rangle = -\frac{Ze}{MB}\langle\mathbf{v}_\perp\cdot\nabla\phi\rangle + \frac{d\langle\mu_2\rangle}{dt}, \tag{C.25}$$

where the gyroaverages of $d\mu_1/dt$ and $d(\mu_2 - \langle\mu_2\rangle)/dt$ vanish. The term $\langle\mathbf{v}_\perp\cdot\nabla\phi\rangle$ can be conveniently rewritten to higher order than in (C.22) by employing equation (C.15) to find

$$\langle\mathbf{v}_\perp\cdot\nabla\phi\rangle = \left(u\hat{\mathbf{b}} - \frac{c}{B}\nabla_{\mathbf{R}}\Psi\times\hat{\mathbf{b}}\right)\cdot\nabla_{\mathbf{R}}\langle\phi\rangle - \frac{\partial}{\partial t}(\Psi - \langle\phi\rangle) - \langle v_{||}\hat{\mathbf{b}}\cdot\nabla\phi\rangle, \tag{C.26}$$

where I used equation (C.17). Employing equation (C.4) and the fact that the difference between $u = \langle v_{||}\rangle$ and $v_{||}$ is order $\delta^2 v_{\mathrm{th}}$ (C.11), I find

$$\langle v_{||}\hat{\mathbf{b}}\cdot\nabla\phi\rangle \simeq \langle v_{||}\hat{\mathbf{b}}\cdot\nabla_{\mathbf{R}}\phi\rangle + u\hat{\mathbf{b}}\cdot\nabla_{\mathbf{R}}(\Psi - \langle\phi\rangle) \simeq u\hat{\mathbf{b}}\cdot\nabla_{\mathbf{R}}\Psi. \tag{C.27}$$

To obtain the second equality, I employ $\hat{\mathbf{b}}\cdot\nabla_{\mathbf{R}}\langle\phi\rangle \gg \hat{\mathbf{b}}\cdot\nabla_{\mathbf{R}}\tilde{\phi}$, which means that $\langle v_{||}\hat{\mathbf{b}}\cdot\nabla_{\mathbf{R}}\phi\rangle \simeq \langle v_{||}\hat{\mathbf{b}}\cdot\nabla_{\mathbf{R}}\langle\phi\rangle + u\hat{\mathbf{b}}\cdot\nabla_{\mathbf{R}}\tilde{\phi}\rangle = u\hat{\mathbf{b}}\cdot\nabla_{\mathbf{R}}\langle\phi\rangle$ to order $\delta^2 T v_{\mathrm{th}}/eL$. Then, equation (C.26) becomes $\langle\mathbf{v}_\perp\cdot\nabla\phi\rangle = -d(\Psi - \langle\phi\rangle)/dt$, where to this order $d/dt = \partial/\partial t + [u\hat{\mathbf{b}} - (c/B)\nabla_{\mathbf{R}}\langle\phi\rangle\times\hat{\mathbf{b}}]\cdot[\nabla_{\mathbf{R}} - (Ze/M)\nabla_{\mathbf{R}}\langle\phi\rangle(\partial/\partial E)]$ and $\partial(\Psi - \langle\phi\rangle)/\partial E = 0$. Finally, imposing $\langle d\mu/dt\rangle = 0$ on equation (C.25), I find

$$\langle\mu_2\rangle = -\frac{Ze}{MB}\left(\Psi - \langle\phi\rangle\right). \tag{C.28}$$

C.5 Comparisons with Dubin *et al*

To compare with reference [21], I first need to write the gyrokinetic equation in the same variables that are used in that reference, i.e., I need to employ u instead of E. The change is easy to carry out. I substitute E_2 and (C.24) into (C.11) to write

$$v_{\|} = \sqrt{2[E - (\mu - \langle\mu_2\rangle)B]} + \frac{c}{B}\hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}}\tilde{\Phi}, \tag{C.29}$$

where I Taylor expand $E_2 - (\mu_2 - \langle\mu_2\rangle)B = -(c/B)u\hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}}\tilde{\Phi}$. Then, gyroaveraging this equation I find

$$\frac{u^2}{2} = E - (\mu - \langle\mu_2\rangle)B. \tag{C.30}$$

Applying the Vlasov operator to this expression and gyroaveraging, I find

$$\left\langle \frac{du}{dt} \right\rangle = -\frac{Ze}{M}\hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}}\Psi, \tag{C.31}$$

where I used equations (C.18), (C.28) and $\langle d\mu/dt\rangle = 0$. With this equation, equation (C.9) and the fact that $\langle d\mu/dt\rangle = 0$, I find the same gyrokinetic Vlasov equation as in reference [21], namely

$$\frac{\partial f}{\partial t} + \left(u\hat{\mathbf{b}} - \frac{c}{B}\nabla_{\mathbf{R}}\Psi \times \hat{\mathbf{b}}\right) \cdot \nabla_{\mathbf{R}} f - \frac{Ze}{M}\hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}}\Psi \frac{\partial f}{\partial u} = 0, \tag{C.32}$$

with $f(\mathbf{R}, u, \mu, t)$. The differences between my function Ψ of (C.10) and the function ψ in reference [21], given in their equation (19b), come from their introduction of the potential function $\phi(\mathbf{R}+\boldsymbol{\rho}, t) \neq \phi(\mathbf{r}, t)$, leading to subtle differences in the definitions of $\langle\phi\rangle$, $\tilde{\phi}$ and $\tilde{\Phi}$. Here, the vector $\boldsymbol{\rho}(\mu, \theta)$ is

$$\boldsymbol{\rho} = \frac{\sqrt{2\mu B}}{\Omega}(\hat{\mathbf{e}}_1 \cos\theta - \hat{\mathbf{e}}_2 \sin\theta), \tag{C.33}$$

with θ the gyrokinetic gyrophase as defined in [21]. The relation between the gyrophase θ and my gyrophase φ is $\theta = -\pi/2 - \varphi$. From now on, I will denote the functions $\langle\phi\rangle$, $\tilde{\phi}$ and $\tilde{\Phi}$ as they are defined in [21] with the subindex D. The definitions in [21] are then

$$\overline{\phi}_D \equiv \overline{\phi}_D(\mathbf{R}, \mu, t) = \frac{1}{2\pi}\oint d\theta\, \phi(\mathbf{R} + \boldsymbol{\rho}, t), \tag{C.34}$$

$$\widetilde{\phi}_D \equiv \widetilde{\phi}_D(\mathbf{R}, \mu, \theta, t) = \phi(\mathbf{R} + \boldsymbol{\rho}, t) - \overline{\phi}_D \tag{C.35}$$

and

$$\widetilde{\Phi}_D \equiv \widetilde{\Phi}_D(\mathbf{R}, \mu, \theta, t) = \int^{\theta} d\theta'\, \widetilde{\phi}_D(\mathbf{R}, \mu, \theta', t) \tag{C.36}$$

such that $\langle \widetilde{\Phi}_D \rangle = 0$. Notice that these definitions coincide with mine to order $\delta T/e$, except for $\widetilde{\Phi}_D$, for which $\widetilde{\Phi}_D \simeq -\widetilde{\Phi}$. The sign is due to the definition of the gyrophase θ. To second order, however, Taylor expanding $\phi(\mathbf{r}, t) = \phi(\mathbf{R} + \boldsymbol{\rho} - \boldsymbol{\rho} - \mathbf{R}_1 - \mathbf{R}_2, t)$ gives

$$\phi \simeq \phi(\mathbf{R} + \boldsymbol{\rho}, t) - (\boldsymbol{\rho} + \mathbf{R}_1 + \mathbf{R}_2) \cdot \nabla_{\mathbf{R}}\phi, \tag{C.37}$$

where

$$\boldsymbol{\rho} + \mathbf{R}_1 \simeq -\frac{Mc\mu_1}{Zev_\perp^2}\mathbf{v} \times \hat{\mathbf{b}} - \frac{\varphi_1}{\Omega}\mathbf{v}_\perp = O(\delta\rho). \tag{C.38}$$

To obtain equation (C.38), I Taylor expand $\boldsymbol{\rho}(\mu, \varphi)$ around μ_0 and φ_0 in equation (C.33). Employing the lowest order results $\partial\phi/\partial\varphi \simeq -\Omega^{-1}\mathbf{v}_\perp \cdot \nabla\phi$ and $\partial\phi/\partial\mu \simeq -(Mc/Zev_\perp^2)(\mathbf{v} \times \hat{\mathbf{b}}) \cdot \nabla\phi$, I write equation (C.37) as

$$\phi \simeq \phi(\mathbf{R} + \boldsymbol{\rho}, t) - \frac{Ze}{MB}\left(\widetilde{\phi}\frac{\partial\phi}{\partial\mu} - \frac{\partial\widetilde{\Phi}}{\partial\mu}\frac{\partial\phi}{\partial\varphi}\right) + \frac{c}{B\Omega}(\nabla_{\mathbf{R}}\widetilde{\Phi} \times \hat{\mathbf{b}}) \cdot \nabla_{\mathbf{R}}\phi, \tag{C.39}$$

where I used the definitions of $\mathbf{R}_2$, μ_1 and φ_1. Then, gyroaveraging gives

$$\langle\phi\rangle \simeq \overline{\phi}_D - \frac{Ze}{MB}\frac{\partial}{\partial\mu}\langle\widetilde{\phi}^2\rangle + \frac{c}{B\Omega}\langle(\nabla_{\mathbf{R}}\widetilde{\Phi} \times \hat{\mathbf{b}}) \cdot \nabla_{\mathbf{R}}\widetilde{\phi}\rangle. \tag{C.40}$$

Substituting this equation into the definition (C.10) of Ψ and employing that to lowest order $\widetilde{\phi} \simeq \widetilde{\phi}_D$ and $\widetilde{\Phi} \simeq -\widetilde{\Phi}_D$, I find

$$\Psi = \overline{\phi}_D - \frac{Ze}{2MB}\frac{\partial}{\partial\mu}\langle\widetilde{\phi}_D^2\rangle - \frac{c}{2B\Omega}\langle(\nabla_{\mathbf{R}}\widetilde{\Phi}_D \times \hat{\mathbf{b}}) \cdot \nabla_{\mathbf{R}}\widetilde{\phi}_D\rangle, \tag{C.41}$$

exactly as in equation (19b) of reference [21].

Finally, I will compare the quasineutrality equations in both methods. Taylor expanding the ion distribution function around $\mathbf{R}_g = \mathbf{r} + \Omega^{-1}\mathbf{v} \times \hat{\mathbf{b}}$, $v_{||}$, μ_0 and φ_0, I find

$$f_i(\mathbf{R}, u, \mu, t) \simeq f_{ig} + \mathbf{R}_2 \cdot \nabla_{\mathbf{R}_g} f_{ig} - \frac{c}{B}\hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \widetilde{\Phi} \frac{\partial f_{ig}}{\partial v_{||}} + (\mu_1 + \mu_2)\frac{\partial f_{ig}}{\partial \mu_0} + \frac{\mu_1^2}{2}\frac{\partial^2 f_{ig}}{\partial \mu_0^2}, \quad \text{(C.42)}$$

where $f_{ig} \equiv f_i(\mathbf{R}_g, v_{||}, \mu_0, t)$. Here I have used equations (C.29) and (C.30) to obtain that $u \simeq v_{||} - (c/B)\hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \widetilde{\Phi}$. The ion density is given by

$$n_i = \int d^3v \, f_i \simeq \int d^3v \left[f_{ig} + \mathbf{R}_2 \cdot \nabla_{\mathbf{R}_g} f_{ig} + (\mu_1 + \langle \mu_2 \rangle)\frac{\partial f_{ig}}{\partial \mu_0} + \frac{\mu_1^2}{2}\frac{\partial^2 f_{ig}}{\partial \mu_0^2} \right]. \quad \text{(C.43)}$$

Here, the integrals of $(c/B)(\hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \widetilde{\Phi})(\partial f_{ig}/\partial v_{||})$ and $(\mu_2 - \langle \mu_2 \rangle)(\partial f_{ig}/\partial \mu_0)$ vanish because $\oint d\varphi_0 \widetilde{\Phi} = 0$ and the only gyrophase dependence is in $\widetilde{\Phi}$ since f_{ig} is assumed to be a smooth function of $\mathbf{r}$ and $\mathbf{v}$ to lowest order, giving $f_{ig} \equiv f_i(\mathbf{R}_g, v_{||}, \mu_0, t) \simeq f_i(\mathbf{r}, v_{||}, \mu_0, t)$. The integral $\oint d\varphi_0 \widetilde{\Phi}$ is performed holding $\mathbf{r}$, $v_{||}$, μ_0 and t fixed, and it vanishes to lowest order as proven at the end of this Appendix. On the other hand, the integral of $\mathbf{R}_2 \cdot \nabla_{\mathbf{R}_g} f_{ig}$ does not vanish. Here the gyrophase dependence of f_{ig} contained in its short wavelength contributions becomes important due to the steep gradient [recall the ordering in (C.1)].

In equation (C.43), I can employ $\widetilde{\phi} \simeq \widetilde{\phi}_D$ and $\widetilde{\Phi} \simeq -\widetilde{\Phi}_D$ in the higher order terms. However, for μ_1 I need the difference between $\widetilde{\phi}$ and $\widetilde{\phi}_D$. Subtracting (C.40) from (C.39), I find

$$\widetilde{\phi} \simeq \widetilde{\phi}_D - \frac{Ze}{MB}\left(\widetilde{\phi}\frac{\partial \phi}{\partial \mu} - \frac{\partial \widetilde{\Phi}}{\partial \mu}\frac{\partial \widetilde{\phi}}{\partial \varphi}\right) + \frac{c}{B\Omega}(\nabla_{\mathbf{R}} \widetilde{\Phi} \times \hat{\mathbf{b}}) \cdot \nabla_{\mathbf{R}} \phi + \frac{Ze}{MB}\frac{\partial}{\partial \mu}\langle \widetilde{\phi}^2 \rangle$$
$$- \frac{c}{B\Omega}\langle (\nabla_{\mathbf{R}} \widetilde{\Phi} \times \hat{\mathbf{b}}) \cdot \nabla_{\mathbf{R}} \widetilde{\phi} \rangle. \quad \text{(C.44)}$$

In this equation, $\widetilde{\phi}_D = \widetilde{\phi}_D(\mathbf{R}, \mu, \varphi, t)$, but for equation (C.43), it is better to use $\widetilde{\phi}_{Dg} = \widetilde{\phi}_D(\mathbf{R}_g, \mu_0, \varphi_0, t)$. By Taylor expanding, I find that

$$\widetilde{\phi}_D \simeq \widetilde{\phi}_{Dg} + \mathbf{R}_2 \cdot \nabla_{\mathbf{R}_g} \widetilde{\phi}_{Dg} + \mu_1 \frac{\partial \widetilde{\phi}_{Dg}}{\partial \mu_0} + \varphi_1 \frac{\partial \widetilde{\phi}_{Dg}}{\partial \varphi_0}. \quad \text{(C.45)}$$

This equation, combined with equation (C.44) and the definitions of $\mathbf{R}_2$, μ_1 and φ_1, leads to

$$\tilde{\phi} \simeq \tilde{\phi}_{Dg} - \frac{Ze\tilde{\phi}_{Dg}}{MB}\frac{\partial\overline{\phi}_{Dg}}{\partial\mu} - \frac{c}{B\Omega}(\nabla_{\mathbf{R}_g}\tilde{\Phi}_{Dg}\times\hat{\mathbf{b}})\cdot\nabla_{\mathbf{R}_g}\overline{\phi}_{Dg} + \frac{Ze}{MB}\frac{\partial}{\partial\mu}\langle\tilde{\phi}^2_{Dg}\rangle + \frac{c}{B\Omega}\langle(\nabla_{\mathbf{R}_g}\tilde{\Phi}_{Dg}\times\hat{\mathbf{b}})\cdot\nabla_{\mathbf{R}_g}\tilde{\phi}_{Dg}\rangle, \quad \text{(C.46)}$$

where I have used that, to lowest order, $\tilde{\phi}_D \simeq \tilde{\phi}_{Dg}$, $\overline{\phi}_D \simeq \overline{\phi}_{Dg} \equiv \overline{\phi}_D(\mathbf{R}_g, \mu_0, t)$ and $\tilde{\Phi}_D \simeq \tilde{\Phi}_{Dg} \equiv \tilde{\Phi}_D(\mathbf{R}_g, \mu_0, \varphi_0, t)$. Substituting equation (C.46) into (C.43) yields

$$n_i \simeq \int d^3v \left\{ f_{ig} + \frac{Ze\tilde{\phi}_{Dg}}{MB}\frac{\partial f_{ig}}{\partial\mu_0} + \frac{c}{B\Omega}(\nabla_{\mathbf{R}_g}\tilde{\Phi}_{Dg}\times\hat{\mathbf{b}})\cdot\nabla_{\mathbf{R}_g} f_{ig} + \frac{Z^2e^2\tilde{\phi}^2_{Dg}}{2M^2B^2}\frac{\partial^2 f_{ig}}{\partial\mu_0^2} + \frac{Ze}{MB}\left[-\frac{Ze\tilde{\phi}_{Dg}}{MB}\frac{\partial\overline{\phi}_{Dg}}{\partial\mu_0} - \frac{c}{B\Omega}(\nabla_{\mathbf{R}_g}\tilde{\Phi}_{Dg}\times\hat{\mathbf{b}})\cdot\nabla_{\mathbf{R}_g}\overline{\phi}_{Dg} + \frac{Ze}{2MB}\frac{\partial}{\partial\mu_0}\langle\tilde{\phi}^2_{Dg}\rangle + \frac{c}{2B\Omega}\langle(\nabla_{\mathbf{R}_g}\tilde{\Phi}_{Dg}\times\hat{\mathbf{b}})\cdot\nabla_{\mathbf{R}_g}\tilde{\phi}_{Dg}\rangle \right]\frac{\partial f_{ig}}{\partial\mu_0} \right\}, \quad \text{(C.47)}$$

where I have used the definitions of $\mathbf{R}_2$ and $\langle\mu_2\rangle$. This result is exactly the same as in equation (20) in reference [21]. For comparison, I give n_i to order $\delta^2 n_i$ with the definitions of $\langle\phi\rangle$, $\tilde{\phi}$ and $\tilde{\Phi}$ in equations (3.16), (3.17) and (3.18),

$$n_i \simeq \int d^3v \left\{ f_{ig} + \frac{Ze\tilde{\phi}_g}{MB}\frac{\partial f_{ig}}{\partial\mu_0} - \frac{c}{B\Omega}(\nabla_{\mathbf{R}_g}\tilde{\Phi}_g\times\hat{\mathbf{b}})\cdot\nabla_{\mathbf{R}_g} f_{ig} + \frac{Z^2e^2\tilde{\phi}^2_g}{2M^2B^2}\frac{\partial^2 f_{ig}}{\partial\mu_0^2} + \frac{Ze}{MB}\left[\frac{Ze\tilde{\phi}_g}{MB}\frac{\partial\tilde{\phi}_g}{\partial\mu_0} - \frac{Ze\tilde{\Phi}_g}{MB}\frac{\partial\tilde{\phi}_g}{\partial\varphi_0} - \frac{c}{B\Omega}(\nabla_{\mathbf{R}_g}\tilde{\Phi}_g\times\hat{\mathbf{b}})\cdot\nabla_{\mathbf{R}_g}\tilde{\phi}_g - \frac{Ze}{2MB}\frac{\partial}{\partial\mu_0}\langle\tilde{\phi}^2_g\rangle + \frac{c}{2B\Omega}\langle(\nabla_{\mathbf{R}_g}\tilde{\Phi}_g\times\hat{\mathbf{b}})\cdot\nabla_{\mathbf{R}_g}\tilde{\phi}_g\rangle \right]\frac{\partial f_{ig}}{\partial\mu_0} \right\}. \quad \text{(C.48)}$$

I have found this equation substituting μ_1, $\mathbf{R}_2$ and $\langle\mu_2\rangle$ into (C.43). From the functions $\tilde{\phi}(\mathbf{R}, \mu, \varphi, t)$ and $\tilde{\Phi}(\mathbf{R}, \mu, \varphi, t)$, I have defined $\tilde{\phi}_g = \tilde{\phi}(\mathbf{R}_g, \mu_0, \varphi_0, t)$ and $\tilde{\Phi}_g = \tilde{\Phi}(\mathbf{R}_g, \mu_0, \varphi_0, t)$. The relationships between $\tilde{\phi}$ and $\tilde{\phi}_g$ and between $\tilde{\Phi}$ and $\tilde{\Phi}_g$ are similar to the one given in (C.45).

The methodology and results of chapter 3 are completely consistent with the results of [21] since they give the same gyrokinetic equation (C.32), generalized potential Ψ (C.41) and quasineutrality condition (C.47).

Integral $\oint d\varphi_0 \widetilde{\Phi}$

To prove that $\oint d\varphi_0 \widetilde{\Phi} = 0$ vanishes, I Fourier analyze $\phi = (2\pi)^{-3} \int d^3k \, \phi_{\mathbf{k}} \exp(i\mathbf{k}\cdot\mathbf{r})$, giving to lowest order

$$\phi(\mathbf{r},t) \simeq \phi(\mathbf{R} - \Omega^{-1}\mathbf{v}\times\hat{\mathbf{b}},t) = \frac{1}{(2\pi)^3}\int d^3k\,\phi_{\mathbf{k}} \exp[i\mathbf{k}\cdot\mathbf{R} - iz\sin(\varphi_0 - \varphi_{\mathbf{k}})], \quad \text{(C.49)}$$

where $z = k_\perp v_\perp/\Omega$. Here, I employ $\mathbf{r} \simeq \mathbf{R} - \Omega^{-1}\mathbf{v}\times\hat{\mathbf{b}}$ and I define $\varphi_{\mathbf{k}}$ such that $\mathbf{k}_\perp = k_\perp(\hat{\mathbf{e}}_1 \cos\varphi_{\mathbf{k}} + \hat{\mathbf{e}}_2 \sin\varphi_{\mathbf{k}})$ to write $\mathbf{k}\cdot\mathbf{r} \simeq \mathbf{k}\cdot\mathbf{R} - z\sin(\varphi_0 - \varphi_{\mathbf{k}})$. Then, I use

$$\exp(iz\sin\varphi) = \sum_{m=-\infty}^{\infty} J_m(z)\exp(im\varphi), \quad \text{(C.50)}$$

with $J_m(z)$ the Bessel function of the first kind, to find

$$\phi \simeq \frac{1}{(2\pi)^3}\int d^3k\,\phi_{\mathbf{k}}\exp(i\mathbf{k}\cdot\mathbf{R}) \sum_{m=-\infty}^{\infty} J_m(z)\exp[-im(\varphi_0 - \varphi_{\mathbf{k}})]. \quad \text{(C.51)}$$

Employing this expression, I obtain $\tilde{\phi}$ by subtracting the average in φ_0 (component $m=0$), and I find $\widetilde{\Phi}$ by integrating $\tilde{\phi}$ over φ_0, giving

$$\widetilde{\Phi} \simeq \frac{1}{(2\pi)^3}\int d^3k\,\phi_{\mathbf{k}}\exp(i\mathbf{k}\cdot\mathbf{R}) \sum_{m\neq 0} \frac{i}{m} J_m(z)\exp[-im(\varphi_0 - \varphi_{\mathbf{k}})], \quad \text{(C.52)}$$

where the summation includes every positive and negative m different from 0. To rewrite $\widetilde{\Phi}$ as a function of $\mathbf{r}$, $v_{||}$, μ_0 and φ_0, I need the expression

$$\exp(i\mathbf{k}\cdot\mathbf{R}) \simeq \exp(i\mathbf{k}\cdot\mathbf{r}) \sum_{p=-\infty}^{\infty} J_p(z)\exp[ip(\varphi_0 - \varphi_{\mathbf{k}})], \quad \text{(C.53)}$$

deduced from $\mathbf{R} \simeq \mathbf{r} + \Omega^{-1}\mathbf{v}\times\hat{\mathbf{b}}$ and (C.50). Then, I find

$$\widetilde{\Phi} \simeq \frac{1}{(2\pi)^3}\int d^3k\,\phi_{\mathbf{k}}\exp(i\mathbf{k}\cdot\mathbf{r}) \sum_{m\neq 0,p} \frac{i}{m} J_m(z) J_p(z)\exp[i(p-m)(\varphi_0 - \varphi_{\mathbf{k}})]. \quad \text{(C.54)}$$

Finally, integrating in φ_0, I obtain

$$\frac{1}{2\pi}\oint d\varphi_0\,\widetilde{\Phi} = \frac{1}{(2\pi)^3}\int d^3k\,\phi_{\mathbf{k}}\exp(i\mathbf{k}\cdot\mathbf{r})\sum_{m\neq 0}\frac{i}{m}[J_m(z)]^2 = 0 \tag{C.55}$$

since $J_{-m}(z) = (-1)^m J_m(z)$.

Appendix D

Quasineutrality equation at long wavelengths

In this Appendix, I obtain the gyrokinetic quasineutrality equation at long wavelengths ($k_\perp \rho_i \ll 1$). In section D.1, the polarization density n_{ip} in (3.55) is calculated for the intermediate scales $\delta_i \ll k_\perp \rho_i \ll 1$. In section D.2, the polarization density n_{ip} and the ion guiding center density $\hat{N}_i$ are computed up to $O(\delta_i^2 n_e)$ for the extreme case of a non-turbulent plasma.

D.1 Polarization density at long wavelengths

For long wavelengths $\delta_i \ll k_\perp \rho_i \ll 1$, the polarization density simplifies to give (3.59). To obtain this result, $\tilde{\phi}$ and hence ϕ must be obtained to $O(\delta_i k_\perp \rho_i T_e/e)$. To this order, the potential is

$$\phi(\mathbf{r}) \simeq \phi(\mathbf{R} - \mathbf{R}_1) \simeq \phi(\mathbf{R}) - \mathbf{R}_1 \cdot \nabla_\mathbf{R} \phi + \frac{1}{2} \mathbf{R}_1 \mathbf{R}_1 : \nabla_\mathbf{R} \nabla_\mathbf{R} \phi, \tag{D.1}$$

with $\mathbf{R}_1 = \Omega_i^{-1} \mathbf{v} \times \hat{\mathbf{b}}$. Here, according to (3.1), $\nabla_\mathbf{R} \phi \sim T_e/eL$ and $\nabla_\mathbf{R} \nabla_\mathbf{R} \phi \sim k_\perp T_e/eL$. With the result in (D.1), the function $\tilde{\phi}$ is

$$\tilde{\phi} = \phi - \langle \phi \rangle \simeq -\mathbf{R}_1 \cdot \nabla_\mathbf{R} \phi + \frac{1}{2} (\mathbf{R}_1 \mathbf{R}_1 - \langle \mathbf{R}_1 \mathbf{R}_1 \rangle) : \nabla_\mathbf{R} \nabla_\mathbf{R} \phi, \tag{D.2}$$

where I neglect $\langle \mathbf{R}_1 \rangle \sim \delta_i \rho_i$. The gyroaverage of the first order correction $\mathbf{R}_1$ is not zero due to the difference between $\mathbf{r}$, μ_0 and φ_0 and the gyrokinetic variables $\mathbf{R}$, μ and φ. This difference will become important in section D.2.

The integral n_{ip} in (3.55) is performed over velocity space holding $\mathbf{r}$ and t fixed. Then, equation (D.2) has to be Taylor expanded around $\mathbf{r}$ to obtain a function that depends on $\mathbf{r}$ and not $\mathbf{R}$. To $O(\delta_i k_\perp \rho_i T_e/e)$, the result is

$$\tilde{\phi} \simeq -\mathbf{R}_1 \cdot \nabla \phi - \frac{1}{2}(\mathbf{R}_1\mathbf{R}_1 + \overline{\mathbf{R}_1\mathbf{R}_1}) : \nabla\nabla\phi, \tag{D.3}$$

where I use $\nabla_{\mathbf{R}}\phi \simeq \nabla\phi + \mathbf{R}_1 \cdot \nabla\nabla\phi$. In the higher order terms, the gyroaverage $\langle \ldots \rangle$ holding the gyrokinetics variable fixed can be approximated by the gyroaverage $\overline{(\ldots)}$ holding $\mathbf{r}$, $v_{||}$ and $v_\perp$ fixed.

Expression (D.3) can be readily substituted into (3.55) and integrated to give

$$n_{ip} \simeq \frac{cn_i}{B\Omega_i}(\overleftrightarrow{\mathbf{I}} - \hat{\mathbf{b}}\hat{\mathbf{b}}) : \nabla\nabla\phi. \tag{D.4}$$

The final result in (3.59) is found by realizing that $\nabla \cdot [(cn_i/B\Omega_i)(\overleftrightarrow{\mathbf{I}} - \hat{\mathbf{b}}\hat{\mathbf{b}})] \cdot \nabla_\perp \phi$ is small by a factor $(k_\perp L)^{-1}$ compared to the term in (D.4).

D.2 Quasineutrality equation for $k_\perp L \sim 1$

In this section, I obtain the long wavelength quasineutrality condition to order $\delta_i^2 n_e$. The derivation is not applicable to turbulent plasmas because I will assume that neither the potential nor the distribution function have short wavelength pieces.

In quasineutrality, the ion distribution function must be written in $\mathbf{r}$, $\mathbf{v}$ variables. In a non-turbulent plasma, the ion distribution function can be expanded around $\mathbf{r}$, E_0 and μ_0 up to $O(\delta_i^2 f_{Mi})$, giving

$$\begin{aligned} f_i(\mathbf{R}, E, \mu, t) = {} & f_{i0} + (\mathbf{R}_1 + \mathbf{R}_2) \cdot \overline{\nabla} f_{i0} + (E_1 + E_2)\frac{\partial f_{i0}}{\partial E_0} + \mu_1 \frac{\partial f_{i0}}{\partial \mu_0} \\ & + \frac{1}{2}\mathbf{R}_1\mathbf{R}_1 : \overline{\nabla}\,\overline{\nabla} f_{Mi} + E_1 \mathbf{R}_1 \cdot \overline{\nabla}\left(\frac{\partial f_{Mi}}{\partial E_0}\right) + \frac{E_1^2}{2}\frac{\partial^2 f_{Mi}}{\partial E_0^2}, \end{aligned} \tag{D.5}$$

with $f_{i0} \equiv f_i(\mathbf{r}, E_0, \mu_0, t)$ and $\overline{\nabla}$ the gradient holding E_0, μ_0, φ_0 and t fixed. In f_i I have not included the collisional gyrophase dependent piece given in (3.38) because it will not contribute to the density. In the higher order terms of the Taylor expansion (D.5), the lowest order distribution function $f_i \simeq f_{Mi}$ must be used.

To find the ion density, the ion distribution function is integrated in velocity space. Some of the terms vanish because the integral over gyrophase is zero; for example, $\int d^3v\, (\mathbf{R}_1 + \mathbf{R}_2) \cdot \overline{\nabla} f_{i0} = 0 = \int d^3v\, E_2(\partial f_{i0}/\partial E_0)$. Then, the ion density becomes

$$n_i \simeq \hat{N}_i + \int d^3v \frac{Ze\tilde{\phi}}{M} \left[\frac{\partial f_{i0}}{\partial E_0} + \frac{1}{B}\frac{\partial f_{i0}}{\partial \mu_0} + \frac{Ze\tilde{\phi}}{2M}\frac{\partial^2 f_{Mi}}{\partial E_0^2} + \mathbf{R}_1 \cdot \overline{\nabla} \left(\frac{\partial f_{Mi}}{\partial E_0} \right) \right], \qquad \text{(D.6)}$$

where $\hat{N}_i(\mathbf{r}, t)$ is the ion gyrocenter density, defined as the portion of the ion density independent of $\tilde{\phi}$ and given by

$$\hat{N}_i = \int d^3v\, f_{i0} - \int d^3v \frac{v_{||} v_\perp^2}{2B\Omega_i} \hat{\mathbf{b}} \cdot \nabla \times \hat{\mathbf{b}} \frac{\partial f_{i0}}{\partial \mu_0} + \int d^3v \frac{\mathbf{R}_1 \mathbf{R}_1}{2} : \overline{\nabla}\,\overline{\nabla} f_{Mi}. \qquad \text{(D.7)}$$

The formula for $\hat{N}_i$ can be simplified. The second term in the right side of the equation is proportional to $\int d^3v\, (v_{||} v_\perp^2/2B)(\partial f_{i0}/\partial \mu_0)$. This integral is simplified by changing to the variables $E_0 = v^2/2$, $\mu_0 = v_\perp^2/2B$ and φ_0 and integrating by parts,

$$-\int d^3v \frac{v_{||} v_\perp^2}{2B} \frac{\partial f_{i0}}{\partial \mu_0} = -B \sum_\sigma \int dE_0\, d\mu_0\, d\varphi_0\, \sigma \mu_0 \frac{\partial f_{i0}}{\partial \mu_0} = \int d^3v\, v_{||} f_{i0}, \qquad \text{(D.8)}$$

where $\sigma = v_{||}/|v_{||}|$ is the sign of the parallel velocity, the summation in front of the integral indicates that the integral must be done for both signs of $v_{||}$, and I have used the equality $d^3v = dE_0\, d\mu_0\, d\varphi_0\, B/|v_{||}|$. The third term in the right side of (D.7) is proportional to

$$M \int d^3v\, (\mathbf{v} \times \hat{\mathbf{b}})(\mathbf{v} \times \hat{\mathbf{b}}) : \overline{\nabla}\,\overline{\nabla} f_{Mi} = (\overleftrightarrow{\mathbf{I}} - \hat{\mathbf{b}}\hat{\mathbf{b}}) : \nabla\nabla p_i. \qquad \text{(D.9)}$$

The final function $\hat{N}_i$ reduces to the result shown in (3.62).

In equation (D.6), $\tilde{\phi}$ appears in several integrals. The expression can be simplified by integrating first in the gyrophase. Two integrals in (D.6) can be done by using the lowest order result $\widetilde{\phi} \simeq -\Omega_i^{-1}(\mathbf{v} \times \hat{\mathbf{b}}) \cdot \nabla\phi$. The integrals are

$$\int d^3v \, \frac{Z^2e^2}{2M^2} \widetilde{\phi}^2 \frac{\partial^2 f_{Mi}}{\partial E_0^2} = \frac{Mc^2 n_i}{2T_i B^2} |\nabla_\perp \phi|^2 \tag{D.10}$$

and

$$\int d^3v \, \frac{c}{B} \widetilde{\phi} (\mathbf{v} \times \hat{\mathbf{b}}) \cdot \nabla \left(\frac{\partial f_{Mi}}{\partial E_0} \right) = \frac{c}{B\Omega_i} \nabla n_i \cdot \nabla_\perp \phi. \tag{D.11}$$

For $\int d^3v \, \widetilde{\phi} (\partial f_{i0}/\partial E_0 + B^{-1} \partial f_{i0}/\partial \mu_0)$, only the gyrophase integral $\int_0^{2\pi} \widetilde{\phi} \, d\varphi_0$ is needed, but $\widetilde{\phi}$ must be written as a function of the variables $\mathbf{r}$, $\mathbf{v}$ to $O(\delta_i^2 T_e/e)$ to be consistent with the order of the Taylor expansion. To do so, first I will write $\phi(\mathbf{r}, t)$ as a function of the gyrokinetic variables by Taylor expansion to $O(\delta_i^2 T_e/e)$ [this Taylor expansion is carried out to higher order than in (D.1)]. The result is

$$\phi(\mathbf{r}, t) \simeq \phi(\mathbf{R}, t) - \mathbf{R}_1 \cdot \nabla_\mathbf{R} \phi + \frac{1}{2} \mathbf{R}_1 \mathbf{R}_1 : \nabla_\mathbf{R} \nabla_\mathbf{R} \phi - \mathbf{R}_2 \cdot \nabla_\mathbf{R} \phi. \tag{D.12}$$

The second term in the right side of the equation needs to be re-expanded in order to express ϕ as a self-consistent function of the gyrokinetic variables to the right order. The function $\mathbf{R}_1$ is, to $O(\delta_i \rho_i)$,

$$\mathbf{R}_1 = \frac{1}{\Omega_i} \mathbf{v} \times \hat{\mathbf{b}} \simeq \frac{\sqrt{2\mu B(\mathbf{R})}}{\Omega_i(\mathbf{R})} [\hat{\mathbf{e}}_1(\mathbf{R}) \sin\varphi - \hat{\mathbf{e}}_2(\mathbf{R}) \cos\varphi] - \Delta\boldsymbol{\rho}, \tag{D.13}$$

with $\Delta\boldsymbol{\rho}(\mathbf{R}, E, \mu, \varphi, t) \sim \delta_i \rho_i$. The function $\Delta\boldsymbol{\rho}$ is found by Taylor expanding $\mathbf{R}_1(\mathbf{r}, \mu_0, \varphi_0)$ around $\mathbf{R}$, μ and φ to $O(\delta_i \rho_i)$, giving

$$\Delta\boldsymbol{\rho} = -\frac{1}{2B\Omega_i^2} (\mathbf{v} \times \hat{\mathbf{b}})(\mathbf{v} \times \hat{\mathbf{b}}) \cdot \nabla B + \frac{Mc}{Zev_\perp^2} \mu_1 (\mathbf{v} \times \hat{\mathbf{b}}) + \frac{1}{\Omega_i} \varphi_1 \mathbf{v}_\perp + \frac{v_\perp}{\Omega_i^2} (\mathbf{v} \times \hat{\mathbf{b}}) \cdot (\sin\varphi_0 \nabla \hat{\mathbf{e}}_1 - \cos\varphi_0 \nabla \hat{\mathbf{e}}_2). \tag{D.14}$$

Combining (D.12) and (D.13), $\widetilde{\phi}$ is found to be

$$\widetilde{\phi} = \phi - \langle \phi \rangle \simeq -\frac{\sqrt{2\mu B(\mathbf{R})}}{\Omega_i(\mathbf{R})} [\hat{\mathbf{e}}_1(\mathbf{R}) \sin\varphi - \hat{\mathbf{e}}_2(\mathbf{R}) \cos\varphi] \cdot \nabla_\mathbf{R} \phi - \mathbf{R}_2 \cdot \nabla_\mathbf{R} \phi$$

$$+(\Delta\boldsymbol{\rho} - \langle\Delta\boldsymbol{\rho}\rangle)\cdot\nabla_{\mathbf{R}}\phi + \frac{1}{2\Omega_i^2}[(\mathbf{v}\times\hat{\mathbf{b}})(\mathbf{v}\times\hat{\mathbf{b}}) - \langle(\mathbf{v}\times\hat{\mathbf{b}})(\mathbf{v}\times\hat{\mathbf{b}})\rangle] : \nabla_{\mathbf{R}}\nabla_{\mathbf{R}}\phi. \quad \text{(D.15)}$$

The function $\tilde{\phi}$ must be written as a function of $\mathbf{r}$, $\mathbf{v}$ and t since the integral in velocity space is done for $\mathbf{r}$ and t fixed. Taylor expanding the first term in (D.15) gives

$$\tilde{\phi} \simeq -\frac{1}{\Omega_i}(\mathbf{v}\times\hat{\mathbf{b}})\cdot\nabla\phi - \langle\Delta\boldsymbol{\rho}\rangle\cdot\nabla\phi - \mathbf{R}_2\cdot\nabla\phi - \frac{v_\perp^2}{4\Omega_i^2}(\overleftrightarrow{\mathbf{I}} - \hat{\mathbf{b}}\hat{\mathbf{b}}) : \nabla\nabla\phi$$
$$-\frac{1}{2\Omega_i^2}(\mathbf{v}\times\hat{\mathbf{b}})(\mathbf{v}\times\hat{\mathbf{b}}) : \nabla\nabla\phi, \qquad \text{(D.16)}$$

where I use $\langle(\mathbf{v}\times\hat{\mathbf{b}})(\mathbf{v}\times\hat{\mathbf{b}})\rangle = (v_\perp^2/2)(\overleftrightarrow{\mathbf{I}} - \hat{\mathbf{b}}\hat{\mathbf{b}})$. Equation (D.16) is more complete than the approximation in (D.3).

Gyroaveraging $\tilde{\phi}$ in (D.16) holding $\mathbf{r}$, $v_{||}$ and $v_\perp$ fixed leads to

$$\frac{1}{2\pi}\int_0^{2\pi}\tilde{\phi}\, d\varphi_0 \simeq -\langle\Delta\boldsymbol{\rho}\rangle\cdot\nabla\phi - \frac{v_\perp^2}{2\Omega_i^2}(\overleftrightarrow{\mathbf{I}} - \hat{\mathbf{b}}\hat{\mathbf{b}}) : \nabla\nabla\phi. \qquad \text{(D.17)}$$

with $\Delta\boldsymbol{\rho}$ the function in (D.14). To simplify equation (D.14), the gradients of the unit vectors $\hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_2$ are expressed as $\nabla\hat{\mathbf{e}}_1 = -(\nabla\hat{\mathbf{b}}\cdot\hat{\mathbf{e}}_1)\hat{\mathbf{b}} - (\nabla\hat{\mathbf{e}}_2\cdot\hat{\mathbf{e}}_1)\hat{\mathbf{e}}_2$ and $\nabla\hat{\mathbf{e}}_2 = -(\nabla\hat{\mathbf{b}}\cdot\hat{\mathbf{e}}_2)\hat{\mathbf{b}} + (\nabla\hat{\mathbf{e}}_2\cdot\hat{\mathbf{e}}_1)\hat{\mathbf{e}}_1$, giving

$$\Delta\boldsymbol{\rho} = -\frac{1}{2B\Omega_i^2}(\mathbf{v}\times\hat{\mathbf{b}})(\mathbf{v}\times\hat{\mathbf{b}})\cdot\nabla B + \frac{Mc}{Zev_\perp^2}\mu_1(\mathbf{v}\times\hat{\mathbf{b}}) + \frac{1}{\Omega_i}\varphi_1\mathbf{v}_\perp$$
$$-\frac{1}{\Omega_i^2}(\mathbf{v}\times\hat{\mathbf{b}})\cdot\nabla\hat{\mathbf{b}}\cdot(\mathbf{v}\times\hat{\mathbf{b}})\hat{\mathbf{b}} - \frac{1}{\Omega_i^2}\mathbf{v}_\perp(\mathbf{v}\times\hat{\mathbf{b}})\cdot\nabla\hat{\mathbf{e}}_2\cdot\hat{\mathbf{e}}_1, \qquad \text{(D.18)}$$

and its gyroaverage

$$\langle\Delta\boldsymbol{\rho}\rangle = -\frac{c}{B\Omega_i}\nabla_\perp\phi - \frac{v_\perp^2}{B\Omega_i^2}\nabla_\perp B - \frac{v_{||}^2}{\Omega_i^2}\hat{\mathbf{b}}\cdot\nabla\hat{\mathbf{b}} - \frac{v_\perp^2}{2\Omega_i^2}(\nabla\cdot\hat{\mathbf{b}})\hat{\mathbf{b}}, \qquad \text{(D.19)}$$

where I use

$$\langle\mu_1(\mathbf{v}\times\hat{\mathbf{b}})\rangle = -\frac{cv_\perp^2}{2B^2}\nabla_\perp\phi - \frac{v_\perp^4}{4B^2\Omega_i}\nabla_\perp B - \frac{v_\perp^2 v_{||}^2}{2B\Omega_i}\hat{\mathbf{b}}\cdot\nabla\hat{\mathbf{b}} \qquad \text{(D.20)}$$

and

$$\langle \varphi_1 \mathbf{v}_\perp \rangle = -\frac{c}{2B}\nabla_\perp \phi - \frac{v_\perp^2}{2B\Omega_i}\nabla_\perp B - \frac{v_\|^2}{2\Omega_i}\hat{\mathbf{b}}\cdot\nabla\hat{\mathbf{b}} + \frac{v_\perp^2}{2\Omega_i}\hat{\mathbf{b}}\times\nabla\hat{\mathbf{e}}_2\cdot\hat{\mathbf{e}}_1. \tag{D.21}$$

These expressions are found by using the definitions of μ_1 and φ_1, given by (3.29) and (3.34), and employing the lowest order expressions $\widetilde{\phi} \simeq -\Omega_i^{-1}(\mathbf{v}\times\hat{\mathbf{b}})\cdot\nabla\phi$, $\widetilde{\Phi} = \int^\varphi \widetilde{\phi}\, d\varphi' \simeq \Omega_i^{-1}\mathbf{v}_\perp\cdot\nabla\phi$ and $\partial\widetilde{\Phi}/\partial\mu \simeq (Mc/Zev_\perp^2)\mathbf{v}_\perp\cdot\nabla\phi$.

Substituting (D.19) in (D.17) gives

$$\frac{1}{2\pi}\int_0^{2\pi}\widetilde{\phi}\, d\varphi_0 = -\frac{v_\perp^2}{2}\nabla\cdot\left(\frac{1}{\Omega_i^2}\nabla_\perp\phi\right) + \left(v_\|^2 - \frac{v_\perp^2}{2}\right)\frac{1}{\Omega_i^2}\hat{\mathbf{b}}\cdot\nabla\hat{\mathbf{b}}\cdot\nabla\phi + \frac{c}{B\Omega_i}|\nabla_\perp\phi|^2, \tag{D.22}$$

where I have used

$$\nabla\cdot\left(\frac{1}{\Omega_i^2}\nabla_\perp\phi\right) = \frac{1}{\Omega_i^2}(\overleftrightarrow{\mathbf{I}} - \hat{\mathbf{b}}\hat{\mathbf{b}}) : \nabla\nabla\phi - \frac{2}{\Omega_i^2}\nabla B\cdot\nabla_\perp\phi - \frac{1}{\Omega_i^2}\hat{\mathbf{b}}\cdot\nabla\hat{\mathbf{b}}\cdot\nabla\phi - \frac{1}{\Omega_i^2}(\hat{\mathbf{b}}\cdot\nabla\phi)\nabla\cdot\hat{\mathbf{b}}. \tag{D.23}$$

Notice that the gyroaverage of $\widetilde{\phi}$ is $O(\delta_i^2 T_e/e)$, which means that the integral is $O(\delta_i^2 n_i)$, and the lowest order distribution function, f_{Mi}, can be used to write $\partial f_{i0}/\partial E_0 \simeq -(M/T_i)f_{Mi}$ and $\partial f_{i0}/\partial\mu_0 \simeq 0$. All these simplifications lead to the final result

$$\frac{Ze}{M}\int d^3v\, \widetilde{\phi}\left(\frac{\partial f_{i0}}{\partial E_0} + \frac{1}{B}\frac{\partial f_{i0}}{\partial\mu_0}\right) = n_i\nabla\cdot\left(\frac{c}{B\Omega_i}\nabla_\perp\phi\right) - \frac{Mc^2 n_i}{T_i B^2}|\nabla_\perp\phi|^2, \tag{D.24}$$

Using (D.10), (D.11) and (D.24), equation (D.6) becomes

$$n_i = \hat{N}_i + \nabla\cdot\left(\frac{cn_i}{B\Omega_i}\nabla_\perp\phi\right) - \frac{Mc^2 n_i}{2T_i B^2}|\nabla_\perp\phi|^2, \tag{D.25}$$

where $\hat{N}_i$ is given by (3.62). Then the quasineutrality condition is as shown in (3.61).

Appendix E

Gyrophase dependent piece of f_i for $k_\perp L \sim 1$

In this Appendix, I show how to obtain the $O(\delta_i^2 f_{Mi})$ gyrophase dependent piece of the ion distribution function from gyrokinetics in a non-turbulent plasma. The general gyrophase dependent piece is found in section E.1. This result is useful because it can be compared to the drift kinetic result [74], proving that the higher order gyrokinetic variables allow us to recover the higher order drift kinetic results. In section E.2, the general gyrophase dependent piece is specialized for the θ-pinch, and it is used in section 5.2 to calculate the radial electric field.

E.1 General gyrophase dependent piece

Part of the gyrophase dependence is in the corrections to the gyrokinetic variables $\mathbf{R}_1$, $\mathbf{R}_2$, E_1, E_2 and μ_1. This gyrophase dependence can be extracted for $k_\perp L \sim 1$ by Taylor expanding $f_i(\mathbf{R}, E, \mu, t)$ around $\mathbf{r}$, E_0 and μ_0, as already done in (D.5). The contribution of the collisional piece $\widetilde{f}_i$, given by (3.38), can be always added later.

Employing (D.5), the gyrophase dependent part of f_i is found to be

$$f_i - \overline{f}_i = \left(\frac{1}{\Omega_i}\mathbf{v} \times \hat{\mathbf{b}} + \mathbf{R}_2\right) \cdot \nabla f_{i0} + \left[\frac{Ze}{M}(\widetilde{\phi} - \overline{\widetilde{\phi}}) + E_2\right] \frac{\partial f_{i0}}{\partial E_0}$$

$$+\frac{1}{4\Omega_i^2}[(\mathbf{v}\times\hat{\mathbf{b}})(\mathbf{v}\times\hat{\mathbf{b}}) - \mathbf{v}_\perp\mathbf{v}_\perp] : \overline{\nabla}\,\overline{\nabla} f_{Mi} + \frac{Z^2e^2}{2M^2}(\widetilde{\phi}^2 - \overline{\widetilde{\phi}^2})\frac{\partial^2 f_{Mi}}{\partial E_0^2}$$
$$+\frac{c}{B}[\widetilde{\phi}(\mathbf{v}\times\hat{\mathbf{b}}) - \overline{\widetilde{\phi}(\mathbf{v}\times\hat{\mathbf{b}})}]\cdot\overline{\nabla}\left(\frac{\partial f_{Mi}}{\partial E_0}\right) + (\mu_1 - \overline{\mu}_1)\frac{\partial f_{i0}}{\partial \mu_0}, \qquad \text{(E.1)}$$

with $f_{i0} \equiv f_i(\mathbf{r}, E_0, \mu_0, t)$. Here, I have employed the lowest order distribution function f_{Mi} in the higher order results, and I have used (A.7) to rewrite $(\mathbf{v}\times\hat{\mathbf{b}})(\mathbf{v}\times\hat{\mathbf{b}}) - \overline{(\mathbf{v}\times\hat{\mathbf{b}})(\mathbf{v}\times\hat{\mathbf{b}})}$.

The function $\widetilde{\phi}$ must be written as a function of the $\mathbf{r}$, $\mathbf{v}$ variables. To do so, I use equation (D.16) to find

$$\widetilde{\phi} - \overline{\widetilde{\phi}} = -\frac{1}{\Omega_i}(\mathbf{v}\times\hat{\mathbf{b}})\cdot\nabla\phi - \mathbf{R}_2\cdot\nabla\phi + \frac{1}{4\Omega_i^2}[\mathbf{v}_\perp\mathbf{v}_\perp - (\mathbf{v}\times\hat{\mathbf{b}})(\mathbf{v}\times\hat{\mathbf{b}})] : (\nabla\nabla\phi). \text{(E.2)}$$

For the higher order terms in (E.1), I can simply use the lowest order result $\widetilde{\phi} \simeq -\Omega_i^{-1}(\mathbf{v}\times\hat{\mathbf{b}})\cdot\nabla\phi$, which leads to

$$\widetilde{\phi}^2 - \overline{\widetilde{\phi}^2} = -\frac{1}{2\Omega_i^2}[\mathbf{v}_\perp\mathbf{v}_\perp - (\mathbf{v}\times\hat{\mathbf{b}})(\mathbf{v}\times\hat{\mathbf{b}})] : (\nabla\phi\nabla\phi) \qquad \text{(E.3)}$$

and

$$\widetilde{\phi}(\mathbf{v}\times\hat{\mathbf{b}}) - \overline{\widetilde{\phi}(\mathbf{v}\times\hat{\mathbf{b}})} = \frac{1}{2\Omega_i}\nabla\phi\cdot[\mathbf{v}_\perp\mathbf{v}_\perp - (\mathbf{v}\times\hat{\mathbf{b}})(\mathbf{v}\times\hat{\mathbf{b}})]. \qquad \text{(E.4)}$$

Using these expressions, the gyrophase dependent part of the distribution function becomes

$$f_i - \overline{f}_i = \mathbf{v}\cdot\mathbf{g}_\perp + (\mu_1 - \overline{\mu}_1)\frac{\partial f_{i0}}{\partial \mu_0} + \mathbf{R}_2\cdot\mathbf{G} + E_2\frac{\partial f_{i0}}{\partial E_0}$$
$$+\frac{1}{4\Omega_i^2}[(\mathbf{v}\times\hat{\mathbf{b}})(\mathbf{v}\times\hat{\mathbf{b}}) - \mathbf{v}_\perp\mathbf{v}_\perp] : \left(\overline{\nabla}\mathbf{G} - \frac{Ze}{M}\nabla\phi\frac{\partial \mathbf{G}}{\partial E_0}\right), \qquad \text{(E.5)}$$

with

$$\mathbf{g}_\perp = \frac{1}{\Omega_i}\hat{\mathbf{b}}\times\left(\overline{\nabla} f_{i0} - \frac{Ze}{M}\nabla\phi\frac{\partial f_{i0}}{\partial E_0}\right) \qquad \text{(E.6)}$$

and

$$\mathbf{G} = \overline{\nabla} f_{i0} - \frac{Ze}{M}\nabla\phi\frac{\partial f_{i0}}{\partial E_0}. \qquad \text{(E.7)}$$

Thus, $\mathbf{g}_\perp = \Omega_i^{-1}\hat{\mathbf{b}} \times \mathbf{G}$. In the long wavelength limit, $\partial/\partial t \ll v_i/L$, so E_2 as given in (3.26) is negligible since it contains a time derivative. Also, the zeroth order Fokker-Planck equation for the ion distribution function is

$$v_{||}\hat{\mathbf{b}} \cdot \mathbf{G} \equiv v_{||}\hat{\mathbf{b}} \cdot \left(\overline{\nabla} f_{i0} - \frac{Ze}{M}\nabla\phi\frac{\partial f_{i0}}{\partial E_0}\right) = C\{f_i\} = 0, \tag{E.8}$$

since the ion distribution function is assumed to be Maxwellian to zeroth order. This condition is important in (E.5) because it implies that the components of $\mathbf{R}_2$ that are parallel to the magnetic field do not enter $f_i - \overline{f}_i$. Therefore, employing the definition of $\mathbf{R}_2$ in (3.15) and using the fact that for long wavelengths $(c/B\Omega_i)\nabla_{\mathbf{R}}\tilde{\Phi} \times \hat{\mathbf{b}} \sim \delta_i^2 k_\perp \rho_i L$ is negligible, I obtain

$$\mathbf{R}_2 \cdot \mathbf{G} = \frac{1}{\Omega_i}\left[\left(v_{||}\hat{\mathbf{b}} + \frac{\mathbf{v}_\perp}{4}\right)\mathbf{v} \times \hat{\mathbf{b}} + \mathbf{v} \times \hat{\mathbf{b}}\left(v_{||}\hat{\mathbf{b}} + \frac{\mathbf{v}_\perp}{4}\right)\right] : \left[\nabla\left(\frac{\hat{\mathbf{b}}}{\Omega_i}\right) \times \mathbf{G}\right] + \frac{v_{||}}{\Omega_i^2}\mathbf{v}_\perp \cdot \nabla\hat{\mathbf{b}} \cdot \mathbf{G}. \tag{E.9}$$

Equation (E.9) can be written in a more recognizable manner by using

$$[\hat{\mathbf{b}}(\mathbf{v} \times \hat{\mathbf{b}}) + (\mathbf{v} \times \hat{\mathbf{b}})\hat{\mathbf{b}}] : \overleftrightarrow{\mathbf{h}} = -\frac{1}{\Omega_i}\hat{\mathbf{b}} \cdot \left(\overline{\nabla}\mathbf{G} - \frac{Ze}{M}\nabla\phi\frac{\partial \mathbf{G}}{\partial E_0}\right) \cdot \mathbf{v}_\perp + [\hat{\mathbf{b}}(\mathbf{v} \times \hat{\mathbf{b}}) + (\mathbf{v} \times \hat{\mathbf{b}})\hat{\mathbf{b}}] : \left[\nabla\left(\frac{\hat{\mathbf{b}}}{\Omega_i}\right) \times \mathbf{G}\right], \tag{E.10}$$

where $\overleftrightarrow{\mathbf{h}}$ is

$$\overleftrightarrow{\mathbf{h}} = \overline{\nabla}\mathbf{g}_\perp - \frac{Ze}{M}\nabla\phi\frac{\partial \mathbf{g}_\perp}{\partial E_0}. \tag{E.11}$$

The first term in the right side of (E.10) can be further simplified by using (E.8) to obtain

$$\hat{\mathbf{b}} \cdot \left(\overline{\nabla}\mathbf{G} - \frac{Ze}{M}\nabla\phi\frac{\partial \mathbf{G}}{\partial E_0}\right) \cdot \mathbf{v}_\perp = \mathbf{v}_\perp \cdot \overline{\nabla}\mathbf{G} \cdot \hat{\mathbf{b}} = -\mathbf{v}_\perp \cdot \nabla\hat{\mathbf{b}} \cdot \mathbf{G}. \tag{E.12}$$

As a result, (E.9) becomes

$$\mathbf{R}_2 \cdot \mathbf{G} = \frac{v_{||}}{\Omega_i}[\hat{\mathbf{b}}(\mathbf{v}\times\hat{\mathbf{b}}) + (\mathbf{v}\times\hat{\mathbf{b}})\hat{\mathbf{b}}] : \overleftrightarrow{\mathbf{h}}$$

$$+\frac{1}{4\Omega_i}[\mathbf{v}_\perp(\mathbf{v}\times\hat{\mathbf{b}}) + (\mathbf{v}\times\hat{\mathbf{b}})\mathbf{v}_\perp] : \left[\nabla\left(\frac{\hat{\mathbf{b}}}{\Omega_i}\right)\times\mathbf{G}\right]. \qquad \text{(E.13)}$$

The gyrophase dependent part of the ion distribution function can now be explicitly written as

$$(f_i - \overline{f}_i)_g = \mathbf{v}\cdot\mathbf{g}_\perp - \left\{\mathbf{v}_d\cdot\mathbf{v} + \frac{v_{||}}{4B\Omega_i}[\mathbf{v}_\perp(\mathbf{v}\times\hat{\mathbf{b}}) + (\mathbf{v}\times\hat{\mathbf{b}})\mathbf{v}_\perp] : \nabla\hat{\mathbf{b}}\right\}\frac{\partial f_{i0}}{\partial\mu_0}$$
$$+\frac{1}{\Omega_i}\left[\left(v_{||}\hat{\mathbf{b}} + \frac{\mathbf{v}_\perp}{4}\right)\mathbf{v}\times\hat{\mathbf{b}} + \mathbf{v}\times\hat{\mathbf{b}}\left(v_{||}\hat{\mathbf{b}} + \frac{\mathbf{v}_\perp}{4}\right)\right] : \overleftrightarrow{\mathbf{h}}, \qquad \text{(E.14)}$$

where the subindex g indicates the non-collisional origin of this gyrophase dependence. Equation (E.14) is exactly the same gyrophase dependent distribution function found in [74].

E.2 Gyrophase dependent piece in a θ-pinch

The solution for f_i found in (4.79) means that for all the terms in (E.14), f_{i0} is approximately f_{M0} from (4.76). Due to the geometry in the θ-pinch, the vector $\mathbf{g}_\perp$ defined in (E.6) is

$$\mathbf{g}_\perp = \frac{1}{\Omega_i}\hat{\boldsymbol{\theta}}\left(\frac{\partial f_{M0}}{\partial r} + \frac{Ze}{T_i}\frac{\partial\phi}{\partial r}f_{M0}\right), \qquad \text{(E.15)}$$

and the matrix $\overleftrightarrow{\mathbf{h}}$ defined in (E.11) is

$$\overleftrightarrow{\mathbf{h}} = \hat{\mathbf{r}}\hat{\boldsymbol{\theta}}\left\{\frac{\partial}{\partial r}\left[\frac{1}{\Omega_i}\left(\frac{\partial f_{M0}}{\partial r} + \frac{Ze}{T_i}\frac{\partial\phi}{\partial r}f_{M0}\right)\right] + \frac{Ze}{\Omega_i}\frac{\partial\phi}{\partial r}\left[\frac{\partial}{\partial r}\left(\frac{f_{M0}}{T_i}\right) + \frac{Ze}{T_i^2}\frac{\partial\phi}{\partial r}f_{M0}\right]\right\}$$
$$-\hat{\boldsymbol{\theta}}\hat{\mathbf{r}}\left[\frac{1}{r\Omega_i}\left(\frac{\partial f_{M0}}{\partial r} + \frac{Ze}{T_i}\frac{\partial\phi}{\partial r}f_{M0}\right)\right], \qquad \text{(E.16)}$$

where I use $\nabla\hat{\boldsymbol{\theta}} = -\hat{\boldsymbol{\theta}}\hat{\mathbf{r}}/r$. Employing this results and taking into account that $\partial f_{i0}/\partial\mu_0 \simeq 0$ in this case, I find (5.18).

Appendix F

Gyrokinetic equation in physical phase space

This Appendix contains the details needed to write equation (4.13) as equation (4.23). In section F.1, equation (4.14) is derived. Equation (4.14) models the finite gyroradius effects in the particle motion through the modified parallel velocity $v_{||0}$ and the perpendicular drift $\tilde{\mathbf{v}}_1$. In section F.2, equation (4.14) is written in conservative form, more convenient to obtain moment equations.

F.1 Gyrokinetic equation in $\mathbf{r}$, E_0, μ_0 and φ_0 variables

In this section, I rewrite part of the gyrokinetic equation as a function of the variables $\mathbf{r}$, E_0, μ_0 and φ_0. The gyrokinetic equation is only valid to $O(\delta_i f_{Mi} v_i/L)$, and the expansions will be carried out only to that order. In particular, I am interested in $\dot{\mathbf{R}} \cdot \nabla_{\mathbf{R}} f_i + \dot{E}(\partial f_i/\partial E)$. In the term $\dot{E}(\partial f_i/\partial E)$, I can make use of the lowest order equality $\partial f_i/\partial E \simeq \partial f_{Mi}/\partial E_0$. Then, employing $\dot{E}$ from (3.27) gives

$$\dot{\mathbf{R}} \cdot \nabla_{\mathbf{R}} f_i + \dot{E}\frac{\partial f_i}{\partial E} \simeq \dot{\mathbf{R}} \cdot \left(\nabla_{\mathbf{R}} f_i - \frac{Ze}{M} \nabla_{\mathbf{R}} \langle \phi \rangle \frac{\partial f_{Mi}}{\partial E_0} \right). \tag{F.1}$$

For $\nabla_{\mathbf{R}} f_i$, changing from $\mathbf{R}$, E, μ and φ to $\mathbf{r}$, E_0, μ_0 and φ_0, I find

$$\nabla_{\mathbf{R}} f_i \simeq \nabla_{\mathbf{R}} \mathbf{r} \cdot \overline{\nabla} f_i + \nabla_{\mathbf{R}} E_0 \frac{\partial f_i}{\partial E_0} + \nabla_{\mathbf{R}} \mu_0 \frac{\partial f_i}{\partial \mu_0} + \nabla_{\mathbf{R}} \varphi_0 \frac{\partial f_i}{\partial \varphi_0}. \tag{F.2}$$

Here, $\partial f_i / \partial \mu_0$ and $\partial f_i / \partial \varphi_0$ are small because the zeroth order distribution function is a stationary Maxwellian. The gradient of E_0 is given by $0 = \nabla_{\mathbf{R}} E \simeq \nabla_{\mathbf{R}} (E_0 + E_1) = \nabla_{\mathbf{R}} E_0 + (Ze/M) \nabla_{\mathbf{R}} \tilde{\phi}$. Similarly, $\nabla_{\mathbf{R}} \mu = 0 = \nabla_{\mathbf{R}} \varphi$ give $\nabla_{\mathbf{R}} \mu_0 \simeq -\nabla_{\mathbf{R}} \mu_1$ and $\nabla_{\mathbf{R}} \varphi_0 \simeq -\nabla_{\mathbf{R}} \varphi_1$. Then, to the required order

$$\nabla_{\mathbf{R}} f_i \simeq \nabla_{\mathbf{R}} \mathbf{r} \cdot \overline{\nabla} f_i - \frac{Ze}{M} \nabla_{\mathbf{R}} \tilde{\phi} \frac{\partial f_i}{\partial E_0}. \tag{F.3}$$

Since $\partial f_i / \partial E \simeq (-M/T_i) f_{Mi}$, and $\tilde{\phi} \nabla_{\mathbf{R}} f_{Mi} \ll f_{Mi} \nabla_{\mathbf{R}} \tilde{\phi}$ because the perpendicular gradient of $\tilde{\phi}$ is steeper and the parallel gradient of f_{Mi} is small, I find

$$\nabla_{\mathbf{R}} f_i \simeq \nabla_{\mathbf{R}} \mathbf{r} \cdot \overline{\nabla} \left(f_i + \frac{Ze\tilde{\phi}}{T_i} f_{Mi} \right) = \nabla_{\mathbf{R}} \mathbf{r} \cdot \overline{\nabla} f_{ig}, \tag{F.4}$$

where I have employed the lowest order result (4.3) and $\nabla_{\mathbf{R}} \tilde{\phi} \simeq \nabla_{\mathbf{R}} \mathbf{r} \cdot \overline{\nabla} \tilde{\phi}$. To prove that $\nabla_{\mathbf{R}} \tilde{\phi} \simeq \nabla_{\mathbf{R}} \mathbf{r} \cdot \overline{\nabla} \tilde{\phi}$ and $\nabla_{\mathbf{R}} \langle \phi \rangle \simeq \nabla_{\mathbf{R}} \mathbf{r} \cdot \overline{\nabla} \langle \phi \rangle$, I follow a similar procedure to the one used for f_i in (F.2). In this case, $\partial \tilde{\phi} / \partial E_0$ and $\partial \langle \phi \rangle / \partial E_0$ are small. Substituting equations (F.4) and $\nabla_{\mathbf{R}} \langle \phi \rangle \simeq \nabla_{\mathbf{R}} \mathbf{r} \cdot \overline{\nabla} \langle \phi \rangle$ into (F.1), I find equation (4.14), where to write $\nabla_{\mathbf{R}} \mathbf{r} \simeq \nabla_{\mathbf{R}_g} \mathbf{r}$ I have used the fact that $\mathbf{R}$ can be replaced by $\mathbf{R}_g$ to lowest order. The only coefficient left to evaluate is $\dot{\mathbf{R}} \cdot \nabla_{\mathbf{R}_g} \mathbf{r}$. Since $\mathbf{r} = \mathbf{R}_g - \Omega_i^{-1} \mathbf{v} \times \hat{\mathbf{b}}$, and $\mathbf{v} \times \hat{\mathbf{b}} = \sqrt{2\mu_0 B(\mathbf{r})} [\hat{\mathbf{e}}_1(\mathbf{r}) \sin\varphi_0 - \hat{\mathbf{e}}_2(\mathbf{r}) \cos\varphi_0]$, I find that $\nabla_{\mathbf{R}_g} \mathbf{r} \simeq \overleftrightarrow{\mathbf{I}} - \overline{\nabla} (\Omega_i^{-1} \mathbf{v} \times \hat{\mathbf{b}})$, where the gradient $\overline{\nabla} (\Omega_i^{-1} \mathbf{v} \times \hat{\mathbf{b}})$ is evaluated holding μ_0 and φ_0 fixed, and it is given by

$$-\overline{\nabla} \left(\frac{1}{\Omega_i} \mathbf{v} \times \hat{\mathbf{b}} \right) = \frac{1}{\Omega_i} \left[\frac{1}{2B} (\nabla B)(\mathbf{v} \times \hat{\mathbf{b}}) + \nabla \hat{\mathbf{b}} \cdot (\mathbf{v} \times \hat{\mathbf{b}}) \hat{\mathbf{b}} + (\nabla \hat{\mathbf{e}}_2 \cdot \hat{\mathbf{e}}_1) \mathbf{v}_\perp \right]. \tag{F.5}$$

In $\dot{\mathbf{R}}$, given by (3.19), the terms $\mathbf{v}_M$ and $\mathbf{v}_E$ are an order smaller than $u \hat{\mathbf{b}}(\mathbf{R})$ so I can use $\nabla_{\mathbf{R}_g} \mathbf{r} \simeq \overleftrightarrow{\mathbf{I}}$ for $\mathbf{v}_M \cdot \nabla_{\mathbf{R}_g} \mathbf{r}$ and $\mathbf{v}_E \cdot \nabla_{\mathbf{R}_g} \mathbf{r}$ to find the result in (4.15). In equation (4.15), I have also used $u \hat{\mathbf{b}}(\mathbf{R}) \simeq u \hat{\mathbf{b}}(\mathbf{r}) + (v_{||}/\Omega_i)(\mathbf{v} \times \hat{\mathbf{b}}) \cdot \nabla \hat{\mathbf{b}}$, $\mathbf{v}_M \simeq \mathbf{v}_{M0}$, $\mathbf{v}_E \simeq \mathbf{v}_{E0}$

and

$$\overline{\nabla} \times \mathbf{v}_\perp = (\mathbf{v} \times \hat{\mathbf{b}}) \cdot \nabla \hat{\mathbf{b}} + \frac{1}{2B} \mathbf{v} \times \hat{\mathbf{b}} (\hat{\mathbf{b}} \cdot \nabla B) + \mathbf{v}_\perp (\hat{\mathbf{b}} \cdot \nabla \hat{\mathbf{e}}_2 \cdot \hat{\mathbf{e}}_1) + \hat{\mathbf{b}} \left[\frac{1}{2B} (\mathbf{v} \times \hat{\mathbf{b}}) \cdot \nabla B - \mathbf{v}_\perp \cdot \nabla \hat{\mathbf{e}}_2 \cdot \hat{\mathbf{e}}_1 \right], \tag{F.6}$$

where I employ $\overline{\nabla} \times \mathbf{v}_\perp = \overline{\nabla} \times [\hat{\mathbf{b}} \times (\mathbf{v} \times \hat{\mathbf{b}})] = \overline{\nabla} \cdot [(\mathbf{v} \times \hat{\mathbf{b}})\hat{\mathbf{b}}] - \overline{\nabla} \cdot [\hat{\mathbf{b}}(\mathbf{v} \times \hat{\mathbf{b}})]$. To find the result in equation (4.16), notice that $v_{||0} = u + (v_{||}/\Omega_i)\hat{\mathbf{b}} \cdot \nabla \hat{\mathbf{b}} \cdot (\mathbf{v} \times \hat{\mathbf{b}}) - (v_{||}/\Omega_i)\hat{\mathbf{b}} \cdot \overline{\nabla} \times \mathbf{v}_\perp$, and the difference $u - v_{||}$ is given in (A.43).

F.2 Conservative form in r, E_0, μ_0 and φ_0 variables

To obtain equation (4.20) from (4.14), I just need to prove that

$$\overline{\nabla} \cdot \left(\frac{B}{v_{||}} \dot{\mathbf{R}} \cdot \nabla_{\mathbf{R}_g} \mathbf{r} \right) - \frac{\partial}{\partial \mu_0} \left(\mathbf{B} \cdot \overline{\nabla} \mu_{10} \right) - \frac{\partial}{\partial \varphi_0} \left(\mathbf{B} \cdot \overline{\nabla} \varphi_{10} \right) - \frac{\partial}{\partial E_0} \left(\frac{B}{v_{||}} \frac{Ze}{M} \dot{\mathbf{R}} \cdot \nabla_{\mathbf{R}_g} \mathbf{r} \cdot \overline{\nabla} \langle \phi \rangle \right) = 0. \tag{F.7}$$

Then, equation (4.20) is found by using $\partial f_{ig}/\partial E_0 \simeq \partial f_{Mi}/\partial E_0$, $\partial f_{ig}/\partial \mu_0 \simeq 0$ and $\partial f_{ig}/\partial \varphi_0 \simeq 0$.

To prove (F.7), I use the value of $\dot{\mathbf{R}} \cdot \nabla_{\mathbf{R}_g} \mathbf{r}$ from (4.15) and the relations $\partial(\mathbf{B} \cdot \overline{\nabla} \mu_{10})/\partial \mu_0 = \overline{\nabla} \cdot [\mathbf{B}(\partial \mu_{10}/\partial \mu_0)]$ and $\partial(\mathbf{B} \cdot \overline{\nabla} \varphi_{10})/\partial \varphi_0 = \overline{\nabla} \cdot [\mathbf{B}(\partial \varphi_{10}/\partial \varphi_0)]$, to find

$$\overline{\nabla} \cdot \left\{ \frac{B}{v_{||}} [(v_{||0} - v_{||})\hat{\mathbf{b}} + \mathbf{v}_{M0} + \mathbf{v}_{E0}] - \mathbf{B} \left(\frac{\partial \mu_{10}}{\partial \mu_0} + \frac{\partial \varphi_{10}}{\partial \varphi_0} \right) \right\} - \frac{\partial}{\partial E_0} \left(\frac{B}{v_{||}} \frac{Ze}{M} \mathbf{v}_{M0} \right) \cdot \overline{\nabla} \langle \phi \rangle = 0. \tag{F.8}$$

Here, I have also employed $\nabla \cdot \mathbf{B} = 0$, $\partial(\mathbf{B} \cdot \overline{\nabla} \langle \phi \rangle)/\partial E = 0$, $\overline{\nabla} \cdot [(B/v_{||})\tilde{\mathbf{v}}_1] = 0$ and $\partial[(B/v_{||})\tilde{\mathbf{v}}_1]/\partial E_0 = 0$ [these last two expressions are easy to prove by using the definition of $\tilde{\mathbf{v}}_1$ from (4.19)]. In the term $\dot{\mathbf{R}} \cdot \nabla_{\mathbf{R}_g} \mathbf{r} \cdot \overline{\nabla} \langle \phi \rangle$, $(v_{||0} - v_{||})\hat{\mathbf{b}} \cdot \overline{\nabla} \langle \phi \rangle$ is negligible because $\hat{\mathbf{b}} \cdot \overline{\nabla} \langle \phi \rangle \simeq 0$ (the zeroth order potential is constant along magnetic field lines). In equation (F.8), it is satisfied that

$$\overline{\nabla} \cdot \left(\frac{B}{v_{||}} \mathbf{v}_{E0} \right) - \frac{\partial}{\partial E_0} \left(\frac{B}{v_{||}} \frac{Ze}{M} \mathbf{v}_{M0} \right) \cdot \overline{\nabla} \langle \phi \rangle = 0, \tag{F.9}$$

where I employ relation (2.8) and $\hat{\mathbf{b}} \cdot \overline{\nabla} \langle \phi \rangle \simeq 0$; and

$$\frac{v_{||0} - v_{||}}{v_{||}} - \frac{\partial \mu_{10}}{\partial \mu_0} - \frac{\partial \varphi_{10}}{\partial \varphi_0} = \frac{v_{||}}{\Omega_i} \hat{\mathbf{b}} \cdot \nabla \times \hat{\mathbf{b}}, \tag{F.10}$$

with $\partial \mathbf{v}_\perp / \partial \mu_0 = (B/v_\perp^2) \mathbf{v}_\perp$, $\partial \mathbf{v}_\perp / \partial \varphi_0 = -\mathbf{v} \times \hat{\mathbf{b}}$ and $v_{||0}$ defined by (4.16). Using these relations, equation (F.8) becomes

$$\frac{Mc}{Ze} \overline{\nabla} \cdot \left[\frac{\mu_0}{v_{||}} \hat{\mathbf{b}} \times \nabla B + v_{||} \nabla \times \hat{\mathbf{b}} \right] = \frac{Mc}{Ze} \overline{\nabla} \cdot [\overline{\nabla} \times (v_{||} \hat{\mathbf{b}})] = 0, \tag{F.11}$$

where the relation (2.8) is used again.

Appendix G

Details of the particle and momentum transport calculation

This Appendix contains the details of the derivations of the gyrokinetic particle conservation equation (4.28) and the momentum conservation equation (4.38) from equation (4.24).

G.1 Details of the particle transport calculation

Equation (4.24) with $G = 1$ leads to equation (4.28), with the integral of the gyroaveraged collision operator giving the term $\nabla \cdot (n_i \mathbf{V}_{iC})$ as shown in Appendix H. The only other integral of some difficulty is $\int d^3v\, f_{ig} \dot{\mathbf{R}} \cdot \nabla_{\mathbf{R}_g} \mathbf{r}$. The integral of $v_{||}\hat{\mathbf{b}}$ is done realizing that $f_i - f_{ig} = (-Ze\tilde{\phi}/T_i) f_{Mi}$ is even in $v_{||}$ to write (4.29). The integral of $(v_{||0} - v_{||})\hat{\mathbf{b}}$ is done by using $\nabla \cdot [\int d^3v\, f_{ig}(v_{||0} - v_{||})\hat{\mathbf{b}}] \simeq \mathbf{B} \cdot \nabla[\int d^3v\, f_{ig}(v_{||0} - v_{||})/B]$. Then f_{ig} can be replaced by f_{Mi} because the gradient is along the magnetic field line, and the slow parallel gradients make the small pieces of the distribution function unimportant. From all the terms in $v_{||0} - v_{||}$, only the gyrophase independent piece $(v_\perp^2/2\Omega_i)\hat{\mathbf{b}} \cdot \nabla \times \hat{\mathbf{b}}$ gives a non-vanishing contribution.

To perform the integral $\nabla \cdot (n_i \tilde{\mathbf{V}}_i) = \nabla \cdot (\int d^3v\, f_{ig} \tilde{\mathbf{v}}_1) = \nabla \cdot [\int d^3v\, f_{ig} (v_{||}/\Omega_i) \overline{\nabla} \times \mathbf{v}_\perp]$, I use the expression of $\overline{\nabla} \times \mathbf{v}_\perp$ in equation (F.6). The contribution of the parallel component of $\overline{\nabla} \times \mathbf{v}_\perp$ is negligible because its divergence only has parallel gradients, and they are small compared to the perpendicular gradients. The integral of $(v_{||}/\Omega_i)\mathbf{v}_\perp(\hat{\mathbf{b}}\cdot\nabla\hat{\mathbf{e}}_2\cdot\hat{\mathbf{e}}_1)$, on the other hand, vanishes because its divergence becomes

$$\int d^3v\,\frac{v_{||}}{\Omega_i}\hat{\mathbf{b}}\cdot\nabla\hat{\mathbf{e}}_2\cdot\hat{\mathbf{e}}_1\mathbf{v}_\perp\cdot\overline{\nabla}f_{ig}=\int d^3v\,v_{||}\hat{\mathbf{b}}\cdot\nabla\hat{\mathbf{e}}_2\cdot\hat{\mathbf{e}}_1\frac{\partial f_{ig}}{\partial\varphi_0}=0. \qquad \text{(G.1)}$$

Here, I neglect the gradients of any quantity that is not f_{ig} because they give contributions of order $\delta_i^2 k_\perp\rho_i n_e v_i/L$. To obtain $\partial f_{ig}/\partial\varphi_0\simeq\Omega_i^{-1}\mathbf{v}_\perp\cdot\overline{\nabla}f_{ig}$, I use that f_{ig}'s only dependence on φ_0 is through $\mathbf{R}_g$. The final result for $n_i\tilde{\mathbf{V}}_i$ is written in (4.35).

G.2 Details of the momentum transport calculation

Equation (4.24) with $G=\mathbf{v}$ gives equation (4.38). The integral of the gyroaveraged collision operator gives $\mathbf{F}_{iC}$. The details are given in Appendix H.

To simplify the integral $\int d^3v\,f_{ig}(\dot{\mathbf{R}}\cdot\nabla_{\mathbf{R}_g}\mathbf{r})M\mathbf{v}$, I use that $\nabla\cdot[\int d^3v\,f_{ig}(v_{||0}-v_{||})\hat{\mathbf{b}}\mathbf{v}]\simeq\mathbf{B}\cdot\nabla[\int d^3v\,f_{ig}\mathbf{v}(v_{||0}-v_{||})/B]$. Then f_{ig} can be replaced by f_{Mi} because the gradient is along the magnetic field line, and the next order corrections can be neglected. The integral $\int d^3v\,f_{Mi}\mathbf{v}(v_{||0}-v_{||})/B$ vanishes because all the terms are either odd in $v_{||}$ or odd in $\mathbf{v}_\perp$. The final result is

$$\int d^3v\,f_{ig}\left(\dot{\mathbf{R}}\cdot\nabla_{\mathbf{R}_g}\mathbf{r}\right)M\mathbf{v}=p_{ig||}\hat{\mathbf{b}}\hat{\mathbf{b}}+\boldsymbol{\pi}_{ig||}\hat{\mathbf{b}}+\overleftrightarrow{\boldsymbol{\pi}}_{ig\times}, \qquad \text{(G.2)}$$

where I use the definitions of $p_{ig||}=\int d^3v\,f_{ig}Mv_{||}^2$, $\boldsymbol{\pi}_{ig||}$ from (4.39), and $\overleftrightarrow{\boldsymbol{\pi}}_{ig\times}$ from (4.40).

To find the integral $\int d^3v\,Mf_{ig}K\{\mathbf{v}\}$, with the linear operator K given in (4.25), I use $K\{\mathbf{v}\}=K\{v_{||}\hat{\mathbf{b}}\}+K\{\mathbf{v}_\perp\}$. For $K\{v_{||}\hat{\mathbf{b}}\}$, I need $\overline{\nabla}v_{||}=-\mu_0\nabla B/v_{||}$, $\partial v_{||}/\partial E_0=v_{||}^{-1}$, $\partial v_{||}/\partial\mu_0=-B/v_{||}$ and $\partial v_{||}/\partial\varphi_0=0$. Then, using the definitions of $p_{ig||}$ and $\boldsymbol{\pi}_{ig||}$ along with

$$(\mathbf{v}_{M0}+\mathbf{v}_{E0})\cdot\left(\frac{\mu_0\nabla B}{v_{||}}+\frac{Ze}{Mv_{||}}\overline{\nabla}\langle\phi\rangle\right)=\frac{v_{||}}{\Omega_i}(\hat{\mathbf{b}}\times\boldsymbol{\kappa})\cdot\left(\mu_0\nabla B+\frac{Ze}{M}\overline{\nabla}\langle\phi\rangle\right), \qquad \text{(G.3)}$$

I find

$$\int d^3v\, M f_{ig} K\{v_{||}\hat{\mathbf{b}}\} = (p_{ig||}\hat{\mathbf{b}} + \boldsymbol{\pi}_{ig||}) \cdot \nabla\hat{\mathbf{b}} - \int d^3v\, f_{ig} M \mu_0 \hat{\mathbf{b}}\hat{\mathbf{b}} \cdot \nabla B$$
$$+ \int d^3v\, f_{Mi} \frac{Mv_\perp^2}{2\Omega_i} \hat{\mathbf{b}}(\hat{\mathbf{b}} \cdot \nabla \times \hat{\mathbf{b}})\hat{\mathbf{b}} \cdot \overline{\nabla} v_{||} + \int d^3v\, f_{Mi} M\hat{\mathbf{b}}(\mathbf{B} \cdot \overline{\nabla}\mu_{10})$$
$$-Ze \int d^3v f_{ig}\hat{\mathbf{b}}(\hat{\mathbf{b}} + \Omega_i^{-1}\overline{\nabla} \times \mathbf{v}_\perp) \cdot \overline{\nabla}\langle\phi\rangle. \qquad \text{(G.4)}$$

Here, I have used that $(v_{||0} - v_{||})\hat{\mathbf{b}} \cdot \nabla\langle\phi\rangle \simeq 0$ and that, in the integrals that include $(v_{||0} - v_{||})\hat{\mathbf{b}} \cdot \overline{\nabla}(v_{||}\hat{\mathbf{b}})$ and $\tilde{\mathbf{v}}_1 \cdot \overline{\nabla} v_{||}$, only the term $(v_\perp^2/2\Omega_i)\hat{\mathbf{b}}(\hat{\mathbf{b}} \cdot \nabla \times \hat{\mathbf{b}})\hat{\mathbf{b}} \cdot \overline{\nabla} v_{||}$ gives a non-vanishing contribution. The integral of $f_{Mi}(v_{||}/\Omega_i)(\hat{\mathbf{b}} \times \boldsymbol{\kappa}) \cdot [\mu_0 \nabla B + (Ze/M)\overline{\nabla}\langle\phi\rangle]$ vanishes because it is odd in $v_{||}$. Equation (G.4) can be further simplified by using

$$\int d^3v\, f_{Mi}\mathbf{B} \cdot \overline{\nabla}\mu_{10} = -\int d^3v\, f_{Mi}\mathbf{B} \cdot \overline{\nabla}\left(\frac{v_{||}v_\perp^2}{2B\Omega_i}\hat{\mathbf{b}} \cdot \nabla \times \hat{\mathbf{b}}\right) =$$
$$-\int d^3v\, f_{Mi}\frac{v_\perp^2}{2\Omega_i}(\hat{\mathbf{b}} \cdot \nabla \times \hat{\mathbf{b}})\hat{\mathbf{b}} \cdot \overline{\nabla} v_{||}, \qquad \text{(G.5)}$$

where I have used that in μ_{10} all the terms but the gyrophase independent piece give vanishing integrals. The last form of (G.5) cancels with a term in (G.4). Using $p_{ig\perp} = \int d^3v\, f_{ig} M v_\perp^2/2$, equation (G.4) then becomes

$$\int d^3v\, M f_{ig} K\{v_{||}\hat{\mathbf{b}}\} = -p_{ig\perp}\hat{\mathbf{b}}\hat{\mathbf{b}} \cdot \nabla \ln B + (p_{ig||}\hat{\mathbf{b}} + \boldsymbol{\pi}_{ig||}) \cdot \nabla\hat{\mathbf{b}}$$
$$-Ze \int d^3v f_{ig}\hat{\mathbf{b}}(\hat{\mathbf{b}} + \Omega_i^{-1}\overline{\nabla} \times \mathbf{v}_\perp) \cdot \overline{\nabla}\langle\phi\rangle. \qquad \text{(G.6)}$$

The integral $\int d^3v\, f_{ig} K\{\mathbf{v}_\perp\}$ is obtained using $\overline{\nabla}\mathbf{v}_\perp = (\nabla B/2B)\mathbf{v}_\perp - \nabla\hat{\mathbf{b}} \cdot \mathbf{v}_\perp\hat{\mathbf{b}} + \nabla\hat{\mathbf{e}}_2 \cdot \hat{\mathbf{e}}_1(\mathbf{v} \times \hat{\mathbf{b}})$, $\partial\mathbf{v}_\perp/\partial E_0 = 0$, $\partial\mathbf{v}_\perp/\partial\mu_0 = (2\mu_0)^{-1}\mathbf{v}_\perp$ and $\partial\mathbf{v}_\perp/\partial\varphi_0 = -\mathbf{v} \times \hat{\mathbf{b}}$. Then, I find

$$\int d^3v\, M f_{ig} K\{\mathbf{v}_\perp\} = \int d^3v\, M f_{ig} v_{||}\hat{\mathbf{b}} \cdot \overline{\nabla}\mathbf{v}_\perp - \frac{Mc}{B}\int d^3v\, f_{Mi}(\overline{\nabla}\langle\phi\rangle \times \hat{\mathbf{b}}) \cdot \overline{\nabla}\mathbf{v}_\perp, \quad \text{(G.7)}$$

where I employ that the integrals of $f_{Mi}[(v_{||0} - v_{||})\hat{\mathbf{b}} + \mathbf{v}_{M0} + \tilde{\mathbf{v}}_1] \cdot \overline{\nabla}\mathbf{v}_\perp$, $f_{Mi}v_{||}\hat{\mathbf{b}} \cdot \overline{\nabla}\mu_{10}(\partial\mathbf{v}_\perp/\partial\mu_0)$ and $f_{Mi}v_{||}\hat{\mathbf{b}} \cdot \overline{\nabla}\varphi_{10}(\partial\mathbf{v}_\perp/\partial\varphi_0)$ vanish because the terms are either odd in $v_{||}$ or in $\mathbf{v}_\perp$. The integral that includes $\overline{\nabla}\langle\phi\rangle$ can be rewritten by realizing that $\overline{\nabla}\langle\phi\rangle = \nabla\phi - \overline{\nabla}\widetilde{\phi}$, to find

$$\int d^3v\, M f_{ig} K\{\mathbf{v}_\perp\} = \int d^3v\, M f_{ig} v_{||} \hat{\mathbf{b}} \cdot \overline{\nabla}\mathbf{v}_\perp + \frac{Mc}{B} \int d^3v\, f_{Mi} (\overline{\nabla}\widetilde{\phi} \times \hat{\mathbf{b}}) \cdot \overline{\nabla}\mathbf{v}_\perp. \quad \text{(G.8)}$$

Combining equations (G.6) and (G.8), I find

$$\int d^3v\, M f_{ig} K\{\mathbf{v}\} = p_{ig\perp} \hat{\mathbf{b}} (\nabla\cdot\hat{\mathbf{b}}) + (p_{ig||}\hat{\mathbf{b}} + \boldsymbol{\pi}_{ig||})\cdot\nabla\hat{\mathbf{b}} - Zen_i \hat{\mathbf{b}}\hat{\mathbf{b}}\cdot\nabla\phi + \tilde{F}_{iE}\hat{\mathbf{b}} + \mathbf{F}_{iB}, \quad \text{(G.9)}$$

where I have used $\hat{\mathbf{b}} \cdot \nabla \ln B = -\nabla \cdot \hat{\mathbf{b}}$, and the definitions of $\tilde{F}_{iE}$ from (4.41) and $\mathbf{F}_{iB}$ from (4.42).

Using (G.2) and (G.9) in (4.24) gives

$$\begin{aligned} \frac{\partial}{\partial t}(n_i M \mathbf{V}_{ig}) + \nabla \cdot (p_{ig||}\hat{\mathbf{b}}\hat{\mathbf{b}} + \boldsymbol{\pi}_{ig||}\hat{\mathbf{b}} + \overleftrightarrow{\boldsymbol{\pi}}_{ig\times}) &= -Zen_i \hat{\mathbf{b}}\hat{\mathbf{b}} \cdot \nabla\phi \\ + p_{ig\perp}\hat{\mathbf{b}}(\nabla \cdot \hat{\mathbf{b}}) + (p_{ig||}\hat{\mathbf{b}} + \boldsymbol{\pi}_{ig||}) \cdot \nabla\hat{\mathbf{b}} &+ \tilde{F}_{iE}\hat{\mathbf{b}} + \mathbf{F}_{iB} + \mathbf{F}_{iC}. \end{aligned} \quad \text{(G.10)}$$

Finally, employing $\nabla \cdot (p_{ig||}\hat{\mathbf{b}}\hat{\mathbf{b}}) = \hat{\mathbf{b}}(\hat{\mathbf{b}} \cdot \nabla p_{ig||} + p_{ig||}\nabla \cdot \hat{\mathbf{b}}) + p_{ig||}\hat{\mathbf{b}} \cdot \nabla\hat{\mathbf{b}}$ and $\nabla \cdot (\boldsymbol{\pi}_{ig||}\hat{\mathbf{b}}) = (\nabla \cdot \boldsymbol{\pi}_{ig||})\hat{\mathbf{b}} + \boldsymbol{\pi}_{ig||} \cdot \nabla\hat{\mathbf{b}}$, I am able to recover equation (4.38). Multiplying equation (4.38) by $\hat{\mathbf{b}}$ and taking into account the cancellation of

$$\mathbf{F}_{iB} \cdot \hat{\mathbf{b}} = -M \int d^3v\, f_{ig} \left(v_{||}\hat{\mathbf{b}} + \frac{c}{B}\overline{\nabla}\widetilde{\phi} \times \hat{\mathbf{b}} \right) \cdot \nabla\hat{\mathbf{b}} \cdot \mathbf{v}_\perp \quad \text{(G.11)}$$

and

$$(\nabla \cdot \overleftrightarrow{\boldsymbol{\pi}}_{ig\times}) \cdot \hat{\mathbf{b}} \simeq -M \int d^3v\, f_{ig} \left(v_{||}\hat{\mathbf{b}} + \frac{c}{B}\overline{\nabla}\widetilde{\phi} \times \hat{\mathbf{b}} \right) \cdot \nabla\hat{\mathbf{b}} \cdot \mathbf{v}_\perp, \quad \text{(G.12)}$$

I find equation (4.44). To obtain relation (G.12), I have employed $\overleftrightarrow{\boldsymbol{\pi}}_{ig\times} \cdot \hat{\mathbf{b}} = 0$ and I have used the lowest order distribution function f_{Mi} for the higher order terms. All the higher order terms, except for $(c/B)\overline{\nabla}\langle\phi\rangle \times \hat{\mathbf{b}}$, cancel because they are either odd in $v_{||}$ or $\mathbf{v}_\perp$.

Appendix H

Finite gyroradius effects in the like-collision operator

In this Appendix, I show how to treat the gyroaveraged like-collision operator, $\langle C\{f_i\}\rangle$. The like-collision operator is

$$C\{f_i\} = \gamma \nabla_v \cdot \left[\int d^3v' \, \nabla_g \nabla_g g \cdot (f_i' \nabla_v f_i - f_i \nabla_{v'} f_i') \right], \tag{H.1}$$

with $f_i = f_i(\mathbf{v})$, $f_i' = f_i(\mathbf{v}')$, $\mathbf{g} = \mathbf{v} - \mathbf{v}'$, $g = |\mathbf{g}|$, $\nabla_g \nabla_g g = (g^2 \overleftrightarrow{\mathbf{I}} - \mathbf{gg})/g^3$ and $\gamma = 2\pi Z^4 e^4 \ln \Lambda / M^2$. Linearizing this equation for $f_i = f_{Mi} + f_{i1}$, with $f_{i1} \ll f_{Mi}$, I find

$$C^{(\ell)}\{f_{i1}\} = \gamma \nabla_v \cdot \mathbf{\Gamma}\{f_{i1}\}, \tag{H.2}$$

with

$$\mathbf{\Gamma}\{f_{i1}\} = \int d^3v' \, f_{Mi} f_{Mi}' \nabla_g \nabla_g g \cdot \left[\nabla_v \left(\frac{f_{i1}}{f_{Mi}} \right) - \nabla_{v'} \left(\frac{f_{i1}'}{f_{Mi}'} \right) \right]. \tag{H.3}$$

The vector $\mathbf{\Gamma}$ can also be written as in (4.34) because $\mathbf{\Gamma}\{f_i\} = \mathbf{\Gamma}\{f_{Mi}\} + \mathbf{\Gamma}\{f_{i1}\} = \mathbf{\Gamma}\{f_{i1}\}$. Using gyrokinetic variables in equation (H.2) and gyroaveraging gives

$$\langle C^{(\ell)}\{f_{i1}\}\rangle = \gamma \frac{u}{B} \left[\frac{\partial}{\partial E} \left(\frac{B}{u} \langle \mathbf{\Gamma} \cdot \nabla_v E \rangle \right) + \frac{\partial}{\partial \mu} \left(\frac{B}{u} \langle \mathbf{\Gamma} \cdot \nabla_v \mu \rangle \right) + \nabla_{\mathbf{R}} \cdot \left(\frac{B}{u} \langle \mathbf{\Gamma} \cdot \nabla_v \mathbf{R} \rangle \right) \right]. \tag{H.4}$$

Here, $B/u \simeq \partial(\mathbf{r}, \mathbf{v})/\partial(\mathbf{R}, E, \mu, \varphi)$ is the approximate Jacobian, and I have used the transformation rule for divergences from one reference system $\{x_i\}$ to another $\{y_j\}$:

$$\nabla_x \cdot \boldsymbol{\Gamma} = \sum_j \frac{1}{J_y} \frac{\partial}{\partial y_j} (J_y \boldsymbol{\Gamma} \cdot \nabla_x y_j) = \sum_j \frac{1}{J_y} \frac{\partial}{\partial y_j} (J_y \Gamma_{y_j}), \tag{H.5}$$

where $J_y = \partial(x_i)/\partial(y_j)$ is the Jacobian of the transformation, $\Gamma_{y_j} = \boldsymbol{\Gamma} \cdot \nabla_x y_j$ and ∇_x is the gradient in the reference system $\{x_i\}$. To rewrite equation (H.4) in terms of the variables $\mathbf{r}$, E_0, μ_0 and φ_0, I need to use (H.5) and the chain rule to find the transformation between the two reference systems $\{y_j\}$ and $\{z_k\}$

$$\frac{1}{J_y} \sum_j \frac{\partial}{\partial y_j} (J_y \Gamma_{y_j}) = \frac{1}{J_z} \sum_k \frac{\partial}{\partial z_k} \left(J_z \sum_j \Gamma_{y_j} \frac{\partial z_k}{\partial y_j} \right). \tag{H.6}$$

Employing this relation to write equation (H.4) as a function of $\mathbf{r}$, E_0, μ_0 and φ_0 gives

$$\begin{aligned} \langle C^{(\ell)}\{f_{i1}\}\rangle \simeq \gamma \frac{v_{||}}{B} \Bigg\{ & \frac{\partial}{\partial E_0} \left(\frac{B}{v_{||}} \langle \boldsymbol{\Gamma} \cdot \nabla_v E \rangle \right) + \frac{\partial}{\partial \mu_0} \left(\frac{B}{v_{||}} \langle \boldsymbol{\Gamma} \cdot \nabla_v \mu \rangle \right) \\ & + \overline{\nabla} \cdot \left[\frac{B}{v_{||}} \left(\langle \boldsymbol{\Gamma} \cdot \nabla_v \mu \rangle \frac{\partial \mathbf{r}}{\partial \mu} + \langle \boldsymbol{\Gamma} \cdot \nabla_v \mathbf{R} \rangle \right) \right] \Bigg\}, \end{aligned} \tag{H.7}$$

where I have used the lowest order gyrokinetic variables $\mathbf{R}_g$, E_0, μ_0 and φ_0. This approximation is justified because the collision operator vanishes to lowest order, and only the zeroth order definitions must be kept. Notice that I keep the first order correction $\mathbf{R}_1 = \Omega_i^{-1} \mathbf{v} \times \hat{\mathbf{b}}$ only within the spatial divergence because the spatial gradients are steep. Employing $\nabla_v E \simeq \mathbf{v}$, $\nabla_v \mu \simeq \mathbf{v}_\perp / B$, $\partial \mathbf{r}/\partial \mu \simeq -\partial \mathbf{R}_1/\partial \mu \simeq -(2\mu_0 \Omega_i)^{-1} \mathbf{v} \times \hat{\mathbf{b}}$ and $\nabla_v \mathbf{R} \simeq \Omega_i^{-1} \overleftrightarrow{\mathbf{I}} \times \hat{\mathbf{b}}$, I find

$$\begin{aligned} \langle C^{(\ell)}\{f_{i1}\}\rangle \simeq \gamma \frac{v_{||}}{B} \Bigg\{ & \frac{\partial}{\partial E_0} \left(\frac{B}{v_{||}} \langle \boldsymbol{\Gamma} \cdot \mathbf{v} \rangle \right) + \frac{\partial}{\partial \mu_0} \left(\frac{1}{v_{||}} \langle \boldsymbol{\Gamma} \cdot \mathbf{v}_\perp \rangle \right) \\ & + \overline{\nabla} \cdot \left[\frac{Mc}{Zev_{||}} \left(\langle \boldsymbol{\Gamma} \rangle \times \hat{\mathbf{b}} - \frac{1}{v_\perp^2} \langle \boldsymbol{\Gamma} \cdot \mathbf{v}_\perp \rangle \mathbf{v} \times \hat{\mathbf{b}} \right) \right] \Bigg\}. \end{aligned} \tag{H.8}$$

In the main text, there are two integrals that involve the gyroaveraged collision operator, $\nabla \cdot (n_i \mathbf{V}_{iC}) = -\int d^3v \, \langle C\{f_i\}\rangle$ and $\mathbf{F}_{iC} = M \int d^3v \, \mathbf{v} \langle C\{f_i\}\rangle$. Using equation (H.8), I obtain equations (4.33) and (4.43). To find (4.43), I have integrated by parts using $\partial \mathbf{v}/\partial E_0 = v_{||}^{-1} \hat{\mathbf{b}}$ and $\partial \mathbf{v}/\partial \mu_0 = (B/v_\perp^2)\mathbf{v}_\perp - (B/v_{||})\hat{\mathbf{b}}$, and $\overline{\nabla} \mathbf{v}$ has been neglected because the gradient is of order $1/L$.

I can prove that the divergence of $n_i \mathbf{V}_{iC}$, given in (4.33), is of order $\delta_i (k_\perp \rho_i)^2 \nu_{ii} n_e$ rather than $\delta_i k_\perp \rho_i \nu_{ii} n_e$. For $k_\perp \rho_i \ll 1$, the function $\mathbf{\Gamma}(\mathbf{r}, E_0, \mu_0, \varphi_0)$ can be Taylor expanded around $\mathbf{R}_g$ to find $\mathbf{\Gamma}(\mathbf{r}, E_0, \mu_0, \varphi_0) \simeq \mathbf{\Gamma}(\mathbf{R}_g, E_0, \mu_0, \varphi_0) - \Omega_i^{-1}(\mathbf{v} \times \hat{\mathbf{b}}) \cdot \nabla_{\mathbf{R}_g} \mathbf{\Gamma}$. Then, the gyroaverage $\langle \ldots \rangle$ holding $\mathbf{R}$, E, μ and t fixed gives

$$\langle \mathbf{\Gamma} \rangle = \frac{1}{2\pi} \oint d\varphi_0 \left[\mathbf{\Gamma}(\mathbf{R}_g, E_0, \mu_0, \varphi_0) - \frac{1}{\Omega_i} (\mathbf{v} \times \hat{\mathbf{b}}) \cdot \nabla_{\mathbf{R}_g} \mathbf{\Gamma} \right], \tag{H.9}$$

where I have employed that to the order of interest holding $\mathbf{R}_g$, E_0 and μ_0 fixed is approximately equal to holding $\mathbf{R}$, E and μ fixed. To rewrite equation (H.9) as a function of $\mathbf{r}$, E_0, μ_0 and φ_0, I Taylor expand $\mathbf{\Gamma}(\mathbf{R}_g, E_0, \mu_0, \varphi_0, t)$ around $\mathbf{r}$ to find

$$\frac{1}{2\pi} \oint d\varphi_0 \, \mathbf{\Gamma}(\mathbf{R}_g, E_0, \mu_0, \varphi_0) \simeq \overline{\mathbf{\Gamma}} + \frac{1}{\Omega_i} (\mathbf{v} \times \hat{\mathbf{b}}) \cdot \overline{\nabla} \, \overline{\mathbf{\Gamma}}, \tag{H.10}$$

with $\overline{\mathbf{\Gamma}} \equiv \overline{\mathbf{\Gamma}}(\mathbf{r}, E_0, \mu_0) = \overline{\mathbf{\Gamma}(\mathbf{r}, E_0, \mu_0, \varphi_0)}$ the gyroaverage holding $\mathbf{r}$, E_0, μ_0 and t fixed. The second term in the right side of (H.9) is higher order and can be simply written as

$$-\frac{1}{2\pi} \oint d\varphi_0 \frac{1}{\Omega_i} (\mathbf{v} \times \hat{\mathbf{b}}) \cdot \nabla_{\mathbf{R}_g} \mathbf{\Gamma} \simeq -\frac{1}{\Omega_i} \overline{(\mathbf{v} \times \hat{\mathbf{b}}) \cdot \overline{\nabla} \mathbf{\Gamma}}. \tag{H.11}$$

Employing equations (H.10) and (H.11) in equation (H.9), I find

$$\langle \mathbf{\Gamma} \rangle = \overline{\mathbf{\Gamma}} + \frac{1}{\Omega_i} (\mathbf{v} \times \hat{\mathbf{b}}) \cdot \overline{\nabla} \, \overline{\mathbf{\Gamma}} - \frac{1}{\Omega_i} \overline{(\mathbf{v} \times \hat{\mathbf{b}}) \cdot \overline{\nabla} \mathbf{\Gamma}}. \tag{H.12}$$

Similarly, letting $\mathbf{\Gamma} \to \mathbf{\Gamma} \cdot \mathbf{v}_\perp$ in (H.12) and ignoring $\overline{\nabla} \mathbf{v}_\perp$ corrections as small, I find

$$\langle \mathbf{\Gamma} \cdot \mathbf{v}_\perp \rangle = \overline{\mathbf{\Gamma} \cdot \mathbf{v}_\perp} + \frac{1}{\Omega_i} (\mathbf{v} \times \hat{\mathbf{b}}) \cdot \overline{\nabla} \left(\overline{\mathbf{\Gamma} \cdot \mathbf{v}_\perp} \right) - \frac{1}{\Omega_i} \overline{(\mathbf{v} \times \hat{\mathbf{b}}) \cdot \overline{\nabla} \mathbf{\Gamma} \cdot \mathbf{v}_\perp}. \tag{H.13}$$

Using these results in (4.33) the integral becomes

$$\nabla \cdot (n_i \mathbf{V}_{iC}) = -\nabla \cdot \left\{ \frac{\gamma}{\Omega_i} \int d^3 v \left[\mathbf{\Gamma} \times \hat{\mathbf{b}} - \frac{1}{\Omega_i} (\mathbf{v} \times \hat{\mathbf{b}}) \cdot \overline{\nabla} \mathbf{\Gamma} \times \hat{\mathbf{b}} - \frac{1}{2\Omega_i} \overline{\nabla} \mathbf{\Gamma} \cdot \mathbf{v}_\perp \right] \right\}, \tag{H.14}$$

where I have used that $\int d^3 v \, \overline{(\ldots)} = \int d^3 v \, (\ldots)$. The integral $\int d^3 v \, \mathbf{\Gamma}$ is zero, as can be proven by exchanging the dummy integration variables $\mathbf{v}$ and $\mathbf{v}'$. The rest of the integral can be written as

$$\nabla\cdot(n_i\mathbf{V}_{iC}) = \nabla\nabla : \left\{\frac{\gamma}{\Omega_i^2}\int d^3v\,\left[(\mathbf{v}\times\hat{\mathbf{b}})(\mathbf{\Gamma}\times\hat{\mathbf{b}}) + \frac{1}{2}(\overleftrightarrow{\mathbf{I}} - \hat{\mathbf{b}}\hat{\mathbf{b}})(\mathbf{\Gamma}\cdot\mathbf{v}_\perp)\right]\right\}, \quad \text{(H.15)}$$

where the spatial gradients of functions different from $\mathbf{\Gamma}$ have been neglected. Using the definition of the linearized collision operator (H.2) and employing $\int d^3v\,[(\mathbf{v}\times\hat{\mathbf{b}})(\mathbf{v}\times\hat{\mathbf{b}}) + (v_\perp^2/2)(\overleftrightarrow{\mathbf{I}} - \hat{\mathbf{b}}\hat{\mathbf{b}})]\nabla_v\cdot\mathbf{\Gamma} = -\int d^3v\,[(\mathbf{\Gamma}\times\hat{\mathbf{b}})(\mathbf{v}\times\hat{\mathbf{b}}) + (\mathbf{v}\times\hat{\mathbf{b}})(\mathbf{\Gamma}\times\hat{\mathbf{b}}) + (\mathbf{\Gamma}\cdot\mathbf{v}_\perp)(\overleftrightarrow{\mathbf{I}} - \hat{\mathbf{b}}\hat{\mathbf{b}})]$, I find

$$\nabla\cdot(n_i\mathbf{V}_{iC}) = -\nabla\nabla : \left\{\frac{1}{\Omega_i^2}\int d^3v\,C^{(\ell)}\{f_{i1}\}\left[\frac{v_\perp^2}{4}(\overleftrightarrow{\mathbf{I}} - \hat{\mathbf{b}}\hat{\mathbf{b}}) + \frac{1}{2}(\mathbf{v}\times\hat{\mathbf{b}})(\mathbf{v}\times\hat{\mathbf{b}})\right]\right\}, \quad \text{(H.16)}$$

where I have also employed $\overline{\nabla}\,\overline{\nabla} : [(\mathbf{v}\times\hat{\mathbf{b}})(\mathbf{\Gamma}\times\hat{\mathbf{b}})] = \overline{\nabla}\,\overline{\nabla} : [(\mathbf{\Gamma}\times\hat{\mathbf{b}})(\mathbf{v}\times\hat{\mathbf{b}})]$. This integral, of order $\delta_i(k_\perp\rho_i)^2\nu_{ii}n_e$, can be simplified by employing that, for $k_\perp\rho_i \ll 1$, the gyrophase dependent part of the distribution function is proportional to $\mathbf{v}_\perp$ to zeroth order [see (4.7)]. Then, the integral becomes

$$\nabla\cdot(n_i\mathbf{V}_{iC}) = -\nabla\nabla : \left[\frac{1}{\Omega_i^2}\int d^3v\,C^{(\ell)}\{f_{i1}\}\frac{v_\perp^2}{2}(\overleftrightarrow{\mathbf{I}} - \hat{\mathbf{b}}\hat{\mathbf{b}})\right], \quad \text{(H.17)}$$

where I used that $\overline{\mathbf{v}_\perp(\mathbf{v}\times\hat{\mathbf{b}})(\mathbf{v}\times\hat{\mathbf{b}})} = 0$ and $\overline{(\mathbf{v}\times\hat{\mathbf{b}})(\mathbf{v}\times\hat{\mathbf{b}})} = (v_\perp^2/2)(\overleftrightarrow{\mathbf{I}} - \hat{\mathbf{b}}\hat{\mathbf{b}})$.

Finally, I prove that $\mathbf{F}_{iC}$ from (4.43) is of order $\delta_i k_\perp\rho_i\nu_{ii}n_e M v_i$. Here, it is important to realize that the first integral in (4.43) must be carried to the next order in $k_\perp\rho_i$, but the integrals in the divergence only need the lowest order expressions. Then, using expressions (H.12) and (H.13) in the first integral of (4.43), and the lowest order expressions $\langle\mathbf{\Gamma}\rangle \simeq \overline{\mathbf{\Gamma}}$ and $\langle\mathbf{\Gamma}\cdot\mathbf{v}_\perp\rangle \simeq \overline{\mathbf{\Gamma}\cdot\mathbf{v}_\perp}$ for the second integral, I find that for $k_\perp\rho_i \ll 1$,

$$\mathbf{F}_{iC} \simeq \nabla\cdot\left\{\frac{M\gamma}{\Omega_i}\int d^3v\,\left[(\mathbf{v}\times\hat{\mathbf{b}})(\mathbf{\Gamma}\cdot\hat{\mathbf{b}})\hat{\mathbf{b}} + (\mathbf{\Gamma}\times\hat{\mathbf{b}})v_{||}\hat{\mathbf{b}} + (\mathbf{\Gamma}\cdot\mathbf{v}_\perp)\,\overleftrightarrow{\mathbf{I}}\times\hat{\mathbf{b}}\right]\right\} =$$
$$-\nabla\cdot\left\{\frac{M}{\Omega_i}\int d^3v\,C^{(\ell)}\{f_{i1}\}\left[(\mathbf{v}\times\hat{\mathbf{b}})v_{||}\hat{\mathbf{b}} + \frac{v_\perp^2}{2}\,\overleftrightarrow{\mathbf{I}}\times\hat{\mathbf{b}}\right]\right\}. \quad \text{(H.18)}$$

To obtain the last result, I have employed that $\int d^3v\,[(\mathbf{v}\times\hat{\mathbf{b}})v_{||}\hat{\mathbf{b}}+(v_\perp^2/2)\overleftrightarrow{\mathbf{I}}\times\hat{\mathbf{b}}]\nabla_v\cdot\mathbf{\Gamma} = -\int d^3v\,[(\mathbf{\Gamma}\times\hat{\mathbf{b}})v_{||}\hat{\mathbf{b}}+(\mathbf{v}\times\hat{\mathbf{b}})(\mathbf{\Gamma}\cdot\hat{\mathbf{b}})\hat{\mathbf{b}}+(\mathbf{\Gamma}\cdot\mathbf{v}_\perp)\overleftrightarrow{\mathbf{I}}\times\hat{\mathbf{b}}]$ and I have used the definition of the linearized collision operator (H.2). Only the gyrophase dependent part of f_{i1} contributes to the first part of integral (H.18), and for $k_\perp\rho_i \ll 1$ the gyrophase dependent part is even in $v_{||}$ [recall (4.7)] so this portion vanishes. As a result, the integral becomes

$$\mathbf{F}_{iC} = -\nabla\cdot\left[\frac{M}{\Omega_i}\int d^3v\,C^{(\ell)}\{f_{i1}\}\frac{v_\perp^2}{2}\overleftrightarrow{\mathbf{I}}\times\hat{\mathbf{b}}\right], \tag{H.19}$$

of order $\delta_i k_\perp \rho_i \nu_{ii} n_e M v_i$.

Appendix I

Gyrokinetic vorticity

In this Appendix, I explain how to obtain the gyrokinetic vorticity equation (4.53) from equations (4.38) and (4.45).

Before adding equations (4.45) and $\nabla \cdot \{(c/B)[\text{equation (4.38)}] \times \hat{\mathbf{b}}\}$, I simplify the perpendicular component of the current density $Zen_i\tilde{\mathbf{V}}_i$. The perpendicular component of $\tilde{\mathbf{v}}_1$, defined in (4.19), is given by

$$\tilde{\mathbf{v}}_{1\perp} \equiv \hat{\mathbf{b}} \times (\tilde{\mathbf{v}}_1 \times \hat{\mathbf{b}}) = \frac{v_{||}}{\Omega_i}\hat{\mathbf{b}} \times (\hat{\mathbf{b}} \cdot \overline{\nabla}\mathbf{v}_\perp + \nabla\hat{\mathbf{b}} \cdot \mathbf{v}_\perp), \tag{I.1}$$

where I use that $(\overline{\nabla} \times \mathbf{v}_\perp) \times \hat{\mathbf{b}} = \hat{\mathbf{b}} \cdot \overline{\nabla}\mathbf{v}_\perp - \overline{\nabla}\mathbf{v}_\perp \cdot \hat{\mathbf{b}}$ and $\overline{\nabla}\mathbf{v}_\perp \cdot \hat{\mathbf{b}} = -\nabla\hat{\mathbf{b}} \cdot \mathbf{v}_\perp$. Employing $(\nabla \times \hat{\mathbf{b}}) \times \mathbf{v}_\perp = \mathbf{v}_\perp \cdot \nabla\hat{\mathbf{b}} - \nabla\hat{\mathbf{b}} \cdot \mathbf{v}_\perp$ and $\hat{\mathbf{b}} \times [(\nabla \times \hat{\mathbf{b}}) \times \mathbf{v}_\perp] = -\mathbf{v}_\perp(\hat{\mathbf{b}} \cdot \nabla \times \hat{\mathbf{b}})$, I find

$$\tilde{\mathbf{v}}_{1\perp} = \frac{v_{||}}{\Omega_i}\hat{\mathbf{b}} \times (\hat{\mathbf{b}} \cdot \overline{\nabla}\mathbf{v}_\perp + \mathbf{v}_\perp \cdot \nabla\hat{\mathbf{b}}) + \frac{v_{||}}{\Omega_i}\mathbf{v}_\perp(\hat{\mathbf{b}} \cdot \nabla \times \hat{\mathbf{b}}). \tag{I.2}$$

Then, the integral $n_i\tilde{\mathbf{V}}_{i\perp}$ becomes

$$n_i\tilde{\mathbf{V}}_{i\perp} = \frac{1}{\Omega_i}\hat{\mathbf{b}} \times \left(\int d^3v\, f_{ig}v_{||}\hat{\mathbf{b}} \cdot \overline{\nabla}\mathbf{v}_\perp + \int d^3v\, f_{ig}v_{||}\mathbf{v}_\perp \cdot \nabla\hat{\mathbf{b}}\right) + \frac{1}{\Omega_i}\hat{\mathbf{b}} \cdot \nabla \times \hat{\mathbf{b}} \int d^3v\, f_{ig}v_{||}\mathbf{v}_\perp. \tag{I.3}$$

Adding equations (4.45) and $\nabla \cdot \{(c/B)[\text{equation (4.38)}] \times \hat{\mathbf{b}}\}$, I find

$$\frac{\partial \varpi_G}{\partial t} = \nabla \cdot \left[J_{||}\hat{\mathbf{b}} + \mathbf{J}_{gd} + \tilde{\mathbf{J}}_i + Zen_i\tilde{\mathbf{V}}_i + Zen_i\mathbf{V}_{iC} + \frac{c}{B}\hat{\mathbf{b}} \times (\nabla \cdot \overleftrightarrow{\boldsymbol{\pi}}_{ig\times}) \right.$$
$$\left. - \frac{c}{B}\hat{\mathbf{b}} \times \mathbf{F}_{iB} - \frac{c}{B}\hat{\mathbf{b}} \times \mathbf{F}_{iC} \right], \qquad \text{(I.4)}$$

with ϖ_G defined in (4.51). In this equation, adding $-(c/B)\hat{\mathbf{b}} \times \mathbf{F}_{iB}$ and the expression in (I.3) for $Zen_i\tilde{\mathbf{V}}_{i\perp}$, two terms cancel to give

$$Zen_i\tilde{\mathbf{V}}_{i\perp} - \frac{c}{B}\hat{\mathbf{b}} \times \mathbf{F}_{iB} = \frac{Mc}{B}\hat{\mathbf{b}} \times \int d^3v\, f_{ig}v_{||}\mathbf{v}_\perp \cdot \nabla\hat{\mathbf{b}}$$
$$+ \frac{Mc}{B}\hat{\mathbf{b}} \cdot \nabla \times \hat{\mathbf{b}} \int d^3v\, f_{ig}v_{||}\mathbf{v}_\perp - \frac{Zec}{B\Omega_i}\hat{\mathbf{b}} \times \int d^3v\, f_{Mi}(\overline{\nabla}\tilde{\phi} \times \hat{\mathbf{b}}) \cdot \overline{\nabla}\mathbf{v}_\perp. \qquad \text{(I.5)}$$

The last term in this equation is absorbed in the definition of $\tilde{\mathbf{J}}_{i\phi}$ in (4.55). I can further simplify by realizing that $\hat{\mathbf{b}} \times \int d^3v\, f_{ig}v_{||}\mathbf{v}_\perp \cdot \nabla\hat{\mathbf{b}} = \hat{\mathbf{b}} \times [\nabla \cdot (\int d^3v\, f_{ig}\mathbf{v}_\perp v_{||}\hat{\mathbf{b}})]$, giving

$$\tilde{\mathbf{J}}_i + Zen_i\tilde{\mathbf{V}}_{i\perp} - \frac{c}{B}\hat{\mathbf{b}} \times \mathbf{F}_{iB} = \tilde{\mathbf{J}}_{i\phi} + \frac{c}{B}\hat{\mathbf{b}} \times \left[\nabla \cdot \left(\int d^3v\, f_{ig}M\mathbf{v}_\perp v_{||}\hat{\mathbf{b}}\right)\right]$$
$$+ \frac{Mc}{B}\hat{\mathbf{b}} \cdot \nabla \times \hat{\mathbf{b}} \int d^3v\, f_{ig}v_{||}\mathbf{v}_\perp. \qquad \text{(I.6)}$$

The integral $\int d^3v\, f_{ig}M\mathbf{v}_\perp v_{||}\hat{\mathbf{b}}$ is part of the definition of $\overleftrightarrow{\boldsymbol{\pi}}_{iG}$ in (4.54), so I can finally write

$$\tilde{\mathbf{J}}_i + Zen_i\tilde{\mathbf{V}}_{i\perp} + \frac{c}{B}\hat{\mathbf{b}} \times (\nabla \cdot \overleftrightarrow{\boldsymbol{\pi}}_{ig\times}) - \frac{c}{B}\hat{\mathbf{b}} \times \mathbf{F}_{iB} = \tilde{\mathbf{J}}_{i\phi} + \frac{c}{B}\hat{\mathbf{b}} \times (\nabla \cdot \overleftrightarrow{\boldsymbol{\pi}}_{iG})$$
$$+ \frac{Mc}{B}\hat{\mathbf{b}} \cdot \nabla \times \hat{\mathbf{b}} \int d^3v\, f_{ig}v_{||}\mathbf{v}_\perp. \qquad \text{(I.7)}$$

Employing this result in equation (I.4) and using the fact that the divergence of $Zen_i\tilde{V}_{i||}\hat{\mathbf{b}}$ and $(Mc/B)\hat{\mathbf{b}} \cdot \nabla \times \hat{\mathbf{b}} \int d^3v\, f_{ig}v_{||}\mathbf{v}_\perp$ is negligible, I recover equation (4.53). The divergence of $Zen_i\tilde{V}_{i||}\hat{\mathbf{b}} \sim \delta_i^2 k_\perp \rho_i e n_e v_i$ is small because the parallel gradient is only order $1/L$, giving $\nabla \cdot (Zen_i\tilde{V}_{i||}\hat{\mathbf{b}}) \sim \delta_i^2 k_\perp \rho_i e n_e v_i / L$; which is negligible with respect to the rest of the terms, the smallest of which is order $\delta_i (k_\perp \rho_i)^2 e n_e v_i / L$. The divergence of $(Mc/B)\hat{\mathbf{b}} \cdot \nabla \times \hat{\mathbf{b}} \int d^3v\, f_{ig}v_{||}\mathbf{v}_\perp \sim \delta_i^2 k_\perp \rho_i e n_e v_i$ has only one term that is of order $\delta_i (k_\perp \rho_i)^2 e n_e v_i / L$, given by

$$\nabla \cdot \left(\frac{Mc}{B} \hat{\mathbf{b}} \cdot \nabla \times \hat{\mathbf{b}} \int d^3v \, f_{ig} v_{||} \mathbf{v}_\perp \right) \simeq \frac{Mc}{B} \hat{\mathbf{b}} \cdot \nabla \times \hat{\mathbf{b}} \int d^3v \, v_{||} \mathbf{v}_\perp \cdot \overline{\nabla} f_{ig}. \tag{I.8}$$

Since the only dependence of f_{ig} on φ_0 is in $\mathbf{R}_g = \mathbf{r} + \Omega_i^{-1} \mathbf{v} \times \hat{\mathbf{b}}$, I find that $\mathbf{v}_\perp \cdot \overline{\nabla} f_{ig} = \Omega_i (\partial f_{ig} / \partial \varphi_0)$. Thus, the divergence of $(Mc/B) \hat{\mathbf{b}} \cdot \nabla \times \hat{\mathbf{b}} \int d^3v \, f_{ig} v_{||} \mathbf{v}_\perp$ vanishes to the relevant order due to the gyrophase integration

$$\nabla \cdot \left(\frac{Mc}{B} \hat{\mathbf{b}} \cdot \nabla \times \hat{\mathbf{b}} \int d^3v \, f_{ig} v_{||} \mathbf{v}_\perp \right) \simeq Ze \hat{\mathbf{b}} \cdot \nabla \times \hat{\mathbf{b}} \int d^3v \, v_{||} \frac{\partial f_{ig}}{\partial \varphi_0} = 0. \tag{I.9}$$

Appendix J

Flux surface averaged gyrokinetic vorticity equation

In this Appendix, I obtain the long wavelength limit of the flux surface averaged vorticity equation (4.62). The long wavelength limit of ϖ_G is given by $\varpi_G \rightarrow \varpi = \nabla \cdot [(Ze/\Omega_i) n_i \mathbf{V}_i \times \hat{\mathbf{b}}]$, as proven in (4.52). Then, upon using (2.16) and integrating once in ψ, equation (4.62) becomes

$$-\frac{\partial}{\partial t}\langle cRn_i M\mathbf{V}_i \cdot \hat{\boldsymbol{\zeta}}\rangle_\psi = \left\langle \tilde{\mathbf{J}}_{i\phi} \cdot \nabla\psi + \frac{cI}{B}(\nabla \cdot \boldsymbol{\pi}_{ig||} - \tilde{F}_{iE}) \right.$$
$$\left. -\frac{c}{B}(\nabla \cdot \overleftrightarrow{\boldsymbol{\pi}}_{iG}) \cdot (\hat{\mathbf{b}} \times \nabla\psi) + Zen_i \mathbf{V}_{iC} \cdot \nabla\psi - cR\mathbf{F}_{iC} \cdot \hat{\boldsymbol{\zeta}} \right\rangle_\psi . \tag{J.1}$$

I will evaluate all the terms on the right side of equation (J.1) to order $\delta_i^2 k_\perp \rho_i e n_e v_i |\nabla\psi|$ for $k_\perp \rho_i \rightarrow 0$.

J.1 Limit of $\langle (cI/B)\nabla \cdot \boldsymbol{\pi}_{ig||}\rangle_\psi$ for $k_\perp \rho_i \rightarrow 0$

The term $\langle (cI/B)\nabla \cdot \boldsymbol{\pi}_{ig||}\rangle_\psi$ is written as

$$\left\langle \frac{cI}{B} \nabla \cdot \boldsymbol{\pi}_{ig||} \right\rangle_\psi \simeq \left\langle \nabla \cdot \left(\frac{cI}{B} \boldsymbol{\pi}_{ig||} \right) \right\rangle_\psi = \frac{1}{V'} \frac{\partial}{\partial \psi} V' \left\langle \frac{cI}{B} \boldsymbol{\pi}_{ig||} \cdot \nabla\psi \right\rangle_\psi . \tag{J.2}$$

The term $\boldsymbol{\pi}_{ig||} \cdot \nabla(cI/B)$ is neglected because it is of order $\delta_i^3 e n_e v_i |\nabla\psi|$. In the definition of $\boldsymbol{\pi}_{ig||}$ in (4.39), one of the terms is $-\int d^3v\, f_{ig}(c/B)(\overline{\nabla}\langle\phi\rangle \times \hat{\mathbf{b}})Mv_{||}$. The difference between $f_{ig}(c/B)\overline{\nabla}\langle\phi\rangle \times \hat{\mathbf{b}}$ and $f_i(c/B)\nabla\phi \times \hat{\mathbf{b}}$ gives rise to the term

$$\frac{cI}{B^2}\int d^3v\, Mv_{||}[f_{ig}(\overline{\nabla}\langle\phi\rangle \times \hat{\mathbf{b}}) - f_i(\nabla\phi \times \hat{\mathbf{b}})] \cdot \nabla\psi =$$
$$\frac{cI}{B^2}\int d^3v\, Mv_{||}\left[\frac{Ze\widetilde{\phi}}{T_i} f_{Mi}(\nabla\phi \times \hat{\mathbf{b}}) - f_{ig}\frac{c}{B}(\overline{\nabla}\widetilde{\phi} \times \hat{\mathbf{b}})\right] \cdot \nabla\psi \tag{J.3}$$

in $(I/B)\boldsymbol{\pi}_{ig||} \cdot \nabla\psi$, where I use $f_i - f_{ig} = -(Ze\widetilde{\phi}/T_i)f_{Mi}$ to obtain the second equality. I will show below that the difference (J.3) is of order $\delta_i^2 k_\perp \rho_i p_i R|\nabla\psi|$ and therefore negligible compared to the other terms in $(I/B)\boldsymbol{\pi}_{ig||} \cdot \nabla\psi$ that are of order $\delta_i^2 p_i R|\nabla\psi|$. Then, in $(I/B)\boldsymbol{\pi}_{ig||} \cdot \nabla\psi$ the difference between $f_{ig}(c/B)\overline{\nabla}\langle\phi\rangle \times \hat{\mathbf{b}}$ and $f_i(c/B)\nabla\phi \times \hat{\mathbf{b}}$ can be neglected to write equation (J.2) as

$$\left\langle \frac{cI}{B}\nabla \cdot \boldsymbol{\pi}_{ig||} \right\rangle_\psi \simeq \frac{1}{V'}\frac{\partial}{\partial\psi}V'\left\langle \frac{cI}{B}\boldsymbol{\pi}'_{ig||} \cdot \nabla\psi \right\rangle_\psi, \tag{J.4}$$

with

$$\boldsymbol{\pi}'_{ig||} = \int d^3v\, f_{ig}(\mathbf{v}_{M0} + \tilde{\mathbf{v}}_1)Mv_{||} - \frac{Mc}{B}\nabla\phi \times \hat{\mathbf{b}}\int d^3v\, f_i v_{||}. \tag{J.5}$$

Equation (J.4) is then seen to be of order $\delta_i^2 k_\perp \rho_i e n_e v_i |\nabla\psi|$.

I will now prove that the difference (J.3) is of order $\delta_i^2 k_\perp \rho_i p_i R|\nabla\psi|$. In equation (J.3), short wavelength components of ϕ, $\widetilde{\phi}$ and f_{ig} can beat nonlinearly to give a long wavelength component, and these functions cannot be Taylor expanded around $\mathbf{r}$. However, the total long wavelength contribution (J.3) can be expanded. The gyrophase dependence in $\mathbf{R}_g$ then gives a contribution of order $\delta_i k_\perp \rho_i p_i R|\nabla\psi|$ that can be ignored when integrating over velocity space to order $\delta_i^2 p_i R|\nabla\psi|$. The result is $(cI/B^2)\int d^3v (Ze\widetilde{\phi}/T_i) f_{Mi} Mv_{||}(\overline{\nabla}\widetilde{\phi} \times \hat{\mathbf{b}}) \cdot \nabla\psi$ because the rest of the terms have vanishing gyroaverages to the order of interest. Since both f_{Mi} and $\widetilde{\phi}$ are even in $v_{||}$, this integral vanishes, and (J.3) is higher order than $\delta_i^2 p_i R|\nabla\psi|$.

J.2 Limit of $\langle \tilde{\mathbf{J}}_{i\phi} \cdot \nabla\psi - (c/B)(\nabla\cdot \overleftrightarrow{\boldsymbol{\pi}}_{iG}) \cdot (\hat{\mathbf{b}} \times \nabla\psi) \rangle_\psi$ for $k_\perp \rho_i \rightarrow 0$

I simplify the terms $\langle \tilde{\mathbf{J}}_{i\phi} \cdot \nabla\psi \rangle_\psi$ and $-\langle (c/B)(\nabla\cdot \overleftrightarrow{\boldsymbol{\pi}}_{iG}) \cdot (\hat{\mathbf{b}} \times \nabla\psi) \rangle_\psi$ by first calculating the divergence of $\tilde{\mathbf{J}}_{i\phi} + (c/B)\hat{\mathbf{b}} \times (\nabla\cdot \overleftrightarrow{\boldsymbol{\pi}}_{iG})$. I employ the long wavelength result for $\nabla \cdot \tilde{\mathbf{J}}_{i\phi}$ in (4.56) to obtain

$$\nabla \cdot \left[\tilde{\mathbf{J}}_{i\phi} + \frac{c}{B}\hat{\mathbf{b}} \times (\nabla\cdot \overleftrightarrow{\boldsymbol{\pi}}_{iG}) \right] = \nabla \cdot \left[\frac{c}{B}\hat{\mathbf{b}} \times (\nabla\cdot \overleftrightarrow{\boldsymbol{\pi}}'_{iG} + \nabla\cdot \overleftrightarrow{\boldsymbol{\pi}}''_{iG}) \right], \tag{J.6}$$

with

$$\overleftrightarrow{\boldsymbol{\pi}}'_{iG} = M \int d^3v\, f_{ig} v_{||} (\hat{\mathbf{b}}\mathbf{v}_\perp + \mathbf{v}_\perp \hat{\mathbf{b}}) \sim \delta_i k_\perp \rho_i p_i \tag{J.7}$$

and

$$\overleftrightarrow{\boldsymbol{\pi}}''_{iG} = \int d^3v\, f_{ig} (\mathbf{v}_{M0} + \tilde{\mathbf{v}}_1) M \mathbf{v}_\perp - \frac{Mc}{B} \nabla\phi \times \hat{\mathbf{b}} \int d^3v\, f_i \mathbf{v}_\perp \sim \delta_i^2 p_i. \tag{J.8}$$

Flux surface averaging equation (J.6) and integrating once in ψ, I find the order $\delta_i^2 k_\perp \rho_i e n_e v_i |\nabla\psi|$ term

$$\left\langle \tilde{\mathbf{J}}_{i\phi} \cdot \nabla\psi - \frac{c}{B}(\nabla\cdot \overleftrightarrow{\boldsymbol{\pi}}_{iG}) \cdot (\hat{\mathbf{b}} \times \nabla\psi) \right\rangle_\psi = -\frac{1}{V'} \frac{\partial}{\partial\psi} V' \left\langle \frac{c}{B} \nabla\psi \cdot \overleftrightarrow{\boldsymbol{\pi}}''_{iG} \cdot (\hat{\mathbf{b}} \times \nabla\psi) \right\rangle_\psi, \tag{J.9}$$

where I neglected $c\, \overleftrightarrow{\boldsymbol{\pi}}''_{iG} : \nabla[(\hat{\mathbf{b}} \times \nabla\psi)/B] \sim \delta_i^3 e n_e v_i |\nabla\psi|$, I used the definition of $\overleftrightarrow{\boldsymbol{\pi}}'_{iG}$ in (J.7) to obtain $\nabla\psi \cdot \overleftrightarrow{\boldsymbol{\pi}}'_{iG} \cdot (\hat{\mathbf{b}} \times \nabla\psi) = 0$, and I will next prove that $\langle \overleftrightarrow{\boldsymbol{\pi}}'_{iG} : \nabla[(\hat{\mathbf{b}} \times \nabla\psi)/B] \rangle_\psi$ vanishes.

To see that $\langle \overleftrightarrow{\boldsymbol{\pi}}'_{iG} : \nabla[(\hat{\mathbf{b}} \times \nabla\psi)/B] \rangle_\psi = 0$, the velocity integral $\int d^3v\, f_{ig} v_{||} \mathbf{v}_\perp$ in $\overleftrightarrow{\boldsymbol{\pi}}'_{iG}$ has to be found to order $\delta_i k_\perp \rho_i n_e v_i^2$. This integral only depends on the gyrophase dependent piece of f_{ig}, given to the required order by

$$f_i - \overline{f}_i \simeq \frac{1}{\Omega_i} (\mathbf{v} \times \hat{\mathbf{b}}) \cdot \overline{\nabla} f_{i0} + \frac{1}{2\Omega_i^2} (\mathbf{v} \times \hat{\mathbf{b}})(\mathbf{v} \times \hat{\mathbf{b}}) : \overline{\nabla}\, \overline{\nabla} f_{i0}, \tag{J.10}$$

with $f_{i0} \equiv f_i(\mathbf{r}, E_0, \mu_0, \varphi_0)$ as defined in (4.8) and, thus, gyrophase independent. The integral involving $\overline{\nabla}\,\overline{\nabla} f_{i0}$ vanishes, leaving

$$\overset{\leftrightarrow}{\boldsymbol{\pi}}{}'_{iG}\colon \nabla\left(\frac{\hat{\mathbf{b}}\times\nabla\psi}{B}\right) =$$
$$\int d^3v\, \frac{Mv_{||}}{\Omega_i}[(\mathbf{v}\times\hat{\mathbf{b}})\cdot\overline{\nabla} f_{i0}](\hat{\mathbf{b}}\mathbf{v}_\perp + \mathbf{v}_\perp\hat{\mathbf{b}}) : \nabla\left(\frac{\hat{\mathbf{b}}\times\nabla\psi}{B}\right) \simeq$$
$$\nabla\cdot\left[\int d^3v\, f_{i0}\frac{Mv_{||}}{\Omega_i}(\mathbf{v}\times\hat{\mathbf{b}})(\hat{\mathbf{b}}\mathbf{v}_\perp + \mathbf{v}_\perp\hat{\mathbf{b}}) : \nabla\left(\frac{\hat{\mathbf{b}}\times\nabla\psi}{B}\right)\right], \tag{J.11}$$

where terms of order $\delta_i^2 p_i$ are neglected. Integrating in gyrophase and flux surface averaging, equation (J.11) reduces to

$$\left\langle \overset{\leftrightarrow}{\boldsymbol{\pi}}{}'_{iG}\colon \nabla\left(\frac{\hat{\mathbf{b}}\times\nabla\psi}{B}\right)\right\rangle_\psi =$$
$$\frac{1}{V'}\frac{\partial}{\partial\psi}V'\left\langle M\int d^3v\, f_{i0}\frac{v_{||}v_\perp^2}{2\Omega_i}[\hat{\mathbf{b}}(\hat{\mathbf{b}}\times\nabla\psi) + (\hat{\mathbf{b}}\times\nabla\psi)\hat{\mathbf{b}}] : \nabla\left(\frac{\hat{\mathbf{b}}\times\nabla\psi}{B}\right)\right\rangle_\psi . \tag{J.12}$$

This expression vanishes because $[\hat{\mathbf{b}}(\hat{\mathbf{b}}\times\nabla\psi) + (\hat{\mathbf{b}}\times\nabla\psi)\hat{\mathbf{b}}] : \nabla[(\hat{\mathbf{b}}\times\nabla\psi)/B] = 0$. To prove this, I employ equation (2.16) to write $\nabla[(\hat{\mathbf{b}}\times\nabla\psi)/B] = \nabla(I\hat{\mathbf{b}}/B) - \nabla(R\hat{\boldsymbol{\zeta}})$. The tensor $\nabla(R\hat{\boldsymbol{\zeta}}) = (\nabla R)\hat{\boldsymbol{\zeta}} - \hat{\boldsymbol{\zeta}}(\nabla R)$ gives zero contribution because it is antisymmetric and it is multiplied by the symmetric tensor $[\hat{\mathbf{b}}(\hat{\mathbf{b}}\times\nabla\psi) + (\hat{\mathbf{b}}\times\nabla\psi)\hat{\mathbf{b}}]$. Then, I am only left with $\nabla(I\hat{\mathbf{b}}/B)$, giving

$$[\hat{\mathbf{b}}(\hat{\mathbf{b}}\times\nabla\psi) + (\hat{\mathbf{b}}\times\nabla\psi)\hat{\mathbf{b}}] : \nabla\left(\frac{\hat{\mathbf{b}}\times\nabla\psi}{B}\right) = (\hat{\mathbf{b}}\times\nabla\psi)\cdot\left[\nabla\left(\frac{I}{B}\right) + \frac{I}{B}\hat{\mathbf{b}}\cdot\nabla\hat{\mathbf{b}}\right]. \tag{J.13}$$

To simplify, I use relation (2.8), with $\hat{\mathbf{b}}\cdot\nabla\hat{\mathbf{b}} = \boldsymbol{\kappa}$, to write $\hat{\mathbf{b}}\cdot\nabla\hat{\mathbf{b}}\cdot(\hat{\mathbf{b}}\times\nabla\psi) = -(\nabla\times\hat{\mathbf{b}})\cdot\nabla\psi = -\nabla\cdot(\hat{\mathbf{b}}\times\nabla\psi)$. Finally, I employ (2.16) and $\hat{\boldsymbol{\zeta}}\cdot\nabla(I/B) = 0 = \nabla\cdot(RB\hat{\boldsymbol{\zeta}})$ due to axisymmetry to obtain

$$[\hat{\mathbf{b}}(\hat{\mathbf{b}}\times\nabla\psi) + (\hat{\mathbf{b}}\times\nabla\psi)\hat{\mathbf{b}}] : \nabla\left(\frac{\hat{\mathbf{b}}\times\nabla\psi}{B}\right) = I\hat{\mathbf{b}}\cdot\nabla\left(\frac{I}{B}\right) - \frac{I}{B}\nabla\cdot(I\hat{\mathbf{b}}) = 0. \tag{J.14}$$

J.3 Limit of $\langle (cI/B)\tilde{F}_{iE}\rangle_\psi$ for $k_\perp \rho_i \to 0$

The function $\tilde{F}_{iE}$, defined in (4.41), is written as

$$\tilde{F}_{iE} = Ze \int dE_0\, d\mu_0\, d\varphi_0 \frac{\partial v_{||}}{\partial E_0} B f_{Mi} \left(\hat{\mathbf{b}} \cdot \nabla\tilde{\phi} + \frac{1}{\Omega_i} \nabla \times \mathbf{v}_\perp \cdot \nabla\tilde{\phi} \right), \tag{J.15}$$

where I use $d^3v = (B/v_{||}) dE_0\, d\mu_0\, d\varphi_0$ and $v_{||}^{-1} = \partial v_{||}/\partial E_0$. Integrating by parts in E_0, and making use of $\partial\tilde{\phi}/\partial E_0 = 0$, $\partial f_{Mi}/\partial E_0 = (-M/T_i) f_{Mi}$ and $\tilde{\mathbf{v}}_1 = (v_{||}/\Omega_i)\nabla \times \mathbf{v}_\perp$, I find

$$\tilde{F}_{iE} = M \int dE_0\, d\mu_0\, d\varphi_0 B \frac{Ze}{T_i} f_{Mi} (v_{||}\hat{\mathbf{b}} + \tilde{\mathbf{v}}_1) \cdot \nabla\tilde{\phi}. \tag{J.16}$$

Multiplying equation (J.16) by cI/B and writing the result as a divergence give

$$\frac{cI}{B} \tilde{F}_{iE} \simeq \nabla \cdot \left(\frac{cI}{B} \boldsymbol{\pi}'_{iE} \right) - \int d^3v\, \frac{Z^2 e^2 \tilde{\phi}}{T_i} f_{Mi} v_{||} \hat{\mathbf{b}} \cdot \nabla \left(\frac{I v_{||}}{\Omega_i} \right), \tag{J.17}$$

with $\boldsymbol{\pi}'_{iE} = \int d^3v\, (Ze\tilde{\phi}/T_i) f_{Mi} (v_{||}\hat{\mathbf{b}} + \tilde{\mathbf{v}}_1) M v_{||}$. Here, I employ $\hat{\mathbf{b}} \cdot \nabla (f_{Mi}/T_i) = 0$, and neglect the integral $Mc \int dE_0\, d\mu_0\, d\varphi_0 Ze\tilde{\phi} \nabla \cdot (\tilde{\mathbf{v}}_1 I f_{Mi}/T_i)$ because it is of order $\delta_i^3 e n_e v_i |\nabla\psi|$. I will now consider the two integrals in equation (J.17). Upon using (2.15), the flux surface average of the first integral gives

$$\left\langle \nabla \cdot \left(\frac{cI}{B} \boldsymbol{\pi}'_{iE} \right) \right\rangle_\psi = \frac{1}{V'} \frac{\partial}{\partial \psi} V' \left\langle Mc \int d^3v\, \frac{Ze\tilde{\phi}}{T_i} f_{Mi} \frac{I v_{||}}{B} \tilde{\mathbf{v}}_1 \cdot \nabla\psi \right\rangle_\psi. \tag{J.18}$$

Multiplying equation (2.16) by $\mathbf{v}$, I find $I v_{||}/B = R\mathbf{v} \cdot \hat{\boldsymbol{\zeta}} + (\mathbf{v} \times \hat{\mathbf{b}}) \cdot \nabla\psi/B$. Substituting this result in equation (J.18), I find that the integral of $\mathbf{v} \times \hat{\mathbf{b}}$ vanishes because $\tilde{\phi}$ and f_{Mi} are even in $v_{||}$, and $\tilde{\mathbf{v}}_1$ is odd. Thus, the first integral of equation (J.17) gives

$$\left\langle \nabla \cdot \left(\frac{cI}{B} \boldsymbol{\pi}'_{iE} \right) \right\rangle_\psi = \frac{1}{V'} \frac{\partial}{\partial \psi} V' \left\langle c \int d^3v\, \frac{Ze\tilde{\phi}}{T_i} f_{Mi} R M (\mathbf{v} \cdot \hat{\boldsymbol{\zeta}}) \tilde{\mathbf{v}}_1 \cdot \nabla\psi \right\rangle_\psi. \tag{J.19}$$

In the second integral of (J.17), I need to keep $\tilde{\phi}(\mathbf{R}_g, \mu_0, \varphi_0, t) \simeq \tilde{\phi}(\mathbf{r}, \mu_0, \varphi_0, t) +$

$\Omega_i^{-1}(\mathbf{v}\times\hat{\mathbf{b}})\cdot\overline{\nabla}\widetilde{\phi}$. Using $\oint d\varphi_0\,\widetilde{\phi}(\mathbf{r},\mu_0,\varphi_0,t)=0$ leaves

$$\int d^3v\,\frac{Z^2e^2\widetilde{\phi}}{T_i}f_{Mi}v_{||}\hat{\mathbf{b}}\cdot\overline{\nabla}\left(\frac{Iv_{||}}{\Omega_i}\right)\simeq$$
$$\frac{Ze}{\Omega_i}\int d^3v\,f_{Mi}\frac{Ze}{T_i}(\mathbf{v}\times\hat{\mathbf{b}})\cdot\overline{\nabla}\widetilde{\phi}\left[v_{||}\hat{\mathbf{b}}\cdot\overline{\nabla}\left(\frac{Iv_{||}}{\Omega_i}\right)\right]\simeq$$
$$\nabla\cdot\left\{\frac{Mc}{B}\int d^3v\,\frac{Ze\widetilde{\phi}}{T_i}f_{Mi}(\mathbf{v}\times\hat{\mathbf{b}})\left[v_{||}\hat{\mathbf{b}}\cdot\overline{\nabla}\left(\frac{Iv_{||}}{\Omega_i}\right)\right]\right\},\tag{J.20}$$

where terms of order $\delta_i^3 en_e v_i|\nabla\psi|$ are neglected to obtain the second equality. Using $v_{||}\hat{\mathbf{b}}\cdot\overline{\nabla}(Iv_{||}/\Omega_i)=\mathbf{v}_{M0}\cdot\nabla\psi$ in equation (J.20) and flux surface averaging, I find

$$\left\langle\int d^3v\,\frac{Z^2e^2\widetilde{\phi}}{T_i}f_{Mi}v_{||}\hat{\mathbf{b}}\cdot\overline{\nabla}\left(\frac{Iv_{||}}{\Omega_i}\right)\right\rangle_\psi=$$
$$\frac{1}{V'}\frac{\partial}{\partial\psi}V'\left\langle Mc\int d^3v\,\frac{Ze\widetilde{\phi}}{T_i}f_{Mi}\frac{(\mathbf{v}\times\hat{\mathbf{b}})\cdot\nabla\psi}{B}\mathbf{v}_{M0}\cdot\nabla\psi\right\rangle_\psi.\tag{J.21}$$

Here, equation (2.16) multiplied by $\mathbf{v}$ gives $(\mathbf{v}\times\hat{\mathbf{b}})\cdot\nabla\psi/B=Iv_{||}/B-R\mathbf{v}\cdot\hat{\boldsymbol{\zeta}}$. The integral of $Iv_{||}/B$ vanishes because both $\widetilde{\phi}$ and f_{Mi} are even in $v_{||}$. Thus, equation (J.21) gives to relevant order

$$\left\langle\int d^3v\,\frac{Z^2e^2\widetilde{\phi}}{T_i}f_{Mi}v_{||}\hat{\mathbf{b}}\cdot\overline{\nabla}\left(\frac{Iv_{||}}{\Omega_i}\right)\right\rangle_\psi\simeq$$
$$-\frac{1}{V'}\frac{\partial}{\partial\psi}V'\left\langle c\int d^3v\,\frac{Ze\widetilde{\phi}}{T_i}f_{Mi}RM(\mathbf{v}\cdot\hat{\boldsymbol{\zeta}})\mathbf{v}_{M0}\cdot\nabla\psi\right\rangle_\psi.\tag{J.22}$$

Substituting equations (J.19) and (J.22) into the flux surface average of equation (J.17), I obtain

$$\left\langle\frac{cI}{B}\tilde{F}_{iE}\right\rangle_\psi\simeq\frac{1}{V'}\frac{\partial}{\partial\psi}V'\left\langle c\int d^3v\,(f_{ig}-f_i)RM(\mathbf{v}\cdot\hat{\boldsymbol{\zeta}})(\mathbf{v}_{M0}+\tilde{\mathbf{v}}_1)\cdot\nabla\psi\right\rangle_\psi,\tag{J.23}$$

where I use $f_i-f_{ig}=-(Ze\widetilde{\phi}/T_i)f_{Mi}$.

J.4 Limit of $\langle Zen_i \mathbf{V}_{iC} \cdot \nabla\psi - cR\mathbf{F}_{iC} \cdot \hat{\boldsymbol{\zeta}} \rangle_\psi$ for $k_\perp \rho_i \rightarrow 0$

This collisional combination vanishes. According to (H.17), for $k_\perp \rho_i \ll 1$, $\langle Zen_i \mathbf{V}_{iC} \cdot \nabla\psi \rangle_\psi$ is given by

$$\langle Zen_i \mathbf{V}_{iC} \cdot \nabla\psi \rangle_\psi = -\frac{1}{V'} \frac{\partial}{\partial\psi} V' \left\langle \frac{c}{B\Omega_i} \int d^3v \, C\{f_i\} |\nabla\psi|^2 \frac{Mv_\perp^2}{2} \right\rangle_\psi . \tag{J.24}$$

Equation (H.19), on the other hand, gives

$$\langle cR\mathbf{F}_{iC} \cdot \hat{\boldsymbol{\zeta}} \rangle_\psi = -\frac{1}{V'} \frac{\partial}{\partial\psi} V' \left\langle \frac{c}{B\Omega_i} \int d^3v \, C\{f_i\} |\nabla\psi|^2 \frac{Mv_\perp^2}{2} \right\rangle_\psi , \tag{J.25}$$

where I use equation (2.1) to obtain $\nabla\psi \cdot (\hat{\mathbf{b}} \times \hat{\boldsymbol{\zeta}}) = |\nabla\psi|^2/RB$.

Finally, since the collisional piece vanishes to relevant order, I just need to substitute equations (J.4), (J.9) and (J.23) into equation (J.1) and employ (2.16) to find (4.63).

Appendix K

Gyroviscosity in gyrokinetics

In this Appendix, I show why the gyroviscosity must take the form given in equation (4.65), and later I simplify that expression for up-down symmetric tokamaks by proving that the collisional piece must vanish.

K.1 Evaluation of equation (4.65)

To prove equation (4.65), I employ that $\mathbf{v}_{M0} \cdot \nabla\psi = v_{||}\hat{\mathbf{b}} \cdot \overline{\nabla}(Iv_{||}/\Omega_i)$ and $\tilde{\mathbf{v}}_1 \cdot \nabla\psi = -v_{||}\hat{\mathbf{b}} \cdot \overline{\nabla}[\Omega_i^{-1}(\mathbf{v} \times \hat{\mathbf{b}}) \cdot \nabla\psi]$, proven below in (K.2), to write

$$(\mathbf{v}_{M0} + \tilde{\mathbf{v}}_1) \cdot \nabla\psi = \frac{Mc}{Ze} v_{||}\hat{\mathbf{b}} \cdot \overline{\nabla}\left[\frac{Iv_{||}}{B} - \frac{(\mathbf{v} \times \hat{\mathbf{b}}) \cdot \nabla\psi}{B}\right] = \frac{Mc}{Ze} v_{||}\hat{\mathbf{b}} \cdot \overline{\nabla}[R(\mathbf{v} \cdot \hat{\boldsymbol{\zeta}})], \quad \text{(K.1)}$$

where I use equation (2.16) to find the last equality. Notice that in $\tilde{\mathbf{v}}_1 \cdot \nabla\psi = (v_{||}/\Omega_i)(\overline{\nabla} \times \mathbf{v}_\perp) \cdot \nabla\psi = (v_{||}/\Omega_i)\overline{\nabla} \cdot (\mathbf{v}_\perp \times \nabla\psi)$, both $\mathbf{v}_\perp$ and $\nabla\psi$ are perpendicular to $\hat{\mathbf{b}}$. Then, $\mathbf{v}_\perp \times \nabla\psi$ must be parallel to $\hat{\mathbf{b}}$, giving $\mathbf{v}_\perp \times \nabla\psi = -\hat{\mathbf{b}}[(\mathbf{v} \times \hat{\mathbf{b}}) \cdot \nabla\psi]$, and

$$\tilde{\mathbf{v}}_1 \cdot \nabla\psi = -\frac{v_{||}}{\Omega_i}\overline{\nabla} \cdot [\hat{\mathbf{b}}(\mathbf{v} \times \hat{\mathbf{b}}) \cdot \nabla\psi] = -v_{||}\hat{\mathbf{b}} \cdot \overline{\nabla}\left[\frac{(\mathbf{v} \times \hat{\mathbf{b}}) \cdot \nabla\psi}{\Omega_i}\right]. \quad \text{(K.2)}$$

Substituting equation (K.1) into equation (4.65) gives

$$\left\langle \int d^3v\, f_i RM(\mathbf{v} \cdot \hat{\boldsymbol{\zeta}})(\mathbf{v}_{M0} + \tilde{\mathbf{v}}_1) \cdot \nabla\psi \right\rangle_\psi =$$

$$\frac{M^2 c}{2Ze}\left\langle \int d^3v\, f_i v_{||}\hat{\mathbf{b}}\cdot\overline{\nabla}[R^2(\mathbf{v}\cdot\hat{\boldsymbol{\zeta}})^2]\right\rangle_\psi = -\frac{M^2 c}{2Ze}\left\langle \int d^3v\, [R^2(\mathbf{v}\cdot\hat{\boldsymbol{\zeta}})^2] v_{||}\hat{\mathbf{b}}\cdot\nabla f_i\right\rangle_\psi . \tag{K.3}$$

Only the axisymmetric, short radial wavelength pieces of f_i contribute to (K.3) because of the flux surface average. To find this portion of f_i, I employ equations (4.10), (4.11), (4.12), (4.13), (4.14) and (4.15) to obtain the gyrokinetic equation

$$\left.\frac{\partial f_{ig}}{\partial t}\right|_{\mathbf{r},\mathbf{v}} + (v_{||}\hat{\mathbf{b}} + \mathbf{v}_{M0} + \tilde{\mathbf{v}}_1)\cdot\left(\overline{\nabla} f_{ig} - \frac{Ze}{M}\overline{\nabla}\langle\phi\rangle\frac{\partial f_{Mi}}{\partial E_0}\right) + \mathbf{v}_{E0}\cdot\overline{\nabla} f_{ig} = \langle C\{f_i\}\rangle. \tag{K.4}$$

In this equation, I will only keep terms of order $\delta_i f_{Mi} v_i / L$ or larger. These terms will give $R\hat{\boldsymbol{\zeta}}\cdot \overleftrightarrow{\boldsymbol{\pi}}_i^{(0)} \cdot\nabla\psi \sim \delta_i^2 p_i R|\nabla\psi|$ as seen from (K.3).

In the nonlinear term $\mathbf{v}_{E0}\cdot\overline{\nabla} f_{ig}$, different components of $\langle\phi\rangle$ and f_{ig} beat together to give an axisymmetric, short radial wavelength contribution to (K.4). The term $\mathbf{v}_{E0}\cdot\overline{\nabla} f_{ig}$ can be treated as the term $\mathbf{v}_E\cdot\nabla_{\mathbf{R}} f_i$ in the axisymmetric piece of equation (4.1), where it was neglected. In equation (K.4), $\mathbf{v}_{E0}\cdot\overline{\nabla} f_{ig} \sim k_\perp\rho_i\delta_i f_{Mi} v_i/L \ll \delta_i f_{Mi} v_i/L$ is negligible, too. This is another form of the conclusion of section 4.1, i.e., the long wavelength, axisymmetric flows are neoclassical.

Neglecting $\mathbf{v}_{E0}\cdot\overline{\nabla} f_{ig}$ and using $f_{ig} = f_i + (Ze\tilde{\phi}/T_i) f_{Mi}$ in equation (K.4), I find that the axisymmetric, short radial wavelength portion of f_i must satisfy

$$\left.\frac{\partial f_i}{\partial t}\right|_{\mathbf{r},\mathbf{v}} + v_{||}\hat{\mathbf{b}}\cdot\left(\nabla f_i + \frac{Ze}{T_i} f_{Mi}\nabla\phi\right) + \left(\frac{\partial f_i}{\partial\psi} + \frac{Ze}{T_i} f_{Mi}\frac{\partial\phi}{\partial\psi}\right)(\mathbf{v}_{M0} + \tilde{\mathbf{v}}_1)\cdot\nabla\psi = \langle C\{f_i\}\rangle. \tag{K.5}$$

Here, the terms that contain $(\mathbf{v}_{M0} + \tilde{\mathbf{v}}_1)\cdot\overline{\nabla}\tilde{\phi}$ and $(\mathbf{v}_{M0} + \tilde{\mathbf{v}}_1)\cdot\overline{\nabla}(\tilde{\phi} f_{Mi})$ are neglected because they are smaller than the rest of the terms by a factor $k_\perp\rho_i$ since $\tilde{\phi} \sim \delta_i T_e/e \sim k_\perp\rho_i\langle\phi\rangle$. Additionally, I used $\nabla_\perp \simeq \nabla\psi(\partial/\partial\psi)$ since only short radial wavelength effects matter in (K.4). Finally, substituting equation (K.1) into (K.5) gives

$$\left.\frac{\partial f_i}{\partial t}\right|_{\mathbf{r},\mathbf{v}} + v_{||}\hat{\mathbf{b}}\cdot\left(\overline{\nabla} f_i + \frac{Ze}{T_i} f_{Mi}\nabla\phi\right) + \frac{Mc}{Ze}\left(\frac{\partial f_i}{\partial\psi} + \frac{Ze}{T_i} f_{Mi}\frac{\partial\phi}{\partial\psi}\right) v_{||}\hat{\mathbf{b}}\cdot\overline{\nabla}[R(\mathbf{v}\cdot\hat{\boldsymbol{\zeta}})] = \langle C\{f_i\}\rangle. \tag{K.6}$$

I substitute $v_{||}\hat{\mathbf{b}} \cdot \overline{\nabla} f_i$ from equation (K.6) into equation (K.3) to find

$$\left\langle \int d^3v\, f_i RM(\mathbf{v}\cdot\hat{\boldsymbol{\zeta}})(\mathbf{v}_{M0}+\tilde{\mathbf{v}}_1)\cdot\nabla\psi \right\rangle_\psi = \frac{M^2c}{2Ze}\left\langle \int d^3v\,[R^2(\mathbf{v}\cdot\hat{\boldsymbol{\zeta}})^2]\left(\left.\frac{\partial f_i}{\partial t}\right|_{\mathbf{r},\mathbf{v}} - \langle C\{f_i\}\rangle\right)\right\rangle_\psi, \qquad \text{(K.7)}$$

where I use $(Ze/T_i)\hat{\mathbf{b}}\cdot\nabla\phi\int d^3v\, f_{Mi}v_{||}[R^2(\mathbf{v}\cdot\hat{\boldsymbol{\zeta}})^2] = 0$ and

$$\frac{M^3c^2}{2Z^2e^2}\left\langle \int d^3v\left(\frac{\partial f_i}{\partial\psi} + \frac{Ze}{T_i}f_{Mi}\frac{\partial\phi}{\partial\psi}\right)v_{||}\hat{\mathbf{b}}\cdot\overline{\nabla}[R(\mathbf{v}\cdot\hat{\boldsymbol{\zeta}})]R^2(\mathbf{v}\cdot\hat{\boldsymbol{\zeta}})^2\right\rangle_\psi = \frac{M^3c^2}{6Z^2e^2}\left\langle \int d^3v\, v_{||}\hat{\mathbf{b}}\cdot\overline{\nabla}\left[\left(\frac{\partial f_i}{\partial\psi} + \frac{Ze}{T_i}f_{Mi}\frac{\partial\phi}{\partial\psi}\right)R^3(\mathbf{v}\cdot\hat{\boldsymbol{\zeta}})^3\right]\right\rangle_\psi = 0. \qquad \text{(K.8)}$$

To obtain the first equality in equation (K.8), terms that contain $\hat{\mathbf{b}}\cdot\overline{\nabla}(\partial f_i/\partial\psi)$ and $\hat{\mathbf{b}}\cdot\nabla(\partial\phi/\partial\psi)$ are neglected.

Finally, using $k_\perp\rho_i \ll 1$ in (K.7), the replacements $f_i \simeq \overline{f}_i$ and $\langle C\{f_i\}\rangle \simeq \overline{C\{f_i\}}$ lead to (4.65).

K.2 Collisional piece in up-down symmetric tokamaks

To prove that the collisional contribution to equation (4.65) vanishes for up-down symmetric tokamaks I employ the drift kinetic equation (4.1) for the long wavelength, axisymmetric component of $\overline{f}_i$. I already proved that the term $\mathbf{v}_E\cdot\nabla_{\mathbf{R}} f_i$ can be safely neglected. Additionally, I assume that the time derivative is small once the statistical equilibrium is reached, and I split the distribution function into $\overline{f}_i = f_{Mi}(\psi, E_0) + \overline{h}_i$, with $\overline{h}_i(\psi,\theta,E_0,\mu_0,t) \ll f_{Mi}$, giving

$$v_{||}\hat{\mathbf{b}}\cdot\nabla\theta\left[\frac{\partial\overline{h}_i}{\partial\theta} + \frac{\partial}{\partial\theta}\left(\frac{Iv_{||}}{\Omega_i}\frac{\partial f_{Mi}}{\partial\psi}\right)\right] = C^{(\ell)}\{\overline{h}_i\}, \qquad \text{(K.9)}$$

where I use $\mathbf{v}_{M0}\cdot\nabla\psi = v_{||}\hat{\mathbf{b}}\cdot\overline{\nabla}(Iv_{||}/\Omega_i)$. Since the tokamak is up-down symmetric,

$Iv_{||}/\Omega_i$ is symmetric in θ and its derivative is antisymmetric.

In equation (K.9), replacing θ by $-\theta$, $v_{||}$ by $-v_{||}$ and $\overline{h}_i$ by $-\overline{h}_i$ does not change the equation. Then, $\overline{h}_i$ changes sign if both θ and $v_{||}$ do. Due to this property, the collisional integral in (4.65) is given by

$$\left\langle \frac{M}{2B\Omega_i} \int d^3v\, C^{(\ell)}\{\overline{h}_i\} \left(|\nabla\psi|^2 \frac{v_\perp^2}{2} + I^2 v_{||}^2 \right) \right\rangle_\psi = 0. \tag{K.10}$$

In the contributions to this integral, the piece of the distribution function with positive $v_{||}$ in the upper half ($\theta > 0$) of the tokamak cancels the piece of the distribution function with negative $v_{||}$ in the lower half ($\theta < 0$). Similarly, the piece with negative $v_{||}$ in the upper half cancels the piece with positive $v_{||}$ in the lower half.

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