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**Citation:** Malakis, A. et al. "Strong violation of critical phenomena universality: Wang-Landau study of the two-dimensional Blume-Capel model under bond randomness." *Physical Review E* 79.1 (2009): 011125. © 2009 The American Physical Society.

**As Published:** <http://dx.doi.org/10.1103/PhysRevE.79.011125>

**Publisher:** American Physical Society

**Persistent URL:** <http://hdl.handle.net/1721.1/51744>

**Version:** Final published version: final published article, as it appeared in a journal, conference proceedings, or other formally published context

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# Strong violation of critical phenomena universality: Wang-Landau study of the two-dimensional Blume-Capel model under bond randomness

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(Received 24 September 2008; published 27 January 2009)

We study the pure and random-bond versions of the square lattice ferromagnetic Blume-Capel model, in both the first-order and second-order phase transition regimes of the pure model. Phase transition temperatures, thermal and magnetic critical exponents are determined for lattice sizes in the range  $L=20-100$  via a sophisticated two-stage numerical strategy of entropic sampling in dominant energy subspaces, using mainly the Wang-Landau algorithm. The second-order phase transition, emerging under random bonds from the second-order regime of the pure model, has the same values of critical exponents as the two-dimensional Ising universality class, with the effect of the bond disorder on the specific heat being well described by double-logarithmic corrections, our findings thus supporting the marginal irrelevance of quenched bond randomness. On the other hand, the second-order transition, emerging under bond randomness from the first-order regime of the pure model, has a distinctive universality class with  $\nu=1.30(6)$  and  $\beta/\nu=0.128(5)$ . These results amount to a strong violation of universality principle of critical phenomena, since these two second-order transitions, with different sets of critical exponents, are between the same ferromagnetic and paramagnetic phases. Furthermore, the latter of these two sets of results supports an extensive but weak universality, since it has the same magnetic critical exponent (but a different thermal critical exponent) as a wide variety of two-dimensional systems with and without quenched disorder. In the conversion by bond randomness of the first-order transition of the pure system to second order, we detect, by introducing and evaluating connectivity spin densities, a microsegregation that also explains the increase we find in the phase transition temperature under bond randomness.

DOI: [10.1103/PhysRevE.79.011125](https://doi.org/10.1103/PhysRevE.79.011125)

PACS number(s): 75.10.Nr, 05.50.+q, 64.60.Cn, 75.10.Hk

## I. INTRODUCTION: STRONG VIOLATION OF UNIVERSALITY

Universality, according to which the same critical exponents occur in all second-order phase transitions between the same two phases, erstwhile phenomenologically established, has been a leading principle of critical phenomena [1]. The explanation of universality, in terms of diverse Hamiltonian flows to a single fixed point, has been one of the crowning achievements of renormalization-group theory [2]. In rather specialized models in spatial dimension  $d=2$ , such as the eight-vertex [3] and Ashkin-Teller [4] models, the critical exponents nevertheless vary continuously along a line of second-order transitions. We shall refer to these cases as the weak violation of universality. We have established in the current study a much stronger and more general instance of universality violation, under the effect of quenched bond randomness. It has been known that quenched bond randomness may or may not modify the critical exponents of second-order phase transitions, based on the Harris criterion [5,6]. It was more recently established that quenched bond randomness always affects first-order phase transitions by conversion to second-order phase transitions, for infinitesimal randomness in  $d=2$  [7,8], and after a threshold amount of randomness in  $d>2$  [8], as also inferred by general arguments [9]. These predictions [7,8] have been confirmed by Monte Carlo simulations [10,11]. Moreover, renormalization-group calculations [12] on tricritical systems have revealed that not only first-order transitions are converted to second-order transitions, but the latter are con-

trolled by a distinctive strong-coupling fixed point.

In the current Wang-Landau (WL) study yielding essentially exact information on the two-dimensional (2d) Blume-Capel model under quenched bond randomness, we find dramatically different critical behaviors of the second-order phase transitions emerging from the first- and second-order regimes of the pure model. These second-order transitions with the different critical exponents are between the same two phases indicating a strong violation of universality, namely different sets of critical exponents on two segments of the same critical line. Moreover, the effect of quenched bond randomness on the critical temperature is opposite in these two regimes, which we are able to explain in terms of a microsegregation mechanism that we observe. Finally, in proving a general strong violation of universality under quenched bond randomness, our study supports a more delicate and extensive weak universality [13,14]: In the random-bond second-order transition emerging from the pure-system first-order transition, the magnetic (but not thermal) critical exponent appears to be the same as that of the pure 2d Ising model, as has also been seen in other random and nonrandom systems.

The Blume-Capel (BC) model [15,16] is defined by the Hamiltonian

$$H_p = -J \sum_{\langle ij \rangle} s_i s_j + \Delta \sum_i s_i^2, \quad (1)$$

where the spin variables  $s_i$  take on the values  $-1, 0, \text{ or } +1$ ,  $\langle ij \rangle$  indicates summation over all nearest-neighbor pairs of

sites, and the ferromagnetic exchange interaction is taken as  $J=1$ . The model given by Eq. (1), studied here in 2d on a square lattice, will be referred to as the pure model. Our main focus, on the other hand, is the case with bond disorder given by the bimodal distribution

$$P(J_{ij}) = \frac{1}{2}[\delta(J_{ij} - J_1) + \delta(J_{ij} - J_2)],$$

$$\frac{J_1 + J_2}{2} = 1, \quad r = \frac{J_2}{J_1}, \quad (2)$$

so that  $r$  reflects the strength of the bond randomness. The resulting quenched disordered (random-bond) version of the Hamiltonian defined in Eq. (1) reads now as

$$H = - \sum_{\langle ij \rangle} J_{ij} s_i s_j + \Delta \sum_i s_i^2. \quad (3)$$

## II. TWO-STAGE ENTROPIC SAMPLING

We briefly describe our numerical approach used to estimate the properties of a large number, 100, of bond disorder realizations, for lattice sizes  $L=20-100$ . The pure-system properties are also obtained, for reference and contrast. We have used a two-stage strategy of a restricted entropic sampling, which is described in our recent study of random-bond Ising spin models in 2d [17], very similar to the one applied also in our numerical approach to the three-dimensional (3d) random-field Ising model [18]. In these papers, we have presented in detail the various sophisticated routes used for the restriction of the energy subspace and the implementation of the WL algorithm [19,20]. The identification of the appropriate energy subspace ( $E_1, E_2$ ) for the entropic sampling of each random-bond realization is carried out by applying our critical minimum energy subspace (CrMES) restriction [21,22] and taking the union subspace at both pseudocritical temperatures of the specific heat and susceptibility. This subspace, extended by 10% from both low- and high-energy sides, is sufficient for an accurate estimation of all finite-size anomalies. Following Ref. [17], the identification of the appropriate energy subspace is carried out in the first multi-range (multi-R) WL stage in a wide energy subspace. The WL refinement levels [ $G(E) \rightarrow fG(E)$ , where  $G(E)$  is the density of states (DOS); for more details see Ref. [17]] used in this stage ( $j=1, \dots, j_i; f_{j+1} = \sqrt{f_j}$ ) were  $j_i=18$  for  $L < 80$  and  $j_i=19$  for  $L \geq 80$ . The same process was repeated several times, typically  $\sim 5$  times, in the newly identified restricted energy subspace. From our experience, this repeated application of the first multi-R WL approach greatly improves accuracy and then the resulting accurate DOS is used for a final redefinition of the restricted subspace, in which the final entropic scheme (second stage) is applied. In this stage, the refinement WL levels  $j=j_i, \dots, j_i+4$  are used in a one-range (one-R) or in a multi-R fashion. For the present model, both approaches were tested and found to be sufficiently accurate, provided that the multi-R uses adequately large energy sub-intervals. This fact will be illustrated in the following section, by presenting the rather sensitive double-peak (DP)

structure of the energy probability density function (PDF) in the first-order regime of the model. Noteworthy, that most of our simulations of the 2d BC model at the second-order regime ( $\Delta=1$ ) were carried out by using in the final stage a one-R approach, in which the WL modification factor was adjusted according to the rule  $\ln f \sim r^{-1}$  proposed recently by Belardinelli and Pereyra [23]. Our comparative tests showed that this alternative approach yields results in agreement with the one-R WL approach.

Let us close this brief outline of our numerical scheme with some appropriate comments concerning statistical errors and disorder averaging. Even for the larger lattice size studied here ( $L=100$ ), and depending on the thermodynamic parameter, the statistical errors of the WL method were found to be of reasonable magnitude and in some cases to be of the order of the symbol sizes, or even smaller. This is true for both the pure version and the individual random-bond realizations. These WL errors have been used for the pure system in our finite-size scaling (FSS) illustrations and fitting attempts. For the disordered version only the averages over the disorder realizations, denoted as  $[\dots]_{av}$ , have been used in the text and their finite-size anomalies, denoted as  $[\dots]_{av}^*$ , have been used in our FSS attempts. Due to very large sample-to-sample fluctuations, mean values of individual maxima ( $[\dots]_{av}^*$ ) have not been used in this study. However, for the finite-size anomalies of the disordered cases, the relevant statistical errors are due to the finite number of simulated realizations. These errors were estimated empirically, from runs of 20 realizations via a jackknife method, and used in the corresponding FSS fitting attempts. These disorder-sampling errors may vary, depending again on the thermodynamic parameter, but nevertheless were also found to be of the order of the symbol sizes. For the case  $\Delta=1$ , these are hardly observable, as illustrated in the corresponding graphs.

## III. PHASE TRANSITIONS OF THE PURE 2d BC MODEL

### A. First-order transition of the pure model

The value of the crystal field at the tricritical point of the pure 2d BC model has been accurately estimated to be  $\Delta_c = 1.965(5)$  [24–28]. Therefore, we now consider the value  $\Delta=1.975$ , for which the pure model undergoes a first-order transition between the ferromagnetic and paramagnetic phases, and carry out a detailed FSS analysis of the pure model. Our first attempt to elucidate the first-order features of the present model will closely follow previous analogous studies carried out on the  $q=5, 8, 10$  Potts model [29–33] and also our study of the triangular Ising model with competing nearest- and next-nearest-neighbor antiferromagnetic interactions [34]. As it is well known from the existing theories of first-order transitions, all finite-size contributions enter in the scaling equations in powers of the system size  $L^d$  [35]. This holds for the general shift behavior (for various pseudotransition temperatures) and also for the FSS behavior of the peaks of various energy cumulants and of the magnetic susceptibility. It is also well known that the DP structure of the energy PDF,  $P(e)$ , where  $e=H/L^2$ , is signaling the emergence of the expected two  $\delta$ -peak behavior in the thermody-

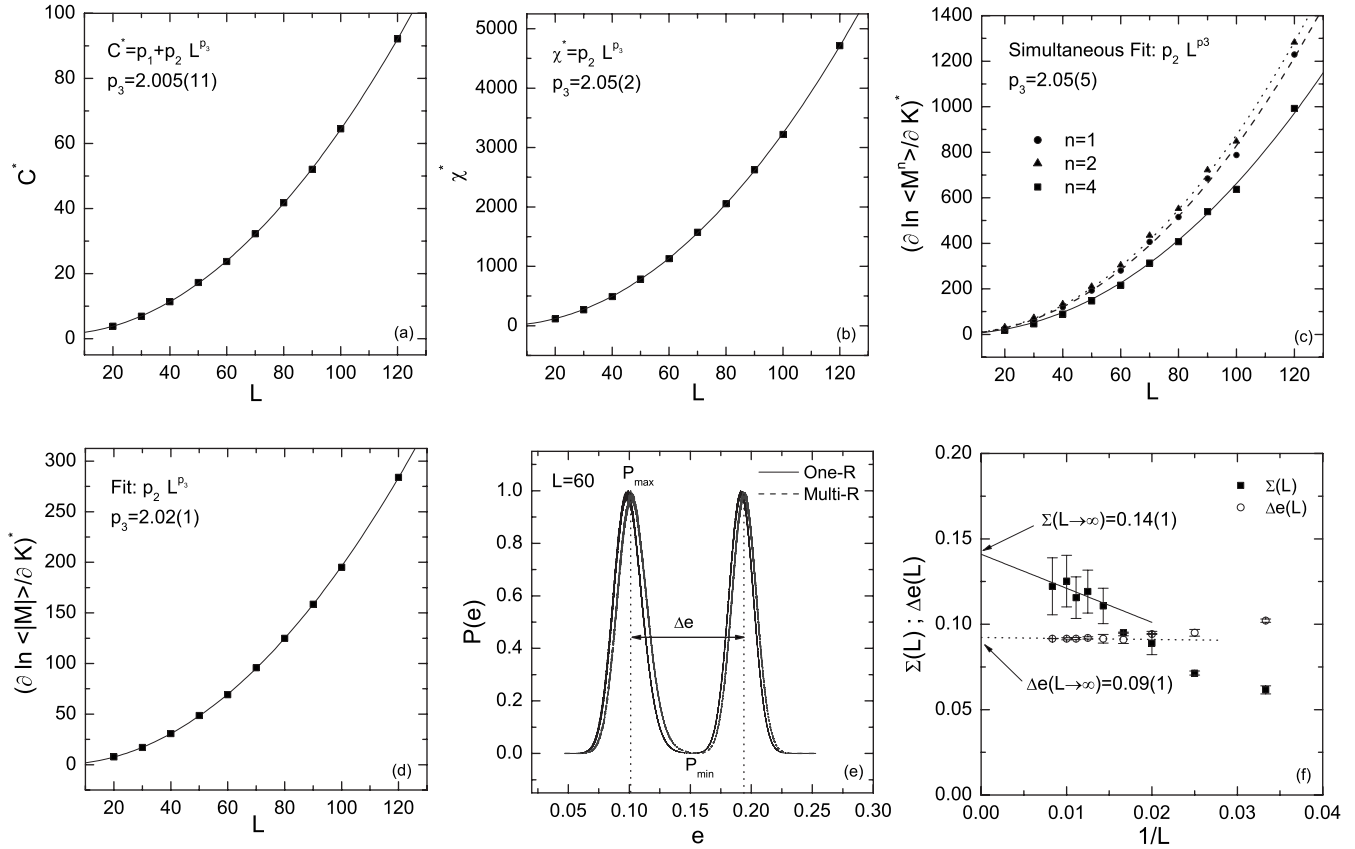


FIG. 1. Behavior of the pure 2d BC model at  $\Delta=1.975$ : (a) FSS behavior of the specific heat peaks giving a clear  $L^2$  divergence. (b) The same for the susceptibility maxima. (c) Simultaneous fitting of the maxima of the averaged logarithmic derivatives of the order parameter defined in Eq. (4). (d) Power-law behavior of the averaged absolute order-parameter derivative in Eq. (5). (e) The DP structure of the energy PDF at  $T=T_h$  via the two different implementations of the WL scheme. The multi-R implementation is displaced very slightly to the right. (f) Limiting behavior for the surface tension  $\Sigma(L)$  defined in the text and the latent heat  $\Delta e(L)$  shown in panel (e). The linear fits shown include only the five points corresponding to the larger lattice sizes.

namic limit, for a genuine first-order phase transition [36,37], and with increasing lattice size the barrier between the two peaks should steadily increase. According to the arguments of Lee and Kosterlitz [30,31] the quantity  $\Delta F(L)/L = [k_B T \ln(P_{\max}/P_{\min})]/L$ , where  $P_{\max}$  and  $P_{\min}$  are the maximum and minimum energy PDF values at the temperature  $T_h$  where the two peaks are of equal height, should tend to a nonzero value. Similarly to the above, the logarithmic derivatives of the powers of the order parameter with respect to the inverse temperature  $K=1/T$ ,

$$\frac{\partial \ln \langle M^n \rangle}{\partial K} = \frac{\langle M^n H \rangle}{\langle M^n \rangle} - \langle H \rangle, \quad (4)$$

and the average absolute order-parameter derivative,

$$\frac{\partial \langle |M| \rangle}{\partial K} = \langle |M| H \rangle - \langle |M| \rangle \langle H \rangle, \quad (5)$$

have maxima that scale as  $L^d$  with the system size in the FSS analysis of a first-order transition. In the case of a second-order transition, the quantities in Eqs. (4) and (5), respectively, scale as  $L^{1/\nu}$  and  $L^{(1-\beta)/\nu}$  [10,11,38], to be used further below.

Figures 1(a) and 1(b) illustrate that the traditionally used

divergences in FSS of the specific heat  $C$  and susceptibility  $\chi$  follow very well a power law of the form  $\sim L^d$ , as expected for first-order transitions [36,37]. Furthermore, Figs. 1(c) and 1(d) demonstrate that the divergences corresponding to the first-, second-, and fourth-order logarithmic derivatives of the order parameter defined in Eq. (4) and the absolute order-parameter derivative defined in Eq. (5) follow also very well the same  $L^d$  behavior, as expected. Figure 1(e) shows the pronounced DP structure of the energy PDF of the model at  $T=T_h$  for  $L=60$ , obtained by the two different implementations of the WL scheme. This graph illustrates that the DP structure is not very sensitive to the multi-R WL process, in contrast to our recent findings for some first-order-like behavior of the 3d random-field Ising model. It also illustrates the accuracy of the implementation schemes. As mentioned above, from these DP energy PDF's one can estimate the surface tension  $\Sigma(L) = \Delta F(L)/L$  and the latent heat  $\Delta e(L)$ , whose values remain finite for a genuine first-order transition. Figure 1(f) shows the limiting behavior of these two quantities and verifies the persistence of the first-order character of the transition at  $\Delta=1.975$ . The limiting values of  $\Sigma(L)$  and  $\Delta e(L)$  are given in the graph by extrapolating at the larger lattice sizes studied. We close this section by noting that the transition temperature  $T^*(\Delta=1.975)$  is estimated to

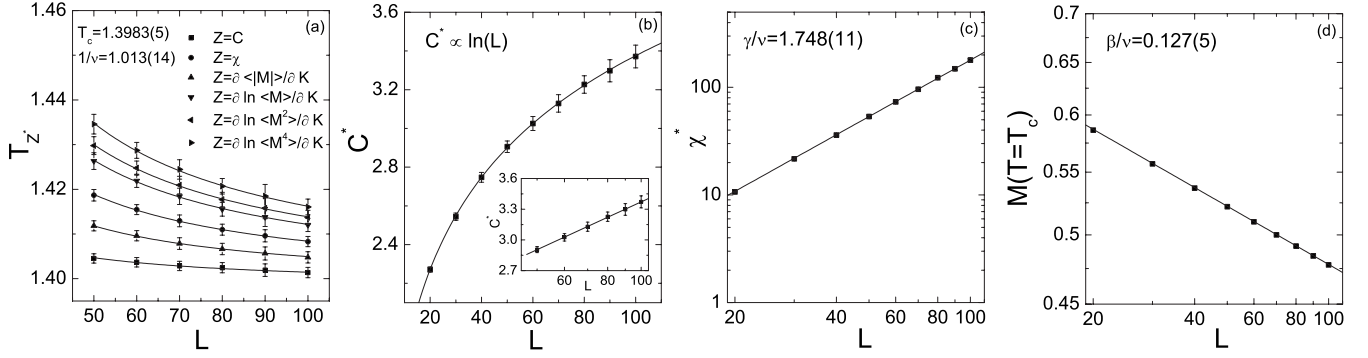


FIG. 2. Behavior of the pure 2d BC model at  $\Delta=1$ : (a) Simultaneous fitting of six pseudocritical temperatures defined in the text for  $L \geq 50$ . (b) FSS of the specific heat peaks. The inset shows a linear fit of the specific heat data on a log scale for  $L \geq 50$ . (c) FSS of the susceptibility peaks on a log-log scale. (d) FSS of the order parameter at the estimated critical temperature also on a log-log scale. Linear fits are applied in panels (c) and (d).

be, in the limit  $L \rightarrow \infty$ ,  $T^* = 0.574(2)$ . This value interpolates and agrees with the general phase diagram points summarized in Ref. [28].

### B. Second-order transition of the pure model

The 2d BC model with no quenched randomness, Eq. (1), at the crystal field value  $\Delta=1$ , undergoes a second-order transition between the ferromagnetic and paramagnetic phases, expected to be in the universality class of the simple 2d Ising model [26]. In the following, we present the FSS analysis of our numerical data for this case, to verify this expectation and to set any contrast with the behavior under quenched randomness, presented further below. Figure 2(a) gives the shift behavior of the pseudocritical temperatures corresponding to the peaks of the following six quantities: Specific heat, magnetic susceptibility, derivative of the absolute order parameter, and logarithmic derivatives of the first, second, and fourth powers of the order parameter. Fitting our data for the larger lattice sizes ( $L=50-100$ ) to the expected power-law behavior  $T = T_c + bL^{-1/\nu}$ , we find that the critical temperature is  $T_c = 1.3983(5)$  and the shift exponent is  $1/\nu = 1.013(14)$ . Almost the same estimate for the critical temperature is obtained when we fix the shift exponent to the value  $1/\nu=1$ . Thus, the shift behavior of the pseudocritical temperatures indicates that the pure 2d BC model with  $\Delta=1$  shares the same correlation length exponent  $\nu$  with the 2d Ising model. Figure 2(b) gives the FSS of the specific heat peaks. Here, the expected logarithmic divergence of the specific heat is very well obtained even from the smaller lattice sizes as shown in the main frame. The inset is a linear fit of the specific heat data on a log scale for  $L \geq 50$ . Finally, Figs. 2(c) and 2(d) present our estimations for the magnetic exponent ratios  $\gamma/\nu$  and  $\beta/\nu$ . In panel (c) we show the FSS behavior of the susceptibility peaks on a log-log scale. The straight line is a linear fit for  $L \geq 50$  giving the estimate  $\gamma/\nu = 1.748(11)$ . For the estimation of  $\beta/\nu$  we use the values of the order parameter at the estimated critical temperature ( $T_c = 1.3983$ ). As shown in panel (d), on a log-log scale, the linear fit provides the estimate  $\beta/\nu = 0.127(5)$ . Thus, our results for the pure 2d BC model at  $\Delta=1$  are in full agreement with the findings of Beale [26] and with universality argu-

ments that place the pure BC model in the Ising universality class.

## IV. PHASE TRANSITIONS OF THE RANDOM-BOND 2d BC MODEL

### A. Second-order transition emerging under random bonds from the first-order transition of the pure model

Figure 3 illustrates the effects, at  $\Delta=1.975$ , induced by bond randomness for different disorder realizations, on the DP structure for lattices with linear size  $L=30$  [Figs. 3(a)–3(c)] and  $L=50$  [Fig. 3(d)]. It is immediately seen that the introduction of bond disorder has a dramatic influence on the DP structure of the energy PDF. The very rough energy PDF of the pure model, with the huge oscillations observed in relatively small lattices, is radically smoothed by the introduction of disorder [Figs. 3(a) and 3(b)] and the energy barrier is highly reduced as the disorder strength is increased. This barrier reduction effect depends of course on the disorder realization, as can be easily observed by comparing Figs. 3(a)–3(c), but its main dependence comes from the value of the disorder strength  $r$  and already for  $r=17/23 \approx 0.74$ , the DP structure is completely eliminated. This is clarified by showing the five realizations of size  $L=50$  in Fig. 3(d). Note here that, for this value of the disorder strength  $r=17/23 \approx 0.74$ , only a very small portion ( $<8\%$ ) of the realizations at the size  $L=30$  shows a DP structure in the energy PDF, but now with a very tiny energy barrier, whereas for the same disorder strength, all realizations at  $L=50$  have a single peak energy PDF. We continue our illustrations of the disorder effects by showing in Figs. 3(e) and 3(f) the energy  $\langle e \rangle - T$  behavior and the order-parameter  $\langle M \rangle - T$  behavior, for the disorder realization with seed 99. It is clear from these figures that, in effect, only the disorder strength  $r=0.9$  resembles a first-order behavior, whereas all other disorder strengths resemble the usual second-order behavior. In other words, only for very weak disorder strengths the finite-size rounded anomaly resembles a discontinuity in the energy and the order parameter.

From the order-parameter behavior shown in Fig. 3(f) it should be observed that the low-temperature behavior, for

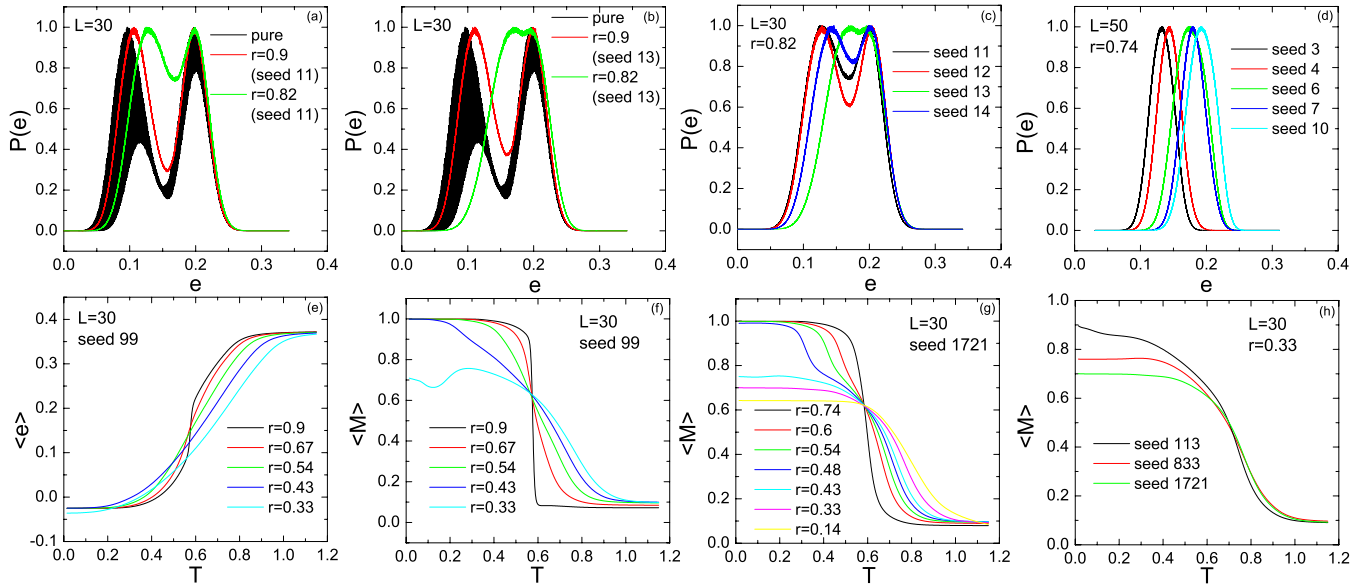


FIG. 3. (Color online) Behavior of the random-bond 2d BC model at  $\Delta=1.975$ : (a)–(d) Softening effects on the first-order transition features of the 2d BC model, induced by bond randomness of various strengths. In (a) and (b), the very rough  $P(e)$  is for the pure model, whereas the smoothed curves are for  $r=0.90$  (deeper) and  $r=0.82$  (shallower). The curves in (c) and (d) are for different seeds. Panels (e)–(h) illustrate the  $\langle e \rangle$ - $T$  behavior and  $\langle M \rangle$ - $T$  behavior, for the size  $L=30$  and various disorder strengths  $r$  for different disorder realizations. The curves in (e) and (f) are for  $r=0.90, 0.67, 0.54, 0.43, 0.33$ , top to bottom on the right of (e), bottom to top on the right of (f), and top to bottom on the left of (f). The curves in (g) are for  $r=0.74, 0.60, 0.54, 0.48, 0.43, 0.33, 0.14$  top to bottom on the left and bottom to top on the right. The curves in (h) are for different seeds.

strong disorder strengths, shows an unexpected and rather complex behavior, which is most prominent for  $r=1/3=0.33 \dots$ . Figure 3(g) further clarifies this low-temperature effect, by presenting one more disorder realization (seed 1721) for several disorder strengths ranging from  $r=1/7 \approx 0.14$  to  $r=17/23 \approx 0.74$ . It is clear from this figure that for these strong disorder strengths the ground state of the model deviates appreciably from the all  $s_i=+1$  or the all  $s_i=-1$  ferromagnetic state. Apparently, this deviation strongly depends on the disorder strength as shown in Fig. 3(h), where the low-temperature behavior of the order parameter is presented for three different disorder realizations. From this illustration it appears that, in the strong disorder regime, the  $T \rightarrow 0$  value of the order parameter averaged over the disorder will depend on the disorder strength. This observation will have direct relevance to the ferromagnetism enhancement (from quenched bond disorder!) and to the conversion of first-order transitions to second-order transitions, through the microsegregation mechanism to be presented and quantified further below. In fact, we have fully verified the above observation for a small  $4 \times 4$  square lattice, for which the exact enumeration of the spin configurations ( $3^{16}$ ) is feasible, using 50 disorder realizations.

It is evident from the DP structures of Fig. 3 that one should avoid working with values of  $J_2$  very close to  $J_2=1$  (pure model), since the first-order characteristics of the pure model may be very strong and finite-size effects will obscure any FSS attempt in relatively small lattices. We therefore carried out extensive simulations at  $r=17/23 \approx 0.74$  and  $r=3/5=0.6$ . Figure 4(a) contrasts the specific heat results for the pure 2d BC model and both disordered cases,  $r=17/23 \approx 0.74$  and  $r=3/5=0.6$ . This figure illustrates that the satu-

ration of the specific heat is very clear in both cases of the disorder strength. However, the presented specific heat behavior for both disorder strengths is unsuitable for any FSS attempt to estimate the exponent ratio  $\alpha/\nu$ , as a result of the early saturation of the specific heat. However, the early saturation of the specific heat definitely signals the conversion of the first-order transition to a second-order transition with a negative critical exponent  $\alpha$ . Furthermore, using our numerical data, we attempted to estimate a complete set of critical exponents for both values of the disorder strength considered here. For  $r=17/23 \approx 0.74$ , our FSS attempts indicated that we are still in a crossover regime for the lattice sizes studied. On the other hand, for the disorder strength  $r=3/5=0.6$ , the FSS attempts, using the larger lattice sizes studied ( $L=50-100$ ), provided an interesting and reliable set of estimates for the critical exponents, which seems to satisfy all expected scaling relations. Figure 4(b) gives the behaviors of five pseudocritical temperatures  $T_{[Z]_{av}}^*$  corresponding to the peaks of the following quantities averaged over the disorder realizations: susceptibility, derivative of the absolute order parameter, as defined in Eq. (5), and first-, second-, and fourth-order logarithmic derivatives of the order parameter, as defined in Eq. (4). The five lines shown are obtained via a simultaneous fitting of the form  $T_{[Z]_{av}}^* = T_c + bL^{-1/\nu}$  for the larger lattice sizes  $L \geq 50$ . The overall shift behavior is very convincing of the accuracy of our numerical method. This accuracy is due to the fact that for each realization, the WL random walk has been repeated in the first stage of the entropic scheme 5 times, thus reducing significantly the statistical errors, which are then further refined in the second stage of the entropic process. Furthermore, since these points are derived from the peaks of the averaged curves and not from

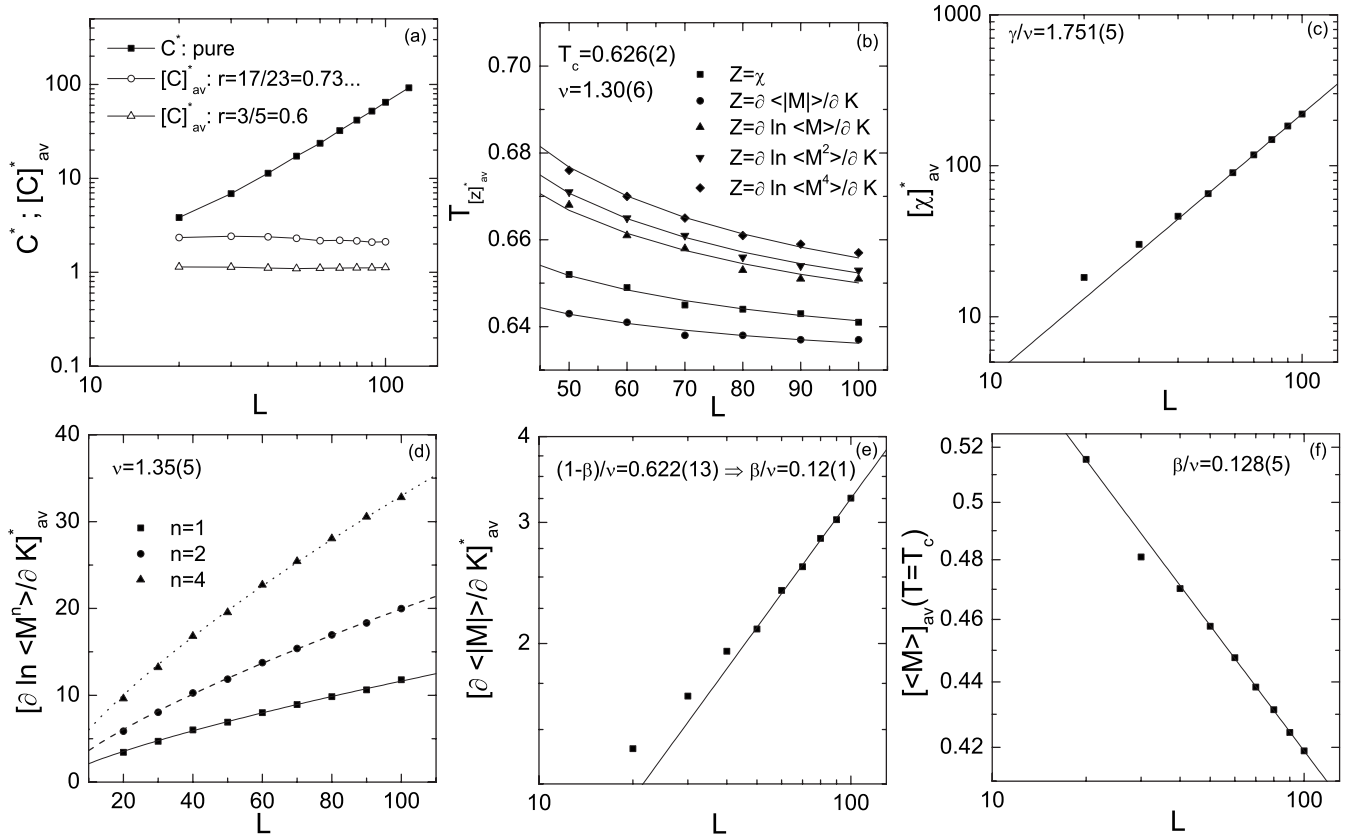


FIG. 4. Behavior of the random-bond 2d BC model at  $\Delta=1.975$ : (a) Illustration of the clear saturation of the specific heat ( $[C]_{av}^*$ ) for the random-bond (open symbols) 2d BC model. (b) FSS behavior of five pseudocritical temperatures defined in the text for  $r=3/5=0.6$ . The lines show a simultaneous fit for the larger lattice sizes  $L \geq 50$ . (c)–(f) Estimation of critical exponents  $\nu$ ,  $\gamma/\nu$ , and  $\beta/\nu$  for the case of  $r=3/5=0.6$ . In all panels the fits have been performed for the larger lattice sizes shown ( $L \geq 50$ ).

the individual maxima of the realizations, they do not suffer from sample-to-sample fluctuations and large statistical errors. This good behavior allows us to estimate, as shown in Fig. 4(b), quite accurately both the critical temperature  $T_c=0.626(2)$  and the correlation length exponent  $\nu=1.30(6)$ . Regarding the latter, we shall see below that it agrees with the estimate obtained via the FSS of the logarithmic derivatives of Eq. (4) and this will be a very strong indication of the self-consistency of our scheme.

Figures 4(c)–4(f) give the FSS behavior of the first-, second-, and fourth-order logarithmic derivatives of the order parameter defined in Eq. (4), the magnetic susceptibility, the absolute order-parameter derivative defined in Eq. (5), and the order parameter at the critical temperature estimated in Fig. 4(b). Figure 4(d) shows a simultaneous fit for three moments and for lattice sizes  $L \geq 50$ . The resulting value for the exponent  $\nu=1.35(5)$  indeed agrees with our earlier estimate from the shift behavior in Fig. 4(b) and also fulfills the Chayes *et al.* inequality  $\nu \geq 2/d$  [39]. Figure 4(c) presents the behavior of the peaks of the average susceptibility on a log-log scale and the solid line shows a linear fit for sizes  $L \geq 50$ . The estimated value for the exponent ratio  $\gamma/\nu$  shown in this panel is very close to 1.75 and it is well known that this value of the ratio  $\gamma/\nu$  is obeyed not only in the simple Ising model but also in several other cases in 2d. In particular, it appears that it is very well obeyed in the cases of disordered models, including the site-diluted, bond-

diluted, and random-bond Ising model [17,40–55]. Furthermore, it has been shown that is also very well obeyed in both the pure and random-bond version of the square Ising model with nearest- and next-nearest-neighbor competing interactions [17,56], as well as the case of the second-order phase transition induced by bond disorder from the first-order behavior in the  $q=8$  Potts model [10,11]. Figure 4(e) is a first estimate for the exponent ratio  $\beta/\nu=0.12(1)$  obtained from the FSS behavior of the maxima of the average absolute order-parameter derivative [Eq. (5)] with the solid line shown being a linear fit, again for  $L \geq 50$ . Finally, Fig. 4(f) shows the conventional FSS method of estimating the ratio  $\beta/\nu$  by considering the scaling behavior of the average order parameter at the estimated critical temperature  $T_c=0.626$ . The solid line is a linear fit for  $L \geq 50$  giving the value  $\beta/\nu=0.128(5)$ . These latter two estimates are very close to the value  $\beta/\nu=0.125$  and combining the above results one finds that the random-bond version of the model appears to satisfy the scaling relation  $2\beta/\nu + \gamma/\nu = d$ . Thus, a kind of weak universality appears [47,57].

### B. Second-order transition emerging under random bonds from the second-order transition of the pure model

We now present our numerical results for the random-bond 2d BC model with  $\Delta=1$  for disorder strength  $r=0.6$ . Bond randomness favoring second-order transitions, this sys-

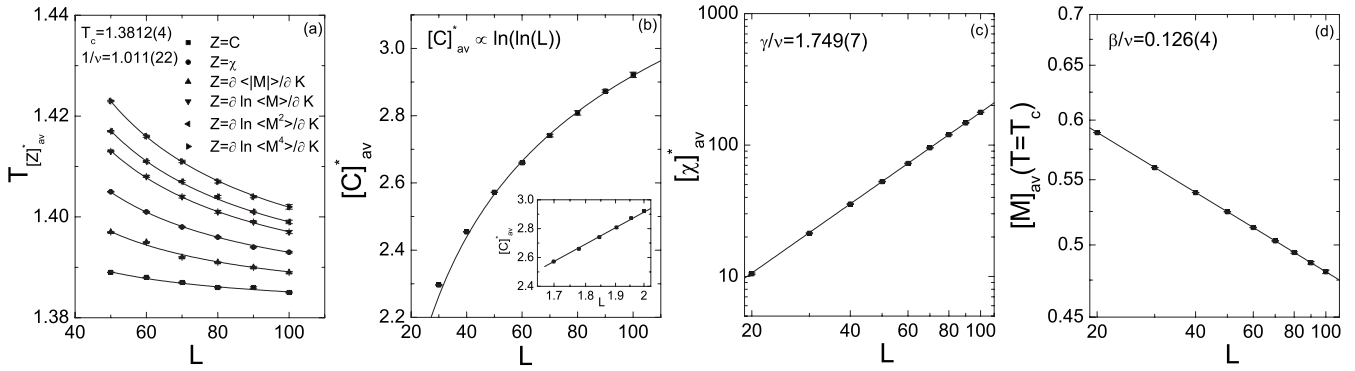


FIG. 5. Behavior of the random-bond ( $r=0.6$ ) 2d BC model at  $\Delta=1$ : (a) Simultaneous fitting of six pseudocritical temperatures defined in the text for  $L \geq 50$ . (b) FSS of the averaged specific heat peaks. A double-logarithmic fit is applied for  $L \geq 50$ . The inset shows a linear fit on a double-logarithmic scale. (c) FSS behavior of the averaged susceptibility peaks on a log-log scale. (d) FSS of the averaged order parameter at the estimated critical temperature, also on a log-log scale. Linear fits are applied for  $L \geq 50$ .

tem is also expected to undergo a second-order transition between the ferromagnetic and paramagnetic phases and it is reasonable to expect that this transition will be in the same universality class as the random-bond 2d Ising model. As far as we know, there has not been any previous attempt to compare the behaviors of the random-bond BC model and the random-bond Ising model. The latter model is a particular case of the more general random Ising model (random-site, random-bond, and bond-diluted) and has been extensively investigated and debated [17,40–55]. Using renormalization-group and conformal field theories, the marginal irrelevance of randomness at the second-order ferromagnetic-paramagnetic transition has been predicted. According to these theories, the effect of infinitesimal disorder gives rise only to logarithmic corrections as the critical exponents maintain their 2d Ising values. On the other hand, there is not full agreement in the literature for the finite disorder regime. Two existing scenarios are mutually exclusive: The first view predicts that the specific heat slowly diverges with a double-logarithmic dependence, with a corresponding correlation-length exponent  $\nu=1$  [42–45]. Another scenario predicts a negative specific heat exponent  $\alpha$  leading to a saturating behavior [47], with a corresponding correlation length exponent  $\nu \geq 2/d$ . Let us now present the FSS analysis of our numerical data.

Figure 5(a) presents again the shift behavior of six pseudocritical temperatures, as defined above for the pure model, but here using the peaks of the corresponding quantities averaged over the disorder realizations. Fitting our data for the larger lattice sizes ( $L=50-100$ ) to the expected power-law behavior,  $T=T_c+bL^{-1/\nu}$ , we find that the critical temperature is  $T_c=1.3812(4)$  and the shift exponent is  $1/\nu=1.011(22)$ . This latter estimate is a first indication that the random-bond 2d BC at  $\Delta=1$  has the same value of the correlation-length critical exponent as the pure version and therefore as the 2d Ising model. Figure 5(b) illustrates the FSS of the specific heat maxima averaged over disorder,  $[C]_{av}^*$ . Using these data for the larger sizes  $L \geq 50$ , we tried to observe the goodness of the fits, assuming a simple logarithmic divergence, a double-logarithmic divergence, or a simple power law. Although there is no irrefutable way of numerically distinguishing between the above scenarios, our fitting attempts

indicated that the double-logarithmic scenario applies better to our data. The double-logarithmic fit is shown in the main panel and also in the inset of Fig. 5(b). Finally, Figs. 5(c) and 5(d) present our estimations for the magnetic exponent ratios  $\gamma/\nu$  and  $\beta/\nu$ . In panel (c) we show the FSS behavior of the susceptibility peaks on a log-log scale. The straight line is a linear fit for  $L \geq 50$  giving the estimate 1.749(7) for  $\gamma/\nu$ . For the estimation of  $\beta/\nu$  we have used the values of the order parameter at the estimated critical temperature  $T_c=1.3812$ . This traditional method, shown in panel (d) on a log-log scale, provides now the estimate  $\beta/\nu=0.126(4)$ . From the above findings, we conclude that, at this finite disorder strength, the random-bond 2d BC model with  $\Delta=1$  belongs to the same universality class as the random Ising model, extending the theoretical arguments based on the marginal irrelevance of infinitesimal randomness. Most strikingly, it is undisputable from our numerical results that the second-order phase transitions emerging, under random bonds, from the first-order and second-order regimes of the pure model, have different critical exponents although they are between the same two phases, thereby exhibiting a strong violation of universality. We note that, since our bond disorder occurs as the variation of the bond strengths that all are in any case nonzero, no Griffiths line [58] divides the paramagnetic phase here.

Finally, we discuss self-averaging properties along the two segments (ex-first-order, Sec. IV A, and still second-order, Sec. IV B) of the critical line. A useful finite-size measure that characterizes the self-averaging property of a system is the relative variance  $R_X=V_X/[X]_{av}^2$ , where  $V_X=[X^2]_{av}-[X]_{av}^2$ , of any singular extensive thermodynamic property  $X$ . A system exhibits self-averaging when  $R_X \rightarrow 0$  as  $L \rightarrow \infty$ , or lack of self-averaging (with broad probability distributions) when  $R_X \rightarrow \text{const} \neq 0$  as  $L \rightarrow \infty$ . The FSS scenario of Aharony and Harris [59] describes self-averaging properties for disordered systems and has been validated by Wiseman and Domany [60–62] in their study of random Ising and Ashkin-Teller models. From these papers, the disordered system resulting in Sec. IV B from the introduction of bond randomness to the marginal case of the second-order transition of the pure 2d BC model is expected to exhibit lack of self-averaging. This expectation also agrees with our recent study



of the self-averaging properties of the 2d random-bond Ising model [63]. In the current work, the FSS behaviors of the relative variances, obtained from the distributions of the magnetic susceptibility maxima ( $X=\chi^*$ ), were observed. Their behavior clearly indicates that the disordered systems exhibit lack of self-averaging along both segments of the critical line, since these relative variances show a monotonic behavior and are still increasing at the maximum lattice sizes studied. For  $\Delta=1$ ,  $R_{\chi^*}(L=90)\approx 0.0011$  and  $R_{\chi^*}(L=100)\approx 0.0014$ , whereas for  $\Delta=1.975$ ,  $R_{\chi^*}(L=90)\approx 0.015$  and  $R_{\chi^*}(L=100)\approx 0.016$ . Thus, the latter case, i.e., the ex-first-order segment, gives a much larger effect, by a factor of  $\sim 12$ . A similarly stronger lack of self-averaging was observed for the disordered system resulting from the case of competing interactions on the square lattice, than for the disordered system resulting from the marginal case of the simple Ising model, in our recent study [63]. Moreover, the discussion in Sec. VII A in the paper by Fisher [64] is relevant here, explaining the expectation for extremely broad distributions near ex-first-order transitions in systems with quenched randomness. This discussion points out also that ex-first-order transitions may have several  $\nu$  exponents [39,64–66] and provides the background for understanding why our finite-size correlation-length exponent obeys the Chayes *et al.* inequality [39].

### C. Contrasting random-bond behavior of critical temperatures, connectivity spin densities, and microsegregation

In most spin models, the introduction of bond randomness is expected to decrease the phase-transition temperature and in several cases the critical temperature goes to zero at the percolation limit of randomness ( $r=0$  and  $J_2=0$ ). For less randomness, only a slight decrease is expected, if the average bond strength is maintained. Indeed, in the second-order regime of the pure 2d BC model,  $\Delta=1$ , the introduction of bond randomness has slightly decreased the critical temperature, by 1% (Sec. IV B). On the other hand, in sharp contrast, for the same disorder strength  $r=0.6$  applied to the first-order regime of the pure model (Sec. IV A), at  $\Delta=1.975$ , we find a considerable increase of the critical temperature, by 9%.

In order to microscopically explain the above observation of ferromagnetic order enhanced by quenched disorder, let us define the following connectivity spin densities,  $Q_n=\langle s_i^2 \rangle_n$ , where the subscript  $n$  denotes the class of lattice sites and is the number of the quenched strong couplings ( $J_1$ ) connecting to each site in this class. Figure 6 illustrates the temperature behavior of these densities averaged over 10 disorder realizations for a lattice of linear size  $L=60$ . For  $\Delta=1.975$ , it is seen that the  $s_i=0$  preferentially occur on the low strong-coupling connectivity sites. The  $s_i=\pm 1$  states preferentially occur with strong-coupling connectivity, which (1) naturally leads to a higher transition temperature and (2) effectively carries the ordering system to higher nonzero spin densities, the domain of second-order phase transitions. Figure 6 constitutes a microsegregation, due to quenched bond randomness, of the  $s_i=\pm 1$  states and of the  $s_i=0$  state. We note that microsegregation is reached by a continuous evolution within the ferromagnetic and paramagnetic phases. A similar

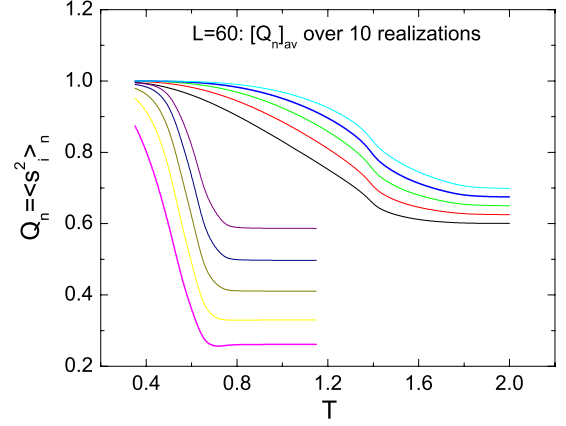


FIG. 6. (Color online) Temperature behavior of the connectivity spin densities  $Q_n$  defined in the text for  $\Delta=1$  (upper curves) and  $\Delta=1.975$  (lower curves) for disorder strength  $r=0.6$  and lattice size  $L=60$ . In each group, the curves are for  $n=0, 1, 2, 3, 4$  from bottom to top.

mechanism has been seen in the low-temperature second-order transition between different ordered phases under quenched randomness [67].

On the other hand, for  $\Delta=1$  and in the neighborhood of the critical temperature, the difference between the smallest and the largest of the connectivity densities is 0.177, whereas for  $\Delta=1.975$ , this difference is 0.449 in the corresponding critical region. Thus, the microsegregation does not occur in the regime of the second-order transition of the pure model and the effect of quenched disorder is the expected slight (the average bond strength is maintained) retrenchment of ferromagnetic order. Microsegregation does occur in the first-order regime of the pure model where macrosegregation occurs in the absence of bond randomness. The result is a local concentration of  $s_i=\pm 1$  states, leading to the enhancement of ferromagnetism. The above-mentioned spreads in the connectivity-density values are very slowly changing with the lattice size. For instance, for  $L=20$  the corresponding values are, respectively, 0.175 and 0.445.

## V. CONCLUSIONS: STRONG VIOLATION OF UNIVERSALITY AND WEAK UNIVERSALITY

In conclusion, the second-order phase transition of the 2d random-bond Blume-Capel model at  $\Delta=1$  appears to belong to the same universality class as the 2d Ising model, having the same values of the critical exponents, i.e., the critical exponents of the 2d Ising universality class. The effect of the bond disorder on the specific heat is well described by the double-logarithmic scenario and our findings support the marginal irrelevance of quenched bond randomness. On the other hand, at  $\Delta=1.975$ , the first-order transition of the pure model is converted to second order, but in a distinctive universality class with  $\nu=1.30(6)$  and  $\beta/\nu=0.128(5)$ .

These results, on the 2d square lattice, amount to a strong violation of universality, since the two second-order transitions mentioned in the preceding paragraph, with different sets of critical exponents, are between the same ferromag-

netic and paramagnetic phases. This result was also obtained by renormalization-group calculations [12] in 2d and 3d that are exact on hierarchical lattices and approximate on square and cubic lattices. The mechanism in these renormalization-group calculations is that the second-order transitions, emerging under random bonds from the first-order transitions of the pure model, have their own distinctive unstable zero-temperature fixed point [12,68].

Furthermore, the latter of these two sets of results supports an extensive but weak universality, since the latter of the two transitions mentioned above has the same magnetic

critical exponent (but a different thermal critical exponent) as a wide variety of 2d systems without [13,14] and with [17,47,56,57] quenched disorder.

#### ACKNOWLEDGMENTS

This research was supported by the special Account for Research Grants of the University of Athens under Grant No. 70/4/4071. N.G.F. acknowledges financial support by the Alexander S. Onassis Public Benefit Foundation. A.N.B. acknowledges support by the Academy of Sciences of Turkey.

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