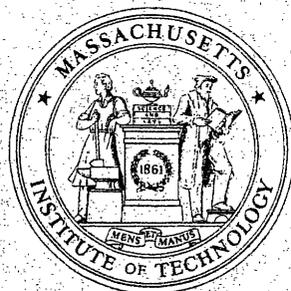


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APPROXIMATION FORMULATIONS FOR THE SINGLE
PRODUCT CAPACITATED LOT SIZE PROBLEM*

by

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APPROXIMATION FORMULATIONS FOR THE SINGLE PRODUCT

CAPACITATED LOT SIZE PROBLEM

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ABSTRACT

We consider two approximation formulations for the single product capacitated lot size problem. They correspond respectively to a restriction of the number of production policies and to the rounding of demands up to multiples of a constant. After briefly reviewing the literature within a new framework, we discuss the relations between these approximation formulations. Next, we provide relative error bounds and algorithms for solving the approximation problems. We demonstrate that these approximation formulations require a significantly smaller number of calculations than the original formulation, and that the relative error bounds are satisfactory for practical purposes.

Key words: Lot size problem, approximation schemes, worst case analysis, computational complexity, pseudopolynomial algorithm.

1. Introduction

In this paper we study the following single product capacitated lot size problem:

$$\begin{aligned}
 \text{(P)} \quad v(\text{P}) &= \text{Min} \left\{ \sum_{t=1}^T [h_t(I_t) + p_t(X_t)] + f(X_1, X_2, \dots, X_T) \right\} \\
 \text{s.t.} \quad I_{t-1} + X_t - I_t &= d_t & t=1, 2, \dots, T \\
 0 \leq X_t &\leq m_t K & t=1, 2, \dots, T \\
 \lambda_t \leq I_t &\leq u_t & t=1, 2, \dots, T \\
 I_0 &= 0
 \end{aligned}$$

where:

X_t = production in period t

I_t = inventory at the end of period t ,

$h_t(I_t)$ = inventory holding cost for $0 \leq I_t \leq u_t$, and shortage cost for

$\lambda_t \leq I_t < 0$ if $\lambda_t < 0$, in period t ,

$p_t(X_t)$ = variable production cost to produce X_t units in period t ,

d_t = demand in period t ,

$m_t K$ = capacity available in period t ; m_t is a given non-negative integer

and K is a known constant,

λ_t = lower bound on inventory in period t ($-\lambda_t$ is the upper bound on the number of backorders in period t if $\lambda_t < 0$),

u_t = upper bound on inventory in period t , and

$f(X_1, \dots, X_T)$ = cost penalty usually associated with changes in production quantities. In section 2 we will discuss several forms of $f(\cdot)$.

Problem (P) is of interest for several reasons. In practice, it is useful in situations where similar items share the same productive resources and can be aggregated into a single product. From a theoretical point of

view, the analysis of the single item problem has been used to study the computational complexity of more general production problems and to gain insight for developing algorithms for such problems. References [1] to [4], [8], [10] to [17], [19], and [22] provide exact algorithms for a variety of single item capacitated lot size problems.

In this paper we concentrate on approximation formulations to (P) because the latter has been shown to be NP-hard ([5]) except for very special instances ([2] and [4]). We provide fast algorithms to solve the approximation problems and show that the worst case error bounds should be acceptable for most practical instances.

Approximation Problems

We analyze two approximation problems. The first is inspired by practical situations where production in any period is restricted to be equal to a multiple of K . That is, in period t , X_t can only assume a value in the set $\{0, K, 2K, \dots, m_t K\}$. This is the case, for example, when there are n machines, numbered 1 to n , whose capacities are multiples of K ; in each period, machine i can be used only if machine j , for $j < i$, is being fully utilized. The restricted production policy problem is written as follows:

$$\begin{aligned}
 \text{(RPP)} \quad v_{\text{RPP}} &= \text{Min} \left\{ \sum_{t=1}^T [h_t(I_t) + p_t(X_t)] + f(X_1, X_2, \dots, X_T) \right\} \\
 \text{s.t.} \quad I_{t-1} + X_t - I_t &= d_t && t=1, 2, \dots, T \\
 l_t \leq I_t \leq u_t &&& t=1, 2, \dots, T \\
 X_t \in \{0, K, 2K, \dots, m_t K\} &&& t=1, 2, \dots, T \\
 I_0 &= 0
 \end{aligned}$$

We provide in section 5, $O(m^2 T^2)$, $O(mT^2)$, $O(mm^3 T^2)$, and $O(m^2 n^2 T^2)$ algorithms

to solve (RPP) depending on the functional forms of the objective function, where m and n are respectively the maximum of m_t for $t=1,2,\dots,T$ and the total number of machines available. Note that $n \leq m$ and that if the capacity of each machine is qK , then $m = qn$.

The second approximation formulation is based on the "softness" of the demand constraints. That is, if the standard deviation of the forecast errors is large enough to justify rounding the demands up to the nearest multiple of K then the resulting problem (RUD), described below, can be interpreted as an approximation of (P).

Let $d'_t = \lceil \sum_{\tau=1}^t d_\tau \rceil - \lceil \sum_{\tau=1}^{t-1} d_\tau \rceil$ for $t=1,2,\dots,T$, where $\sum_{\tau=1}^0 d_\tau = 0$ and $\lceil a \rceil$ denotes the smallest multiple of K greater than or equal to a . The approximation formulation with demands rounded-up to the nearest multiple of K is written as follows:

$$\begin{aligned}
 \text{(RUD)} \quad v_{\text{RUD}} &= \text{Min} \left\{ \sum_{t=1}^T [h_t(I_t) + p_t(X_t)] + f(X_1, X_2, \dots, X_T) \right\} \\
 \text{s.t.} \quad I_{t-1} + X_t - I_t &= d'_t && t=1,2,\dots,T \\
 l_t \leq I_t \leq u_t &&& t=1,2,\dots,T \\
 0 \leq X_t \leq m_t K &&& t=1,2,\dots,T \\
 I_0 &= 0
 \end{aligned}$$

If K is smaller than or is of the same magnitude as the standard deviation of the forecast error in every period, to solve (RUD) may be as meaningful as to solve (P). As we show in section 3, (RUD) and (RPP) are equivalent under mild assumptions. Therefore, the same fast algorithm to solve (RPP) can be used to derive a solution to (RUD).

The plan of this paper is as follows. In section 2 we briefly review the literature categorized in terms of the different functional forms of $f(\cdot)$

and present a list of assumptions and measures of computational complexity of the existing algorithms. In particular we introduce a unified functional form of $f(\cdot)$ that subsumes the three forms previously used in the literature. In section 3 we establish the desired relations between (RPP) and (RUD). In section 4 we compute the worst case error bound when (RPP) is used as an approximation to (P). Finally, in section 5 we provide algorithms to solve (RPP) and compute a measure of their computational complexity. Concluding this section, we make the following assumptions that will hold throughout the paper:

$p_t(X_t)$ is non-decreasing for $t=1,2,\dots,T$;

$h_t(I_t)$ is non-decreasing for $I_t \geq 0$ and non-increasing for $I_t \leq 0$ for $t=1,2,\dots,T$.

2. Brief Review of the Literature

In this section we review and classify a representative sample of the research done on problem (P) in terms of the functional form of $f(\cdot)$. Four categories are identified. They are not necessarily mutually exclusive; rather, they reflect an increase in generality.

(i) Sequence-independent Set-up Costs

$$(F1) \quad f(X_1, X_2, \dots, X_T) = \sum_{t=1}^T s_t \delta(X_t)$$

where
$$\delta(X_t) = \begin{cases} 1 & \text{if } X_t > 0 \\ 0 & \text{otherwise} \end{cases}$$

and s_t is the set-up cost in period t .

Problems in this category are formulated by using in (P) the form (F1) of $f(\cdot)$. Florian and Klein [3] characterized the optimal solutions of problems in this class assuming $\ell_t = 0$, $u_t = \infty$, and $p_t(\cdot)$ and $h_t(\cdot)$ concave. For $m_t = 1$, i.e. constant capacity, they derived an $O(T^4)$ algorithm. For the same class of problems, but with $m_t \neq 1$, Lambert and Luss [12] presented an $O(m^2 T^4)$ algorithm (where $m = \max_{t=1, \dots, T} m_t$).

Jagannathan and Rao [8] derived an $O(m^2 T^4)$ algorithm to solve (P) with $p_t(\cdot)$ piecewise concave, reflecting the difference in costs between regular and overtime labor, where the breakpoints are multiples of a constant. Also, for a piecewise concave $p_t(\cdot)$ arising out of the use of multiple identical machines, Lipman [15] reported an $O(T^5)$ algorithm for (P) with $\ell_t = 0$ and $u_t = K = \infty$. Assuming $h_t(\cdot)$ is concave on $(-\infty, 0]$ and $[0, \infty)$, Florian and Klein [3] and Jagannathan and Rao [8] showed that their algorithms with no backorder allowed can be easily extended for the case where backorders must be satisfied within α periods. This condition has been previously used by Zangwill [22] for (P) with $\ell_t = 0$ and $u_t = K = \infty$. Further, for $h_t(\cdot)$ concave on $(-\infty, 0]$ and $[0, \infty)$,

$K = \infty$ and either $u_t = \infty$ and $l_t = 0$ or $u_t = 0$ and $l_t = -\infty$, Love [16] provided a characterization of optimal solutions and an $O(T^3)$ algorithm. Swoveland [19] generalized Love's characterization to the capacitated lot size problem with piecewise concave $p_t(\cdot)$ and $h_t(\cdot)$. When the breakpoints of $p_t(\cdot)$ are written as multiples of a constant, Swoveland showed that the computational complexity of his algorithm is $O(m^2 Q^2 T^4)$, where Q is the maximum number of breakpoints of $h_t(\cdot)$.

When the production capacities are neither constant nor multiples of a constant, problem (P) is usually NP-hard. Florian et al [4] discussed the computational complexity of this problem. Bitran and Yanasse [2] studied several cases with $p_t(\cdot)$ and $h_t(\cdot)$ linear and proposed polynomial algorithms when the parameters of problem (P) satisfied certain special conditions.

Exponential algorithms have also been developed for the problem with general capacity constraints on production. Baker et al [1] provided an $O(2^T)$ branch and bound algorithm for $l_t = 0$, $u_t = \infty$, $p_t(X_t) = pX_t$, $h_t = hI_t$, $X_t \leq C_t$, $t=1,2,\dots,T$. A similar type of algorithm was developed by Lambrecht and Vander Eeken [13] with almost identical conditions except that $p_t(X_t) = p_t X_t$ and $h_t(I_t) = h_t I_t$.

(ii) Single Machine/Sequence-dependent Set-up Costs

Problems of type (P) in this category assume the existence of a single machine and have the following form of $f(\cdot)$ which incorporates sequence-dependent as well as as sequence-independent set-up costs:

$$(F2) \quad f(X_1, X_2, \dots, X_T) = \text{Min} \sum_{t=1}^T (s_t Y_t + e_t Z_t)$$

$$\text{s.t.} \quad Y_t \geq \delta(X_t) \quad t=1,2,\dots,T$$

$$Z_t \geq Y_t - Y_{t-1} \quad t=1,2,\dots,T$$

$$Y_t \text{ and } Z_t \in \{0,1\} \quad t=1,2,\dots,T$$

$$Y_0 = 0$$

where

$$Y_t = \begin{cases} 1 & \text{if the machine is on in period } t, \text{ and} \\ 0 & \text{otherwise,} \end{cases}$$

$$Z_t = \begin{cases} 1 & \text{if the machine state changes from off to on in period } t, \text{ and} \\ 0 & \text{otherwise,} \end{cases}$$

e_t = sequence-dependent start-up cost in period t , and

s_t = cost of having the machine on in period t ; this is a sequence-independent cost. For example, s_t may represent a rental fee.

Karmarkar et al [10] introduced this model and pointed out two advantages of defining $Y_t \leq \delta(X_t)$ rather than $Y_t = \delta(X_t)$ in (F2). First, in this formulation, the machine may be on whether or not there is production. This strategy can be more economical in some cases since the company need not incur the sequence-dependent costs. Second, unlike the case where Y_t is required to be equal to $\delta(X_t)$, $f(X_1, X_2, \dots, X_T)$ is concave.

(iii) Production Smoothing

Instead of considering sequence-dependent set-up costs, the third category of functions $f(\cdot)$ incorporates a penalty for changes in the production level. (F3) is written as follows:

$$(F3) \quad f(X_1, X_2, \dots, X_T) = \sum_{t=1}^T [s_t \delta(X_t) + f_t(X_t - X_{t-1})]$$

If X_t is measured in labor hours, $f_t(\cdot)$ penalizes variations of the labor force. Usually, such costs are associated with training, hiring and firing.

Problems of type (P) with the functional form (F3) for $f(\cdot)$ are extensively studied in the literature. Authors have frequently assumed linear objective functions. Readers are referred to Silver [18] for a tutorial on this problem category.

For concave objective functions, Zangwill [23] addressed the production

smoothing problem with $l_t = 0$ and $u_t = K = \infty$. He characterized optimal solutions when $f_t(\cdot)$ is concave on $(-\infty, 0]$ and $[0, \infty)$ and demands are non-decreasing, and provided a dynamic programming algorithm using a penalty function of the form

$$f_t(X_t - X_{t-1}) = \begin{cases} a_i & \text{if } X_t > X_{t-1}, \\ 0 & \text{if } X_t = X_{t-1}, \\ b_i & \text{if } X_t < X_{t-1}. \end{cases}$$

Korganker [11] extended Zangwill's results for problem (P) to the case with constant production capacities, non-decreasing demands, $l_t = -\infty$ and $u_t = \infty$, and presented an exponential time algorithm.

(iv) Multiple Machines/Sequence-dependent Set-up Costs

Lasdon and Terjung [14] formulated problem (P) assuming the existence of multiple identical machines and sequence-dependent set-up costs. In their model, they used the following functional form for $f(\cdot)$:

$$(F4') \quad f(X_1, X_2, \dots, X_T) = \sum_{t=1}^T C_t \text{Max}(X_t - X_{t-1}, 0)$$

where C_t is the cost of starting a machine in period t , $X_0 = 0$, and X_t is an integer quantity representing the number of machines utilized in period t .

Schrage [17] provided an $O(m^3T^2)$ algorithm to solve that problem.

In this paper we consider a more general form of $f(\cdot)$, designated as (F4), and defined below; (F4) is more general in the sense that its feasible set contains those of (F1), (F2), and (F4') and that of (F3) whenever the production quantities are restricted to integral numbers. It allows for fractional production and for keeping the machines on even if they are not being used. It is useful to note that this type of formulation does not preclude the shut off of a machine, if it is not being used, in case such strategy is more economical.

$$\begin{aligned}
 \text{(F4)} \quad f(X_1, X_2, \dots, X_T) &= \text{Min} \left[\sum_{t=1}^T \sum_{i=1}^n s_{ti} Y_{ti} + \sum_{t=1}^T f_t \left(\sum_{i=1}^n Y_{t-1,i}, \sum_{i=1}^n Y_{ti} \right) \right] \\
 \text{s.t.} \quad \sum_{i=1}^n q_i K Y_{ti} &\geq [X_t] \quad t=1, 2, \dots, T \\
 Y_{t1} &\geq Y_{t2} \geq \dots \geq Y_{tn} \quad t=1, 2, \dots, T \\
 Y_{ti} &\in \{0, 1\} \quad \begin{matrix} i=1, 2, \dots, n; \\ t=1, 2, \dots, T \end{matrix}
 \end{aligned} \tag{2.1}$$

where

n = number of machines, $n \leq m$,

$q_i K$ = capacity of the i^{th} machine where q_i is a positive integer,

$$Y_{ti} = \begin{cases} 1 & \text{if machine } i \text{ is on in period } t, \text{ and} \\ 0 & \text{otherwise} \end{cases}$$

s_{ti} = sequence-independent cost of keeping machine i on in period t , and

$f_t(a, b)$ = cost of changing the number of machines on from a to b .

Note that constraints (2.1) require machines $1, 2, \dots, i$ to be on whenever machine $i+1$ is on for $i=1, 2, \dots, n-1$.

The functional form (F4) of $f(\cdot)$ will be used in sections 3 and 4. The following proposition will be useful in our development. Let $J_{j_t}^{j_t-1}$ denote the interval $[\sum_{i=1}^{j_t-1} q_i K, \sum_{i=1}^{j_t} q_i K]$ for $j_t = 1, 2, \dots, n$ and $t=1, 2, \dots, T$, where

$\sum_{i=1}^0 q_i K = 0$. By using an argument similar to the one used by Karmarkar et al in [10] we can prove that:

Proposition 2.1: The functional form (F4) of $f(\cdot)$ is concave on $\prod_{\tau=1}^T J_{j_\tau}^{j_\tau}$ for $j_t=1, 2, \dots, n$ and $t=1, 2, \dots, T$. 

Assume that $l_t = 0$, $u_t = \infty$ for $t=1, 2, \dots, T$, and the objective function of (P) is given by $\sum_{t=1}^T s_t \delta(X_t) + \sum_{t=1}^T v_t X_t + \sum_{t=1}^T h_t I_t$ where the v_t 's are non-increasing, then there is an optimal solution which satisfies $I_{t-1} X_t (m_t K - X_t) = 0$

for $t=1,2,\dots,T$ (see [1] and [2]). The following proposition is an extension of this theorem to problem (P) where $f(X_1, X_2, \dots, X_T)$ takes the functional form (F4).

Proposition 2.2: Assume that for $t=1,2,\dots,T$, $l_T = 0$, $u_t = \infty$, $P_t(X_t) = v_t X_t$, the v_t 's are non-increasing, and $f(X_1, X_2, \dots, X_T)$ takes the functional form (F4) in (P). Then there is an optimal solution which satisfies:

$$I_{t-1} X_t \pi \left(\sum_{j=1}^n \sum_{i=1}^j q_i K - X_t \right) = 0 \quad \text{for } t=1,2,\dots,T.$$

To conclude this section we summarize in Table 2.1 the main characteristics of the problems addressed and the algorithms presented in this paper and in those cited in our brief review of the literature.

Table 2.1: Assumptions and Orders of Calculations of Algorithms in the Literature

Authors	Inv. costs	Shortage costs	Production costs		Capacity	Inventory limits		Order of calculation
	$h_t(I_t)$ $I_t > 0$	$h_t(I_t)$ $I_t < 0$	$P_t(X_t)$	$f(x)$	C_t	l_t	u_t	
Baker et al [1]	hI_t	∞	vX_t	F1	G	Z	∞	$O(2^T)$
Bitran & Yanasse [2]	$h_t I_t$	∞	$v_t X_t$	F1	G	Z	∞	
	G	∞	NI	NI	ND	Z	∞	$O(T^4)$
	G	∞	NI	NI	C	Z	∞	$O(T^3)$
	Z	∞	C	C	G	Z	∞	$O(T \log T)$
	Z	∞	ND	ND	NI	Z	∞	$O(T)$
Florian & Klein [3]	CC	∞	CC	F1	C	Z	∞	$O(T^4)$
Florian & Rinnooy Kan [4]	(a) PL	∞	(a) PL	F1	G & Int.	Z	∞	$O(\sum_{t=1}^T d_t \sum_{t=1}^T C_t)$
Jagannathan & Rao [8]	ND, NN & CC	NI, NN & CC	(b)	F1	$m_t K$	$-\infty$	∞	$O(m^2 T^4)$
Lambert & Luss [12]	CC	∞	CC	F1	$m_t K$	Z	∞	$O(m^2 T^4)$
Swoveland [19]	PCC	PCC	(c) PCC	F1	$m_t K$	G	G	(d) $O(T^4 Q^2 m^2)$
Karmarkar et al [10]	$h_t I_t$	∞	$v_t X_t$	F2	G	Z	∞	(e)
Schrage [17]	$h_t I_t$	∞	$v_t X_t$	F2	C	Z	∞	(f) $O(T^4)$
Korganker [11]	CC	∞	CC	F3	C	Z	∞	(g)

3. Relation Between (RPP) and (RUD)

In this section we explore relations between (RPP) and (RUD). In particular, we show that under mild assumptions the two problems have common optimal solutions. Throughout this section and the next we assume that the functional form of $f(\cdot)$ is given by (F4).

As a vehicle to study the relation between (RPP) and (RUD) we introduce the following problem which is the restricted-production version of (RUD):

$$\begin{aligned}
 \text{(RUD')} \quad v_{\text{RUD}'} &= \text{Min} \left\{ \sum_{t=1}^T [h_t(I_t) + p_t(X_t)] + f(X_1, \dots, X_T) \right\} \\
 \text{s.t.} \quad I_{t-1} + X_t - I_t &= d_t' & t=1, 2, \dots, T \\
 \ell_t &\leq I_t \leq u_t & t=1, 2, \dots, T \\
 X_t &\in \{0, K, 2K, \dots, m_t K\} & t=1, 2, \dots, T \\
 I_0 &= 0
 \end{aligned}$$

For convenience of notation we assume that the following substitutions have been made

$$I_t = \sum_{\tau=1}^t (X_\tau - d_\tau) \text{ in (P) and (RPP) and}$$

$$I_t = \sum_{\tau=1}^t (X_\tau - d_\tau') \text{ in (RUD) and (RUD')}.$$

Hence, the feasible set S of (RUD) can be written as

$$\begin{aligned}
 S = \{X \in \mathbb{R}^T \mid & \sum_{\tau=1}^t d_\tau' + \ell_t \leq \sum_{\tau=1}^t X_\tau \leq \sum_{\tau=1}^t d_\tau' + u_t, X_t \leq m_t K, X_t \geq 0 \\
 & \text{for } t=1, 2, \dots, T\}.
 \end{aligned}$$

Assume that ℓ_t and u_t are multiples of K for $t=1, 2, \dots, T$ and let

$$W = \{w \in \mathbb{R}^{2T} \mid w = (a_1, a_2, \dots, a_T, b_1, b_2, \dots, b_T) \text{ where } a_t \in \{1, 2, \dots, m_t\},$$

$$b_t \in \{1, 2, \dots, (u_t - \ell_t)/K\} \text{ if } u_t > \ell_t, \text{ and } b_t = 0 \text{ if } u_t = \ell_t \text{ for } t=1, 2, \dots, T\}$$

$$S_w = \{X \in \mathbb{R}^T \mid K(a_t - 1) \leq X_t \leq Ka_t,$$

$$\sum_{\tau=1}^t d'_\tau + \ell_t + K(b_t - 1) \leq \sum_{\tau=1}^t X_\tau \leq \sum_{\tau=1}^t d'_\tau + \ell_t + Kb_t$$

$$\text{if } b_t \geq 1, \text{ and } \sum_{\tau=1}^t X_\tau = \sum_{\tau=1}^t d'_\tau + \ell_t \text{ if } b_t = 0 \text{ for } t=1, 2, \dots, T$$

and $(a_1, a_2, \dots, a_T, b_1, b_2, \dots, b_T) = w\}$.

Therefore, $S = \bigcup_{w \in W} S_w$. The next proposition will play an important role in establishing the relationships between (RUD), (RUD'), and (RPP).

Proposition 3.1: If ℓ_t and u_t are multiples of K for $t=1, 2, \dots, T$, then the extreme points of S_w are multiples of K .

Proof: Let $S'_w = \{X' \in \mathbb{R}^T \mid X' = X/K \text{ for } X \in S_w\}$. S'_w can be written as $S'_w = \{X' \in \mathbb{R}^T \mid AX \leq b, X' \geq 0\}$. It is not difficult to show that A is a totally unimodular matrix and b is an integer vector. Therefore the extreme points of S_w are multiples of K (see [6], [7], and [21]). Since there is a one to one correspondence between the extreme points of S'_w and S_w the desired results follows. 

Proposition 3.2: Assume that ℓ_t and u_t are multiples of K and that $h_t(I_t)$ and $p_t(X_t)$ are piecewise concave functions whose breakpoints are multiples of K . Then, $v_{\text{RUD}} = v_{\text{RUD}'}$, and the optimal solutions of (RUD') are also optimal in (RUD).

Proof: Problem (RUD) can be written as:

$$\min_{w \in W} \min_{X \in S_w} \left\{ \sum_{t=1}^T h_t \left[\sum_{\tau=1}^t (X_\tau - d'_\tau) \right] + \sum_{t=1}^T p_t(X_t) + f(X_1, X_2, \dots, X_T) \right\}$$

By proposition 2.1, $f(X_1, X_2, \dots, X_T)$ is concave on S_w . Therefore, the objective function of (RUD) is concave on S_w . This fact together with proposition 3.1 implies that there is an optimal solution X in (RUD) whose components are multiples of K . Hence, X is feasible in (RUD') and since both problems have the same objective function, and the feasible set of (RUD') is contained in the feasible set of (RUD), it follows that $v_{RUD} = v_{RUD'}$; hence the optimal solutions of (RUD') are also optimal in (RUD). 

In order to establish the relation between (RPP) and (RUD) we first prove the following result that relates (RPP) and (RUD').

Proposition 3.3: Assume that $h_t(I_t) = h_t I_t$, ℓ_t is a multiple of K and $u_t = \infty$ for $t=1, 2, \dots, T$. Then, X is optimal in (RUD') if and only if it is optimal in (RPP). Moreover,

$$v_{RUD'} = v_{RPP} - \sum_{t=1}^T h_t \left(\sum_{\tau=1}^t d_{\tau} - \sum_{\tau=1}^t d_{\tau}' \right).$$

Proof: Suppose that $X \in R^T$ is feasible in (RUD'). Then $\sum_{\tau=1}^t X_{\tau} \geq \sum_{\tau=1}^t d_{\tau}' + \ell_t$. By the definition of d_t' , $\sum_{\tau=1}^t X_{\tau} \geq \sum_{\tau=1}^t d_{\tau}' + \ell_t \geq \sum_{\tau=1}^t d_{\tau} + \ell_t$. Hence, X is feasible in (RPP). Conversely, suppose X is feasible in (RPP). Then, $\sum_{\tau=1}^t X_{\tau} \geq \sum_{\tau=1}^t d_{\tau} + \ell_t$. By the definition of d_t' , $\sum_{\tau=1}^t d_{\tau}' + \ell_t < \sum_{\tau=1}^t d_{\tau} + K + \ell_t \leq \sum_{\tau=1}^t X_{\tau} + K$. Since X_t , d_t' , and ℓ_t are multiples of K for $t=1, 2, \dots, T$, $\sum_{\tau=1}^t X_{\tau} \geq \sum_{\tau=1}^t d_{\tau}' + \ell_t$. Therefore, X is feasible in (RUD') and the feasible set of (RPP) and (RUD') are the same. Since the objective function values of (RPP) and (RUD') differ by the constant $\sum_{t=1}^T h_t \left(\sum_{\tau=1}^t d_{\tau} - \sum_{\tau=1}^t d_{\tau}' \right)$ for any given feasible solution, the desired result follows. 

Corollary 3.1: Assume that for $t=1, 2, \dots, T$, ℓ_t is a multiple of K , $u_t = \infty$, $h_t = h_t I_t$, and $p_t(X_t)$ is a piecewise concave function whose breakpoints are multiples of K . Then, if $X \in R^T$ is optimal in (RPP), X is optimal in (RUD)

and

$$v_{RUD} = v_{RPP} - \sum_{t=1}^T \left(\left| \sum_{\tau=1}^t d_{\tau} \right| - \sum_{\tau=1}^t d_{\tau} \right) h_t.$$

The results of this section show that under mild conditions, (RUD), (RUD') and (RPP) are essentially equivalent. Therefore, if the forecast errors of the demand are of the same magnitude as K , the practitioner may opt to solve one of these three approximation problems instead of (P). We show in the next section that even when this is not the case, these problems are good approximations to (P).

4. Error Bounds for Approximations to Problem (P)

In this section we relate the optimal values of (RPP), (RUD'), (RUD) to the optimal value of (P). Suppose ℓ_t and u_t are multiples of K for $t=1,2,\dots,T$. Let $H_t(P_t)$ be the maximum value of the differences in $h_t(\cdot)$ ($p_t(\cdot)$) for two consecutive multiples of K , i.e.,

$$H_t = \max\{|h_t(Kq_t+K) - h_t(Kq_t)| \text{ for } q_t \in \{\ell_t/K, (\ell_t+K)/K, \dots, (u_t-K)/K\}\}$$

$$P_t = \max\{p_t(Kq_t+K) - p_t(Kq_t) \text{ for } q_t \in \{0,1,\dots,m_t-1\}\} \text{ for } t=1,2,\dots,T$$

and let

$$Q = \max\left\{\sum_{t=1}^T [p_t(X_t) - p_t(X_t - Y_t)] \text{ for } X_t \in \{K, 2K, \dots, m_t K\} \text{ for } t=1, \dots, T \text{ and } \sum_{t=1}^T Y_t \leq K\right\}$$

Denote by $v(X)$ the objective function value of (P) or (RPP) when that function is expressed in terms of the production quantities.

The next proposition provides an error bound when an optimal solution of (RPP) is used as a solution to (P).

Proposition 4.1: Assume that (P) is feasible, and that ℓ_t and u_t are multiples of K for $t=1,2,\dots,T$. Then, if X_{RPP}^* is an optimal solution to (RPP), $v(X_{RPP}^*) - v_P \leq \sum_{t=1}^T (H_t + P_t)$.

Proof: Since (P) is feasible and has a bounded feasible set, it follows that (P) has an optimal solution, say X . Let

$$X'_t = \left[\sum_{\tau=1}^t X_\tau \right] - \left[\sum_{\tau=1}^{t-1} X_\tau \right] \text{ for } t=1,2,\dots,T.$$

It suffices to show that $X' = (X'_1, \dots, X'_T)$ is feasible in (RPP) and that

$$v(X') \leq v_P + \sum_{t=1}^T (H_t + P_t) \text{ since } v(X_{RPP}^*) \leq v(X') \text{ trivially implies that } v(X_{RPP}^*) \leq v_P + \sum_{t=1}^T (H_t + P_t).$$

We first prove that X' is feasible in (RPP). Note that, for $t=1,2,\dots,T$,

$$\sum_{\tau=1}^t X'_\tau = \left[\sum_{\tau=1}^t X_\tau \right] \geq \sum_{\tau=1}^t X_\tau \geq \sum_{\tau=1}^t d_\tau + \ell_t. \text{ Since } \sum_{\tau=1}^t X'_\tau \text{ and } \ell_t \text{ are multiples}$$

of K , $\sum_{\tau=1}^t X'_\tau \geq \lceil \sum_{\tau=1}^t d_\tau + \ell_t \rceil = \lceil \sum_{\tau=1}^t d_\tau \rceil + \ell_t = \sum_{\tau=1}^t d'_\tau + \ell_t$. To show that

$\sum_{\tau=1}^t X'_\tau \leq \sum_{\tau=1}^t d'_\tau + u_t$, suppose that $\sum_{\tau=1}^t X'_\tau \geq \sum_{\tau=1}^t d'_\tau + u_t + K$. Then,

$\sum_{\tau=1}^t X_\tau > \lceil \sum_{\tau=1}^t X_\tau \rceil - K = \sum_{\tau=1}^t X'_\tau - K \geq \sum_{\tau=1}^t d'_\tau + u_t \geq \sum_{\tau=1}^t d_\tau + u_t$, contradicting

the feasibility of X in (P). Since $\sum_{\tau=1}^t X'_\tau$, $\sum_{\tau=1}^t d'_\tau$ and u_t are multiples of K ,

it follows that $\sum_{\tau=1}^t X'_\tau \leq \sum_{\tau=1}^t d'_\tau + u_t$. It remains to show that $X'_t \leq Km_t$

for $t=1,2,\dots,T$. Suppose that $X'_t \geq K(m_t+1)$ for $t=1,2,\dots,T$. Then,

$$X'_t = \sum_{\tau=1}^t X'_\tau - \sum_{\tau=1}^{t-1} X'_\tau > \lceil \sum_{\tau=1}^t X'_\tau \rceil - \sum_{\tau=1}^{t-1} X'_\tau - K \geq \lceil \sum_{\tau=1}^t X'_\tau \rceil - \lceil \sum_{\tau=1}^{t-1} X'_\tau \rceil - K =$$

 $= X'_t - K \geq Km_t$ for $t=1,2,\dots,T$, contradicting the feasibility of X in (P).

Therefore, X' is feasible in (RPP).

It remains to show that $v(X') \leq v_P + \sum_{t=1}^T (H_t + P_t)$. By construction of X' , $\lceil X'_t \rceil \leq \lceil X_t \rceil$ for $t=1,2,\dots,T$. Hence, by (F4), $f(X'_1, X'_2, \dots, X'_T) \leq f(X_1, X_2, \dots, X_T)$. Since $p_t(\cdot)$ is non-decreasing and $X'_t \leq X_t + K$ with X'_t being a multiple of K , $p_t(X'_t) \leq p_t(X_t) + P_t$ holds for $t=1,2,\dots,T$. Still by construction of X' ,

$$\sum_{\tau=1}^t X'_\tau \leq \sum_{\tau=1}^t X'_\tau < \sum_{\tau=1}^t X_\tau + K \text{ implying that } h_t \left(\sum_{\tau=1}^t X'_\tau - \sum_{\tau=1}^t d_\tau \right) \leq$$

$$\leq h_t \left(\sum_{\tau=1}^t X_\tau - \sum_{\tau=1}^t d_\tau \right) + H_t. \text{ Therefore, } v(X') \leq v_P + \sum_{t=1}^T (H_t + P_t). \quad \square$$

Proposition 3.3 and Proposition 4.1 imply the following corollary.

Corollary 4.1: Assume that (P) is feasible and that $h_t(I_t) = h_t I_t$, ℓ_t is a multiple of K and $u_t = \infty$ for $t=1,2,\dots,T$. Then if X_{RUD}^* is an optimal solution of (RUD'),

$$v(x_{RUD}^*) - v_P \leq \sum_{t=1}^T (H_t + P_t).$$

The error bound for the case when an optimal solution to (RUD) is used as a solution to (P) can be computed as follows. Let X_{RUD}^* be an optimal solution to (RUD).

Proposition 4.2: If (P) is feasible, h_t is a multiple of K , $u_t = \infty$, and $h_t(I_t) = h_t I_t$ for $t=1,2,\dots,T$, then $v(x_{RUD}^*) - v_P \leq K(\sum_{t=1}^T h_t) + Q$ and the bound is tight.

Proof: Since (P) is feasible and its feasible set is bounded, it has an optimal solution X . To prove the result we will construct a feasible solution X' to (RUD) from X and show that $v(X') - v_P \leq K(\sum_{t=1}^T h_t) + Q$. Since $h_t(I_t)$ is linear for $t=1,2,\dots,T$ and X' is feasible in (RUD), $v(x_{RUD}^*) \leq v(X')$. Hence, it will follow that $v(x_{RUD}^*) - v_P \leq K(\sum_{t=1}^T h_t) + Q$.

Let $(x_{t_1}, x_{t_2}, \dots, x_{t_q})$ be the sequence of positive production quantities of X . Set $X'_t = 0$ if $t \notin \{t_1, t_2, \dots, t_q\} \triangleq R$. Determine $X'_{t_1}, X'_{t_2}, \dots, X'_{t_q}$ by the following recursion equations.

- (a) $a_{t_1} = K$
- (b) $X'_{t_i} = \min\{X_{t_i} + a_{t_i}, [X_{t_i}]\}$ and
- (c) $a_{t_{i+1}} = a_{t_i} - (X'_{t_i} - X_{t_i})$ for $i=1,2,\dots,q$.

Then, the following relations hold for $t=1,2,\dots,T$.

- (i) $X'_t \geq X_t$
- (ii) $[X'_t] = [X_t]$
- (iii) $0 \leq X'_t \leq m_t K$,
- (iv) $\sum_{\tau=1}^t X'_\tau \leq \sum_{\tau=1}^t X_\tau + K$, and
- (v) If $\sum_{\tau=1}^t X_\tau + K > \sum_{\tau=1}^t X'_\tau$, then $[\sum_{\tau=1}^t X_\tau] = \sum_{\tau=1}^t X'_\tau$.

(i) through (iv) are easy to show. To prove (v), suppose that

$\sum_{\tau=1}^t X_{\tau} + K > \sum_{\tau=1}^t X'_{\tau}$ for some t . Let $i = \max\{j \mid t_j \leq t\}$. Since $X'_t = X_t = 0$ for $t \notin R$, it suffices to show that $\lceil \sum_{j=1}^i X_{t_j} \rceil = \sum_{j=1}^i X'_{t_j}$. By (a) and (c),

$$a_{t_{i+1}} = a_{t_i} - (X'_{t_i} - X_{t_i}) = K - \sum_{j=1}^i (X'_{t_j} - X_{t_j}) > 0. \text{ By (c) and (i),}$$

$0 < a_{t_{i+1}} \leq a_{t_i} \leq \dots \leq a_{t_1}$. (c) and $a_{t_{j+1}} > 0$ imply $X'_{t_j} < X_{t_j} + a_{t_j}$ for $j=1,2,\dots,i$. Therefore, (b) implies $X'_{t_j} = \lceil X_{t_j} \rceil$ for $j=1,2,\dots,i$. Note that

$$\sum_{j=1}^i X_{t_j} + K > \sum_{j=1}^i X'_{t_j} = \sum_{j=1}^i \lceil X_{t_j} \rceil. \text{ Since } K > \sum_{j=1}^i \lceil X_{t_j} \rceil - \sum_{j=1}^i X_{t_j} \geq 0,$$

$$\lceil \sum_{j=1}^i X_{t_j} \rceil = \sum_{j=1}^i \lceil X_{t_j} \rceil = \sum_{j=1}^i X'_{t_j}.$$

Next, we prove that X' is feasible in (RUD). Since (iii) holds, it suffices to show that $\sum_{\tau=1}^t X'_{\tau} \geq \sum_{\tau=1}^t d'_{\tau} + \ell_t$. From (iv), $\sum_{\tau=1}^t X'_{\tau} \leq \sum_{\tau=1}^t X_{\tau} + K$ for $t=1,2,\dots,T$. Hence, two cases are possible:

Case 1: $\sum_{\tau=1}^t X_{\tau} + K = \sum_{\tau=1}^t X'_{\tau}$. Since $\sum_{\tau=1}^t X_{\tau} \geq \sum_{\tau=1}^t d_{\tau} + \ell_t > \sum_{\tau=1}^t d'_{\tau} + \ell_t - K$

holds, $\sum_{\tau=1}^t X'_{\tau} = \sum_{\tau=1}^t X_{\tau} + K > \sum_{\tau=1}^t d'_{\tau} + \ell_t$ for $t=1,2,\dots,T$.

Case 2: $\sum_{\tau=1}^t X_{\tau} + K > \sum_{\tau=1}^t X'_{\tau}$. By (v), $\sum_{\tau=1}^t X'_{\tau} = \lceil \sum_{\tau=1}^t X_{\tau} \rceil$. Since

$$\sum_{\tau=1}^t X_{\tau} \geq \sum_{\tau=1}^t d_{\tau} + \ell_t \text{ and } \ell_t \text{ is a multiple of } K, \lceil \sum_{\tau=1}^t X_{\tau} \rceil \geq \lceil \sum_{\tau=1}^t d_{\tau} \rceil + \ell_t =$$

$$= \sum_{\tau=1}^t d'_{\tau} + \ell_t. \text{ Therefore, } \sum_{\tau=1}^t X'_{\tau} \geq \sum_{\tau=1}^t d'_{\tau} + \ell_t.$$

It remains to show that $v(X') - v_p \leq K(\sum_{t=1}^T h_t) + Q$. (ii) implies

$f(X'_1, X'_2, \dots, X'_T) = f(X_1, X_2, \dots, X_T)$. (iv), the linearity of $h_t(\cdot)$ and the definition of H_t imply $h_t(\sum_{\tau=1}^t X'_\tau - \sum_{\tau=1}^t d_\tau) \leq h_t(\sum_{\tau=1}^t X_\tau - \sum_{\tau=1}^t d_\tau) + Kh_t$.

Also, (i), (ii), (iv), the monotonicity of $P_t(\cdot)$ and the definition of Q imply that $\sum_{t=1}^T P_t(X'_t) \leq \sum_{t=1}^T P_t(X_t) + Q$. Therefore, $v(X') \leq v_P + K(\sum_{t=1}^T h_t) + Q$.

The following example shows that this bound is tight. Suppose that $p_t(X_t) = vX_t$, $m_t = m$ for $t=1, 2, \dots, T$, $d_1 = \varepsilon$, and $d_t = 0$ for $t \neq 1$. Then, $Q = Kv$. Suppose ε is less than K . Then, $X_1 = \varepsilon$, $X_t = 0$ for $t \neq 1$ is optimal to (P) while $X_1 = K$, $X_t = 0$ for $t \neq 1$ is optimal to (RUD).

$$\begin{aligned} v(X_{RUD}^*) - v_P &= \sum_{t=1}^T h_t(K - \varepsilon) + v(K - \varepsilon) \\ &= K(\sum_{t=1}^T h_t) + Q - \varepsilon(\sum_{t=1}^T h_t + v). \end{aligned}$$

As $\varepsilon \rightarrow$ zero, $v(X_{RUD}^*) - v_P - K(\sum_{t=1}^T h_t) - Q$ tends to zero. 

If Q is hard to calculate, then a weaker bound can be computed as follows: $v(X_{RUD}^*) - v_P \leq K(\sum_{t=1}^T h_t) + \sum_{t=1}^T P_t$, since $Q \leq \sum_{t=1}^T P_t$. Under stronger conditions than those of Proposition 4.1 a tighter bound can be computed on the difference between $v(X_{RPP}^*)$ and v_P .

Proposition 4.3: Assume that for $t=1, 2, \dots, T$, ℓ_t is a multiple of K , $u_t = \infty$, $h_t(I_t) = h_t I_t$, and $P_t(X_t)$ is a piecewise concave function with breakpoints which are multiples of K . Then

$$v(X_{RPP}^*) - v_P \leq K(\sum_{t=1}^T h_t) + Q.$$

Proof: The proposition follows immediately from Corollary 3.1 and Proposition 4.2. 

It is important to note that the error bounds described do not depend on set-up related costs. This means that the approximations are particularly

good if the set-up costs are significant.

The relative error bound implied by proposition 4.3 can be fairly tight in practical settings. Assume that for $t=1,2,\dots,T$, $l_t = 0$, $u_t = \infty$, $m_t = m$, $h_t(I_t) = h_t I_t$, $p_t(X_t) = v_t X_t$, $f(X_1, X_2, \dots, X_T)$ takes the form of (F4), and denote the capacity utilization rate, $\sum_{t=1}^T d_t / mKT$, by q . In practice it is common to compute the holding costs as a fraction of v_t , i.e., $h_t = i v_t / T$ where i is the inventory carrying factor over the planning horizon T . It reflects the opportunity cost of the capital tied up in inventory as well as other costs like handling, insurance, pilferage, etc. We have that

$$\begin{aligned} v_P &\geq \left(\text{Min}_{t=1,\dots,T} v_t \right) \sum_{t=1}^T d_t = \left(\text{Min}_{t=1,\dots,T} h_t T / i \right) \sum_{t=1}^T d_t = \\ &= \left(\text{Min}_{t=1,\dots,T} h_t \right) qmKT^2 / i . \end{aligned}$$

By proposition 4.3,

$$\begin{aligned} v(X_{RPP}^*) - v_P &\leq K \left(\sum_{t=1}^T h_t \right) + Q = \\ &= K \sum_{t=1}^T h_t + K \text{Max}_{t=1,\dots,T} v_t \leq KT(1 + 1/i) \text{Max}_{t=1,\dots,T} h_t . \end{aligned}$$

Therefore

$$\frac{v(X_{RPP}^*) - v_P}{v_P} \leq \frac{\text{Max}_{t=1,\dots,T} h_t}{\text{Min}_{t=1,\dots,T} h_t} \frac{KT(1+1/i)i}{qmKT^2} = \frac{\text{Max}_{t=1,\dots,T} h_t}{\text{Min}_{t=1,\dots,T} h_t} \frac{1+i}{qmT} .$$

Typically $i = 0.3$, $T = 12$, and $Q = .8$. In this case

$$[v(X_{RPP}^*) - v_P] / v_P \leq \left(\text{Max}_{t=1,\dots,T} h_t / \text{Min}_{t=1,\dots,T} h_t \right) 0.14 / m .$$

Note that, if $h_t = h$ for $t=1,2,\dots,T$, then the relative error bound is proportional to $1/T$. If the planning horizon is partitioned into months, then the relative error bound is given by $.14/m$. However, if the planning

horizon is partitioned in weeks, then the relative error bound becomes $0.031/m$. Therefore, the finer the partition of the planning horizon, the tighter the bound will be.

5. Algorithms to Solve (RPP)

To simplify the explanation we first provide an algorithm to solve (RPP) with the functional form (F2) for $f(\cdot)$.

Let a point (t,y,w) correspond to a state where the time period is t , the cumulative production is Ky and

$$w = \begin{cases} 1 & \text{if the machine is on in period } t, \\ 0 & \text{otherwise.} \end{cases}$$

In the following algorithm S_t denotes the set of feasible points, in period t , and $g(t,y,w)$ denotes the cumulative costs from $(0,0,0)$ to (t,y,w) . Consider the following layered network (N,A) where $N = \bigcup_{t=1}^T S_t$ and A consists of the following arcs:

- (i) from $(t,y,0)$ to $(t+1,y',1)$ if $y < y' \leq y + m_{t+1}$,
- (ii) from $(t,y,0)$ to $(t+1,y,0)$,
- (iii) from $(t,y,1)$ to $(t+1,y',1)$ if $y \leq y' \leq y + m_{t+1}$,
- (iv) from $(t,y,1)$ to $(t+1,y,0)$,

where the origin of each arc must be in S_t and the destination in S_{t+1} . Note that arc $((t,y,1),(t+1,y,1))$ corresponds to the alternative that there is no production in period $t+1$ but the machine is on. The costs associated with the arcs defined in (i) - (iv) are, respectively:

- (i) $h_{t+1}(Ky' - \sum_{\tau=1}^{t+1} d_{\tau}) + p_{t+1} [(y'-y)K] + s_{t+1} + e_{t+1}$,
- (ii) $h_{t+1}(Ky - \sum_{\tau=1}^{t+1} d_{\tau})$,
- (iii) $h_{t+1}(Ky' - \sum_{\tau=1}^{t+1} d_{\tau}) + p_{t+1} [(y'-y)K] + s_{t+1} \delta(y'-y)$,
- (iv) $h_{t+1}(Ky - \sum_{\tau=1}^{t+1} d_{\tau})$.

The following algorithm solves a shortest path problem in (N,A) .

Step 0. Set $S_0 = \{(0,0,0)\}$, $g(0,0,0) = 0$ and $h_0 = p_0 = s_0 = e_0 = 0$. Let $S_t = \{(t,y,w) \mid 0 \leq y \leq \sum_{\tau=1}^t m_\tau, \ell_t \leq Ky - \sum_{\tau=1}^t d_\tau \leq u_t, y \text{ integer and } w \in \{0,1\}\}$ for $t=1,2,\dots,T-1$. Also, let $S_T = \{(T,y,w) \mid y \in \{\lceil \sum_{t=1}^T d_t + \ell_T \rceil / K, \dots, \lceil \sum_{t=1}^T d_t \rceil / K\}$ and $w \in \{0,1\}\}$ if $\ell_T \leq 0$ and $S_T = \{(T, \lceil \sum_{t=1}^T d_t + \ell_T \rceil / K, w) \mid w \in \{0,1\}\}$ if $\ell_T > 0$.

Step 1. For $t = 0$ to $T-1$, compute

$$g(t+1,y',w') = \text{Min} \{g(t,y,w) + h_{t+1}(Ky - \sum_{\tau=1}^{t+1} d_\tau) + p_{t+1}[(y'-y)K] + s_{t+1}\delta(y'-y) + e_{t+1}(1-w)w'\}$$

s.t. $(t,y,w) \in S_t$ and $y' - m_{t+1} \leq y \leq y'$,

for any $(t+1,y',w') \in S_{t+1}$.

It is not difficult to show that

$$v_{RPP} = \text{Min}_{(T,y,w) \in S_T} g(T,y,w) .$$

The number of nodes in graph (N,A) is at most $1 + \sum_{t=1}^T [2 \cdot (\sum_{\tau=1}^t m_\tau) + 1] \leq 1 + mT(T+1) + 2T$. The number of arcs incident to a node is at most $m+1$.

Therefore, the total number of steps in the above algorithm is $O(m^2T^2)$.

Assume that $f(X_1, X_2, \dots, X_T)$ takes the functional form (F4). Essentially, the algorithm for this case is the same as the one just described for (F2). Let w represent the number of machines kept on in period t , i.e. $w=0,1,2,\dots,n$. Then, the number of nodes is $O(mnT^2)$. Since the number of arcs incident to a node is $O(mn)$, the total number of calculation is $O(m^2n^2T^2)$. Note that if there are no sequence-dependent set-up costs in (F4), w can be dropped from the description of the state in the dynamic recursions. Therefore, the number of calculations for this case is $O(m^2T^2)$.

Assume that for $t=1,2,\dots,T$, $\ell_t = 0$, $u_t = \infty$ and $p_t(X_t) = v_t X_t$ and the v_t 's are non-increasing. Then, by proposition 2.2, there is an optimal solution

which satisfies $I_{t-1} X_t \prod_{j=1}^n (\sum_{i=1}^j q_i K - X_t) = 0$ for $t=1,2,\dots,T$ for the functional form (F4) and $I_{t-1} X_t (m_t K - X_t) = 0$ for $t=1,2,\dots,T$ for the functional form (F2). Since we need to consider only $O(n^2)$ arcs incident to a node for (F4) and two arcs for (F2), the total number of calculations is reduced to $O(mn^3 T^2)$ and $O(mT^2)$ for (F4) and (F2), respectively.

Assume the sequence-dependent set-up costs are zero in (F4). For $t=1,2,\dots,T$, let $C_t(X_t)$ be the cost associated with X_t and let $C'_t(X_t)$ denote the piecewise linear function whose breakpoints occur on multiples of K with $C'_t(X_t) = C_t(X_t)$ for any X_t which is a multiple of K . If $C'_t(X_t)$ is convex for $t=1,2,\dots,T$, we can use the algorithm proposed by Johnson [9] and Veinott [20] to solve (RPP) optimally. The algorithm takes $O(mT^2)$ calculations if $\lambda_t \geq 0$ and $O(mT^3)$ calculation if $\lambda_t < 0$.

6. Conclusions

The proposed approximation formulations cover a large class of problems and require a significantly smaller number of calculations than those required to solve optimally the original problem. The approximation formulation (RUD) is motivated by fluctuations and inaccuracy of demand forecasting. We claim that if the standard deviation of demands is larger than a scaling constant, to solve (RUD) is as meaningful as to solve the original problem (P). Since (RUD) and the other approximation formulation (RPP) are equivalent under mild conditions, (RUD) can be solved by the same algorithms provided for (RPP). We provide a validation of the approximation formulations (RPP), (RUD), and (RUD') through worst case analysis of the error bounds under the assumption that all inputs are accurate. Note that the algorithms for (RPP) are pseudopolynomial in the sense that they are polynomial with respect to a constant m which is closely related to the scaling of the unit of demand quantities. A point to observe is that the relative error bound is proportional to $1/m$ while the order of calculations is proportional to m or m^2 . For practical purposes, we showed that there exists a satisfactory trade-off between them. This suggests that the combination of pseudopolynomial algorithms and the worst case analysis is fairly effective for the single product capacitated lot size problem. Our current research indicates that this approach can be extended to other practically important problems such as the multiple product capacitated lot size problem and the multistage lot size problem.

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References

1. Baker, K. R., Dixon P., Magazine, M. J., and Silver, E. A. (1978). An Algorithm for the Dynamic Lot-Size Problem with Time-Varying Production Capacity Constraints. *Management Sci.* 24, 1710-1720.
2. Bitran, G. R., and Yanasse, H. H. (1982). Computational Complexity of the Capacitated Lot Size Problem. *Management Sci.* 28, 1174-1186.
3. Florian, M., and Klein, M. (1971). Deterministic Production Planning with Concave Costs and Capacity Constraints. *Management Sci.* 18, 12-20.
4. Florian, M., Lenstra, J. K., and Rinnooy Kan, A. H. G. (1980). Deterministic Production Planning: Algorithms and Complexity. *Management Sci.* 26, 669-679.
5. Garey, M. R., and Johnson, D. S. (1979). *Computers and Intractability: A Guide to the Theory of NP-Completeness*. Freeman, San Francisco.
6. Heller, I., and Tompkins, C. B. (1956). An Extension of a Theorem of Dantzig's, In *Linear Inequalities and Related Systems*, ed. H. W. Kuhn and A. W. Tucker, Princeton, N.J.: Princeton University Press, 247-252.
7. Hoffman, A. J., and Kruskal, J. B. (1956). Integral Boundary Points of Convex Polyhedra, In *Linear Inequalities and Related Systems*, ed. H. W. Kuhn and A. W. Tucker, Princeton, N.J.: Princeton University Press, 223-246.
8. Jagannathan, R., and Rao, M. R. (1973). Class of Deterministic Production Planning Problems. *Management Sci.* 19, 1295-1300.
9. Johnson, S. M. (1957). Sequential Production Planning Over Time at Minimum Cost. *Management Sci.* 3, 435-437.
10. Karmarkar, U. S., Kekre, S., and Kekre, S. (1981). The Dynamic Lot Sizing Problem with Startup and Reservation Costs. Working Paper Series No. 8125, The Graduate School of Management, The University of Rochester.
11. Korgaonker, M. G. (1977). Production Smoothing Under Piecewise Concave

- Costs, Capacity Constraints and Nondecreasing Requirements. *Management Sci.* 24, 302-311.
12. Lambert, A., and Luss, H. (1982). Production Planning with Time-Dependent Capacity Bounds. *European J. Operational Res.* 9, 275-280.
 13. Lambrecht, M., and Vander Eecken, J. (1978). A Capacity Constrained Single-Facility Dynamic Lot-Size Model. *European J. Operational Res.* 2, 132-136.
 14. Lasdon, L. S., and Terjung, R. C. (1971). An Efficient Algorithm for Multi-Item Scheduling. *OR.* 19, 946-969.
 15. Lippman, S. A. (1969). Optimal Inventory Policy with Multiple Set-up Costs. *Management Sci.* 16, 118-138.
 16. Love, S. F. (1973). Bounded Production and Inventory Models with Piecewise Concave Costs. *Management Sci.* 20, 313-318.
 17. Schrage, L. The Multiproduct Lot Scheduling Problem. To appear.
 18. Silver, E. A. (1967). A Tutorial on Production Smoothing and Work Force Balancing. *OR.* 15, 985-1010.
 19. Swoveland, C. (1975). A Deterministic Multi-Period Production Planning Model with Piecewise Concave Production and Holding-Backorder Costs. *Management Sci.* 21, 1007-1013.
 20. Veinott, A. F., Jr. (1964). Production Planning with Convex Costs: A Parametric Study. *Management Sci.* 10, 441-460.
 21. Veinott, A. F., Jr., and Dantzig, G. B. (1968). Integral Extreme Points. *SIAM Rev.* 10, 371-372.
 22. Zangwill, W. I. (1966). A Deterministic Multi-Period Production Scheduling Model with Backlogging. *Management Sci.* 13, 105-119.
 23. Zangwill, W. I. (1966). Production Smoothing of Economic Lot Sizes with Non-Decreasing Requirements. *Management Sci.* 13, 191-209.