Cube versus Torus Models for Combinatorial Optimization Problems and the Euclidean Minimum Spanning Tree Constant

by

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Abstract

For a sample of points drawn uniformly from either the d-dimensional torus or the d-cube, $d \geq 2$, we define a class of random processes with the property of being asymptotically equivalent in expectation in the two models. Examples include the traveling salesman problem (TSP), the minimum spanning tree problem (MST), etc. Application of this result helps closing down one open question: We prove that the analytical expression recently obtained by Avram and Bertsimas for the MST constant in the d-torus model is in fact valid for the traditional d-cube model. For the MST, we also extend our result and show that stronger equivalences hold. Finally we present some remarks on the possible use of the d-torus model for exploring rates of convergence for the TSP in the square.

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1 Introduction

In Beardwood et al. [2], the authors prove that for any bounded i.i.d. random variables $\{X_i : 1 \leq i < \infty\}$ with values in $\mathbf{R}^d$, $d \geq 2$, the length of the shortest tour through $\{X_1, \ldots, X_n\}$ is asymptotic to a constant times $n^{(d-1)/d}$ with probability one (the same being true in expectation). This theoretical result has been the inspiration for a growing interest in the area of probabilistic analysis of combinatorial optimization problems. An important contribution is contained in Steele [8] in which the author uses the theory of independent subadditive processes to obtain strong limit laws for a class of problems in geometrical probability which exhibit non-linear growth. Examples include the traveling salesman problem (TSP), the Steiner tree problem, and the minimum weight matching problem. Among other problems, not in this class, but with a similar asymptotical behavior, is the minimum spanning tree problem (MST) and some weighted versions of it (see Steele [9]).

For most of these problems, the results concern the almost sure convergence of a sequence of normalized random variables, say L_n/n^{c_d}, to a constant β_d, as well as the convergence of the normalized means. One of the persistently important open problems in this area is the determination of the exact value of the constant β_d for any interesting functional. In fact progress has been made by Avram and Bertsimas [1] who have recently obtained an exact expression (as a series expansion) for the MST constant when the points are drawn uniformly from the d-dimensional torus. The use of the d-torus was to avoid boundary effects but they conjectured that the resulting constant was in fact the same for the traditional d-cube model.

In this paper, we prove this conjecture for the MST and give general conditions under which the d-torus and d-cube models are asymptotically equivalent in expectation for a large class of problems. In addition to the MST, this class contains many of the well-known combinatorial optimization problems such as the TSP and minimum weight matching problem. We also show that, for the MST, the equivalence can be expressed almost surely and that the optimal trees themselves (and not only their weight) are in some sense very close. Note that, for comparison with related results on the d-torus versus d-cube model, it has been shown, in Steele and Tierney [10], that, when $d \geq 3$, the limiting distribution for the largest of the nearest-neighbor links is different in the two models.

The paper is structured as follows. In the next section we introduce a set of conditions defining a class of graph problems with the property of being asymptotically equivalent in expectation in the d-cube and d-torus models. In Section 3 we prove that the MST, the TSP and the matching problem are all examples of this class. In Section 4 we extend our result on the MST to cover equivalence with probability one, and we give a structural comparison of the problem in the two models. In a concluding section, we present some open questions and discuss the applicability of our results for studying rates of convergence.

2 Equivalent class

Definition of the edge-set problems:

We are concerned here with a class of combinatorial optimization problems defined on an undirected graph (G, V) with positive weighted edges. Typically, these problems will be to find among all feasible subsets of edges (of given cardinality) a subset of minimum weight. The weight of a subset of edges will be a function of the weight of the edges belonging to this subset. The only restriction we impose on the objective function is the following: *The weight of a feasible solution is assumed to be a non-decreasing function with respect to the weight of any edge belonging to the feasible solution.* Note that we do not restrict ourselves to linear or even separable function. Let us call a member of such a class of problems an **edge-set problem**. Let us now consider two special models for our weighted graph.

The d-cube model:

Let $\{x_i : 1 \leq i < \infty\}$ be an arbitrary infinite sequence of points in $[0,1]^d$ (the unit cube in $\mathbf{R}^d$, $d \geq 2$, the d-dimentional space of real numbers, with the Euclidean metric and the Lebesgue measure), and let $x^{(n)} = \{x_1, x_2, \ldots, x_n\}$ denote its first n points. For each finite n, $x^{(n)}$ will be the vertex set, and $K_n(x) = \{(x_i, x_j); 1 \leq i < j \leq n\}$ the edge set of our graph. The weight of an edge (x_i, x_j) will be the Euclidean distance $||x_i - x_j||$ from x_i to x_j.

The d-torus model:

In order to eliminate the boundary effects of the previous model, consider the previous sequence $x_1, x_2, \ldots, x_n, \ldots$ modulo 1 in each component. Alternatively, one can imagine a sequence on the d-torus $T^d = ([0,1] \bmod 1)^d$ (the metric space with its Lebesgue measure and Euclidean d-torus metric). Note that the weight of an edge (x_i, x_j) is now taken to be $||x_i - x_j(\mathrm{mod}1)^d||$ (for $y \in [-1,1]$, $y(\mathrm{mod}1)$ is the minimum of $|y|$ and $1 - |y|$).

Set of conditions:

Consider a generic edge-set problem. We will write $L_{cube}(x^{(n)})$ for the value of an optimal solution (described by its set of edges $\mathcal{A}_{cube}(x^{(n)})$) for the problem in the d-cube. With a slight abuse of notation, we will use $L_{torus}(x^{(n)})$ and $\mathcal{A}_{torus}(x^{(n)})$ for the corresponding quantity in the d-torus. Also, from now on, $|\{.\}|$ will stand for the cardinality of the set $\{.\}$. We can now consider the following set of conditions for an edge-set problem.

1. (Sublinear growth). There exist two constants $\alpha_d \geq 0$ and $0 < c_d \leq 1$ such that $L_{cube}(x^{(n)}) \leq \alpha_d n^{c_d}$ for any sequence $\{x_i : 1 \leq i < \infty\}$ in the d-cube.

2. (Bounded degree). For any $x^{(n)} = \{x_1, x_2, \ldots, x_n\}$ the degree of the points in $\mathcal{A}_{torus}(x^{(n)})$ is bounded by a constant D_d.

3. (Bounded passage from torus to cube). Among $\mathcal{A}_{torus}(x^{(n)})$, let k be the number of edges (x_i, x_j) such that $||x_i - x_j(\bmod 1)^d|| < ||x_i - x_j||$. Then there exists a feasible solution to the d-cube problem, of weight bounded by $L_{torus}(x^{(n)}) + \gamma_d k^{c_d}$, where $\gamma_d \geq 0$ and c_d is the same constant as in Condition 1.

4. (Probabilistically small maximal edge).
Let $\text{MAX}(x^{(n)}) = \sup\left\{||x_i - x_j(\text{mod} 1)^d||; (x_i, x_j) \in \mathcal{A}_{torus}(x^{(n)})\right\}$. Then for $(X_i)_i$ a sequence of points independently and uniformly distributed over $[0,1]^d$, there exists a sequence of real numbers $(r_n)_n$, such that $\lim_{n\to\infty} r_n = 0$ and $\lim_{n\to\infty} \mathbf{P}(\text{MAX}(X^{(n)}) > r_n) = 0$.

The most restrictive conditions are obviously the last two. We can now state our main result.

Theorem 1 *Let us consider an edge-set problem verifying Conditions 1–4. Then if* $(X_i)_i$ *is a sequence of points independently and uniformly distributed over* $[0,1]^d$ *we have*

$$\limsup_{n\to\infty} \frac{\mathbf{E}L_{cube}(X^{(n)})}{n^{c_d}} = \limsup_{n\to\infty} \frac{\mathbf{E}L_{torus}(X^{(n)})}{n^{c_d}}, \tag{2.1}$$

and

$$\liminf_{n\to\infty} \frac{\mathbf{E}L_{cube}(X^{(n)})}{n^{c_d}} = \liminf_{n\to\infty} \frac{\mathbf{E}L_{torus}(X^{(n)})}{n^{c_d}}. \tag{2.2}$$

Proof:

First let us consider an arbitrary sequence $(x_i)_i$. For all edges (x_i, x_j) of $\mathcal{A}_{cube}(x^{(n)})$, replace $||x_i - x_j||$ by $||x_i - x_j(\bmod 1)^d||$. From the restriction imposed on an edge-set problem's objective function, we obtain a feasible solution to the d-torus model of weight less than $L_{cube}(x^{(n)})$. Hence we have

$$L_{torus}(x^{(n)}) \leq L_{cube}(x^{(n)}). \tag{2.3}$$

Now let $\mathcal{F}(x^{(n)}) = \left\{(x_i, x_j) \in \mathcal{A}_{torus}(x^{(n)}) : ||x_i - x_j(\text{mod} 1)^d|| < ||x_i - x_j||\right\}$. Also, for $r < 1/2$, let $Q(r) = [0,1]^d \setminus [r, 1-r]^d$ be a layer of width r on the inside of the d-cube. Partition $\mathcal{F}(x^{(n)})$ in two sets $\mathcal{F}_1^{(r)}(x^{(n)}) = \{(x_i, x_j) \in \mathcal{F}(x^{(n)}) : x_i \in [r, 1-r]^d, x_j \in [r, 1-r]^d\}$, and $\mathcal{F}_2^{(r)}(x^{(n)}) = \mathcal{F}(x^{(n)}) \setminus \mathcal{F}_1^{(r)}(x^{(n)})$. Call their respective cardinality $k_1(r)$ and $k_2(r)$. From Condition 3 we then have

$$L_{cube}(x^{(n)}) \leq L_{torus}(x^{(n)}) + \gamma_d \left(k_1(r) + k_2(r)\right)^{c_d}. \tag{2.4}$$

From Condition 2, it is easy to see that

$$k_1(r) \leq \left|\mathcal{A}_{torus}(x^{(n)})\right| \leq D_d n, \tag{2.5}$$

and

$$k_2(r) \leq D_d |\{x_i \in Q(r)\}|. \tag{2.6}$$

Let us now consider a sequence $(X_i)_i$ of points independently and uniformly distributed over $[0,1]^d$. We then have from (2.5)

$$\mathbf{E}K_1(r) = \sum_{k=0}^{D_d n} k\mathbf{P}(K_1(r) = k) \leq D_d n \mathbf{P}(K_1(r) \geq 1), \tag{2.7}$$

and since the event $\{K_1(r) \geq 1\}$ is included in the event $\{\mathrm{MAX}(X^{(n)}) > r\}$ we get

$$\mathbf{E}K_1(r) \leq D_d n \mathbf{P}(\mathrm{MAX}(X^{(n)}) > r). \tag{2.8}$$

From (2.6), we also have

$$\mathbf{E}K_2(r) \leq D_d \mathbf{E}|\{X_i \in Q(r)\}| = D_d n[1 - (1-2r)^d]. \tag{2.9}$$

Finally we obtain, from (2.3),

$$\mathbf{E}L_{torus}(X^{(n)}) \leq \mathbf{E}L_{cube}(X^{(n)}), \tag{2.10}$$

and, from (2.4), (2.8), (2.9) and the concavity of $f(y) = \gamma_d y^{c_d}$, for $c_d \leq 1$,

$$\begin{aligned} \mathbf{E}L_{cube}(X^{(n)}) \;&\leq\; \mathbf{E}L_{torus}(X^{(n)}) \\ &+\; \gamma_d \left(D_d n \mathbf{P}(\mathrm{MAX}(X^{(n)}) > r) + D_d n[1 - (1-2r)^d]\right)^{c_d}. \end{aligned} \tag{2.11}$$

It is now easy to conclude from (2.10), (2.11), Conditions 1 and 4 (taking $r = r_n$, where $(r_n)_n$ is the sequence of Condition 4).

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Consequence: As a corollary to Theorem 1, if $\mathbf{E}L_{cube}(X^{(n)})/n^{c_d}$ happens to converge to the constant β_d, it will be true of $\mathbf{E}L_{torus}(X^{(n)})/n^{c_d}$, and vice-versa.

3 Examples

3.1 The minimum spanning tree problem and its constant

The minimum spanning tree problem consists of finding a spanning tree through a given set of points of minimum total length.

It is clear that the MST is a member of the class of edge-set problems. Now Condition 1 with $c_d = (d-1)/d$ is well-known for the MST (see for example [9]). From geometric considerations it is also easy (and well-known) that the MST in Euclidean metric spaces verifies Condition 2 with D_d bounded by the number of spherical caps with angle $\pi/3$ which are needed to cover the unit sphere in $\mathbf{R}^d$. In the proposition below we show that Condition 3 is also satisfied.

Proposition 1 *For the MST, Condition 3 holds.*

Proof:
From $\mathcal{A}_{torus}(x^{(n)})$, delete the edges (x_i, x_j) such that $||x_i - x_j(\bmod 1)^d|| < ||x_i - x_j||$. If k is the number of such edges, we end up with a forest of $k+1$ components. Pick one representative from each component, and construct the MST (in the d-cube) through these $k+1$ points. From Condition 1 the length of such a tree is bounded by $\alpha_d(k+1)^{(d-1)/d}$. Now the forest together with this tree form a spanning tree of $x^{(n)}$ in the d-cube, and this shows the validity of Condition 3.

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In order to show that Condition 4 is verified we need two intermediate lemmas.

Lemma 1 *Let $(Q_j)_{1\le j\le m^d}$ be a partition of the d-cube $[0,1]^d$ into cubes with edges parallel to the axle and of length $1/m$. If for a sequence of points $\{x_i : 1 \le i < \infty\}$, $x^{(n)} \cap Q_j$ is not empty for all j, then the MST in the d-torus is such that*

$$MAX(x^{(n)}) \le \frac{\sqrt{d+3}}{m}. \tag{3.12}$$

Proof:

This proof is a generalization of an argument used in [4] for the MST in the square. It goes as follows. Let e be an edge of $\mathcal{A}_{torus}(x^{(n)})$ so that its weight is $\mathrm{MAX}(x^{(n)})$. By discarding e we end up with a forest with two components, with point sets, say V_e and W_e such that for all $x_i \in V_e$ and all $x_j \in W_e$ we have $||x_i - x_j(\bmod 1)^d|| \ge \mathrm{MAX}(x^{(n)})$ (by definition of an optimal MST). We will now prove the lemma by contradiction. Let us assume that $\mathrm{MAX}(x^{(n)}) > \sqrt{d+3}/m$. Then $\mathrm{MAX}(x^{(n)}) > \sqrt{d}/m$ and thus each Q_j either contains points from V_e or from W_e but not from both. So the partition of the points into V_e and W_e leads to a partition of the cubes in two sets, I and J such that for all $i \in I$ we have $x^{(n)} \cap Q_i \in V_e$, and for all $j \in J$ we have $x^{(n)} \cap Q_j \in W_e$. Now, since all cubes are non-empty, we can always find a pair of adjacent (i.e, sharing a facet) cubes Q_i and Q_j with $i \in I$ and $j \in J$. But now, the largest possible edge connecting these two squares is bounded from above by $\sqrt{d+3}/m$ and thus, using our working hypothesis, by $\mathrm{MAX}(x^{(n)})$. This leads to a contradiction (see the beginning of our proof). Note that the same arguments hold for the problem in the d-cube.

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Lemma 2 *Let $(X_i)_i$ be a sequence of points independently and uniformly distributed over $[0,1]^d$, and let $(Q_i)_{1\le i\le m^d}$ be a partition of the d-cube $[0,1]^d$ into cubes with edges* *parallel to the axle and of length* $1/m$*. If* N_j *denotes the cardinality of* $X^{(n)} \cap Q_j$*, then, with* $p = 1/m^d$*, we have, for* $h \geq 12$ *and* $n \geq 3$*,*

$$\mathbf{P}\left(\forall j : N_j > np - \sqrt{hnp\log n}\right) \geq 1 - \frac{1}{2pn^{h/4}}. \tag{3.13}$$

Proof:

For all j let $\mathcal{B}_{n,j}$ be the event $\left\{N_j \leq np - \sqrt{hnp\log n}\right\}$. We obviously have

$$\mathbf{P}\left(\exists j : \mathcal{B}_{n,j}\right) \leq \sum_{j=1}^{m^d} \mathbf{P}\left(\mathcal{B}_{n,j}\right) = m^d \mathbf{P}\left(\mathcal{B}_{n,1}\right) = \mathbf{P}\left(\mathcal{B}_{n,1}\right)/p. \tag{3.14}$$

Now N_1 is a binomial random variable with n trials and parameter p. Using classical bounds on the tail of a binomial distribution (see [3, Corollary 4, p.11]) we have, with $q = 1 - p$,

$$\mathbf{P}\left(N_1 \leq np - \sqrt{hnp\log n}\right) \leq \frac{1}{2}\exp\left\{-h\log n/3q + 1/q\right\} \leq \frac{1}{2n^{h/4}}, \tag{3.15}$$

the last inequality being valid for $h \geq 12$, and $n \geq 3$. Now the lemma follows from (3.14) and (3.15).

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We are now in position to prove that Condition 4 holds for the MST.

Proposition 2 *Let* $(X_i)_i$ *be a sequence of points independently and uniformly distributed over* $[0,1]^d$*. Then for the MST we have*

$$\mathbf{P}\left(MAX(X^{(n)}) > \lambda_d \left(\frac{\log n}{n}\right)^{1/d}\right) \leq \frac{1}{n^2 \log n}, \tag{3.16}$$

where $\lambda_d = 12^{1/d}\sqrt{d+3}$*.*

Proof:

If we take $m^d (= 1/p) < n/(h\log n)$, then we have from Lemma 2

$$\mathbf{P}\left(\forall j : N_j > 0\right) > 1 - \frac{1}{2hn^{h/4-1}\log n}, \tag{3.17}$$

and from Lemma 1 we also have, for any $\varepsilon > 0$,

$$\mathbf{P}\left(\mathrm{MAX}(X^{(n)}) \leq (h+\varepsilon)^{1/d}\sqrt{d+3}\,(\log n/n)^{1/d}\right) \geq \mathbf{P}\left(\forall j : N_j > 0\right). \qquad (3.18)$$

The proposition follows from (3.17) and (3.18) by taking $h = 12$. Note again that the same arguments hold for the problem in the d-cube.

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Combining Theorem 1, Propositions 1 and 2, and the convergence in expectation of the MST in the d-cube (see for example [9]), we finally obtain our main result for the MST:

Theorem 2 *Let $(X_i)_i$ be a sequence of points independently and uniformly distributed over $[0,1]^d$. Then for the MST we have*

$$\lim_{n\to\infty} \frac{\mathbf{E}L_{torus}(X^{(n)})}{n^{(d-1)/d}} = \lim_{n\to\infty} \frac{\mathbf{E}L_{cube}(X^{(n)})}{n^{(d-1)/d}} = \beta_d(mst). \qquad (3.19)$$

Consequence: As a corollary to Theorem 2, the series expansion recently obtained for the MST constant in the d-torus (see [1]) is also valid for the classical Euclidean model of the MST. This is then one rare example, among this class of problems, for which we have been able to characterize the limiting constant analytically.

Before concluding this section, let us mention that in [9], the author studies the asymptotics of generalizations of the minimum spanning tree problem in which the distance between points are some fixed power of the Euclidean distance. It is quite clear that Theorem 2 can be readily extended to cover this case as well.

3.2 The traveling salesman and matching problems

The traveling salesman problem consists of finding a tour through a given set of points of minimum total length, and the minimum weight matching problem consists of finding a matching (a pairing) of these points of minimum total length. The analyses of these two problems are quite similar, so we will concentrate mainly on the TSP, giving, afterward, the specificities of the matching problem.

Here again, it is clear that the TSP belongs to the class of edge-set problems. Also, Condition 1 is quite well-known with $c_d = (d-1)/d$ (see for example [2]) and Condition 2 holds obviously. The verification of Condition 3 for the TSP is slightly more involved than for the MST but remains easy.

Proposition 3 *For the TSP, Condition 3 holds.*

Proof:
Again delete from $\mathcal{A}_{torus}(x^{(n)})$ all edges (x_i, x_j) such that $||x_i - x_j(\bmod 1)^d|| < ||x_i - x_j||$. If k is the number of such edges, we end up with k components, each of them being a path (possibly degenerated to a point). Pick one extremity (a point with a single adjacent edge) from each component, and construct a MST (in the d-cube) through these k points. From the previous section, the length of such a tree is bounded by $\alpha_d^{mst} k^{(d-1)/d}$. We have now a connected graph with less than $2k$ points with odd degree. We find a minimum weight matching (in the d-cube) on these points (it is easy to see that the number of such points is even). The extra weight will then be bounded by $\alpha_d^{mat}(2k)^{(d-1)/d}$ (since the matching problem verifies Condition 1, see for example [8]). Now the resulting graph is a spanning walk (see [6]) of total weight (in the d-cube) not exceeding $L_{torus}(x^{(n)}) + \alpha_d^{mst} k^{(d-1)/d} + \alpha_d^{mat}(2k)^{(d-1)/d}$. Since a tour of lesser weight can be obtained from this walk, this shows the validity of Condition 3.

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Now we will show that Condition 4 holds in a slightly modified model in which we use a Poisson point process. More precisely, let π_n denote a Poisson point process in $[0,1]^d$ with intensity equal to n times the Lebesgue measure λ. For any bounded Borel set $A \subset [0,1]^d$, let $\pi_n(A)$ denote the random set of points in A (almost surely a finite set of points) and $N_n(A)$ the cardinality of $\pi_n(A)$ (a Poisson random variable with parameter $n\lambda(A)$). When A is $[0,1]^d$, we will simply write Π_n and N_n. Now we have the following result.

Proposition 4 *Let π_n be a Poisson point process in $[0,1]^d$ with intensity equal to n times the Lebesgue measure, then for the TSP we have*

$$\lim_{n\to\infty} \mathbf{P}\left(MAX(\Pi_n) > \frac{\log n}{n^{1/d}} \right) = 0. \tag{3.20}$$

In order to prove this proposition we will need a series of intermediate results. In some of the results, we will use ε as a positive universal constant meant to be close to zero, but whose value might change from equation to equation. The first lemma is a geometrical property that exploits the property of a 2-change procedure (we recall that a 2-change procedure consists of replacing any two edges of a tour by two other edges so that we still have a tour). The lemma intuitively says that if we have a sufficient concentration of points in a given area, then for any optimal solution to the TSP, one can always find three points x_i, x_j, and x_k *in this area* such that (x_i, x_j) and (x_j, x_k) belong to the optimal solution. More precisely, we have the following.

Lemma 3 *Let $\{x_i : 1 \leq i < \infty\}$ be an arbitrary infinite sequence of points in $[0,1]^d$. For any $r < 1/8$ let $B(0,r) = \{y \in [0,1]^d : ||y(\text{mod}1)^d|| \leq r\}$ be the (d-torus) ball with radius r and center 0. Now if $|B(0,r) \cap x^{(n)}| \geq 12$, then , for any optimal solution to the TSP in the d-torus, there exists a point in $B(0,r) \cap x^{(n)}$ such that both its adjacent points (along the tour) belong to $B(0,4r)$.*

Proof:

This lemma follows from the fact that in an optimal solution, a 2-change procedure cannot improve the solution. Indeed let us consider a tour together with an arbitrary orientation. For each point x_i, let $x_{a(i)}$ and $x_{b(i)}$ be respectively its predecessor and successor along the tour. It is then easy to show that if we have 6 points (or more) in $B(0,r)$, each of them having its successor outside of $B(0,4r)$, then there is at least two of these points, say x_i and x_j, such that a 2-change procedure replacing $(x_i, x_{b(i)})$ and $(x_j, x_{b(j)})$ by (x_i, x_j) and $(x_{b(i)}, x_{b(j)})$ will give a saving greater than or equal to $(3r + 3r) - (4r + r) = r$ (i.e., will shorten the length of the tour by this quantity). Indeed, note that the 'worst case' configuration against the validity of our statement is (using the pigeonhole principle) given by 6 points evenly spread on the boundary of $B(0,r)$, with each successor $x_{b(i)}$ being on the boundary of $B(0,4r)$ at the intersection with the semi infinite line from 0 through x_i. Now the same argument (with 6 points) holds with predecessors. Now with very generous bounds, we can be sure that among 12 points, there is at least one point for which *both* its successor and predecessor belong to $B(0,4r)$.

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The next lemma is an adaptation of a technical result proved by Rhee [7, Lemma 3]. For any points x, y and z in $[0,1]^d$, let $S(x,y,z) = ||x - y(\text{mod}1)^d|| + ||y - z(\text{mod}1)^d|| - ||x - z(\text{mod}1)^d||$.

Lemma 4 *Let $X^{(n)} = \{X_1, \ldots, X_n\}$ be n points independently and uniformly distributed over $[0,1]^d$. For a given constant $c > 0$, consider $B(0, c/n^{1/d})$. Then, for any $\varepsilon_1 > 0$, there exists a constant $\delta > 0$ such that, for all pair of points X_i and X_j in $B(0, c/n^{1/d}) \cap X^{(n)}$, we have, for all n,*

$$\mathbf{P}\left(S(X_i, 0, X_j) \geq \frac{\delta}{n^{1/d}}\right) \geq 1 - \varepsilon_1. \tag{3.21}$$

Proof:

Let $N(c) = |B(0, c/n^{1/d}) \cap X^{(n)}|$. Then there exists a constant $k(c, \varepsilon)$, such that, for all n,

$$\mathbf{P}(N(c) \leq k(c,\varepsilon)) \geq 1 - \varepsilon. \tag{3.22}$$

Hence it suffices to show that for two independent points X and Y uniformly distributed on $B(0, c/n^{1/d})$, we have

$$\lim_{\eta \to 0} \mathbf{P}(S(X, 0, Y) \leq \eta/n^{1/d}) = 0, \tag{3.23}$$

where the limit is uniform in n. But this is obvious since the probability in the limiting expression is independent of n.

■

The two previous lemmas can now be put to work in order to prove the following key result.

Lemma 5 *Let $X^{(n)} = \{X_1, \ldots, X_n\}$ be n points independently and uniformly distributed over $[0,1]^d$ and, for any c, let $X(c) = B(0, c/n^{1/d}) \cap X^{(n)}$. Then for any $\varepsilon_2 > 0$, there exists two positive constants r and μ so that the event $\mathcal{H}(r, \mu) = \left\{\exists X_i \in X(r) : X_{a(i)}, X_{b(i)} \in B(0, 4r/n^{1/d}) \;\&\; S(X_{a(i)}, X_i, X_{b(i)}) \geq \mu/n^{1/d}\right\}$ verifies*

$$\mathbf{P}\left(\mathcal{H}(r, \mu)\right) \geq 1 - \varepsilon_2. \tag{3.24}$$

Moreover this statement remains true if one replaces the random variables $X^{(n)}$ by π_n, a Poisson point process in $[0,1]^d$ with intensity equal to n times the Lebesgue measure.

Proof:

First let $N(r) = |B(0, r/n^{1/d}) \cap X^{(n)}|$. By conditionning on the value of $N(r)$, and using Lemma 1, we have

$$\mathbf{P}\left(\mathcal{H}(r, \mu)\right) \geq \mathbf{P}\left(\mathcal{H}(r, \mu) | N(r) \geq 12\right) \mathbf{P}\left(N(r) \geq 12\right). \tag{3.25}$$

But $N(r)$ is a binomial random variable of expectation not depending on n and then one can always choose r large enough so that $\mathbf{P}\left(N(r) \geq 12\right) \geq 1 - \varepsilon$ for all $n \geq 12$. Now from Lemma 3, we know that if $N(r) \geq 12$, then there exists a point X_i, such that both $X_{a(i)}$ and $X_{b(i)}$ belong to $B(X_i, 5r/n^{1/d})$. Now, the application of Lemma 4 with $c = 5r$ implies that there exists a constant μ small enough so that $\mathbf{P}\left(S(X_{a(i)}, X_i, X_{b(i)}) \geq \mu/n^{1/d}\right) \geq 1-\varepsilon$. Indeed, note that in the d-torus, everything is unchanged through translation. The lemma is then proved for the uniform case. For the Poisson point process the argument is basically the same if one notices that we can always successively find three constants r, $K(r)$, and $M(K(r))$ large enough so that

$$\mathbf{P}\left(\{12 \leq N_n(r) \leq K(r)\} \cap \{N_n(5r) \leq M(K(r))\}\right) \geq 1 - \varepsilon. \tag{3.26}$$

where $N_n(c)$ stands for $N_n(B(0, c/n^{1/d}))$ (a Poisson random variable with a parameter not depending on n). Note that the requirement of $N_n(5r)$ being bounded from above by a constant is necessary to have a valid version of Lemma 4.

∎

We are now ready to complete the proof of Proposition 4.

Proof of Proposition 4:

First note, from Lemma 5, that if the event $\mathcal{H}(r,\mu)$ is true then we get a saving of at least $\mu/n^{1/d}$ by skipping point X_i from the tour. Let $\mathcal{A}_{torus}(\Pi_n)$ be a given optimal solution to the TSP through the points of Π_n. Let us look at the probability that a given edge of the tour has a length, say D (in the d-torus), greater than or equal to $\log n/n^{1/d}$. Take one such edge, say (X, Y), and divide it into three equal segments. Now take the middle segment and further divide it into $K - 1$ equal segments. Let $(z_j)_{1 \leq K}$ be the K endpoints defining the small segments and consider K adjacent d-balls of same radius and centered at these points. We choose K so that the common radius is $4r/n^{1/d}$ for a given r (we will then have $K \geq 1 + \log n/12r$). Now suppose that for at least one of the ball, say $B(z_j, r/n^{1/d})$ the event $\mathcal{H}(r,\mu)$ is true with a given point, say Z. Then one can transform the current solution by connecting Z to X and Y instead of its previous adjacent points. Since $Z \in B(z_j, r/n^{1/d})$ and $\min\{||z_j - X(\bmod 1)^d||, ||z_j - Y(\bmod 1)^d||\} \geq \log n/3n^{1/d}$ the extra cost of going from X to Y via Z will be of order $\Omega(1/n^{1/d}\log n)$ and the saving in the ball of order

$\Omega(1/n^{1/d})$. Hence, for n sufficiently large, we obtain a shorter tour. In conclusion, with the help of Lemma 5, we have

$$\mathbf{P}(D \geq \log n/n^{1/d}) \leq \varepsilon_2^K \leq \varepsilon_2^{\log n/12r+1} = \varepsilon_2 n^{\log \varepsilon_2/12r}. \tag{3.27}$$

Now for n sufficiently large, one can find a constant m so that $N_n \leq mn$ with a probability arbitrarily close to one, say $1-\varepsilon$. Hence we finally have

$$\mathbf{P}\left(\mathrm{MAX}(\Pi_n) > \frac{\log n}{n^{1/d}}\right) \leq mn(\varepsilon_2 n^{\log \varepsilon_2/12r}) + \varepsilon. \tag{3.28}$$

Now we can always find ε_2 so that the proposition is true.

∎

We are now in a position to give the main result for the TSP.

Theorem 3 *Let $(X_i)_i$ be a sequence of points independently and uniformly distributed over $[0,1]^d$. Then for the TSP we have*

$$\lim_{n\to\infty} \frac{\mathbf{E}L_{torus}(X^{(n)})}{n^{(d-1)/d}} = \lim_{n\to\infty} \frac{\mathbf{E}L_{cube}(X^{(n)})}{n^{(d-1)/d}} = \beta_d(tsp). \tag{3.29}$$

Proof:

From well-known results about the TSP (see for example [2] or [8]) it suffices to show that (3.29) holds for a Poisson point process in $[0,1]^d$ with intensity equal to n times the Lebesgue measure. But this a consequence of Propositions 3 and 4, and an adaptation of the proof of Theorem 1 to a Poisson process. The easiest way of adapting these arguments is first by conditionning on N_n so that we get

$$\begin{aligned} \mathbf{E}L_{cube}(\Pi_n|N_n) &\leq \mathbf{E}L_{torus}(\Pi_n|N_n) \\ &+ \gamma_d\left(D_d N_n \mathbf{P}(\mathrm{MAX}(\Pi_n > r|N_n) + D_d N_n[1-(1-2r)^d]\right)^{c_d}, \end{aligned}$$

and then by remarking that there exists a constant m so that

$$\mathbf{E}[N_n \mathbf{P}(\mathrm{MAX}(\Pi_n > r|N_n)] \leq mn\mathbf{P}(\mathrm{MAX}(\Pi_n) > r) + o(1). \tag{3.30}$$

All the other arguments in the proof of Theorem 1 remain valid for the Poisson point process.

■

Now let us consider the minimum weight matching problem. It is an edge-set problem and Conditions 1, 2, and 3 are either well-known (see [8]) or obvious. For the proof of Condition 4 we proceed exactly as before. Indeed, here again, a close analysis of the 2-change procedure show that if there are more than 6 points in $B(0,r) \cap x^{(n)}$ then there is at least one point among them such that its matching point is inside $B(0,4r)$. Now a lemma such as Lemma 4 is not needed anymore and we can prove directly a result similar to Lemma 5. This result will basically say that with high probability one can find a point among $\pi_n\left(B(0,c/n^{1/d})\right)$ such that if we delete its matching edge, we gain at least $\mu/n^{1/d}$ for a positive constant μ. The proof of Condition 4 is then identical to the one given for the TSP.

4 Stronger equivalences for the MST

For the MST the equivalence in expectation can be strengthened into an equivalence with probability one.

Theorem 4 *Let $(X_i)_i$ be a sequence of points independently and uniformly distributed over $[0,1]^d$. Then for the MST we have*

$$\lim_{n\to\infty} \frac{L_{torus}(X^{(n)})}{n^{(d-1)/d}} = \lim_{n\to\infty} \frac{L_{cube}(X^{(n)})}{n^{(d-1)/d}} = \beta_d(mst) \ (a.s.). \tag{4.31}$$

Proof:

From the proof of Theorem 1 we recall that, for the MST, we have

$$L_{torus}(X^{(n)}) \le L_{cube}(X^{(n)}) \le L_{torus}(X^{(n)}) + \gamma_d \left(K_1(r) + D_d|\{X_i \in Q(r)\}|\right)^{c_d}, \tag{4.32}$$

where $Q(r)$ is the layer of width r on the inside of the d-cube and $K_1(r)$ is the cardinality of $\mathcal{F}_1^{(r)}(X^{(n)})$ (see the proof of Theorem 1 for definition). Now from the strong law of large numbers we have

$$\lim_{n\to\infty} \frac{|\{X_i \in Q(r)\}|}{n} = 1 - (1-2r)^d \text{ (a.s.)}. \tag{4.33}$$

Also from Proposition 2 we have

$$\sum_{n=1}^{\infty} \mathbf{P}(K_1(r_n) > 0) \leq \sum_{n=1}^{\infty} \mathbf{P}(\text{MAX}(X^{(n)}) > r_n) \leq \sum_{n=1}^{\infty} \frac{1}{n^2 \log n} < \infty, \tag{4.34}$$

where $r_n = \lambda_d(\log n/n)^{1/d}$. From the Borel-Cantelli lemma this implies that

$$\lim_{n\to\infty} K_1(r_n) = 0 \text{ (a.s.)}. \tag{4.35}$$

The result (4.31) then follows from (4.32), (4.33), (4.35), and the almost sure convergence of $L_{cube}(X^{(n)})/n^{(d-1)/d}$ to $\beta_d(mst)$ (see [9]).

∎

Let us end this section by a stronger version of Condition 3 for the MST. This result might prove valuable for comparing the d-torus and the d-cube in terms of asymptotic distribution for the MST. The result is the following.

Lemma 6 *Let $\{x_i : 1 \leq i < \infty\}$ be an arbitrary infinite sequence of points in $[0,1]^d$ and $\mathcal{A}_{torus}(x^{(n)})$ be an optimal MST in the d-torus through $x^{(n)}$. Let $\mathcal{E}_{torus}(x^{(n)})$ be the set of edges (x_i, x_j) of this solution such that $||x_i - x_j(\text{mod}1)^d|| = ||x_i - x_j||$ (i.e., that do not 'cross' the boundary of the d-cube). Then there exists an optimal solution for this problem in the d-cube, say $\mathcal{A}^*{}_{cube}(x^{(n)})$, such that:*

$$\textit{If } (x_i, x_j) \textit{ belongs to } \mathcal{E}_{torus}(x^{(n)}), \textit{ then } (x_i, x_j) \textit{ belongs to } \mathcal{A}^*{}_{cube}(x^{(n)}) \tag{4.36}$$

Proof:

Take any edge $(x_i, x_j) \in \mathcal{E}_{torus}(x^{(n)})$ and consider a given solution $\mathcal{A}_{cube}(x^{(n)})$ in the d-cube. Suppose that (x_i, x_j) does not belong to this solution. Then $\mathcal{A}_{cube}(x^{(n)}) \cup \{(x_i, x_j)\}$ contains a unique cycle, say $\mathcal{C}_{cube}$, such that for all edges $(x_k, x_l) \in \mathcal{C}_{cube} \setminus \{(x_i, x_j)\}$ we have

$$||x_k - x_l|| < ||x_i - x_j||. \tag{4.37}$$

(Note that we can discard the easy case for which there is an edge (x_k, x_l) such that $||x_k - x_l|| = ||x_i - x_j||$. Indeed we would then exchange the two edges and obtain an optimal solution in the d-cube that verifies (4.36) for the edge (x_i, x_j) under consideration). Now among the edges of $\mathcal{C}_{cube} \setminus \{\mathcal{A}_{torus}(x^{(n)}) \cap \mathcal{C}_{cube}\}$ there is at least one edge, say (x_k, x_l) such that $\mathcal{A}_{torus}(x^{(n)}) \cup \{(x_k, x_l)\}$ has a cycle containing (x_i, x_j). The proof of this key result goes as follows. Let $Z = (z_1, \ldots, z_m)$ be the points (other than x_i and x_j) along the cycle $\mathcal{C}_{cube}$ and numbered as they appear from x_i to x_j. By definition of a tree there is a unique path in $\mathcal{A}_{torus}(x^{(n)})$ going from x_i to each of these points. Color a point of Z red if this path does not go through x_j and blue otherwise. Note that the blue points can alternatively be defined as the points reached from x_j without going through x_i. Also color x_i red and x_j blue. Now any edge of $\mathcal{C}_{cube} \setminus \{(x_i, x_j)\}$ with adjacent points of opposite color, if added, would form a cycle in $\mathcal{A}_{torus}(x^{(n)})$ containing (x_i, x_j). Along the cycle starting from x_i and in the opposite direction of x_j we must find such an edge, since these two points are of opposite color. It is now easy to conclude. Indeed if we remove (x_i, x_j) from $\mathcal{A}_{torus}(x^{(n)})$ and replace it by (x_k, x_l), we end up (from (4.37)) with a spanning tree of weight less than $L_{torus}(x^{(n)})$: A contradiction.

∎

5 Concluding remarks

In the course of proving the main theorems of this paper we have obtained several results of independent interests. For example, in Proposition 2, we have proved that for n points i.i.d. uniform on $[0,1]^d$, the length of the largest edge of the optimal MST solutions (in the d-cube or d-torus) is *almost surely* asymptotically bounded from above by $\lambda_d(\log n/n)^{1/d}$. Now it is easy to show (see for example [4]) that, for a Poisson point process π_n with intensity n times the Lebesgue measure on $[0,1]^d$, there exists a constant ξ_d, such that this last quantity is asymptotically bounded from below by $\xi_d(\log n/n)^{1/d}$ with a probability going to 1 as n goes to infinity. So in the Poisson case the growth of the largest edge is $\Omega((\log n/n)^{1/d})$ in probability. Is it true for the *uniform case*, and with *in probability* replaced by *almost surely* ? We think so. Also the same questions arise for the TSP (or matching problem). In this paper (see Lemma 5) we have shown that the largest edge in the Poisson case is asymptotically bounded from above by a constant times $\log n/n^{1/d}$ (in probability). We feel that our techniques could be sharpen so that we get $(\log n/n)^{1/d}$ instead (but how ?). More generally we conjecture that in the *uniform case* the growth of the largest edge of a TSP (in the d-cube or d-torus) is *almost surely* $\Omega((\log n/n)^{1/d})$.

Let us finally conclude with some remarks on the problem of finding, for the TSP in the plane, the rate of convergence of the normalized mean $\mathbf{E}L_{cube}(X^n)/\sqrt{n}$ to β_{tsp}. In [5], we have shown that

$$\left|\mathbf{E}L_{cube}(X^n)/\sqrt{n} - \beta_{tsp}\right| \le c_1/\sqrt{n}, \tag{5.38}$$

for a positive constant c_1, but we have not been able to show that this result is best in the sense that $|\mathbf{E}L_{cube}(X^n)/\sqrt{n} - \beta_{tsp}| = \Omega(1/\sqrt{n})$. Now let us consider the easier case of the Poisson point process model in the d-cube. Under this assumption, if one follows the usual subadditivity argument (see, for example [2, 8]), one can easily get the following *one-sided* bound

$$\mathbf{E}L_{cube}(\Pi_n)/\sqrt{n} \ge \beta_{tsp} - d/\sqrt{n}, \tag{5.39}$$

for a positive constant d. Let us improve (5.39) by replacing the traditional *partionning and patching* way of getting the subadditivity inequality by a *recursive* way. We divide $[0,1]^2$ into 4 squares with edges parallel to the axle and of side length $1/2$ and we solve the TSP in each of them. Now we simply connect each small tours to the center of $[0,1]^2$ (by a double link of smallest length). Starting with π_{4n} in $[0,1]^2$, we then obtain $\mathbf{E}L_{cube}(\Pi_{4n}) \leq 2\mathbf{E}L_{cube}(\Pi_n) + k/\sqrt{n}$, where k is a positive constant. By using this inequality recursively we finally get

$$\mathbf{E}L_{cube}(\Pi_n)/\sqrt{n} \geq \beta_{tsp} - 4k/3n. \tag{5.40}$$

Can we do better ? We feel that the techniques of Subsection 3.2 (in particular Lemma 5) could (?) now play a role. The idea is to improve the feasible solution (obtained from the connection of the 4 TSP tours) by considering potential savings along the borderline of two given adjacent (i.e., sharing a facet) small squares. More precisely, in the 2-torus, if we consider a small ball of radius $\Omega(1/\sqrt{n})$ and centered on this borderline, we conjecture that, with a positive probability p, there exists a point in the ball such that if we skip it along its current small TSP tour and connect it to the TSP in the other square, we get a saving of $\Omega(1/\sqrt{n})$. Note that if this is true, by placing $\Omega(\sqrt{n})$ such balls centered on the borderline (much as in the proof of Proposition 4), we would have an expected total saving of $O(1)$. This in turn would imply that, for large n,

$$\mathbf{E}L_{torus}(\Pi_{4n}) \leq 2\mathbf{E}L_{torus}(\Pi_n) - a, \tag{5.41}$$

for a positive constant a. As a conclusion, note that if (5.41) is true (no matter how it is proved), then, from Theorem 3, we would have $\mathbf{E}L_{cube}(\Pi_n)/\sqrt{n} \geq \beta_{tsp} + c_2/\sqrt{n}$, for a positive constant c_2. Together with (5.38), the final conclusion would then be that

$$\mathbf{E}L_{cube}(\Pi_n)/\sqrt{n} = \beta_{tsp} + \Omega(1/\sqrt{n}). \tag{5.42}$$

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