Prof. E. A. Guillemin
Dr. M. V. Cerrillo
E. F. Bolinder
N. DeClaris
P. M. Lewis II
J. C. Pinson

A. SECOND-ORDER SADDLEPOINTS ON THE RIEMANN SPHERE

Second-order saddlepoints play a fundamental role in methods of approximate integration. In these methods not only the trajectories of the saddlepoints in the complex frequency plane (s-plane) but also the varying velocities of the saddlepoints along the trajectories are of great interest. Two engaging questions are: With what speed do two second-order saddlepoints coalesce, thereby creating a third-order saddlepoint? With what speed do they afterward separate in directions 90° from the incoming directions? These matters can be studied easiest by giving an example.

The Bessel function $J_o(t)$ has the Laplace transform

$$F(s) = (s^2 + 1)^{-1/2}$$

The trajectories of the two second-order saddlepoints that belong to the inverse Laplace transform have been calculated (1)

$$s_{s_{1,2}} = \frac{1}{2t} \pm \left[\left(\frac{1}{2t}\right)^2 - 1\right]^{1/2}$$

The trajectories are sketched in Fig. XVI-1. A third-order saddlepoint is created at $s = 1$ for $t = 1/2$. A simple calculation shows that one of the two saddlepoints starts out at infinity (primary saddlepoint) and the other at $s = 0$ (secondary saddlepoint) with initial velocities $\left(v_{s_1}\right)_{t=0} = (-)\infty$ and $\left(v_{s_2}\right)_{t=0} = (+)1$. While the primary saddlepoint slows down and then accelerates to $v_{s_1} = (-)\infty$ at $s = 1$, the secondary saddlepoint accelerates continuously to $v_{s_2} = (+)\infty$ at $s = 1$. After coalescence, the two primary saddlepoints move along the unit circle with a decreasing speed that varies from infinity at $s = 1$ to zero at $s = \pm i$.

An interesting representation of the saddlepoint movements is obtained by mapping the complex frequency plane stereographically on the Riemann unit sphere. See Fig. XVI-2.

Simple calculations show that the line L_1 through the saddlepoints at $t = 0$ coincides with the z-axis. For $t > 0$, L_1 moves along the x-axis with a velocity $dx/dt = 2$; at the same time being parallel to the z-axis. Its polar, L_2, starts out with infinite velocity at infinity and slows down according to the law $dx/dt = (-)1/2t^2$. At $t = 1/2$, the two lines L_1 and L_2 are both tangent to the sphere at the point (1, 0, 0); the velocities are the same $\left[(+)2 \text{ and } (-)2\right]$. For $t > 1/2$, the two saddlepoints are obtained as points at which the line L_2 cuts the surface of the sphere. The velocity of L_2 approaches zero as L_2

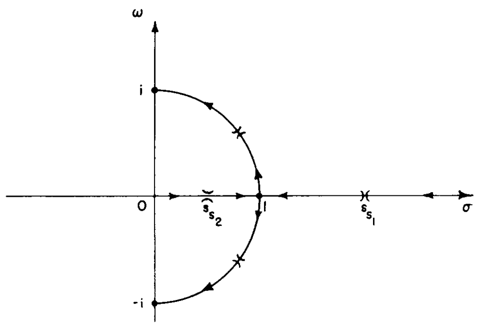

Fig. XVI-1. Saddlepoint trajectories in the s-plane.

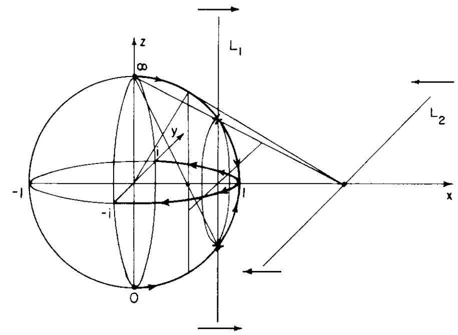

Fig. XVI-2. Saddlepoint trajectories on the Riemann unit sphere.

approaches the y-axis; simultaneously, its polar L_1 moves with constant speed toward infinity.

The movements of the lines L_1 and L_2 in three-dimensional (hyperbolic) space and the interpretation of the saddlepoints as being the points where L_1 or L_2 cuts the surface of the sphere furnish the explanation for the infinity of the saddlepoint velocities at a third-order saddlepoint in the complex frequency plane. The saddlepoint movements, which seem to be highly discontinuous in the s-plane, can actually be considered quite smooth when they are on the surface of a sphere. An interesting analogy exists between the movements of saddlepoints with time on the surface of the Riemann sphere and the movements of fixed points of impedance transformations with the variation of, let us say, an element of a network in network theory (see Sec. XVII-C).

E. F. Bolinder

References

1. M. V. Cerrillo, Technical Report 55:2a, Research Laboratory of Electronics, M.I.T., May 3, 1950.