XXI. MICROWAVE THEORY

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RESEARCH OBJECTIVES

The aim of this work is to investigate means by which modern (higher) geometry can be used in solving microwave problems and in simplifying solutions that are already being applied to these problems.

The branches of mathematics that are useful in dealing with microwave problems are analysis, algebra, and geometry. Of these, the first two have found extensive application in microwave theory. Although both physicists and engineers favor graphic representation of the problems they are trying to solve, geometry seems to have been applied to a limited extent. Perhaps an investigator, who, by the nature of his training, is able to use both analysis and higher algebra, often considers elementary geometrical treatment difficult to understand. It is important to stress that modern (higher) geometry has advanced beyond the graphical constructions that can be performed with ruler and compass. This will be understood by a quick glance at the geometrical portions of the collected works of Gauss, Riemann, Cayley, Klein, Lie, Clifford, and Poincaré.

Two papers stand out as having been of fundamental importance in the development of modern geometry. These are Riemann's "Über die Hypothesen, welche der Geometrie zu Grunde liegen" (completed in 1854, published in 1868), and Felix Klein's "Vergleichende Betrachtungen über neuere geometrische Forschungen" (1872). In the first paper, which initiated the development of Differential Geometry, Riemann discusses, among other things, manifolds of n dimensions of constant curvature. If this curvature is negative, and if $n = 3$, we have the non-Euclidean geometry of Gauss, Bolyai, and Lobachevsky, to which Klein gave the name "hyperbolic geometry." If the curvature is zero, we have Euclidean geometry, which he called "parabolic geometry." Finally, for a positive curvature, Riemann created another non-Euclidean geometry, which Klein called "elliptic geometry." (We cannot be sure, however, whether Riemann thought of his geometry as spherical or elliptical.) In elliptic geometry, space is unbounded but finite.

In the second paper, Klein proposed a program, the famous "Erlangen Program," for the unification of the principal geometries. He classified geometric properties and assigned them to different geometries according to the invariant properties of corresponding transformation groups.

Some results from the first period of our investigation are: A graphical method, called "the isometric circle method," was introduced for impedance transformations through bilateral, two terminal-pair networks (Quarterly Progress Report, April 15, July 15, 1956, and Section XXI-A below). The method was extended and generalized in the two- and three-dimensional hyperbolic spaces (Quarterly Progress Report, April 15, 1956). By using three-dimensional hyperbolic space with the unit sphere as the fundamental surface, two terminal-pair networks can be analyzed and the efficiency for arbitrary loading can be determined (Quarterly Progress Report, Oct. 15, 1956, and Section XXI-B below).

A natural approach for carrying out our research program is to base it on the principal geometrical works that are pertinent and extend the study to some geometrical works on network and microwave theory – for example, those of Feldtkeller, Weissfloch, Van Slooten, and Deschamps.

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A. THE ISOMETRIC CIRCLE METHOD IN ANALYTICAL FORM

In the Quarterly Progress Report of April 15, 1956, a graphical method was presented for transforming impedances through bilateral, two terminal-pair networks. If the same notations are used, the different steps for performing the graphical constructions can be expressed in analytical form (a star indicates a complex-conjugate quantity).

$$Z' = \frac{aZ + b}{cZ + d} \qquad D = ad - bc \tag{1}$$

$$Z_1^* = -\frac{\dfrac{d^*}{c^*} Z + \dfrac{|d|^2 - |D|}{|c|^2}}{Z + \dfrac{d}{c}} \equiv \text{inversion in } C_d \tag{2a}$$

$$Z_2 = \frac{-\dfrac{a+d}{c} Z_1^* + \dfrac{|a|^2 - |d|^2}{|c|^2}}{\dfrac{a^* + d^*}{c^*}} \equiv \text{reflection in L} \tag{2b}$$

$$Z' - \frac{a}{c} = \left(Z_2 - \frac{a}{c}\right) \exp[-j(2 \arg (a+d) + \arg D)] \equiv \text{rotation around } O_i \tag{2c}$$

Usually the complex constants (the frequency is assumed to be fixed) a, b, c, and d in Eq. 1 are multiplied by a constant, so that D = 1. In order to show how the reciprocal part is "filtered" out by the use of the isometric circle method, this procedure is split up, so that the absolute value of D appears in Eq. 2 and the phase angle in Eq. 4.

If the network is lossless (nonloxodromic case) the equations simplify to

$$Z' = \frac{a' Z + jb''}{jc'' Z + d'} \qquad D = a'd' + b''c'' = \text{real} \tag{3}$$

$$Z_1^* = \frac{d' Z - j \dfrac{d'^2 - D}{c''}}{jc'' Z + d'} \equiv \text{inversion in } C_d \tag{4a}$$

$$Z' = Z_1^* - j \frac{a' - d'}{c''} \equiv \text{reflection in L} \tag{4b}$$

The corresponding transformations in the complex reflection-coefficient plane, the Smith chart, are:

$$\Gamma' = \frac{A\Gamma + C^*}{C\Gamma + A^*} \qquad |A|^2 - |C|^2 = 1 \tag{5}$$

$$\Gamma_1 = \frac{-\dfrac{A^*}{C} \Gamma^* - 1}{\Gamma^* + \dfrac{A}{C^*}} \equiv \text{inversion in } C_d \tag{6a}$$

$$\Gamma' = -\frac{C^*}{C} \Gamma_1^* \equiv \text{reflection in L} \tag{6b}$$

In the nonloxodromic case, the rotation around the center of the isometric circle of the inverse transformation disappears. Equations 2a, 2b, 4a, 4b, 6a, and 6b are all of the form:

$$Z' = \frac{aZ^* + b}{cZ^* + d} \qquad \Gamma' = \frac{A\Gamma^* + B}{C\Gamma^* + D} \tag{7}$$

Following Cartan (1), Eq. 1 can be denoted a homograph and Eq. 11 an antihomography. The antihomographies are of two kinds: the anti-involutions of the first kind, which are characterized by having at least one fixed point (examples: $Z' = 1/Z^*$, $Z' = Z^*$), and the anti-involutions of the second kind, which have no fixed point (example: $Z' = -1/Z^*$). In the complex plane every anti-involution of the first kind is represented by an inversion with positive power or by a symmetry in relation to a straight line.

From Cartan (1) we select the following points:

1. Every homography can be considered as a product of two involutions.

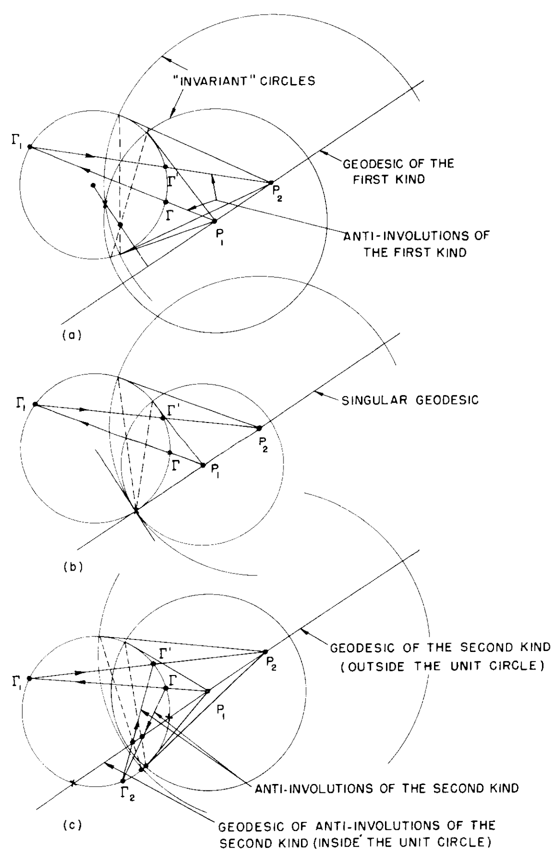

Fig. XXI-1. Transformations composed of two anti-involutions: (a) elliptic; (b) parabolic; (c) hyperbolic.

2. Every involution is the product of two exchangeable anti-involutions of the first kind.

3. Every homography can be considered as a product of four anti-involutions of the first kind.

4. The anti-involutions of the first kind can be represented by the points that are exterior to the sphere; the ones of the second kind by the points that are interior to the sphere.

5. The product of two anti-involutions of the first kind is an elliptic, hyperbolic or parabolic homography, depending on whether the invariant circles intersect, do not intersect or are all tangent.

An involution is a loxodromic transformation with a multiplier = -1 (example: ideal gyrator, $Z' = 1/Z$). Since the rotation in Eq. 4 is a product of two reflections, the isometric circle method is basically composed of four anti-involutions of the first kind. In the nonloxodromic case the number is reduced to two.

The theory stated in point 5 above, as exemplified by the complex reflection-coefficient plane in Fig. XXI-1a, b, and c, is the connecting link between the works of Deschamps (2), Van Slooten (3), de Buhr (4), and the writer (5). The isometric circle method for nonloxodromic transformations can be illustrated by Fig. XXI-1 for the special case of $P_2 \to \infty$ and $P_1 = O_d$. See reference 5.

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References

1. E. Cartan, Lecons sur la Géométrie Projective Complexe (Gauthier-Villars, Paris, 1931).

2. G. A. Deschamps, Proc. Symposium on Modern Network Synthesis, Polytechnic Institute of Brooklyn, April 1952, pp. 277-295.

3. J. Van Slooten, Meetkundige Beschouwingen in Verband met de Theorie der Electrische Vierpolen (W. D. Meinema, Delft, 1946).

4. J. de Buhr, NTZ, 80-84 (Feb. 1956).

5. E. F. Bolinder, Acta Polytechnica, Electrical Engineering Series No. 202, 1956.

B. GRAPHICAL DETERMINATION OF THE EFFICIENCY OF BILATERAL, TWO TERMINAL-PAIR NETWORKS

In the Quarterly Progress Report of October 15, 1956, page 131, it was shown that the image circle, obtained in the complex-impedance plane as an image of the imaginary axis, after a stereographic mapping on the unit sphere, can be moved until it is symmetric with the xz- and xy-planes. The transformations that are involved correspond to impedance transformations through bilateral, lossless, two terminal-pair networks

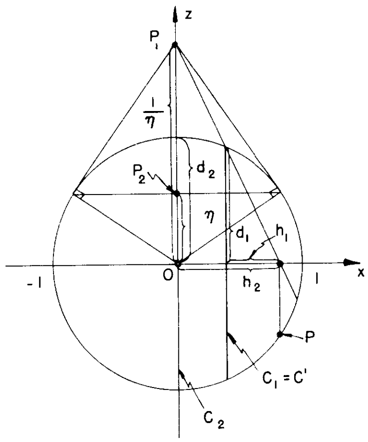

Fig. XXI-2. Graphical determination of the efficiency η.

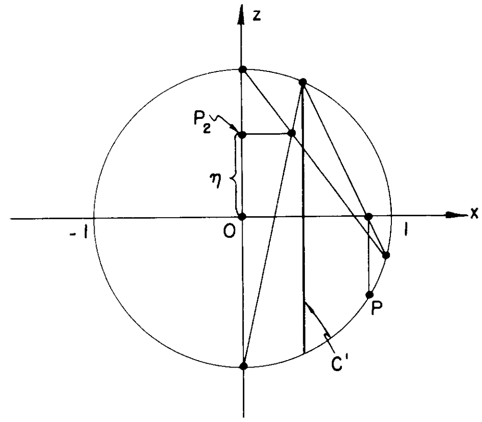

Fig. XXI-3. Simplified determination of the efficiency η.

which do not change the efficiency of the network.

Deschamps (1) has shown that the efficiency of a two terminal-pair network can be expressed as the ratio of two "pseudo distances" from a point P, corresponding to an arbitrary loading impedance transformed through the network, to two circles C_1 and C_2. In Fig. XXI-2, C_1 is the projection on the xz-plane of the transformed image circle C', and C_2 is the projection of the great circle on the yz-plane. For passive networks, P is always situated at the right of C_1. With the notations of Fig. XXI-2, the efficiency η of the network can be written as

$$\eta = \frac{h_1}{d_1} \frac{d_2}{h_2} \tag{1}$$

where, in our case, $d_2 = 1$, because it is the radius of the unit circle. Uniform triangles yield the distance $OP_1 = 1/\eta$ on the z-axis, so that the polar to P_1 cuts the z-axis at $OP_2 = \eta$. Figure XXI-3 shows a simple construction for obtaining P_2 without making any constructions outside the unit circle.

It is interesting to check that if the point P is in (1, 0), then $\eta = \eta_{max}$. See Fig. XXI-4. It is clear that P_2 can be constructed in the usual way or it is simply the crossover point between the z-axis and a line joining (-1, 0) and the upper point of C_1.

$$\eta = \eta_{max} = \frac{h_1}{d_1} \tag{2}$$

Since the yz-plane constitutes the complex reflection-coefficient plane, Fig. XXI-4

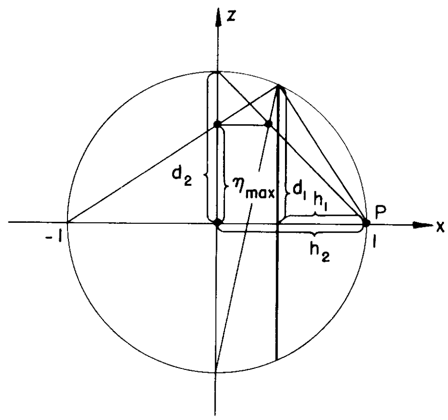

Fig. XXI-4. Graphical determination of the maximum efficiency η_{max}.

reveals that η_{max} is the radius of the image circle C' stereographically mapped on the yz-plane from (-1, 0). This checks nicely with reference 2.

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References

1. G. A. Deschamps, Proc. Symposium on Modern Network Synthesis, Polytechnic Institute of Brooklyn, April 1952, pp. 277-295.
2. H. W. Wheeler and D. Dettinger, Wheeler Monograph No. 9, March 1949.