XXIII. MATHEMATICS

A. ON BINARY RELATIONS

It is well known that propositional logic has a wide application to digital computors and automata. Theoretically, propositional logic, which can be considered as a Boolean lattice, is also equivalent to a Boolean ring; hence, the choice of a system for a given purpose is merely a matter of convenience. The Boolean lattice is a set theory with operations of union, intersection, and complement or difference. We can establish the theory of relations in set theory, because every relation can be taken as a set of finite sequences of elements. Conversely, we can express every set M in terms of the binary relation $R = \{\langle x, x\rangle : x \in M\}$, and if the sets M and N are expressed by the binary relations R and S, respectively, then $R \cup S$, $R \cap S$, and $R - S$ express sets $M \cup N$, $M \cap N$, and $M - N$, respectively. Therefore, theoretically, the theory of binary relations can perform the same role for computors and automata as propositional logic. Moreover, it has other useful operations, such as the relative product and converse, and there are many applications that can be expressed in terms of relations. I believe that the theory of binary relations, which is one of the most interesting fields in logic, should be cultivated for practical purposes also (1).

Any relation can be taken as a set of ordered pairs $\langle x, y\rangle$. The set $R(X)$, obtained by operating the relation R on the set X, is defined by $R(X) = \{y; \langle y, x\rangle \in R,\ x \in X\}$. For any relations R and S, there is a unique relation T, which satisfies $T(X) = R[S(X)]$ identically; hence we would denote by T the relative product RS. The relation $\{\langle x, y\rangle : \langle y, x\rangle \in R\}$ is called the converse R^{-1} of R. $R(X)$, RS, and R^{-1} are expressed by $R``X$, R/S, $\breve{R}$, respectively, by Whitehead and Russell (2). We can use these operations universally, whatever relations R and S may be, and whatever set X may be. These operations, together with union, intersection, and difference, have simple properties, some of which originate from the operational properties of relations that operate on atomic domains.

In one approach, we let L be a complete atomic lattice. Small letters will be used for atoms of the lattice. "L is complete atomic" means here that $A = \bigcup_{a \leq A} a$ holds true for every A in L. Now, we call the mapping R of L into another complete lattice M a generalized relation when, and only when, $R(X) = \bigcup_{x \leq X} R(x)$. For two generalized relations R and S, we can define $R \leq S$ by $(x)\,[R(x) \leq S(x)]$. The totality $\mathscr{L}$ of generalized relation form L to M forms a complete lattice with the following properties:

1. $\mathscr{L}$ is atomic, if and only if M is atomic.
2. $\mathscr{L}$ is Boolean, if and only if M is Boolean.
3. $\mathscr{L}$ and M have the same polynomial identities between polynomials composed by "$\cup$" and "$\cap$".

As a consequence of property 3, we conclude that $\mathscr{L}$ is distributive or modular if, and only if, M is distributive or modular. We can define the relative products of the generalized relations, but the product fails to satisfy the associative law.

Another approach is to establish the system of propositions to which we can reduce all the theory of relations (3, 4). If we take relative product as the fundamental operation in the lattice of relations, which is assumed to be complete, atomic, and Boolean, we obtained the following simple system of postulates:

1. $AB = \bigcup_{a \leq A,\ b \leq B,\ p = ab} p.$
2. $(ab)c = a(bc)$.
3. $abc = 0$ implies $ab = 0$ or $bc = 0$.
4. There exists one, and only one, x that satisfies $axb > 0$.

All the properties of binary relations can be derived from these simple postulates.

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References

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