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*Sufficient conditions for finite-time
stability of impulsive dynamical systems*

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APPENDIX F
CONSERVATIVENESS OF THE DISTURBANCE BOUND

We prove the assertion from Remark 4. We indicate the changes needed in the proof of Theorem 2. We argue as in the earlier proof up through (31), and we let $M \geq 2$ be any constant. Consider two cases.

Case 1M: If $|\tilde{s} + \tilde{x}_1 + \tilde{x}_2| \geq Ms_{in}$, then $\tilde{v} = |\tilde{s}| + |\tilde{x}_1| + |\tilde{x}_2| \leq 2|\tilde{s} + \tilde{x}_1 + \tilde{x}_2|$, by Lemma 2 since $M \geq 2$. We deduce that $\Delta\kappa_1 \leq \Delta\kappa_1(\tilde{s} + \tilde{x}_1 + \tilde{x}_2)^2/\{M^2s_{in}^2\}$, $\Delta C_2[2\kappa_1 + \tilde{v}] \leq \Delta C_2[\{(2\kappa_1)/(Ms_{in})\}|\tilde{s} + \tilde{x}_1 + \tilde{x}_2| + \tilde{v}] \leq \Delta C_2[\{(2\kappa_1)/(Ms_{in})\} + 2]|\tilde{s} + \tilde{x}_1 + \tilde{x}_2|$, $\Delta\bar{\alpha}|\tilde{x}_1| \leq 2\Delta\bar{\alpha}(\tilde{s} + \tilde{x}_1 + \tilde{x}_2)^2/\{Ms_{in}\}$, and $\Delta\bar{\beta}|\tilde{x}_2| \leq 2\Delta\bar{\beta}(\tilde{s} + \tilde{x}_1 + \tilde{x}_2)^2/\{Ms_{in}\}$, by multiplying by $|\tilde{s} + \tilde{x}_1 + \tilde{x}_2|/\{Ms_{in}\} \geq 1$. Substituting into (31), noting that $\Delta \leq 1/\{2\kappa_2\}$, and grouping terms gives $V_2 \leq -W_3(\tilde{s}, \tilde{\xi}_1, \tilde{\xi}_2) + \Delta\kappa_2\tilde{Z}^2 \leq -W_4(\tilde{s}, \tilde{\xi}_1, \tilde{\xi}_2)$ where κ_2 is now as in Remark 4.

Case 2M: If $|\tilde{s} + \tilde{x}_1 + \tilde{x}_2| \leq Ms_{in}$, then a slight variant of the proof of Lemma 3 gives $|\tilde{s}| + |\tilde{x}_1| + |\tilde{x}_2| \leq (2+M)s_{in}$. Therefore, (31) gives $\dot{V}_2 \leq -W_4(\tilde{s}, \tilde{\xi}_1, \tilde{\xi}_2) + C_3\Delta$ where C_3 is now redefined to be $C_3 = 2\kappa_1 + C_2M(2+M)s_{in}^2 + [2MC_2\kappa_1 + \{\bar{\alpha} + (x_{1*}/x_{2*})\bar{\beta}\}(2+M)]s_{in}$. The rest of the proof is exactly as before, because we again have (32).

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**Sufficient Conditions for Finite-Time Stability
of Impulsive Dynamical Systems**

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Gianmaria De Tommasi

Abstract—The finite-time stability problem for state-dependent impulsive dynamical linear systems (SD-IDLS) is addressed in this note. SD-IDLS are a special class of hybrid systems which exhibit jumps when the state trajectory reaches a *resetting set*. A sufficient condition for finite-time stability of SD-IDLS is provided. *S*-procedure arguments are exploited to obtain a formulation of this sufficient condition which is numerically tractable by means of Differential Linear Matrix Inequalities. Since such a formulation may be in general more conservative, a procedure which permits to automate its verification, without introduce conservatism, is given both for second order systems, and when the resetting set is *ellipsoidal*.

Index Terms—Finite-time stability (FTS), state-dependent impulsive dynamical linear systems (SD-IDLS).

I. INTRODUCTION

The concept of finite-time stability (FTS) dates back to the Sixties, when it was introduced in the control literature [1]. A system is said to be finite-time stable if, given a bound on the initial condition, its state does not exceed a certain threshold during a specified time interval. It is important to recall that FTS and Lyapunov Asymptotic Stability (LAS) are independent concepts. In particular, due to possible elongations of the system trajectories, LAS is not sufficient to guarantee FTS. Moreover, while LAS deals with the behavior of a system within an infinite time interval, FTS studies the behavior of the system within a finite (possibly short) interval. It follows that an unstable system can be FTS if the considered time interval is sufficiently small. It is worth noticing that Lyapunov stability becomes necessary for FTS of linear systems if the considered time interval becomes infinite.

In [2], [3] sufficient conditions for FTS and finite-time stabilization of continuous-time linear systems have been provided; such conditions are based on the solution of a feasibility problem involving either Linear Matrix Inequalities (LMIs [4]) or Differential Linear Matrix Inequalities (DLMI [5]). The former approach is less demanding from the computational point of view, while the latter is less conservative.

The increasing interest that the researchers have devoted in the last decade to the theory and application of hybrid systems represents a natural stimulus to the extension of the FTS concept to such context, which is the objective of the present work. Indeed, in this note, we will focus on a class of hybrid systems, namely *state-dependent* impulsive dynamical linear systems (SD-IDLS) [6], where the state jumps occur when the trajectory reaches an assigned subset of the state space, the so-called *resetting set*. Many results concerning the classical Lyapunov

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asymptotic stability for hybrid systems have been proposed in the literature (see for instance [6], [7] and references therein).

In this note, as in [8] and [9], we exploit S -procedure arguments [10], in order to end up with numerically tractable analysis conditions formulated as DLMI. The main result of the present work is a sufficient condition which guarantees the FTS of a given SD-IDLS. Moreover it is shown that, either in presence of second order systems or when the resetting set is *ellipsoidal*, the S -procedure does not introduce conservatism in the FTS analysis.

For the sake of completeness, it should be noticed that a more recent notion of finite-time stability, which is strictly related to LAS, has been given in [11] for continuous autonomous systems and in [12] for nonlinear impulsive dynamical systems. This different concept of finite-time stability requires convergence of system trajectories to an equilibrium state in finite-time. Hence, the notion in [12] is unrelated to the notion of FTS here adopted, since the former implies LAS and does not require to specify any bounding regions nor the time interval.

The present work is structured as follows: the next section presents the notation, the class of state-dependent impulsive dynamical linear systems, together with some preliminary results exploited throughout this note. The main results are given in Section III, while in Section IV the conservatism introduced by the S -procedure is discussed, and two procedures are presented to reduce such conservatism. The applicability of the proposed results is illustrated in Section V through a numerical example.

II. PRELIMINARIES

The notation used throughout this note is presented in this section, together with the FTS problem statement for SD-IDLS. Preliminary results on quadratic forms are also provided.

A. Problem Statement

Let us consider the time-varying SD-IDLS described by

$$\dot{x}(t) = A(t)x(t), \quad x(t_0) = x_0, \quad x(t) \in \mathbb{R}^n \setminus \bigcup_{k=1}^N S_k \quad (1a)$$

$$x(t^+) = A_{d,k}x(t), \quad x(t) \in S_k, \quad k = 1, \dots, N \quad (1b)$$

where $A(\cdot) : \mathbb{R}_0^+ \mapsto \mathbb{R}^{n \times n}$, $A_{d,k} \in \mathbb{R}^{n \times n}$, $k = 1, \dots, N$. The sets $S_k \subseteq \mathbb{R}^n$, $k = 1, \dots, N$, are connected and closed pairwise disjoint sets (i.e. $S_i \cap S_j = \emptyset$, $\forall i \neq j$), such that $0 \notin S_k$. We refer to the differential (1a) as the *time-varying continuous-time dynamics*, to the difference (1b) as the *resetting laws*, and to the sets S_k as the *resetting sets* [6]. If $x(\cdot)$ is a solution of (1a), (1b), it is then possible to define the correspondent *resetting times* set as follows:

$$\mathcal{T}_{x(\cdot)} = \{t \in \mathbb{R}_0^+ | x(t) \in S_k, k = 1, \dots, N\}.$$

It turns out that $x(\cdot)$ is left-continuous, i.e. it is continuous $\forall t \notin \mathcal{T}_{x(\cdot)}$ and $x(t) = \lim_{\tau \rightarrow t^-} x(\tau)$, $\forall t \in \mathcal{T}_{x(\cdot)}$. In order to guarantee the well-posedness of the resetting times we make the following assumption, which prevents $x(\cdot)$ from intersecting the interior of the resetting sets.

Assumption 1: For all $t \in \mathbb{R}_0^+ : x(t) \in S_k$, $k = 1, \dots, N$, $\exists \varepsilon > 0 : x(t + \delta) \notin \bigcup_k S_k$, $\forall \delta \in (0, \varepsilon]$. \diamond

Definition 1 (FTS of SD-IDLS): Given an initial time t_0 , a positive scalar T , a positive definite matrix R , a positive definite matrix-valued function $\Gamma(\cdot)$ defined over $[0, T]$, with $\Gamma(0) < R$, system (1) is said to be finite-time stable with respect to $(t_0, T, R, \Gamma(\cdot))$ if

$$x_0^T R x_0 \leq 1 \Rightarrow x(t)^T \Gamma(t - t_0) x(t) < 1 \quad \forall t \in [t_0, t_0 + T]$$

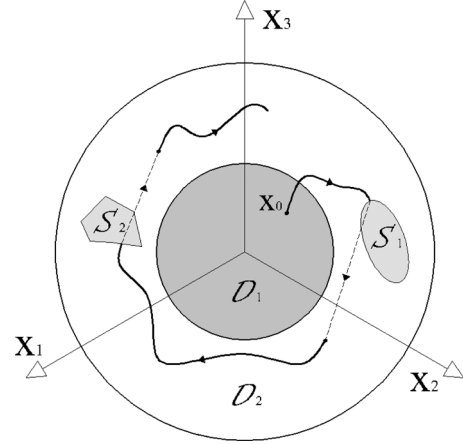


Fig. 1. Example of evolution of a state-dependent impulsive dynamical linear system. The trajectory starts inside the sphere \mathcal{D}_1 defined by a positive definite matrix R , and remains inside the sphere \mathcal{D}_2 defined by a positive definite matrix Γ , $\forall t \in [t_0, t_0 + T]$. When the trajectory reaches one of the two resetting sets the state jumps.

\diamond

Fig. 1 shows an example of FTS trajectory.

Remark 1: If a system is finite-time stable in the sense of [12], its state trajectory is confined in an open neighborhood of the origin (moreover it converges to zero in finite-time and the origin is Lyapunov stable). Neither the bounding region (denoted by \mathcal{N} in [12]) nor the time interval, however, can be prespecified, as in the FTS definition given above. \diamond

B. Some Useful Definitions

The following definitions will be useful throughout this note.

Definition 2 (Conical Hull [13], p. 28): Given a set $S \subseteq \mathbb{R}^n$, the set

$$\text{cone}(S) := \{\lambda_1 x_1 + \dots + \lambda_k x_k : \{x_1, \dots, x_k\} \subseteq S, \lambda_i \geq 0\}$$

is said to be the *conical hull* of S [13]. The notation $\text{cone}(-S)$ denotes the set

$$\text{cone}(-S) := \{\lambda_1 x_1 + \dots + \lambda_k x_k : \{x_1, \dots, x_k\} \subseteq S, \lambda_i \leq 0\}.$$

\diamond

Definition 3 (Projection wrt the Origin): Consider a hyper-surface $H \subset \mathbb{R}^n$ and a set $S \subseteq \mathbb{R}^n$. The projection of S on H with respect to the origin is defined as

$$S_H := \{y \in H : y = \lambda x, \lambda \in \mathbb{R}, x \in S\}.$$

\diamond

The projection wrt the origin is a perspective projection, where the point of perspective is the origin of the state space, and the projection surface is the hyper-surface H .

C. Preliminary Results on Quadratic Forms

As it will be shown later, the main result of this work requires to check whether, given a connected and closed set $S \subseteq \mathbb{R}^n$ and a symmetric matrix $Q_0 \in \mathbb{R}^{n \times n}$, the inequality

$$x^T Q_0 x < 0, \quad x \in S \setminus \{0\} \quad (2)$$

is satisfied.

In the following, our goal is to find some numerically tractable conditions which guarantee the satisfaction of (2). Exploiting \mathcal{S} -procedure arguments ([4], p. 24), it is readily seen that Q_0 satisfies (2) if the following feasibility problem admits a solution.

Problem 1: Given a connected and closed set $\mathcal{S} \subset \mathbb{R}^n$, a symmetric matrix $Q_0 \in \mathbb{R}^{n \times n}$ and symmetric matrices $Q_i \in \mathbb{R}^{n \times n}$ satisfying

$$x^T Q_i x \leq 0, \quad x \in \mathcal{S}, \quad i = 1, \dots, p \quad (3a)$$

find nonnegative scalars $c_i, i = 1, \dots, p$, such that

$$Q_0 - \sum_{i=1}^p c_i Q_i < 0. \quad (3b)$$

◇

The usefulness of Problem 1 relies in the fact that it can be recast in the LMIs framework, where the coefficients c_i are the optimization variables of the LMI (3b). Clearly, a method to choose the matrices Q_i is needed. In the next section we provide a procedure to build a suitable set of matrices Q_i , which can be exploited when the set \mathcal{S} satisfies some assumptions.

As mentioned above, if Problem 1 admits a feasible solution, then (2) is satisfied. In general, the converse is not true. Therefore it makes sense to investigate under which conditions solving Problem 1 is equivalent to check condition (2); the answer is given by the following lemma.

Lemma 1: Given a connected and closed set $\mathcal{S} \subset \mathbb{R}^n$ and a symmetric matrix $Q_0 \in \mathbb{R}^{n \times n}$, assume there exists a symmetric matrix $\bar{Q} \in \mathbb{R}^{n \times n}$ such that

$$x^T \bar{Q} x \leq 0 \quad \forall x \in (\text{cone}(\mathcal{S}) \cup \text{cone}(-\mathcal{S})) \setminus \{0\} \quad (4a)$$

$$x^T \bar{Q} x > 0 \quad \forall x \in \mathbb{R}^n \setminus (\text{cone}(\mathcal{S}) \cup \text{cone}(-\mathcal{S})) \quad (4b)$$

$$\exists \tilde{x} \in \mathcal{S} \setminus \{0\} : \tilde{x}^T \bar{Q} \tilde{x} < 0 \quad (4c)$$

then condition (2) is equivalent to the feasibility Problem 1 with $p = 1$ and $Q_1 = \bar{Q}$.

Proof: The proof is trivial once it is recognized that:

1) $x^T Q_0 x < 0$ for all $x \in \mathcal{S}$ iff $x^T Q_0 x < 0$ for all $x \in \text{cone}(\mathcal{S}) \cup \text{cone}(-\mathcal{S})$ (see Lemma 2 in the Appendix).

2) solving Problem 1 with $p = 1$ and $Q_1 = \bar{Q}$, is equivalent to applying lossless \mathcal{S} -procedure, since \bar{Q} satisfies (12) of Lemma 3 reported in the Appendix. ■

In Section IV it will be shown that, when the set \mathcal{S} satisfies certain assumptions, the hypotheses of Lemma 1 are fulfilled and the approach via Problem 1 does not add conservatism in the FTS analysis.

III. MAIN RESULTS

The following theorem gives a sufficient condition for FTS of system (1).

Theorem 1: The SD-IDLS (1) is FTS wrt $(t_0, T, R, \Gamma(\cdot))$ if the following coupled differential/difference Lyapunov inequalities with terminal and initial conditions:

$$\dot{P}(t) + A(t)^T P(t) + P(t)A(t) < 0 \quad (5a)$$

$$x^T \left(A_{d,k}^T P(t) A_{d,k} - P(t) \right) x < 0 \quad (5b)$$

$$\forall x \in \mathcal{S}_k, \quad k = 1, \dots, N \quad (5b)$$

$$P(t) \geq \Gamma(t - t_0) \quad (5c)$$

$$P(t_0) < R \quad (5d)$$

admit a continuously differentiable symmetric solution $P(\cdot)$ over the interval $[t_0, t_0 + T]$.

Proof: Let consider $V(t, x) = x^T P(t)x$. Given a system trajectory, if $t \notin \mathcal{T}_{x(\cdot)}$, i.e. $x(t)$ does not touch any resetting set, then the time derivative of $V(t, x)$ is defined and it yields

$$\dot{V}(t, x) = x^T \left(\dot{P}(t) + A(t)^T P(t) + P(t)A(t) \right) x$$

which is negative by virtue of (5a).

Moreover if $t \in \mathcal{T}_{x(\cdot)}$, i.e. when the system trajectory touches a resetting set, we have

$$V(t^+, x) - V(t, x) = x^T (A_{d,k} P(t) A_{d,k} - P(t)) x$$

which is negative in view of (5b). We can conclude that $V(t, x)$ is strictly decreasing along the trajectories of system (1a), (1b); hence, given x_0 such that $x_0^T R x_0 \leq 1$, we have, for all $t \in [t_0, t_0 + T]$

$$\begin{aligned} x(t)^T \Gamma(t - t_0) x(t) &\leq x(t)^T P(t) x(t) \quad \text{by (5c)} \\ &< x_0^T P(t_0) x_0 \\ &< x_0^T R x_0 \leq 1 \quad \text{by (5d)}. \end{aligned}$$

■

Note that, for a given k and t , condition (5b) is equal to (2) if we let $Q_0 = A_{d,k}^T P(t) A_{d,k} - P(t)$ and $\mathcal{S} = \mathcal{S}_k$. Therefore, by exploiting the machinery introduced in Section II, we can relax inequality (5b) and replace it with (see Problem 1)

$$A_{d,k}^T P(t) A_{d,k} - P(t) - \sum_{i=1}^p c_{i,k}(t) Q_{i,k} < 0$$

where $Q_{i,k}$ are given symmetric matrices satisfying $x^T Q_{i,k} x \leq 0$, for all $x \in \mathcal{S}_k$, and $c_{i,k}(t) \geq 0$, with $i = 1, \dots, p_k$ and $t \in [t_0, t_0 + T]$.

On the basis of this consideration, we can immediately derive the following theorem.

Theorem 2: Given a set of symmetric matrices $Q_{i,k}, i = 1, \dots, p_k, k = 1, \dots, N$, satisfying

$$x^T Q_{i,k} x \leq 0, \quad x \in \mathcal{S}_k, \quad i = 1, \dots, p_k, \quad k = 1, \dots, N \quad (6)$$

assume there exist a continuously differentiable symmetric matrix function $P(\cdot)$ and nonnegative scalar functions $c_{i,k}(\cdot), i = 1, \dots, p_k, k = 1, \dots, N$, such that, for all $t \in [t_0, t_0 + T]$,

$$\dot{P}(t) + A(t)^T P(t) + P(t)A(t) < 0, \quad (7a)$$

$$A_{d,k}^T P(t) A_{d,k} - P(t) - \sum_{i=1}^{p_k} c_{i,k}(t) Q_{i,k} < 0$$

$$k = 1, \dots, N \quad (7b)$$

$$P(t) \geq \Gamma(t - t_0) \quad (7c)$$

$$P(t_0) < R \quad (7d)$$

then the SD-IDLS (1) is FTS wrt $(t_0, T, R, \Gamma(\cdot))$.

Remark 2: In view of the results given in Theorem 2, it is now possible to clarify the usefulness of the formulation introduced in Problem 1. Such formulation, indeed, allows us to replace condition (5b) with condition (7b). Note that in principle the former requires to solve an infinite number of time-varying inequalities over the sets \mathcal{S}_k , the latter is just a set of LMIs, which can be easily solved in an efficient way. ◇

IV. ANALYSIS OF SOME CASES OF INTEREST

Theorem 2 may introduce conservatism with respect to Theorem 1 since, in general, the \mathcal{S} -procedure is lossy. However, if for every resetting set \mathcal{S}_k there exists a symmetric matrix Q_k which satisfies conditions (4), then Theorem 2 is equivalent to Theorem 1. In this section

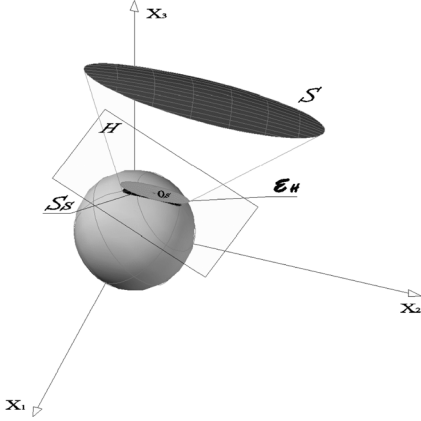


Fig. 2. Construction of the set \mathcal{E}_H .

we will discuss two cases where Theorem 2 does not introduce conservatism: resetting sets in \mathbb{R}^2 , and *ellipsoidal* resetting sets; we prove that the conservatism can be eliminated in both cases, except for the following trivial cases:

- 1) $S \subseteq \mathbb{R}^n$ lies on a hyperplane which intersect the origin;
- 2) $S \subseteq \mathbb{R}^n$ has dimension less than $n - 1$.

The definition of ellipsoidal resetting set is based on the following constructive geometrical procedure.

Procedure 1 (Construction of \mathcal{E}_H): Given a connected and closed set $S \subset \mathbb{R}^n$, construct the set \mathcal{E}_H as follows:

- 1) denote with S_s the projection, with respect to the origin, of S on the unit sphere $x^T x = 1$;
- 2) denote with 0_s the Chebyshev center of S_s ;¹
- 3) denote with H the hyper-plane of dimension $n - 1$ orthogonal to the line that joins the origin to 0_s , and such that $0_s \in H$;
- 4) \mathcal{E}_H is the projection, with respect to the origin, of S on the hyper-plane H .

An example of construction of the set \mathcal{E}_H is shown in Fig. 2. \diamond

Definition 4 (Ellipsoidal Resetting Set): Consider a *non-trivial* resetting set $S \subset \mathbb{R}^n$ and construct the set \mathcal{E}_H using Procedure 1. If \mathcal{E}_H is an hyper-ellipsoid of dimension $n - 1$, then S is called *ellipsoidal resetting set*. \diamond

Remark 3: Since \mathcal{E}_H is constructed using two projections with respect to the origin it follows that $\text{cone}(S) = \text{cone}(\mathcal{E}_H)$. \diamond

A. Resetting set S in \mathbb{R}^2

The following theorem provides a necessary and sufficient condition which enables to find a symmetric matrix $Q \in \mathbb{R}^{2 \times 2}$ that verifies conditions (4).

Theorem 3: Every *non-trivial* resetting set S in \mathbb{R}^2 admits a symmetric matrix $Q \in \mathbb{R}^{2 \times 2}$ that verifies conditions (4).

Proof: To prove our statement, we provide a procedure to calculate a matrix Q satisfying conditions (4). Let $s_1, s_2 \in S$ such that, said $\bar{S} = \text{conv}(\{s_1, s_2\})$, we have² $\text{cone}(\bar{S}) = \text{cone}(S)$. Then, taking into account Lemma 1, condition (4) can be equivalently evaluated on the set \bar{S} . In particular, considering the properties of the quadratic forms, it is easy to verify that such condition can be replaced by the following:

$$x^T Q x < 0 \quad \forall x \in \text{int}(\bar{S}) \quad (8a)$$

$$x^T Q x = 0 \quad \text{for } x = s_1, s_2 \quad (8b)$$

$$x^T Q x > 0 \quad \forall x \in H \setminus \bar{S}. \quad (8c)$$

¹The Chebyshev center of a set $S \subseteq \mathbb{R}^n$ is defined as $0_S := \arg \min_{x \in \mathbb{R}^n} (\max_{\theta \in S} \|\theta - x\|_\infty)$.

²Given a set of points $K = \{x^{(1)}, \dots, x^{(k)}\} \subset \mathbb{R}^n$, $\text{conv}(K)$ denotes the *convex hull* of K , i.e. the smallest convex set containing K [13].

where H is the hyperplane on which \bar{S} lies, and $\text{int}(\bar{S})$ denotes the interior of the set \bar{S} . Letting $s_m = (s_1 + s_2)/2$, a symmetric matrix $Q \in \mathbb{R}^{2 \times 2}$ such that $s_1^T Q s_1 = 0$, $s_2^T Q s_2 = 0$, and $s_m^T Q s_m < 0$ verifies conditions (8). \blacksquare

B. Ellipsoidal Resetting Sets S

The following theorem provides a sufficient condition to find a matrix $Q \in \mathbb{R}^{n \times n}$ that verifies conditions (4).

Theorem 4: If S is an ellipsoidal resetting set, then there exists a matrix $Q \in \mathbb{R}^{n \times n}$ that verifies conditions (4).

Proof: If S is an ellipsoidal resetting set then $\text{cone}(S) = \text{cone}(\mathcal{E}_H)$ (see Remark 3). Taking into account Lemma 1, it follows that conditions (4) can be equivalently evaluated on the set \mathcal{E}_H . In particular, considering the properties of the quadratic forms, it is easy to verify that such conditions can be replaced by the following:

$$x^T Q x < 0 \quad \forall x \in \text{int}(\mathcal{E}_H) \quad (9a)$$

$$x^T Q x = 0 \quad \forall x \in \partial \mathcal{E}_H \quad (9b)$$

$$x^T Q x > 0 \quad \forall x \in H \setminus \mathcal{E}_H. \quad (9c)$$

To conclude the proof we need to show that the assumption of ellipsoidal set \mathcal{E}_H is sufficient to find a matrix Q which verifies conditions (9). In the sequel of the proof we assume that:

- 0_s is on the n -th coordinated axis, i.e.

$$0_s = (0 \quad \dots \quad 0 \quad r)^T, \quad r \in \mathbb{R}.$$

- The hyper-plane H is orthogonal to the n -th coordinated axis.

As a matter of fact, it is always possible, by means of opportune rotations, to satisfy these assumptions. In view of the assumptions made above, it is possible to describe the set $\partial \mathcal{E}_H$ by the two equations

$$\frac{x_1^2}{a_1^2} + \dots + \frac{x_{n-1}^2}{a_{n-1}^2} = 1, \quad x_n = r$$

where $a_i \geq 0, i = 1, \dots, n - 1$. It is then straightforward to check that the following matrix $Q = \text{diag}(1/a_1^2 \dots 1/a_{n-1}^2 - (1/r^2))$, satisfies conditions (9). \blacksquare

V. EXAMPLE

The following example illustrate the effectiveness of the proposed procedure. Let consider the second order SD-IDLS defined by the following matrices:

$$A = \begin{pmatrix} 0 & 1 \\ -1 + 0.3 \sin 10t & 0.5 \end{pmatrix}, \quad A_{d,1} = \begin{pmatrix} 1.2 & 0 \\ 0 & -0.75 \end{pmatrix},$$

$$A_{d,2} = \begin{pmatrix} -0.72 & 0.16 \\ 0.13 & -0.78 \end{pmatrix}$$

where the two resetting sets S_1 and S_2 are given by

$$S_1 = \text{conv} \left(\begin{pmatrix} -0.4 \\ 0.6 \end{pmatrix}, \begin{pmatrix} 0.8 \\ 0.7 \end{pmatrix} \right),$$

$$S_2 = \text{conv} \left(\begin{pmatrix} -0.8 \\ -0.3 \end{pmatrix}, \begin{pmatrix} 0.4 \\ -0.6 \end{pmatrix} \right).$$

Note that the continuous dynamic is unstable, while the discrete dynamic $A_{d,1}$ is not Schur stable.³ We want to analyze the FTS for such impulsive system, for $t_0 = 5$ s, $T = 2.5$ s, and

$$\Gamma = \begin{pmatrix} 0.20 & -0.10 \\ -0.10 & 0.18 \end{pmatrix}, \quad R = \begin{pmatrix} 7.5 & 4.5 \\ 4.5 & 9.0 \end{pmatrix}.$$

³A matrix is said to be Schur stable if all the roots of its characteristic polynomial lie in the open unit disk.

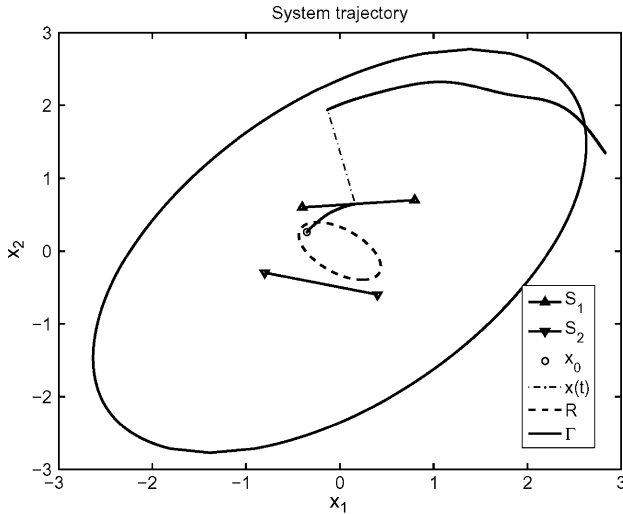


Fig. 3. Trajectory which is not FTS wrt (T, R, Γ) . The two resetting laws are given by $\hat{A}_{d,1}$ and $\hat{A}_{d,2}$, respectively.

In order to recast the conditions provided in Theorem 2 in terms of LMIs, the matrix function $P(\cdot)$ is assumed piecewise linear, that is

$$P(t) = \begin{cases} P_0 + \Theta_1(t - t_0), & t \in [t_0, t_0 + T_s], \\ P_0 + \sum_{h=1}^j \Theta_h T_s + \Theta_{j+1}(t - jT_s - t_0), & t \in [t_0 + jT_s, t_0 + (j+1)T_s] \\ & j = 1, \dots, J \end{cases}$$

where $J = \max\{j \in \mathbb{N} : j < T/T_s\}$, $T_s \ll T$ and $P_0, \Theta_l, l = 1, \dots, J + 1$ are the optimization variables. It is straightforward to recognize that such a piecewise function can approximate a generic continuous $P(\cdot)$ with adequate accuracy, provided that the length of T_s is sufficiently small. Theorem 3 assures that for each of the considered resetting sets in \mathbb{R}^2 there exists a matrix Q which verifies (4). Applying the procedure proposed in the proof of Theorem 3, the following matrices have been found:

$$Q_1 = \begin{pmatrix} -0.2909 & 0.0693 \\ 0.0693 & 0.2216 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} -0.2000 & 0.2000 \\ 0.2000 & 0.3556 \end{pmatrix}.$$

Using matrices Q_1 and Q_2 , it is possible to find a piecewise linear matrix $P(\cdot)$ and two nonnegative scalar functions $c_i(\cdot), i = 1, 2$, which verify conditions in Theorem 2. Hence the considered system is FTS with respect to (t_0, T, R, Γ) . Let now change the two resetting laws as follows:

$$\hat{A}_{d,1} = \begin{pmatrix} -0.8 & 0 \\ 0 & 3.0 \end{pmatrix}, \quad \hat{A}_{d,2} = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix}.$$

It turns out that the considered SD-IDLS is not FTS wrt (t_0, T, R, Γ) , as shown by the trajectory in Fig. 3. Moreover, in this case it is not possible to satisfy the conditions in Theorem 2.

VI. CONCLUSION

An extension of the finite-time stability concept to a class of hybrid systems has been presented in this note, together with a sufficient condition for FTS of state-dependent impulsive dynamical linear systems. A DLMI formulation of this condition has been provided as well, in order to check it in a numerically tractable way. Such a formulation has been obtained exploiting S -procedure arguments, and it may be in general more conservative than the original sufficient condition. Moreover it requires the definition of a set of specific symmetric matrices

for each resetting set, which is not a straightforward task. To deal with these problems, a procedure which allows us to automate the building of the symmetric matrices, without introducing conservatism, is provided both for ellipsoidal resetting sets in \mathbb{R}^n and when we deal with second order systems.

APPENDIX

The two results presented in this appendix are needed to prove Lemma 1 in Section II.

Lemma 2: Consider a nonempty, connected and closed set $S \subseteq \mathbb{R}^n$ and a symmetric matrix $Q_0 \in \mathbb{R}^{n \times n}$; then (2) is satisfied if and only if

$$x^T Q_0 x < 0, \quad \forall x \in (\text{cone}(S) \cup \text{cone}(-S)) \setminus \{0\}. \quad (10)$$

Proof: Trivial. ■

Lemma 3 (S-Procedure [10]): Let $Q_0, Q_1, \dots, Q_p \in \mathbb{R}^{n \times n}$ be $p + 1$ symmetric matrices. Consider the following condition on Q_0, Q_1, \dots, Q_p :

$$x^T Q_0 x < 0 \quad \forall x : x \neq 0 \wedge x^T Q_i x \leq 0, \quad i = 1, \dots, p.$$

It is obvious that if

$$\exists c_1 \geq 0, \dots, c_p \geq 0 : Q_0 - \sum_{i=1}^p c_i Q_i < 0 \quad (11)$$

then condition (11) holds. It is not trivial that when $p = 1$, the converse holds, provided that

$$\exists \tilde{x} \text{ such that } \tilde{x}^T Q \tilde{x} < 0. \quad (12)$$

Lemma 3 implies that condition (12) must be satisfied, in order to do not introduce conservatism when applying the S -procedure to check (2). ■

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