Time-optimal generation of cluster states

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The definition of a cluster state naturally suggests an implementation scheme: find a physical system with an Ising coupling topology identical to that of the target state and evolve freely for a time of $\frac{1}{2J}$. Using the tools of optimal control theory, we address the question of whether or not this implementation is time-optimal. We present some examples where it is not and provide an explanation in terms of geodesics on the Bloch sphere.

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I. INTRODUCTION

The one-way quantum computation model is an approach to quantum information processing where the evolution is driven by local operations and measurements only [1,2]. In the experimental realization of such a model, preparation of the highly entangled initial state is therefore of primary concern. These initial states—so-called cluster states—are also interesting in their own right due to the favorable scaling of their entanglement properties [3,4]. To date, cluster states of four to six qubits have been realized experimentally in photonic systems [5,6] and are also actively pursued in other architectures, e.g., ion traps [7].

In this paper, the efficient generation of cluster states is studied using techniques from optimal control theory [8,9]. Given an experimental framework, the aim of these techniques is to find the optimal set of controls to steer the system so that a desired target state or unitary gate is implemented. For systems consisting of two qubits, general analytical solutions exist for the construction of time-optimal unitary transformations [10,11] and state-to-state transfers [12,13] if fast local controls are available. For three or more qubits, analytical solutions are only known in some special cases. In [14–17] it was shown that the time-optimal generation of indirect couplings and trilinear Hamiltonians and the efficient transfer of order along Ising spin chains [18–20] can be reduced to the problem of computing singular geodesics. In addition, powerful numerical methods [21] are available that make it possible to explore the physical limits of time-optimal control experiments in cases where no analytical solutions are known. The GRAPE algorithm [21] has been used for the generation of quantum gates [22] and state-to-state transfers in NMR [23], as well as for superconducting qubits [24]. Here, analytical and numerical techniques are used to investigate the problem of time-optimal cluster state preparation.

A cluster state is defined by a graph. To prepare the n-qubit cluster state corresponding to a graph G:

(1) Prepare (locally) the initial state

$$|I_n\rangle := \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)^{\otimes n}.$$

(2) Evolve under the Ising Hamiltonian

$$H_d = \frac{\pi J(t)}{2} \sum_{\langle a,a'\rangle} (\mathbb{1} + \sigma_z^{(a)})(\mathbb{1} - \sigma_z^{(a')}), \tag{1}$$

for a time such that $\int_0^T J(t)dt = \frac{1}{2}$, the sum is over all edges of G, where each edge connects the qubit pair $\langle a,a'\rangle$.

We consider n-qubit systems of the form

$$H = H_d + \sum_j u_j H_c^{(j)},$$

where the u_j are time-dependent functions to be chosen, and the $H_c^{(j)}$ characterize the available controls. Two different control settings will be considered:

(i) Local x and y control on each qubit:

$$H_c^{(2j-1)} = \frac{1}{2}\sigma_x^{(j)}, \quad H_c^{(2j)} = \frac{1}{2}\sigma_y^{(j)}, \tag{2}$$

where $j \in \{1,2,\ldots,2n\}$.

(ii) A single global x control:

$$H_c^{(1)} = F_x := \frac{1}{2}\sum_{j=1}^{n} \sigma_x^{(j)}. \tag{3}$$

In the following, we allow for fast local controls, i.e., the functions u_j are unrestricted.

II. THREE COMPLETELY COUPLED QUBITS

To shed some light on when a speedup may be possible, we begin with a symmetric three-qubit system which will prove analytically tractable. The qubits are Ising coupled according to the complete coupling graph K_3 [25], illustrated in Fig. 1. Without loss of generality we will drop the local terms in Eq. (1), as only the entangling part of the operation contributes to the time required. Furthermore J is assumed to be constant. The K_3 cluster state is then defined as

$$|T_3\rangle := \exp\left(-i\frac{1}{2J}H_d\right)|I_3\rangle,$$

where

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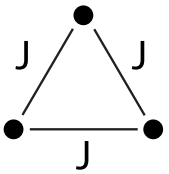

FIG. 1. K_3 coupling graph.

$$H_d = \frac{\pi J}{2}(\sigma_z^{(1)}\sigma_z^{(2)} + \sigma_z^{(2)}\sigma_z^{(3)} + \sigma_z^{(1)}\sigma_z^{(3)}).$$

The preparation of this state poses the following control problem: maximize

$$F(U) := |\langle T_3|(U|I_3\rangle)| \tag{4}$$

subject to equation of motion $\dot{U} = -iHU$. In the first instance we will allow for full local control on the qubits and thus specify H_c according to Eq. (2).

To maximize the fidelity defined in Eq. (4) in the shortest possible time, we first apply a numerical gradient-ascent algorithm, as detailed in [21]. Over a comprehensive range of initial conditions the controls are updated iteratively, incrementing F to a local maximum. The minimal time found by the algorithm to generate $|T_3\rangle$ is approximately $0.77 \times \frac{1}{2J}$, as seen in Fig. 2(a). The significance of this particular value will become clearer later. Figure 2(b) shows a numerically optimized solution close to the minimal time. We find in all observed cases that the maximum fidelity does not depend on whether the controls are specified according to Eq. (2) or (3).

Motivated by these symmetric solutions, we now restrict ourselves to control setting (ii), specifying H_c according to Eq. (3). This control Hamiltonian, in addition to the drift Hamiltonian and the initial and target states, is symmetric under the cyclic permutation operator,

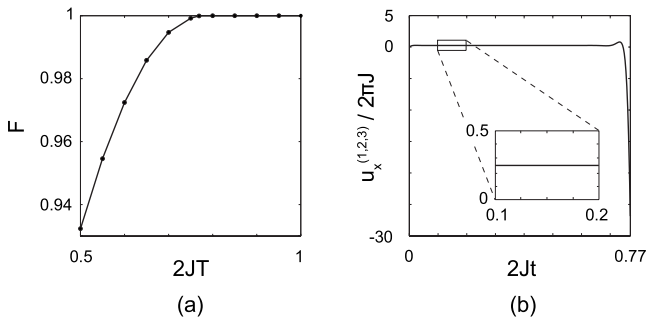

FIG. 2. (a) Maximum achievable fidelity as a function of the transfer time. (b) A sample solution at the minimal time for initial controls $u_x^j(t) = u_y^j(t) = 0$. The y controls are omitted as they remain zero, and the x controls are identical on each qubit due to the permutation symmetry of the drift Hamiltonian and the initial and target states.

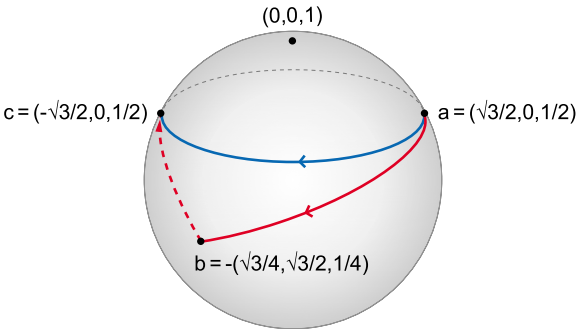

FIG. 3. (Color online) Transfer of Eq. (5) on the Bloch sphere. The $u=0$ solution (blue upper line) transfers a to c in a time of $\frac{1}{2J}$. A time-optimal solution (red lower line) transfers a to b in $\frac{2}{3\sqrt{3}J}$, followed by a hard pulse (dashed red lower line) from b to c in negligible time.

$$S := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

and the persymmetry operator

$$P := (\sigma_x)^{\otimes 3},$$

which themselves commute. The dynamics are thus restricted to the simultaneous eigenspace of S and P corresponding to the eigenvalue pair $\{1, 1\}$. Transforming to a new basis composed of the simultaneous eigenstates of S and P makes this explicit [26]. The Hamiltonians are now diagonalized into 2×2 blocks. As we need only consider the $\{1, 1\}$ block, the state transfer problem can be reduced to

$$|I_3'\rangle = \frac{1}{2}\begin{bmatrix} \sqrt{3} \\ 1 \end{bmatrix} \rightarrow |T_3'\rangle = \frac{1}{2}\begin{bmatrix} \sqrt{3} \\ -1 \end{bmatrix}, \tag{5}$$

under the Hamiltonian $H' = H_d' + u(t)H_c'$, where

$$H_d' = \frac{\pi J}{2}\begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}, \quad H_c' = \frac{1}{2}\begin{bmatrix} 2 & \sqrt{3} \\ \sqrt{3} & 0 \end{bmatrix}. \tag{6}$$

In order to consider the problem geometrically, we represent the transfer on the Bloch sphere by projecting onto the axes $I_j := \sigma_j/2$, as illustrated in Fig. 3. The initial and final states $|I_3'\rangle$ and $|T_3'\rangle$ are identified with vectors $a = \frac{1}{2}(\sqrt{3}, 0, 1)$ and $b = \frac{1}{2}(-\sqrt{3}, 0, 1)$, respectively, while the Hamiltonians H_d' and H_c' correspond to (un-normalized) rotation axes $(0, 0, -2\pi J)$ and $(\sqrt{3}, 0, 1)$, respectively. The trivial solution is to set $u(t) = 0$ and rotate about H_d' for $\frac{1}{2J}$ units of time but faster solutions may exist. Motivated by the numerical results in Fig. 2(b), we first restrict $u(t)$ to the following form:

(1) Constant pulse u over time interval $[0, T]$.

(2) Hard pulse of angle ϕ at time T.

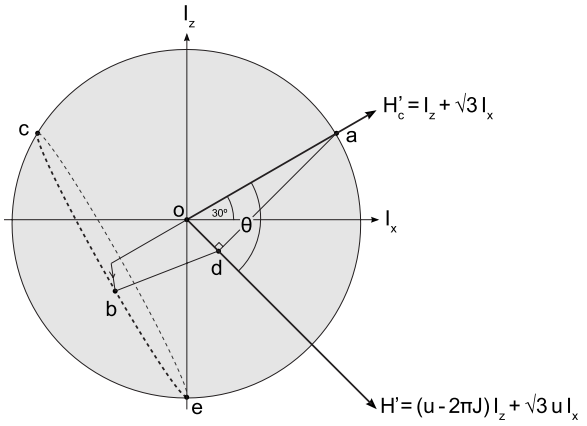

FIG. 4. Geometric constructions used. All points shown here lie in the I_x-I_z plane *except* for b, which lies above it on the upper surface of the sphere.

After deriving a time-optimal solution in this setting, we will show that it remains time-optimal when the restrictions are removed and general time-varying pulses are considered. The solution is

$$(u,\phi,T)=\left(\frac{\pi J}{2},\frac{-\pi}{4},\frac{2}{3\sqrt{3}J}\right), \tag{7}$$

which explains the value of $T\approx 0.77\times\frac{1}{2J}$ obtained numerically. To demonstrate the time-optimality of this solution, we can consider the geometric constructions in Fig. 4. Starting from a, the task is to transfer the state to any point b on the circle $\widehat{bce}$ obtained by rotating about H_c'. The hard pulse then transfers b to c in an arbitrarily small time. The choice of constant u specifies a rotation axis H', at an angle θ to H_c'. The transfer time to be minimized is

$$T=\frac{\angle adb}{|H'|},$$

where $\angle adb$ is the angle swept out by the Bloch vector and the length $|H'|$ gives its angular velocity of rotation. Using simple geometry these quantities are expressed in terms of θ as

$$\angle adb=2\arcsin\left(\frac{\sqrt{3}}{2\sin\theta}\right),\quad |H'|=\frac{\sqrt{3}\pi J}{\sin\theta}$$

so that the time is

$$T=\frac{2}{\sqrt{3}\pi J}\sin\theta\arcsin\left(\frac{\sqrt{3}}{2\sin\theta}\right).$$

Noting that θ must lie in the interval $[\frac{\pi}{3},\frac{2\pi}{3}]$ for intersection with the circle, we find the maximum at $\theta=\frac{\pi}{2}$. The time required is $T_{\min}=\frac{2}{3\sqrt{3}J}$. The angle of the final hard pulse is $-\frac{\pi}{2}$ in this picture, but since $|H_c'|=2$ the angle around a normed axis is $\phi=-\frac{\pi}{4}$.

It remains to show that $\widehat{bce}$ cannot be reached in less than $T_{\min}$ when allowing for time-varying pulses. For this we introduce

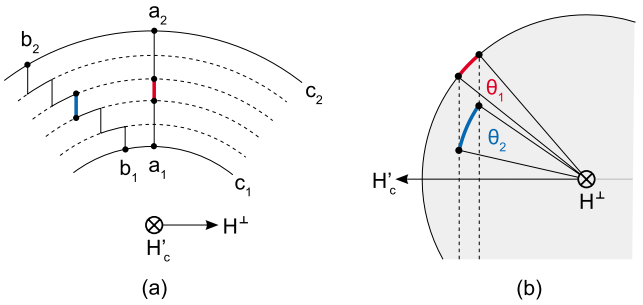

FIG. 5. (Color online) (a) An arbitrary path can be decomposed into rotation about H_c' and rotation about the orthogonal axis $H^\perp$ via the Trotter decomposition. (b) Rotating our viewpoint by 90°, we see that the optimal trajectory from $a_1\rightarrow a_2$ minimizes the angle rotated through in each segment (red) when compared to a generic trajectory (blue), i.e., $\theta_1<\theta_2$.

$$H^\perp:=\frac{\sqrt{3}\pi J}{2}(-\sqrt{3}I_z+I_x),$$

which is simply the rotation axis orthogonal to H_c'. We consider a generic time-varying control $u(t)$. Our aim is to compare each segment of this generic path to the optimal one. Let c_1 and c_2 be two circles generated by rotating about H_c', chosen to be close enough so that $u(t)$ is well approximated by a constant in the interval between them. We consider the time required to travel from c_1 to c_2 along two different paths: our proposed optimal solution $a_1\rightarrow a_2$, obtained by rotating purely about $H^\perp$, and a generic path $b_1\rightarrow b_2$. Suppose the evolution along $b_1\rightarrow b_2$ takes a time Δ. This evolution can be decomposed according to the Trotter formula,

$$e^{-i\Delta(H^\perp+vH_c')}=\lim_{n\rightarrow\infty}(e^{-i(\Delta/n)H^\perp}e^{-i(v\Delta/n)H_c'})^n, \tag{8}$$

which is represented graphically in Fig. 5(a). Note that the time required for the operation on the right-hand side of Eq. (8) is still Δ, as the evolutions along H_c' can be arbitrarily fast. The time $\frac{\Delta}{n}$ in a single segment is equal to the angle swept out divided by the norm of $H^\perp$. This angle is minimized in every segment when traveling from $a_1\rightarrow a_2$, as Fig. 5(b) illustrates. The time-optimal solution is therefore to rotate purely about $H^\perp$, which corresponds exactly to solution (7). The speedup here arises from the fact that H_d' and H_c' are not orthogonal, which can be carried over into higher dimensions.

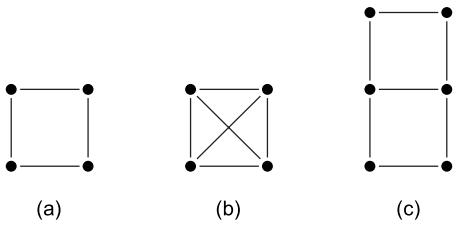

FIG. 6. Some of the graphs considered: (a) C_4, (b) K_4, and (c) $G_{2,3}$.

TABLE I. Minimal times calculated by the GRAPE algorithm with an estimated numerical accuracy of ± 1 on the last digit. As for the three-qubit example under control setting (ii), included in the first row as a reference, the drift and control can be reduced to matrices H_d' and H_c' of size $d \times d$. $\hat{H}'$ indicates a rescaling to unit norm.

Graph	d	$\lvert\langle \hat{H}_d' \rvert \hat{H}_c' \rangle\rvert$	$T_{\min}\left(\frac{1}{2J}\right)$
K_3	2	0.2	0.77
L_3	3	0	1.00
K_4	3	0	0.91
C_4	4	0	1.00
K_5	3	0.1	0.70
K_6	4	0	1.00
$G_{2,3}$	14	0	1.00
K_7	10	0.07	0.60

III. HIGHER DIMENSIONAL CASES

Finally we provide the minimal times for cluster state preparation on a variety of graphs, some of which are illustrated in Fig. 6 to clarify our notation. The times are evaluated numerically using the GRAPE algorithm and are included in Table I. For all of the four-qubit graphs, both control schemes (i) and (ii) were considered, yielding exactly the same minimal times in each case. For the larger graphs only control scheme (ii) was considered, allowing us to reduce the problem to a dimension d using a symmetry-adapted basis, as in Sec. II. The times listed here hold not just for the target state but its entire local unitary orbit, which may include other entangled states of interest [27]. While a Bloch sphere analysis is not possible for $d > 2$, we find that a connection persists between the nonorthogonality of H_d' and H_c' and the minimal time—in the cases considered, a speedup is possible when H_d' and H_c' are nonorthogonal. Minimal time solutions found by the algorithm for K_5 and K_7 are shown in Fig. 7. We observe that the effective hard rotation pulse at time T is still present, while the preceding pulse shape is no longer constant.

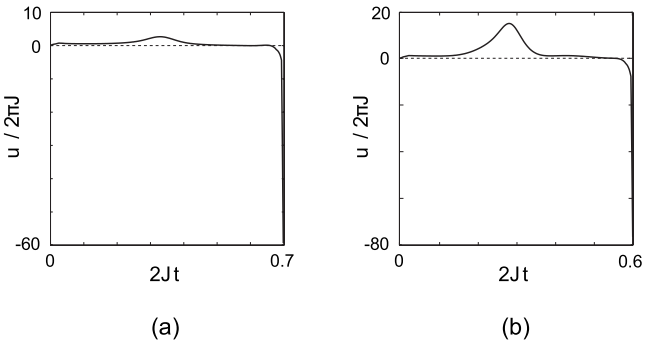

FIG. 7. Numerically optimized sequences for (a) K_5 and (b) K_7.

IV. SUMMARY

For several known two-qubit quantum gates, the "do nothing" operation, i.e., the evolution under a given coupling Hamiltonian, is time-optimal. For example, this is the case for a SWAP gate in the presence of an isotropic Heisenberg coupling [28]. Hence it may be surprising to find that it is possible to create cluster states faster than the time required by the straightforward implementation their definition implies. Here we provide examples where this is the case. In particular for three Ising-coupled qubits with identical coupling constants, the problem of time-optimal cluster state generation could be solved analytically via geodesics on a sphere. The numerical techniques used here can also be applied to the problem of cluster state generation in an arbitrary coupling topology, where real experimental values for the coupling constants can be used. This will be addressed in a future work.

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