Continuous Trajectory Planning of Mobile Sensors for Informative Forecasting

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Continuous Trajectory Planning of Mobile Sensors for Informative Forecasting

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Abstract

This paper addresses planning of continuous paths for mobile sensors to reduce uncertainty in some quantities of interest in the future. The mutual information between the continuous measurement path and the future verification variables defines the information reward. Two expressions for computing this mutual information are presented: the filter form extended from the state-of-the-art and the smoother form inspired by the conditional independence structure. The key properties of the approach using the filter and smoother strategies are presented and compared. The smoother form is shown to be preferable because it provides better computational efficiency, facilitates easy integration with existing path synthesis tools, and most importantly, enables correct quantification of the rate of information accumulation. A spatial interpolation technique is used to relate the motion of the sensor to the evolution of the measurement matrix, which leads to the formulation of the optimal path planning problem. A gradient-ascent steering law based on the concept of information potential field is also presented as a computationally efficient suboptimal strategy. A simplified weather forecasting example is used to compare several planning methodologies and to illustrate the potential performance benefits of using the proposed planning approach.

Key words: Sensor networks, mutual information, trajectory planning, mobile robots.

1 Introduction

One key problem for (mobile) sensor networks is to create plans for maneuvering/locating sensing resources in order to extract information from the environment. In this research, the plans are often generated to reduce uncertainty in some quantity of interest (e.g., position and velocity of targets [10–12, 14, 15, 23, 29, 36], the pose of the sensor platform itself, the forecast of weather over some region of interest [1, 3, 22, 28], physical quantities under the ocean [9, 17], or the distribution of radioactive materials [4]) at some point in time – called the verification time and denoted here as $T$. Note that in most previous work [10–12, 14, 15, 17, 23, 29, 36], this verification time was taken to be the final time of the plan $\tau$, so that $T = \tau$.

This paper investigates a similar planning problem, but the goal is modified to determine the best measurement plan to minimize the uncertainty in some verification variables in the future. In particular, the problem formulation is extended to the more general setting wherein the verification time can be significantly larger than the final time of the plan i.e., $T \gg \tau$. In addition, the quantities of interest, called verification variables, can be variables distributed over a subset of the entire environment, not necessarily being the whole set of state variables. These two modifications enable the solution of additional classes of planning problems, e.g., planning to reduce uncertainty in forecasts, and also provide additional insights on the standard problems solved in the literature [2]. One way to address this generalized planning problem is to simply augment a null measurement between $\tau$ and $T$ to some measurement path, and optimize the solution over this augmented measurement space. This approach is considered, but several potential issues are discussed. An alternative approach that is inspired by the well-known conditional independence of the past and the future for a given present state [5] is then proposed and several advantages, including improved computational efficiency and correct quantification of the amount of information accumulated, are clearly demonstrated in section 2.5.

1.1 Quantification of Mutual Information

Mutual information is used herein to define the uncertainty reduction of the quantity of interest. The mutual information between the verification variables (in the future) and some measurement sequence over the time window $[0, \tau]$ represents the difference between the prior and the posterior
entroy of the verification variables when conditioned on this sequence of measurements. Thus, mutual information explicitly quantifies the impact of sensing on the entropy reduction of the verification variables. For the continuous trajectory planning problems of interest in this paper, this measurement sequence is represented by continuous random processes instead of a finite number of random variables.

In the information theory literature, there has been a well-established theory on computing the mutual information between the signal and observation in the continuous-time domain. Duncan [7] showed that the mutual information between the signal history and observation history (i.e., signal during \([0, \tau]\) and observation during \([0, \tau]\)) can be expressed as a function of the estimation error when the signal is Gaussian and the observation is taken through an additive Gaussian channel. Similar quantification is performed for non-Gaussian signals [13, 18] and fractional Gaussian channel [8]. On the other hand, Tomita et al. [31] showed that the optimal filter for a linear system that maximizes the mutual information between the observation history for \([0, \tau]\) and the state value at \(\tau\) is the Kalman-Bucy filter; Mayer-Wolf and Zakai [24] related this mutual information to the Fisher information matrix. Mitter and Newton [25] presented an expression for the mutual information between the signal path during \([t, \tau]\), \(t < \tau\) and the observation history during \([0, \tau]\), with a statistical mechanical interpretation of this expression.

However, these previous results cannot be directly used to quantify the mutual information of interest in this work because they all consider the case where the verification time is equal to the final time, \(T = \tau\) (or there is a verification interval \([t, T]\) with \(T = \tau\)) and the verification variables are the whole state variables. Newton [26, 27] recently extended his prior results [25] to quantify the mutual information between the future signal path during \([\tau, T]\), \(\tau < T\) and the past measurement history during \([0, \tau]\). But, the results in [26, 27] (combined with those in [25]) can only be used when the verification interval overlaps with the measurement interval and the verification variables are the whole state variables. Therefore, a new expression of the mutual information is required to address the more general planning problem in this paper. Also, although there have been several studies on continuous decision making for sensors [10, 11, 17, 21], none of these explicitly quantified the mutual information. In contrast, this work presents an explicit way of computing the mutual information for continuous-time planning problems.

The straightforward approach, which augments a null measurement after \(\tau\), results in the mutual information being computed by integrating matrix differential equations during the time window \([0, T]\), generalizing the expression of mutual information in [24, 25, 31]. The alternative approach based on the conditional independence that is presented here enables the quantification of the mutual information between the variables at \(T\) and a measurement path over \([0, \tau]\) as the difference between the unconditioned and conditioned mutual information between the state at \(\tau\) and the respective measurement path. The expressions of the mutual information obtained by these two approaches will be called the filter form and the smoother form, respectively. These two forms are shown to be equivalent under the standard assumptions that the system dynamics are Markov and that the sensing noise at a given time is independent of the future process noise. However, the smoother form will be shown to possess the following advantages over the filter form (section 2.5):

1. for the smoother form, the associated matrix differential equations are integrated over a much shorter time window than for the filter form, which reduces the computational cost;
2. the smoother form projects the decision space from the forecast horizon onto the planning horizon which facilitates the integration of this technique into various path planning algorithms;
3. analysis of the time derivatives of the presented expressions of mutual information (in section 2.4) shows that only the smoother form correctly predicts the rate of information accumulation in the generalized planning problem in this work.

Theoretical results in this study are developed with linear time-varying systems, but key insights can be extended to general nonlinear cases. In particular, extending the results to discrete-time linear systems is trivial.

1.2 Continuous Trajectory Planning for Environmental Forecasting

This paper designs continuous trajectories for mobile sensors, specifically, in the context of environmental sensing. As illustrated in Fig. 1, a mobile sensor continuously observes environmental field variables (e.g., temperature, pressure, concentration of chemicals, diffusivity) along a continuous path during the time interval \([0, \tau]\). The goal is to reduce the uncertainty in the forecast of the environmental variables over the verification region (red squares) at the verification time \(T\), with \(T \gg \tau\). Weather forecasting is one motivating application of this planning problem, the goal being the design of an adaptive sensor network that supplements fixed observation networks [1, 3, 22, 28].

![Fig. 1. Continuous motion planning of a sensor for informative forecast: a mobile sensor senses some environmental variable represented by the contours along the path designed to achieve best forecast for the verification region in a two-dimensional space.](image-url)
The rest of the paper is organized as follows. Section 2 develops theoretical results on the quantification of mutual information. Section 3 describes spatial interpolation techniques used to develop a tractable representation of continuous trajectories. In section 4, the optimal trajectory planning problem and a gradient-ascent steering law are presented. The paper concludes with numerical studies using a simplified weather forecasting problem.

2 Information by Continuous Measurement

2.1 Linear System Model

Consider the linear (time-varying) dynamics of state variables $X_t \in \mathbb{R}^{n_x}$ subject to additive process noise $W_t \in \mathbb{R}^{n_x}$:

$$\dot{X}_t = A(t)X_t + W_t.$$  \hspace{1cm} (1)

$W_t$ is zero-mean Gaussian, independent of $X_t$, and $\mathbb{E}[W_tW_t'] = \Sigma_W \delta(t - s)$, $\Sigma_W \succeq 0$ where the prime sign (’) denotes the transpose of a matrix. The initial condition of the state $X_0$ is normally distributed as $X_0 \sim \mathcal{N}(\mu_0, P_0)$, $P_0 > 0$. In the trajectory planning problem described in section 1.2, the state vector $X_t$ represents the environmental variables at a finite number of grid points. In the context of environmental forecasting, the linear time-varying dynamics in (1) are often used to approximate the propagation of a perturbation through the nonlinear dynamics [28].

Also, consider a linear measurement model for $Z_t \in \mathbb{R}^m$ with additive sensing noise $N_t \in \mathbb{R}^m$:

$$Z_t = C(t)X_t + N_t.$$  \hspace{1cm} (2)

$N_t$ is zero-mean Gaussian, independent of $X_t$ and $W_t$, $\forall s$, and $\mathbb{E}[N_tN_t'] = \Sigma_N \delta(t - s)$, $\Sigma_N > 0$. This linear sensing model is a good representation of the observations of directly measurable environmental variables (e.g., temperature, pressure) that are distributed in the field. In the later part of this paper, the time argument will be omitted if no confusion is expected.

With this environmental system model, this work determines the impact of a measurement path in the near future on the uncertainty reduction of some verification variables in the far future. A measurement path up to time $t$ is defined as

$$Z_t = \{Z_\sigma : \sigma \in [0, t]\}.$$  \hspace{1cm} (3)

The verification variables are a subset of the state variables that can be expressed as

$$V_t = M_V X_t \in \mathbb{R}^{n_V}$$  \hspace{1cm} (4)

where $M_V \in \{0, 1\}^{n_V \times n_x}$, $n_V \leq n_x$ with every row-sum of $M_V$ being unity. Although this work is specifically focused on the case where entries of $M_V$ are zero or one, the results can be easily extended to a general $M_V \in \mathbb{R}^{n_V \times n_x}$.

Entropy is employed as a metric of uncertainty because it represents the degree of randomness of generic random entities (e.g., random variables [5], random processes [19], stochastic systems [37]). Then, the uncertainty reduction of one random quantity by another random quantity is expressed as the mutual information between them. Therefore, the information reward by a measurement path is defined as:

$$J_V(T, \tau) \triangleq I(V_T; Z_\tau), \hspace{1cm} 0 \leq \tau < T,$$  \hspace{1cm} (5)

where $I(A_1; A_2)$ denotes the mutual information between two random entities $A_1$ and $A_2$. This reward represents the entropy reduction of the verification variables in the far future time $T$ by the measurement history up to the near future time $\tau$. This work focuses on the case where $\tau < T$ because the results of previous work [24, 25, 31] can be used for the case when $\tau = T$.

2.2 Filter Form

For linear Gaussian systems, there are known expressions for the mutual information between the state variables at a given time and a measurement history up to that time [24, 25, 31]. Therefore, one way to compute the information reward is to consider the filtering problem that estimates $X_T$ based on the measurement history up to time $T$ denoted as $Z_T \triangleq Z_\tau \cup \{0, \tau, T\}$ where $\{0, \tau, T\}$ means that no measurement is taken during $(\tau, T]$. Then, $I(X_T; Z_\tau) = I(X_T; Z_T)$, because no information is gathered by a null measurement. This procedure of obtaining $I(X_T; Z_\tau)$ can be extended to computing $I(V_T; Z_\tau)$, as outlined in the following proposition:

Proposition 1. (Filter Form) For the linear system described by (1) and (2), the information reward can be computed as

$$J_V^F(T, \tau) \triangleq I(V_T; Z_\tau, \emptyset[\tau, T])$$

$$= \frac{1}{2} \det(M_V P_X(T) M_V') - \frac{1}{2} \det(M_V Q_X(T) M_V')$$  \hspace{1cm} (6)

where $\det$ denotes log det of a symmetric positive definite matrix, and $P_X(T)$ and $Q_X(T)$ are obtained by integrating the following matrix differential equations:

$$\dot{P}_X(t) = A(t)P_X(t) + P_X(t)A'(t) + \Sigma_W$$  \hspace{1cm} (7)

$$\dot{Q}_X(t) = A(t)Q_X(t) + Q_X(t)A'(t) + \Sigma_W - \mathbb{I}_{[0, \tau]}(t) Q_X(t) C(t) \Sigma_N^{-1} C(t)' Q_X(t)$$  \hspace{1cm} (8)

with initial conditions $P_X(0) = Q_X(0) = P_0 > 0$, and $\mathbb{I}_{[0, \tau]}(t) : \mathbb{R} \mapsto \{0, 1\}$ is defined by

$$\mathbb{I}_{[0, \tau]}(t) = \begin{cases} 1, & t \in [0, \tau] \\ 0, & \text{otherwise} \end{cases}$$

Furthermore, Eqs. (7) and (8) are well-defined for finite $T$ with $P_0 > 0$. 

3
smoothing problem that incorporates the continuous measurement history $Z_r$ and the discrete noise-free measurement of the verification variables at time $T$:

**Proposition 3. (Smoother Form)** Suppose that $P_{0|V} \triangleq \text{Cov}(X_0|V_T) > 0$ is given. Then, for the linear system described by (1) and (2), the information reward can be computed as

\[
J^S_{V}(T, \tau) \triangleq I(X_r; Z_r) - I(X_r; Z_r|V_T) = J_0(\tau) - \frac{1}{2} \text{ldet} [I + Q_X(\tau) S_X(\tau)]
\]

where $J_0(\tau) \triangleq \frac{1}{2} \text{ldet} S_{X|V}(\tau) - \frac{1}{2} \text{ldet} S_X(\tau)$, $\Delta_S(\tau) \triangleq S_{X|V}(\tau) - S_X(\tau)$, and $J^S_{V}(T, \tau)$ are determined by the following matrix differential equations:

\[
\dot{S}_X = -S_X A - A^T S_X - S_X \Sigma W S_X + S_{X|V} \Sigma W S_{X|V}
\]

\[
\dot{S}_{X|V} = -S_{X|V} (A + \Sigma W S_X) - (A + \Sigma W S_X)^T S_{X|V} + S_{X|V} \Sigma W S_{X|V}
\]

\[
\dot{Q}_X = A Q_X + Q_X A^T + \Sigma W - Q_X C^T \Sigma^{-1} C Q_X
\]

with initial conditions $S_X(0) = P_0^{-1}$, $S_{X|V}(0) = P_0^{-1}$ and $Q_X(0) = P_0$.

**Proof.** See Appendix A.

**Remark 4. (Computation of Conditional Initial Covariance)** To apply Proposition 3, $P_{0|V}$ must be available. For the linear setting as in this work, $P_{0|V}$ can be computed by the covariance update formula:

\[
P_{0|V} = P_0 - P_0 \Phi(T, 0) M^T \Sigma M \Phi(T, 0)^T P_0
\]

where $\Phi(T, 0)$ is the state transition matrix from time 0 to $T$, which is $e^{AT}$ in the linear time-invariant case. Note that the inverse on the right-hand side exists for finite $T$ with $P_0 > 0$. For a time-varying case, a fixed-point smoothing using state augmentation can be easily applied to find $P_{0|V}$. When the linear system is used to approximate the short-term behavior of a nonlinear system whose long-term behavior is tracked by some nonlinear estimation scheme, $P_{0|V}$ can be provided by this nonlinear estimator. For instance, in the ensemble-based estimation framework, the ensemble augmentation technique presented by the present authors [3] can be used for this purpose.

**Corollary 5.** The filter form information reward $J^F_{V}(T, \tau)$ in Proposition 1 and the smoother form information $J^S_{V}(T, \tau)$ in Proposition 3 are identical, because they are two different expressions for $I(V_T; Z_r)$.

Notice that the smoother form utilizes an additional piece of backward information $S_{X|V}$, while the filter form only uses the forward information captured in $P_X$ (or, equivalently...
$S_X$ and $Q_X$. One aspect for which this additional backward information plays a key role is the information available on the fly, which is discussed in the following.

2.4 On-the-fly Information and Mutual Information Rate

This section discusses the on-the-fly information available in the process of computing the filter form and the smoother form mutual information, and identifies important features of the smoother form in terms of information supply and dissipation. Moreover, this analysis facilitates building an information potential field that can be used to visualize the spatial distribution of information quantities and to develop a gradient-ascent steering law for a mobile sensor in section 4.2.

2.4.1 Filter-Form On-the-fly Information (FOI)

Since the Lyapunov equation in (7) and the Riccati equation in (8) are integrated forward from time 0, $P_X(t)$ and $Q_X(t)$ are available at arbitrary $t < \tau$ in the process of computing the mutual information in (6). With these, the mutual information between the current (i.e., at $t$) state variables and the measurement thus far (FOI) can be evaluated as

$$
\mathcal{I}(X_t; Z_t) = \frac{1}{2} \det P_X(t) - \frac{1}{2} \det Q_X(t).
$$

(17)

The expression for the time derivative of FOI was first presented in [24], and its interpretation as information supply and dissipation was presented in [25]. The rate of FOI can be derived as

$$
\frac{d}{dt} \mathcal{I}(X_t; Z_t)
= \frac{d}{dt} \left[ \frac{1}{2} \det P_X(t) - \frac{1}{2} \det Q_X(t) \right]
= \frac{1}{2} \tr \{ P_X^{-1} \dot{P}_X - Q_X^{-1} \dot{Q}_X \}
= \frac{1}{2} \tr \{ \Sigma_X^{-1} Q_X C' \} - \frac{1}{2} \tr \{ \Sigma_W (Q_X^{-1} - P_X^{-1}) \}
$$

(18)

where $\tr$ denotes the trace of a matrix, and every matrix is evaluated at $t$. The first term in (18) depends on the measurement and represents the rate of information supply, while the second term depends on the process noise and represents the rate of information dissipation [25]. It can be shown that the supply and the dissipation terms are non-negative:

$$
\tr \{ \Sigma_X^{-1} Q_X C' \} \geq 0,
\tr \{ \Sigma_W (Q_X^{-1} - P_X^{-1}) \} \geq 0,
$$

since $C_X Q_X C' \geq 0$, $Q_X^{-1} - P_X^{-1} \succeq 0$, and trace of the product of two symmetric positive definite matrices is non-negative [20]. Thus, measurement tends to increase FOI while the process noise tends to decrease it. Observe that FOI can be decreasing over time if the information dissipation dominates the information supply.

Remark 6. The approach in [10] considered the entropy of the current state $\mathcal{H}(X_t) = -\frac{1}{2} \det J_X(t) + \frac{\alpha}{2} \log(2\pi e)$ where $J_X(t) \triangleq Q_X^{-1}(t)$, rather than the mutual information as in (17). The resulting rate calculation is similar to (18) in that the first terms are the same (information supply), but the second terms (information dissipation) are quite different. Therefore, the procedure in [10] cannot be used to accurately compute the rate of FOI.

2.4.2 Projected Filter-Form On-the-fly Information (PFOI)

Similar to FOI, the mutual information between the current verification variables and the measurement thus far (PFOI) can also be computed on the fly, while computing the filter form mutual information:

$$
\mathcal{I}(V_t; Z_t) = \frac{1}{2} \det P_V(t) - \frac{1}{2} \det Q_V(t).
$$

(19)

where $P_V(t) \triangleq M_V P_X(t) M_V'$ and $Q_V(t) \triangleq M_V Q_X(t) M_V'$.

The time derivation of PFOI can also be expressed in terms of $P_X(t)$ and $Q_X(t)$ as follows.

$$
\frac{d}{dt} \mathcal{I}(V_t; Z_t)
= \frac{1}{2} \tr \{ P_V^{-1} \dot{P}_V - Q_V^{-1} \dot{Q}_V \}
= \frac{1}{2} \tr \{ \Sigma_N^{-1} C Q_X M_V' Q_V^{-1} M_V Q_X C' \} + \beta(t),
$$

Direct Supply

(20)

where $\beta(t)$ represents all the remaining terms that do not depend on the observation matrix $C$. The first term, underbraced as “Direct Supply” represents the immediate influence of the measurement on the current verification variables. The remaining term $\beta(t)$ captures all of the correlated effect due to coupling in the dynamics on the information supply/dissipation. Observe that sign of $\beta(t)$ is indefinite, while the direct supply term is non-negative as $C Q_X M_V' Q_V^{-1} M_V Q_X C' \succeq 0$.

2.4.3 Smoother-Form On-the-fly Information (SOI)

In the smoother form framework, the mutual information between the future verification variables $V_T$ and the measurement up to the current time $t$ (SOI) can be calculated as

$$
\mathcal{I}(V_T; Z_t) = \mathcal{I}(X_t; Z_t) - \mathcal{I}(X_t; Z_t | V_T)
= J_0(t) - \frac{1}{2} \det(I + Q(t) \Delta_S(t)).
$$

(21)

The values of matrices $J_0(t)$, $Q(t)$, and $\Delta_S(t)$ are calculated in the process process of the forward integration (14)–(16).

The temporal derivative of the smoother form mutual information can be written as follows.

Proposition 7. (Smoother-form Information Rate) For the temporal derivative of the smoother form on-the-fly information, the following holds:

$$
\frac{d}{dt} \mathcal{I}^s_T(t, t) = \frac{1}{2} \tr \{ \Sigma_N^{-1} C(t) \Pi(t) C(t)' \} \geq 0
$$

(22)
where $\Pi(t) \triangleq Q_X(t)(S_{X|V}(t) - S_X(t))[I + Q_X(t)(S_{X|V}(t) - S_X(t))]^{-1}Q_X(t)$.

**Proof.** See Appendix B.

Since the influence of the future process noise has already been captured in $S_{X|V}$, the mutual information rate for the smoother form is non-negative regardless of the process noise, as stated in Proposition 7. If one stops taking measurement at time $t$, the information reward stays constant.

To summarize, the smoother form information rate in (22) correctly accounts for the influence of the process noise and the coupling through dynamics occurring over $(t, T]$, and correctly identifies the net impact of sensing at time $t$ on the entropy reduction of the verification variables at $T$. However, the two expressions in (18) and (20) based on the filter form, only look at the information being accumulated in the current variables (state or verification) without accounting for their evolution and diffusion at the future time; therefore, these expressions do not correctly quantify the impact of the current measurement on the future verification variables.

**Remark 8. (Information Rate for Multiple Sensors)** Consider the case when there are multiple sensor platforms, and the observation matrix of the $i$-th sensor is $C_i$, constituting the overall observation matrix of $C = [C_1 \ldots C_n]$. Then, the smoother-form mutual information rate in Proposition 7 can be written as

$$\frac{d}{dt} J_{\text{S}}^S(T, t) = \sum_{i=1}^{n_s} \frac{1}{2} \text{tr} \left\{ \Sigma_{N_i}^{-1} C_i(x_i, y_i) \Pi(t) C_i(x_i, y_i)' \right\}$$

where $\Sigma_{N_i}$ is the $(i, i)$-th block entry of $\Sigma_N$, and $(x_i, y_i)$ is the location of the $i$-th sensor. In other words, the total rate of change of mutual information is the sum of the rate of change of mutual information of individual sensor platforms.

### 2.5 Comparison of Filter and Smoother Forms

#### 2.5.1 Correct On-the-fly Information and Information-Accumulation Rate

One benefit of the smoother form is that on-the-fly information based on the smoother form correctly captures the amount and the rate of information gathering by pre-incorporating the effect of future process noise and the correlation via future dynamics, as discussed in section 2.4. These properties are illustrated in the following example.

**Example 9.** Fig. 2 compares the time histories of three on-the-fly quantities: the smoother-form on-the-fly information $J_{\text{S}}^S(T, t) = I(V_T; Z_t)$, the filter-form on-the-fly information $I(X_t; Z_t)$ and the projected filter-form on-the-fly information $I(V_t; Z_t)$. In this example, the following system matrices are used with $\tau = 2$ and $T = 5$:

$$A = \begin{bmatrix} 0.1 & 1 \\ -1 & -0.5 \end{bmatrix}, \quad \Sigma_W = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.01 \end{bmatrix}, \quad P_0 = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 0.5 & 0.5 \end{bmatrix}, \quad \Sigma_N = 0.01, \quad M_V = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

There are three key points to discuss about Fig. 2. First, observe how each on-the-fly information changes over time. For $J_{\text{S}}^S(T, t)$, it is found that information increases in the presence of measurement (before $\tau$) and stays constant in the absence of measurement (after $\tau$). In the history of $I(X_t; Z_t)$, the information supply over $[0, \tau]$ increases the accumulated information while the information dissipation over $(\tau, T]$ decreases the accumulated information. The history of $I(V_t; Z_t)$ is fluctuating; it can decrease with measurement (around $t = 0.5$) and can increase without measurement (at $t = 2$), because information can be supplied/dissipated from/to the other state variables $X_t \setminus V_t$ via the system dynamics.

Second, notice that $J_{\text{S}}^S(T, t)$ at $t = \tau$ agrees with $I(V_t; Z_t)$ at $t = T$ in Fig. 2, and this agreement numerically confirms the equivalence of the filter form and the smoother form, i.e., $I(V_T; Z_T) = I(V_T; Z_T)$ with a null measurement during $(\tau, T]$.

Third, when comparing three quantities at some arbitrary time $t$, it is found that $I(X_t; Z_t)$ overestimates $I(V_T; Z_t)$, and $I(V_t; Z_t)$ may overestimate or underestimate $I(V_T; Z_t)$ except when $t = T$. Thus, the filter form quantities, $I(X_t; Z_t)$ and $I(V_t; Z_t)$, are not good indicators of the accumulated information $I(V_T; Z_T)$; only the smoother form quantity $J_{\text{S}}^S(T, t)$ accurately represents the accumulated information.
2.5.2 Computational Efficiency

Optimal trajectory planning typically requires computation of the information reward for many different measurement choices, i.e., many options of $Z_r$. If utilizing the filter form in (6) for trajectory planning, $P_X(T)$ can be calculated independent of what $Z_r$ is. But, $Q_X(T)$ needs to be calculated for each candidate measurement path; thus, the Riccati equation in (8) has to be integrated over time window $[0, T]$ numerous times. Since the computational cost of integrating a matrix differential equation grows with the integration time interval, the reward calculation using the filter form can be computationally intensive when $T \gg \tau$.

On the other hand, when computing the information reward using the smoother form in (13) for multiple candidates of $Z_r$, the only quantity that needs to be calculated many times is $Q_X(\tau)$, because $S_X(\tau)$ and $P_{X|0|V}$ can be precomputed independent of $Z_r$. Thus, for trajectory planning using the smoother form, one integrates the same matrix differential equation (i.e., Riccati equation) for a shorter time window $[0, \tau]$. Therefore, compared to the filter form, the computation cost of evaluating rewards for different measurement options is reduced by the factor of approximately $T/\tau$.

2.5.3 Easy Integration with Existing Synthesis Techniques

The smoother form facilitates easy integration with existing path synthesis algorithms (established to solve the problem of $T = \tau$), since it projects the decision space from the forecast time window onto the planning time window. Previous sensor planning work [10, 15] has established synthesis techniques for the conventional problem of min $\det Q_X(\tau)$. With the smoother form, the optimal forecasting problem is written as min $\det(I + Q_X(\tau)\Delta_S(\tau))$. Thus, the forecasting problem can be treated as a similar type of optimization problem with a possibly different weighting in the objective function. Of course, depending on $\Delta_S(\tau)$, the solutions of the two problems can be very different (see section 5). However, existing synthesis tools for the conventional decision making problem can be easily integrated into the forecasting problem, because the only change needed is a slight modification of the objective function. This provides flexibility in the choice of a synthesis method in the planning problem.

3 Continuous Path Representation

Section 2 presented a formula to quantify the information reward for a continuous measurement path in a finite-dimensional linear system framework. This section shows how to relate the motion of a sensor in continuous space to a measurement trajectory in the time domain.

The approach employs spatial interpolation techniques (e.g., Kriging [6], Gaussian processes regression (GPR) [35]) that are used to describe continuous environmental variables at an arbitrary location in terms of a finite number of variables at some specified grid points. These techniques assume that the environmental variables at location $\mathbf{r}$ can be represented as a linear combination of those at a finite number of grid points $\mathbf{r}_i$’s:

$$\phi_t(\mathbf{r}) = \sum_{i=1}^{n_G} \lambda_i(\mathbf{r}, \mathbf{r}_i) \phi_t(\mathbf{r}_i),$$  \hspace{1cm} (24)

where $n_G$ is the number of grid points, $\phi_t(\mathbf{r}_i) \in \mathbb{R}^{n_E}$ represents the environmental variables at $\mathbf{r}_i$ at time $t$ for $i \in \{1, 2, \ldots, n_G\}$, and $n_E$ denotes the number of environmental variables associated with a single grid point. In determining the coefficients $\lambda_i$’s, this paper considers the zero-mean GPR method [35] with squared exponential covariance functions, which leads to:

$$\lambda_i(\mathbf{r}, \mathbf{r}_i) = \sum_{r=1}^{n_G} \alpha_{ij} \rho(\mathbf{r}, \mathbf{r}_j)$$ \hspace{1cm} (25)

where $\rho(\mathbf{r}, \mathbf{r}_j) \triangleq \exp \left[-\frac{1}{2\sigma_f^2}(x-x_j)^2 - \frac{1}{2\sigma_l^2}(y-y_j)^2 \right]$ in the two-dimensional space, and $\alpha_{ij}$ is the $(i, j)$-th element of the matrix $[\rho(\mathbf{r}_i, \mathbf{r}_j)]^{-1}$. The parameters $l_x$ and $l_y$ represent the correlation length scales in each direction.

Under the assumption in (24), the environmental dynamics over the whole continuous space can be fully described by the dynamics of the finite number of variables at grid points. The state vector $X_t \in \mathbb{R}^{n_X}$ where $n_X = n_G \times n_E$, is defined as

$$X_t = [\phi_t(\mathbf{r}_1)^	op \cdots \phi_t(\mathbf{r}_{n_G})^	op]^	op,$$

and this work considers linear dynamics for $X_t$ as in (1). Consider a sensor located at $\mathbf{r}$ at time $t$ that receives measurement of $\phi_t(\mathbf{r})$. Since $\phi_t(\mathbf{r})$ is a linear combination of $\phi_t(\mathbf{r}_i)$’s, the observation equation for this sensor can be expressed as $Z_t = C(t)X_t + N_t$ where

$$C(t) = [\lambda_1(\mathbf{r}, \mathbf{r}_1)I_{n_E} \cdots \lambda_{n_G}(\mathbf{r}, \mathbf{r}_{n_G})I_{n_E}] \in \mathbb{R}^{n_E \times n_X}.$$

If a sensor is continuously moving, its motion is fully described by the time history of the location vector $\mathbf{r}(t)$. Thus, the effect of the sensor’s motion on the uncertainty dynamics is through the evolution of the observation matrix $C(t)$ due to changes in $\lambda_i(\mathbf{r}, \mathbf{r}_i)$’s in time. Consider a sensor moving along a specified path $p_r = \{\mathbf{r}(t) : t \in [0, \tau]\}$ where $\mathbf{r}(t)$ is known for all $t \in [0, \tau]$. Then, the evolution of observation matrix $C_r \triangleq \{C(t) : t \in [0, \tau]\}$ can be derived by relating $C(t)$ and $\mathbf{r}(t)$. The information reward associated with this path, denoted as $J_r(T; \tau; p_r)$, can be computed by evaluating $Q_X(\tau; C_r)$, which is the final value of the Riccati equation corresponding to observation matrix history $C_r$, while $J_0(\tau)$ and $\Delta_S(\tau)$ in (13) have been computed in advance independently of $p_r$.

To account for the limited mobility of the sensor, the path is, in general, represented as a set of equations of the location vector and its time derivatives: $g_{dyn}(\mathbf{r}(t), \dot{\mathbf{r}}(t), \ddot{\mathbf{r}}(t), \mathbf{u}(t)) = 0$ where $\mathbf{u}$ is the control input for the sensor motion. For
instance, a two-dimensional holonomic motion of a sensor platform with constant speed \( v \) can be written as

\[
\dot{x}(t) = v \cos \theta(t), \quad \dot{y}(t) = v \sin \theta(t)
\]  

(26)

where \( \theta(t) \) is the heading angle, which is treated as a control input in this model.

4 Path Planning Formulations

4.1 Optimal Path Planning

The optimal trajectory planning determines the path \( p_r \), or equivalently the time history of the control input, that maximizes the smoother form information reward \( J_0^S(T, \tau) = J_0(\tau) - \frac{1}{2} \det(I + Q_X(\tau; C_r) \Delta_S(\tau)) \). The prior and posterior initial covariance \( P_0 \) and \( P_0 \mid V \), respectively, are computed first, which enables calculation of \( J_0(\tau) \) and \( \Delta_S(\tau) \). Then, an optimization problem involving only the computation of \( Q_X(\tau; C_r) \) is posed. This optimization problem is indeed a nonlinear optimal control problem (OCP) with a terminal cost functional. The control variables for this OCP are the controls for the sensor motion (e.g., \( \theta(t) \)) for the two-dimensional holonomic motion, while there are two types of state variables: the vehicle position variables, \( x \) and \( y \), and the entries of the \( Q_X(t) \) matrix. The optimal path planning problem for a two-dimensional holonomic mobile sensor is stated as

\[
\theta^*(t) \in \arg \max_{\theta(t)} J_0(\tau) - \det(I + Q_X(\tau; C_r) \Delta_S(\tau))
\]  

(27)

subject to

\[
\dot{Q}_X = A Q_X + Q_X A' + \Sigma_W - Q_X C(x, y)' \Sigma_N^{-1} C(x, y) Q_X \\
\dot{x} = v \cos \theta, \quad \dot{y} = v \sin \theta \\
Q_X(0) = P_0, \quad x(0) = x_0, \quad y(0) = y_0
\]

where \( C(x, y) \) is expressed as a function of \( x \) and \( y \) to emphasize that its dependency on time is only through the evolution of \( x \) and \( y \). Regarding the size of this OCP, there is one control variable and the number of state variables is \( n_X(n_X + 1)/2 + 2 \). Constraints in the sensor’s motion such as specified final locations and those induced by nonholonomy can be easily incorporated by modifying the vehicle’s dynamics and imposing additional constraints. Also, multiple sensor problems can be dealt with by adding associated dynamic/kinematic constraints and by modifying the expression of the observation matrix.

4.2 Information Potential Field and Gradient-Ascent Steering

Optimal path planning gives a motion plan for maximum information reward, but, solving a nonlinear optimal control problem can take a substantial amount of computational effort, especially, when \( n_X \) is large. Thus it is beneficial in practice to devise a computationally efficient suboptimal steering law. One approach is to build some potential field and to move along the gradient of that field. The mutual information rate discussed in section 2.4 can be utilized to construct an information potential field, which provides a visualization of how information is distributed or concentrated. This type of information potential field extends a similar notion presented in [10], which derived the expression of time derivative of entropy (as discussed in Remark 6), and neglected terms unrelated to the observation matrix to build a potential field. Note that since the potential field in [10] was developed for a different problem formulation that focused on reducing the entropy of the state at a given time, it cannot be directly used for the problem formulation in this work. Thus, this section builds a potential field using the smoother form information rate in (22) that correctly identifies the influence of a measurement taken at time \( t \) on the entropy reduction of the verification variables at \( T \).

For the two-dimensional holonomic sensor motion in (26), the gradient-ascent steering law is

\[
\theta_G(t) = \arctan2\left( \frac{\partial}{\partial p_x} \left( \frac{\partial}{\partial p_y} J_0^S(T, t) \right), \frac{\partial}{\partial p_y} \left( \frac{\partial}{\partial p_x} J_0^S(T, t) \right) \right)
\]

where \( \frac{\partial}{\partial p_x} J_0^S(T, t) \) is the smoother form mutual information rate, and \( \arctan2 \) denotes the four-quadrant inverse tangent function. Since the relationship between \( C(x, y) \) and \( (x, y) \) is known, the mutual information rate in (22) can be written as a function of spatial coordinates, and its gradient can be evaluated accordingly. When \( C(x(t), y(t)) \in \mathbb{R}^{1 \times n_X} \), namely, there is only one environmental variable of interest, the spatial derivative can be written as

\[
\frac{\partial}{\partial p_x} \left( \frac{\partial}{\partial p_y} J_0^S(T, t) \right) = \Sigma_N^{-1} C(x(t), y(t)) \Pi(t) d(p), \quad p = x, y
\]

where \( d(p) \) is an \( n_X \)-dimensional column vector whose \( i \)-th element is \( d(p)_i = -\Sigma(p, r_j)(p - p_j) \), \( p = x, y \). In case \( C \) is not a row vector, the relation in (23) can be used to derive the expressions for the mutual information rate and its gradient by treating each row of the observation matrix as a separate sensor.

5 Numerical Examples

This section compares several planning methodologies that are based on different quantifications of the information reward. The results of the simulation study show the potential advantages of the new technologies proposed in this paper for forecasting problems when compared with traditional planning techniques.

5.1 Scenarios

A simplified weather forecasting problem is considered for numerical simulations. The two-dimensional Lorenz-2003
model [1] is employed to describe the nonlinear environmental dynamics. The system equations are

\[ \phi_{ij} = -\phi_{ij} - \zeta_{i-4,j} \phi_{i-2,j} + \frac{1}{3} \sum_{k \in \{-1,1\}} \zeta_{i-2+k,j} \phi_{i+2+k,j} - \mu \eta_{i,j-4} \eta_{i,j-2} + \frac{4}{3} \sum_{k \in \{-1,1\}} \zeta_{i,j-2+k} \phi_{i,j+2+k} + \phi_0 \]

where \( \zeta_{i,j} \) denotes the west-to-eastern grid index, while \( j \) denotes the south-to-northern grid index. The boundary conditions of \( y_{i+L_i,j} = y_{i-L_i,j} = y_{i,j} \) and \( y_{i,0} = y_{i,-1} = 3 \), \( y_{i,L_i+1} = 0 \) in advection terms, are applied to model the mid-latitude area of the northern hemisphere as an annulus. The parameter values are \( L_i = 72 \), \( L_j = 17 \), \( \mu = 0.66 \) and \( \phi_0 = 8 \). The size of \( 1 \times 1 \) grid corresponds to \( 347 \) km \( \times 347 \) km in real distance, and 1 time unit in this model is equivalent to 5 days in real time. The overall system is tracked by a nonlinear estimation scheme, specifically an ensemble square-root filter (EnSRF) [34] data assimilation scheme, that incorporates measurements from a fixed observation network of size 186.

The path planning problem is posed for the linearized model over some \( 4 \times 3 \) local region (therefore, \( N = 12 \)) in the entire \( L_i \times L_j \) grid space. A linear time-invariant model is obtained by deriving the Jacobian matrix of the dynamics around the nonlinear estimate for \( \phi_{ij} \)'s at the grid points in the local region. Thus, the state vector \( X_t \) represents the perturbation of \( \phi_{ij} \)'s from the ensemble mean. In this linear model, the dependence of the local dynamics on the evolution of the external dynamics is ignored in deriving the Jacobian matrix (or \( A \) matrix). Instead, this effect is incorporated in the process noise term, i.e., the states on the boundary of the local region, which may be affected by external dynamics more substantially, are assumed to be subject to larger process noise. The goal is to design a 6-hr flight path (\( \tau = 6 \) hrs) for a single UAV sensor platform to improve the forecast over the three grid points (red squares in Fig. 3) in the eastern part of the local region in 72 hrs (\( T = 72 \) hrs).

The motion of the sensor is described as 2-D holonomic motion in (26) and it flies at constant speed \( v \) grid/hr (= 347e km/hr). The prior and posterior initial covariance matrices, \( P_0 \) and \( P_{0|V} \) are provided by the EnSRF data assimilation scheme, where \( P_{0|V} \) is computed by the ensemble augmentation method in [3]. Two scenarios with different local region, correlation length scale parameters \((l_x, l_y) = (1, 0.7) \) or \((1.5, 1) \), and vehicle speed \((v = 1/3 \) or \(1/2) \) are considered, and the sensing noise intensity is set at \( \Sigma_N = 0.0025 \).

\[ 5.2 \text{ Results} \]

Two proposed path planning methods, optimal path planning and gradient-ascent steering, are compared with the shortsighted versions of them. Shortsighted path planning takes into account \( I'(X_v; \mathcal{Z}_v) \) instead of \( I'(V_T'; \mathcal{Z}_v) \) to represent the implementation of the traditional planning approach (i.e., \( T = \tau \) and \( M_V = I \)) to a forecasting problem. The optimal shortsighted solution minimizes \( I'Q_X(\tau) \), and the shortsighted gradient-ascent law utilizes the filter form information rate in (18) to construct an information potential field. Since the information dissipation term of the filter form information rate in (18) is not an explicit function of \((x, y)\), the shortsighted gradient-ascent law results in a formula that has \( Q_X \) instead of \( I \) in the gradient expression.

Each of the two optimal control problems is formulated as a nonlinear program (NLP) by parameterizing the control history as a piecewise linear function consisting of 12 linear segments of equal time span. TOMLAB/SNOPT v6.0 [32] is used to solve the NLPs; gradient-ascent solutions, and various straight-line solutions are used as initial guess for the optimization. Both optimized solutions are obtained within two minutes (per initial guess) and satisfy first-order optimality criteria with tolerance of \( 10^{-4} \). Also, as a reference, the best and the worst straight-line paths are also considered. The best straight line solves an NLP to find a constant \( \theta_0 \) that maximizes the smoother form information reward, assuming the vehicle dynamics of \( \dot{x} = v \cos \theta_0, \ \dot{y} = v \sin \theta_0 \).

Table 1 represents the information rewards \( J_{SV}(T, \tau) \) for all the methods considered. It is first found that the gradient-ascent steering provides relatively good performance for both scenarios, while the two shortsighted strategies provide very poor performance in scenario 2. Figs. 3(a) and 3(b) illustrate the sensor trajectories from the six strategies for scenario 1 overlaid with the smoother-form and the filter-form information potential field at the initial time. In both potential fields, a dark region represents an information-rich area. In Figs. 3(a) and 3(b), the shape of the smoother-form information and filter-form information fields are similar in terms of the locations of information-rich regions, which leads to reasonable performance of shortsighted planning in this case. In contrast, Figs 3(c) and 3(d) illustrate the large difference between the initial smoother form and filter form information fields in scenario 2. As a consequence, while the paths generated based on the smoother form head southwards, those for the two shortsighted decisions head north, leading to significant performance deterioration.

In summary, this example illustrates that planning solutions based on \( I(V_T'; \mathcal{Z}_v) \) can be very different from those based on \( I(X_v; \mathcal{Z}_v) \), depending on the problem. In this example, the future process noise turns out not to be a dominant factor that causes the difference, but the dominating factors are the fact that \( T \gg \tau \) and \( M_V \neq I \).

\[ 6 \text{ Conclusions} \]

A methodology for continuous motion planning of sensors for informative forecasting was presented. The key contribution of this work is to provide a framework for quantifying the information obtained by a continuous measurement path to reduce the uncertainty in the long-term forecast for a subset of state variables. The smoother form of the information reward that projects the decision space from the long forecast
horizon onto a short planning horizon is shown to improve computational efficiency, enables the correct evaluation of the rate of information gathering, and offers flexibility in the choice of path synthesis tools. An optimal path planning formulation and a gradient-ascent steering law were presented using spatial interpolation for path representation. A numerical example for a simplified weather forecast compared several planning methodologies and illustrated the performance degradation of the traditional planning approach for some scenarios.

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References

\[ P_{X}(\tau) = \frac{1}{2} \log \det \Sigma_{X}(\tau) - \frac{1}{2} \log \det \Sigma_{Z}(\tau) \] where \( P_{X}(\tau) \) is the solution to the Lyapunov equation, \( \dot{P}_{X} = A P_{X} + P_{X} A^{*} + \Sigma_{W} \) with \( P_{X}(0) = P_{0} \), and \( Q_{X}(t) \) is the solution to the Riccati equation in (16). Regarding the conditional mutual information term, note that
\[
\mathcal{I}(X_{\tau} ; Z_{\tau} | V_{T}) = \frac{1}{2} \| \log \det P_{1} - \log \det P_{2} \| \quad (A.1)
\] where \( P_{1} \) is the covariance of \( \bar{X}_{1} \equiv X_{\tau} - \mathbb{E}[X_{\tau} | V_{T}] \) and \( P_{2} \) is the covariance of \( \bar{X}_{2} \equiv X_{\tau} - \mathbb{E}[X_{\tau} | V_{T}, Z_{\tau}] \). Namely,
$P_2$ is the error covariance of the fixed-interval smoothing with the past measurement $Z_t$, and the future measurement $V_T$. Wall et al. [33] suggested the expression for the error covariance for the fixed-interval smoothing $Q_{X|V}(t) \triangleq \text{Cov}(X_t|V_T, Z_t)^{-1} = Q_X^{-1}(t) + P_X^{-1}(t) - P_X^{-1}(t)$ where $P_X^{-1}(t)$ is the estimation error covariance accounting for the future measurement plus a priori information. $P_{X|V}(t)$ is computed as a solution of a Riccati-like equation that is integrated backwards. Since, in this work’s setting, there is no future measurement except a discrete measurement at $T$, $P_{X|V}(\tau)$ is the same as $P_1$ and can be computed by the following Lyapunov-like equation integrated backwards:

$$
\dot{P}_{X|V} = (A + \Sigma W P_X^{-1}) P_{X|V} + P_{X|V}(A + \Sigma W P_X^{-1})' - \Sigma W \\
\text{with terminal condition } P_{X|V}(T) = \text{Cov}(X_T|V_T), \text{ to time } \tau.
$$

Note that $\text{Cov}(X_T|V_T)$ is all zero except the part corresponding to $\text{Cov}(X_T \setminus V_T|V_T)$. Observe that (A.2) need not to be integrated backwards, because every quantity on the right-hand side is available at time $t$ by the past knowledge. It can be integrated forward with initial condition $P_{X|V}(0) = P_0|V$, which is assumed to be available. Thus, $P_2$ and the information reward $J^g_{S}(T, \tau)$ can be computed by the forward integration of three matrix differential equations: a Lyapunov equation for $P_X$, a Riccati equation for $Q_X$, and a Lyapunov-like equation for $P_{X|V}$.

In addition, equations for $P_X$ and $P_{X|V}$ can be written in terms of the information matrices, $S_X \equiv P_X^{-1}$ and $S_{X|V} \equiv P_{X|V}^{-1}$; this removes the need for performing matrix inversion in (A.2). Using $\frac{d}{dt} \left( M_1^{-1} \right) = -M_1^{-1} \left( \frac{d}{dt} M_1 \right) M_1^{-1}$ for any non-singular square matrix $M_1$, equations for $S_X$ and $S_{X|V}$ are obtained as in (14) and (15). Finally, using the properties of the determinant function: $\text{Idet} M_1^{-1} = -\text{Idet} M_1$ and $\text{det}(M_1 M_2) = \text{det} M_1 \text{det} M_2$ for square matrices $M_1$ and $M_2$, we have

$$
J^g_{S}(T, \tau) = \frac{1}{2} \left[ \text{Idet} S_{X|V}(\tau) - \text{Idet} S_X(\tau) \right] \\
- \frac{1}{2} \left[ \text{Idet} \left( Q_X(\tau) Q_X^{-1}(\tau) + S_{X|V}(\tau) - S_X(\tau) \right) \right],
$$

which is identical to (13).

**B Proof of Theorem 7**

Using the expression for the time derivative of $\text{Idet}$ of a symmetric positive definite matrix,

$$
\frac{d}{dt} \text{Idet} S_X(T, t) = \frac{d}{dt} \left[ \frac{1}{2} \left( \text{Idet} S_{X|V} - \text{Idet} S_X \right) - \frac{1}{2} \text{Idet}(I + Q_X \Delta_S) \right] \\
= \frac{1}{2} \text{tr} \left[ \left( S_{X|V}^{-1} - S_X^{-1} \right) S_X \right] \\
- \frac{1}{2} \text{tr} \left[ \left( I + Q_X \Delta_S \right)^{-1} \left( \dot{Q}_X \Delta_S + Q_X \Delta_S \right) \right]
$$

where $\Delta_S \triangleq \dot{S}_{X|V} - \dot{S}_X$. Using the expressions of $\dot{S}_X$ and $\dot{S}_{X|V}$ in (14) and (15), and the cyclic property of the trace function $\text{tr}(M_1 M_2 M_3) = \text{tr}(M_2 M_3 M_1) = \text{tr}(M_3 M_1 M_2)$, we have:

$$
\text{tr} \left[ \left( S_{X|V}^{-1} - S_X^{-1} \right) S_X \right] = \text{tr} \left( I + Q_X \Delta_S \right)^{-1} \left( \dot{Q}_X \Delta_S + Q_X \Delta_S \right).
$$

In addition, utilizing expressions of $\dot{S}_X$ and $\dot{S}_{X|V}$, the cyclic property of the trace function, and the matrix inversion lemma [16], we have

$$
\text{tr} \left[ \left( I + Q_X \Delta_S \right)^{-1} \left( \dot{Q}_X \Delta_S + Q_X \Delta_S \right) \right] \\
= \frac{1}{2} \text{tr} \left[ \Sigma W \left( I + Q_X \Delta_S \right)^{-1} Q_X C' \Sigma W^{-1} C Q_X \Delta_S \right].
$$

From (B.1) and (B.2), $\frac{d}{dt} J^g_{S}(T, t)$ becomes

$$
\frac{d}{dt} J^g_{S}(T, t) = \frac{1}{2} \text{tr} \left[ \left( I + Q_X \Delta_S \right)^{-1} Q_X C' \Sigma W^{-1} C Q_X \Delta_S \right] \\
= \frac{1}{2} \text{tr} \left[ \Sigma W \left( I + Q_X \Delta_S \right)^{-1} Q_X \right].
$$

The Wigner’s theorem [30] states that a product of three symmetric positive definite matrices is positive definite if the product is symmetric. Note that $\Pi$ can be written as a product of three positive definite matrices:

$$
\Pi = Q_X \Delta_S \left( I + Q_X \Delta_S \right)^{-1} Q_X \\
= Q_X (\Delta_S^{-1} + Q_X)^{-1} Q_X,
$$

because $Q_X$ and $\Delta_S$ are positive definite. Also, using the matrix inversion lemma [16], it can be shown that $\Pi$ is symmetric:

$$
\Pi = Q_X \Delta_S Q_X - Q_X \Delta_S (Q_X^{-1} + \Delta_S)^{-1} \Delta_S Q_X = \Pi'.
$$

From (B.3) and (B.4), $\Pi > 0$. This leads to $CTIC' \geq 0$, and finally $\text{tr} \left[ \Sigma X^{-1} CTIC' \right] \geq 0$, because the trace of the product of two positive definite matrices is non-negative [20].