

**ECONOMIC LOT-SIZING WITH
START-UP COSTS:
THE CONVEX HULL**

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Abstract

A partial description of the convex hull of solutions to the economic lot-sizing problem with start-up costs (ELSS) has been derived recently. Here a larger class of valid inequalities is given and it is shown that these inequalities describe the convex hull of ELSS. This in turn proves that a plant location formulation as a linear program solves ELSS. Finally a separation algorithm is given.

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1. Introduction.

Although the Economic Lot Sizing problem (ELS) as defined by Wagner and Whitin [11], has proved to be useful in many production environments, it does not capture all the properties of problems arising in this area. Due to this, many generalisations and variants of ELS have been studied in the literature, for instance ELS with backloging and ELS in a multi-echelon structure (see Zangwill [13]).

Here we consider a model with costs included for switching on a machine or changing over between different items, the so-called startup costs. Production problems in which these costs appear have been studied by Van Wassenhove and Vanderhenst [10], Karmarkar and Schrage [6] and Fleischmann [3]. The standard dynamic programming formulation of the Economic Lot-Sizing problem with startup costs (ELSS) can be solved in $O(n \log n)$ time, where n is the length of the planning horizon (see van Hoesel [4]).

Research on the polyhedral structure of ELSS was initiated by Wolsey [12]. He derived a partial description for ELSS by generalizing the (l,S) -inequalities for ELS, developed in Barany et al. [2]. Here we generalize these inequalities further to the so-called (l,S,T) -inequalities and we show that these inequalities give a complete description of the convex hull of ELSS. The proof technique that we use is due to Lovasz [7]: the set of optimal solutions with respect to an arbitrary objective function is shown to be contained in a hyperplane defined by one of the inequalities of the model. This proof technique appears to be especially suitable for problems where a greedy algorithm solves the dual linear program arising from a complete polyhedral description of the problem.

A related formulation of ELSS is the plant location model (PL), in which the production variables are split. The inequalities for PL derived in [12] are shown to imply the (l,S,T) -inequalities, thereby proving that the linear programming relaxation of PL solves ELSS.

In addition we discuss a separation algorithm for the (l,S,T) -inequalities of ELSS by formulating the separation problem as a set of shortest path problems. This algorithm has a running time of $O(n^3)$.

2. Formulation of ELSS; the (l,S,T) -inequalities

Consider a planning horizon consisting of the periods $1, \dots, n$. The nonnegative demand in period i is denoted by d_i . The unit costs in period i are the production costs p_i , and the holding costs h_i . The fixed costs in period i are the setup costs and the startup costs, denoted by f_i and g_i respectively. The startup cost is incurred in period i if a setup takes place in period i and not in period $i-1$. ELSS is modeled using the following variables:

$$\begin{aligned}
 z_i \ (i=1, \dots, n): & \quad \begin{cases} 1 & \text{if a startup is incurred in period } i \\ 0 & \text{otherwise} \end{cases} \\
 y_i \ (i=1, \dots, n): & \quad \begin{cases} 1 & \text{if a setup is incurred in period } i \\ 0 & \text{otherwise} \end{cases} \\
 x_i \ (i=1, \dots, n): & \quad \text{the production in period } i \\
 s_i \ (i=1, \dots, n): & \quad \text{the inventory at the end of period } i
 \end{aligned}$$

In the following we denote the cumulative demand of the periods $\{i, \dots, j\}$ by d_{ij} , i.e. $d_{ij} := \sum_{t=i}^j d_t$. This notation is also adopted for the cost

parameters i.e. $h_{ij} := \sum_{t=i}^j h_t$.

The standard mixed integer program is

$$(ELSS) \quad \min \sum_{i=1}^n (g_i z_i + f_i y_i + p_i x_i + h_i s_i) \quad (2.1)$$

$$\text{s.t.} \quad x_i + s_{i-1} = d_i + s_i \quad (i=1, \dots, n) \quad (s_0 := 0) \quad (2.2)$$

$$x_i > 0 \Rightarrow y_i = 1 \quad (i=1, \dots, n) \quad (2.3)$$

$$y_i = 1 \wedge z_i = 0 \Rightarrow y_{i-1} = 1 \quad (i=1, \dots, n) \quad (y_0 := 0) \quad (2.4)$$

$$s_n = 0 \quad (2.5)$$

$$y_i, z_i \text{ binary} \quad (i=1, \dots, n)$$

$$x_i, s_i \text{ nonnegative} \quad (i=1, \dots, n)$$

It is a straightforward matter to eliminate the inventory variables s_i . From (2.2) it follows that $s_i = \sum_{j=1}^i x_j - d_{1i}$. Using the nonnegativity of s_i this gives $\sum_{j=1}^i x_j \geq d_{1i}$. Moreover, since the ending inventory should be zero by (2.5), we have $\sum_{j=1}^n x_j = d_{1n}$. Finally, since the setup and startup variables are binary, the inequalities (2.3) can be replaced by $x_i \leq d_{in} y_i$ ($i=1, \dots, n$) and the inequalities (2.4) can be replaced by $y_i \leq y_{i-1} + z_i$ ($i=1, \dots, n$). Using $c_i := p_i + h_{in}$, this leads to

$$(ELSS) \quad \min \sum_{i=1}^n (g_i z_i + f_i y_i + c_i x_i) \quad (2.6)$$

$$\text{s.t.} \quad \sum_{i=1}^n x_i = d_{1n} \quad (2.7)$$

$$\sum_{i=1}^n x_i \geq d_{1i} \quad (i=1, \dots, n-1) \quad (2.8)$$

$$x_i \leq d_{in} y_i \quad (i=1, \dots, n) \quad (2.9)$$

$$y_i \leq y_{i-1} + z_i \quad (i=1, \dots, n) \quad (y_0 := 0) \quad (2.10)$$

$$y_i, z_i \text{ binary} \quad (i=1, \dots, n)$$

$$x_i \text{ nonnegative} \quad (i=1, \dots, n)$$

An important structural property of the fixed cost variables in ELSS is the following: if $y_i=1$ for some $i \in \{1, \dots, n\}$ then there is a period $j \leq i$ such that $z_j=1; y_j=\dots=y_i=1$. This follows by inductively applying (2.4). It leads to the following simple but useful lemma.

Lemma 2.1.

Suppose $y_i=1$ for some i in a feasible production plan. For any $k < i$ at least one of the following variables $\{y_k, z_{k+1}, z_{k+2}, \dots, z_i\}$ has value one.

Proof Let $j \leq i$ be as above so that $z_j=1, y_j=\dots=y_i=1$. If $j \leq k$ then $y_k=1$ and if $j > k$, $z_j=1$. The claim follows.

□

The remainder of this section is devoted to the description of the (l, S, T) -inequalities and a proof of their validity. Take an arbitrary period $l \leq n$, and let $L = \{1, \dots, l\}$. Now let $S \subseteq L$ and $T \subseteq S$, such that the first element in S is also in T . We define the corresponding (l, S, T) -inequality as follows:

$$\sum_{i \in L \setminus S} x_i + \sum_{i \in T} d_{il} y_i + \sum_{i \in S \setminus T} d_{il} (z_{p(i)+1} + \dots + z_i) \geq d_{1l} \quad (2.11)$$

where $p(i) = \max\{j \in S: j < i\}$. If $S \cap \{1, \dots, i-1\} = \emptyset$ then $p(i) = 0$.

Example: $l = 14$; $S = \{4, 7, 8, 10, 12, 13\}$; $T = \{4, 10, 12\}$:

The coefficients of the inequality are given in the following table:

i	1	2	3	4	5	6	7	8	9	10	11	12	13	14
x_i	1	1	1		1	1			1		1			1
y_i				$d_{4,l}$						$d_{10,l}$		$d_{12,l}$		
z_i					$d_{7,l}$	$d_{7,l}$	$d_{7,l}$	$d_{8,l}$					$d_{13,l}$	

It should be noted that the inequalities derived in Wolsey [12] are a special case of the (l,S,T) -inequalities in which all elements of $\{p(i), \dots, i\}$ lie in S . Therefore the above example is not included, because periods 5 and 6 lie in $L \setminus S$ and $p(7) = 4$.

Lemma 2.2

The (l,S,T) -inequalities are valid.

Proof. Take an arbitrary (l,S,T) -inequality, and denote a feasible solution to ELSS by $\{x_i, y_i, z_i | i = 1, \dots, n\}$.

Case 1: S does not contain a period with positive production, i.e. $x_i = 0$ for

$$i \in S. \text{ Then } \sum_{i \in L \setminus S} x_i = \sum_{i=1}^l x_i \geq d_{ll}.$$

Case 2: S contains a period with positive production. Let j be the first period in S , such that $y_j = 1$. Now

$$\sum_{i \in L \setminus S} x_i \geq \sum_{i \in \{1, \dots, j-1\} \setminus S} x_i = \sum_{i \in \{1, \dots, j-1\}} x_i \geq d_{1,j-1}$$

If $j \in T$ then $\sum_{i \in L \setminus S} x_i + d_{j,l} y_j \geq d_{1,j-1} + d_{j,l} = d_{ll}$.

If $j \in S \setminus T$ then denote the last element in $\{1, \dots, j-1\} \cap T$ by k . Note that k exists, since by definition the first element of S is in T . It is easily seen that the following is part of the left-hand side of the (l,S,T) -inequality.

$$d_{kl}y_k + \sum_{i=k+1}^j d_{t(i),l} z_i \quad (2.12)$$

Here $t(i) \in S$ is such that $i \in \{p(t(i))+1, \dots, t(i)\}$. Since $j \in S$ it follows that $t(i) \leq j$ for $i \in \{k+1, \dots, j\}$. Therefore (2.12) is greater than or equal to $d_{jl}(y_k + z_{k+1} + \dots + z_j)$. Since $y_j = 1$ it follows from lemma 2.1. that $y_k + z_{k+1} + \dots + z_j \geq 1$ and therefore $\sum_{i \in L \setminus S} x_i + d_{kl}y_k + \sum_{i=k+1}^j d_{t(i),l} z_i \geq d_{1,j-1} + d_{jl} = d_{1l}$.

□

3. The convex hull of ELSS

Consider the following sets of valid inequalities:

$$\sum_{i=1}^n x_i = d_{1n} \quad (3.1)$$

$$z_i \leq 1 \quad (i=1, \dots, n) \quad (3.2)$$

$$y_i \leq 1 \quad (i=1, \dots, n) \quad (3.3)$$

$$y_i \leq y_{i-1} + z_i \quad (y_0 := 0) \quad (i=1, \dots, n) \quad (3.4)$$

$$x_i \geq 0 \quad (i=1, \dots, n) \quad (3.5)$$

$$y_i \geq 0 \quad (i=1, \dots, n) \quad (3.6)$$

$$z_i \geq 0 \quad (i=1, \dots, n) \quad (3.7)$$

$$\sum_{i \in L \setminus S} x_i + \sum_{i \in T} d_{il} y_i + \sum_{i \in S \setminus T} d_{il}(z_{p(i)+1} + \dots + z_i) \geq d_{1l} \quad (3.8)$$

(For all $l=1, \dots, n$, $S \subseteq L$ and $T \subseteq S$ such that the first element of S is in T .)

Note that the inequalities (2.8) are special cases of the (l, S, T) -inequalities, where $S = \emptyset$. The inequalities (2.9) can be derived from (3.1) and (3.8), where $S = T = \{i\}$. The remainder of this section will be devoted to proving the following theorem:

Theorem 3.1.

The set of inequalities (3.1)–(3.8) describes the convex hull of ELSS.

The technique we will use to prove the theorem is somewhat different from the usual one. Such a proof has been proposed by Lovasz [7] for the matching polytope. Basically the idea is to show that for any objective function the set of optimal solutions to ELSS satisfies one of the inequalities (3.2)–(3.8) at equality. Thus (3.2)–(3.8) must include all facets of the convex hull of solutions.

In addition as we specify for each objective function which inequality is satisfied at equality, one can use this proof technique to derive a greedy algorithm that solves the linear programming dual of (3.1)–(3.8)

We consider an arbitrary cost function $\sum_{i=1}^n(\alpha_i x_i + \beta_i y_i + \gamma_i z_i)$ and the resulting set of optimal solutions $M(\alpha, \beta, \gamma)$ to ELSS.

Case 0: $\min\{\alpha_i | i = 1, \dots, n\} = \delta \neq 0$

As $\sum_{i=1}^n x_i = d_{1n}$ we can remove δ times the inequality (3.1) from the objective function without changing the set of optimal solutions. Thus we can assume that $\min\{\alpha_i | i = 1, \dots, n\} = 0$.

Case 1: $\gamma_i < 0$ for some $i \in \{1, \dots, n\}$.

Any solution with $z_i = 0$ can be improved by setting $z_i = 1$. Thus $M(\alpha, \beta, \gamma) \subseteq \{(x, y, z) | z_i = 1\}$.

Case 2: $\gamma_i \geq 0$ for all i , $\beta_i < 0$ for some i .

i) $\beta_i + \gamma_i < 0$ for some i .

Any solution with $y_i=0$ can be improved by setting $y_i=z_i=1$. Thus

$$M(\alpha, \beta, \gamma) \subseteq \{(x, y, z) | y_i = 1\}.$$

ii) $\beta_i + \gamma_i \geq 0$ for all i .

Let $j = \min\{i | \beta_i < 0\}$. We show that $M(\alpha, \beta, \gamma) \subseteq \{(x, y, z) | y_{j-1} + z_j = y_j\}$. As $\gamma_j > 0$, any solution with $y_{j-1} = z_j = 1$ can be improved by setting $z_j = 0$. Also any solution with $y_{j-1} + z_j = 1$ and $y_j = 0$ can be improved by setting $y_j = 1$. Thus the claim follows.

We are left with objective functions satisfying $\min\{\alpha_i | i = 1, \dots, n\} = 0$; $\beta_i \geq 0$ ($i = 1, \dots, n$); $\gamma_i \geq 0$ ($i = 1, \dots, n$). Thus all solutions have nonnegative objective value. In the rest of the proof it is important to look at the "zero/positive" structure of the coefficients α_i , β_i , γ_i .

We now look for the first period l having the property that the cost of satisfying the demands d_1, \dots, d_l only equals the cost of satisfying all the demands d_1, \dots, d_n . Observe that if $\gamma_m = \beta_m = \beta_{m+1} \dots = \beta_{k+1} = \alpha_{k+1} = 0$ for some $m \leq k+1$, we certainly have that $l \leq k$.

If $d_{l+1} = \dots = d_k = 0$, $d_{k+1} > 0$ we must have $\gamma_m = \beta_m = \dots = \beta_{j+1} = \alpha_{j+1} = 0$ for some $l+1 \leq j \leq k+1$ and $m \leq k+1$.

Choice of l : Define $\alpha_{n+1} = 0$, $\beta_{n+1} = 0$, $\gamma_{n+1} = 0$. First take l minimal such that there exists an $m \leq l+1$ with $\gamma_m = \beta_m = \dots = \beta_{l+1} = \alpha_{l+1} = 0$.

Case 3:

- i) If $\gamma_i > 0$ for $i > m$, $M(\alpha, \beta, \gamma) \subseteq \{(x, y, z) | z_i = 0\}$.
- If $\beta_i > 0$ for $i > l+1$, $M(\alpha, \beta, \gamma) \subseteq \{(x, y, z) | y_i = 0\}$.
- If $\alpha_i > 0$ for $i > l+1$, $M(\alpha, \beta, \gamma) \subseteq \{(x, y, z) | x_i = 0\}$.

ii) If $d_l=0$, then from the definition of l we have $\alpha_l+\beta_l+\gamma_l>0$. The argument applies as in case 3i) with $i=l$.

Now $L=\{1,\dots,l\}$ is determined with $d_l>0$. For these objective functions we now need to choose the sets S and T .

Choice of S : $S:=\{i\leq l|\alpha_i=0\}$. Note that if $l=n$ then S is not empty, since there exists an $i\in\{1,\dots,n\}$ with $\alpha_i=0$.

Case 4: If $S=\emptyset$, $M(\alpha,\beta,\gamma)\subseteq\{(x,y,z)|\sum_{i=1}^l x_i=d_{ll}\}$.

Any solution with $s_l>0$ can be improved by reducing s_l to zero as $\alpha_l>0$ for $i=1,\dots,l$. Note that $l<n$ if $S=\emptyset$, since there is an $i\in\{1,\dots,n\}$ with $\alpha_i=0$.

For $i\in S$, let $q(i)=\max\{j: j\leq i \text{ and } \beta_j>0\}$ with $q(i)=0$ if $\beta_1=\dots=\beta_i=0$.

Case 5: Suppose $q(i)=0$ for some $i\in S$. Let j be the first period with this property. If $y_1+\dots+y_j=0$ then $\sum_{i=1}^{j-1} x_i=0=d_{1,j-1}$. If $y_1+\dots+y_j\geq 1$ then we can produce in j at no cost and thus, since $\alpha_1,\dots,\alpha_{j-1}>0$ from the minimality of j we have $s_{j-1}=0$ and therefore $\sum_{i=1}^{j-1} x_i=d_{1,j-1}$.

Case 6: If $\alpha_{q(i)}>0$ for some $i\in S$ consider the smallest such i and let $k=q(i)$. Then $M(\alpha,\beta,\gamma)\subseteq\{(x,y,z)|y_k+z_{k+1}+\dots+z_i=y_i\}$.

First suppose that $y_k+z_{k+1}+\dots+z_i\geq 2$. Let t,u be the first two periods in which the variables $\{y_k,z_{k+1},\dots,z_i\}$ take value 1. Setting $y_t=\dots=y_u=1$ and $z_u=0$ leads to a cost reduction as $\beta_{t+1}=\dots=\beta_u=0$, $\gamma_u>0$, from the minimality of l .

Second suppose that $y_k+z_{k+1}+\dots+z_i=1$ and $y_i=0$. Then as $\beta_k,\gamma_{k+1},\dots,\gamma_i>0$ from the definition of i , k and l , either the solution can be improved by

setting $y_k + z_{k+1} + \dots + z_i = 0$ or production occurs in the interval $k, \dots, i-1$, and thus $\sum_{t=k}^{i-1} x_t > 0$. In the latter case consider s_{i-1} . If $s_{i-1} > 0$ the solution can be improved by reducing s_{i-1} by $\min\{\sum_{t=k}^{i-1} x_t, s_{i-1}\}$ as $\alpha_k, \dots, \alpha_{i-1} > 0$ and completing all production by setting $y_i = 1$ and producing in period i at zero cost. If $s_{i-1} = 0$ and $y_i = 0$ then $d_i = 0$. But now $d_{i+1, l} > 0$ cannot be satisfied at zero cost in periods $i+1, \dots, l$ as otherwise l would be smaller. Therefore again $y_i = 1$.

Now we have that $\alpha_{q(i)} = 0$ and $q(i) \geq 1$ for all $i \in S \neq \emptyset$. As $\alpha_{q(i)} = 0$ we have $q(i) \in S$.

Choice of T: $T = \{q(i) | i \in S\} \neq \emptyset$.

Note that the first element in S is also in T because if $k = \min\{i | i \in S\}$, $\alpha_{q(i)} \in S$ implies $k = q(k) \in T$.

Case 7: We claim that the following (l, S, T) -inequality

$$\sum_{i \in L \setminus S} x_i + \sum_{i \in T} d_{il} y_i + \sum_{i \in S \setminus T} d_{il} (z_{p(i)+1} + \dots + z_i) \geq d_{1l}$$

is satisfied at equality for all points in $M(\alpha, \beta, \gamma)$.

i) Suppose $y_{p(i)} + z_{p(i)+1} + \dots + z_i = 0$ for all $i \in S$. So $y_i = 0$ for all $i \in S$. If $s_i > 0$, then as $\alpha_i > 0$ for all $i \in L \setminus S$, the solution can be improved by reducing s_i to zero.

Now suppose there is an $i \in T$ with $y_i = 1$ or an i with $z_i = 1$ for $i \in P(k)$ where $P(k) = \{p(k)+1, \dots, k\}$, for some $k \in S \setminus T$. Let j be the first such period i .

ii) Suppose $j \in T$ and $y_j = 1$. Then as $\alpha_j = 0$, all production for periods j, \dots, l can take place in period j at zero cost. Therefore necessarily $x_i = 0$ for all $i \in L \setminus S \cap \{j+1, \dots, l\}$, $y_i = 0$ for all $i \in T \cap \{j+1, \dots, l\}$ and $z_i = 0$ for all $i \in \left[\bigcup_{k \in S \setminus T} \{p(k)+1, \dots, k\} \right] \cap \{j+1, \dots, l\}$. In addition by the same argument as in i) $\sum_{t \in L \setminus S} x_t = d_{1,j-1}$, and the claim follows.

iii) Suppose $z_j = 1$ for $j \in P(k)$ with $k \in S \setminus T$. Then as $\beta_j = \dots = \beta_k = \alpha_k = 0$, all production for periods k, \dots, l can take place in period k at zero cost. Therefore necessarily $x_i = 0$ for all $i \in L \setminus S \cap \{k+1, \dots, l\}$, $y_i = 0$ for all $i \in T \cap \{j+1, \dots, l\}$ and $z_i = 0$ for all $i \in \left[\bigcup_{k \in S \setminus T} \{p(k)+1, \dots, k\} \right] \cap \{j+1, \dots, l\}$. In addition by the same argument as in i) $\sum_{t \in L \setminus S} x_t = d_{1,k-1}$, and the claim follows.

This ends the proof of theorem 3.1. and this section.

4. The Plant Location Reformulation

In this section we consider a reformulation in which the production variables x_i ($i=1, \dots, n$) are split as follows: for each $i \in \{1, \dots, n\}$ and for each $t \in \{i, \dots, n\}$ the variable x_{it} denotes the part of the demand d_t that is produced in period i . The connection with the original variables is simply $x_i = \sum_{t=i}^n x_{it}$. The major advantage of this reformulation is that the model allows for tighter constraints, since the production is disaggregated:

$$(PL) \quad \min \sum_{i=1}^n (g_i z_i + f_i y_i + c_i \left[\sum_{t=i}^n x_{it} \right]) \quad (4.1)$$

$$\text{s.t.} \quad \sum_{i=1}^t x_{it} = d_t \quad (t=1, \dots, n) \quad (4.2)$$

$$x_{it} \leq d_t y_i \quad (i=1, \dots, n; t=i, \dots, n) \quad (4.3)$$

$$y_i \leq y_{i-1} + z_i \quad (y_0 = 0) \quad (i=1, \dots, n) \quad (4.4)$$

$$y_i, z_i \text{ binary} \quad (i=1, \dots, n)$$

$$x_{it} \text{ nonnegative} \quad (i=1, \dots, n; t=i, \dots, n)$$

The LP-relaxation of PL is not tight in the sense that it still allows fractional solutions. By adding the following constraints, the so-called (i_1, i_2, t) -inequalities, we get a reformulation of ELSS which is at least as strong as the formulation given in the previous section.:

Let $1 \leq i_1 \leq i_2 \leq t \leq n$.

$$d_t(y_{i_1} + z_{i_1+1} + \dots + z_{i_2}) \geq \sum_{i=i_1}^{i_2} x_{it} \quad (4.5)$$

These inequalities can be found in Wolsey [12]. The proof that these inequalities, together with the inequalities (4.1)-(4.4), are at least as strong as the (l, S, T) -inequalities constitutes the main part of this section. This is shown by proving that the (l, S, T) -inequalities are implied by nonnegative linear combinations of the inequalities (4.2)-(4.5). We take an arbitrary (l, S, T) -inequality, thus let $l \leq n$; $S \subseteq L$; $T \subseteq S$ such that the first element of S is in T .

Now take an arbitrary $t \in L$. The elements in $T \cap \{1, \dots, t\}$ are denoted by $t_1 < t_2 < \dots < t_K < t_{K+1} := t+1$. Now for each $t_k \in T$, let s_k be the largest element in $\{t_k, \dots, t_{k+1}-1\} \cap S$. Note that $t_k \in S$ and therefore s_k is well defined. Summing all (t_k, s_k, t) -inequalities for $k=1, \dots, K$ gives:

$$\sum_{k=1}^K d_t y_{t_k} + \sum_{k=1}^K \left[\sum_{i=t_k+1}^{s_k} d_t z_i \right] \geq \sum_{k=1}^K \left[\sum_{i=t_k}^{s_k} x_{it} \right] \quad (4.6)$$

The first term in the left-hand side of (4.6) sums over the elements of $T \cap \{1, \dots, t\}$. The second term in the left-hand side of (4.6) sums over the elements of $Q(t)$ which is defined as $\left[\bigcup_{j \in S \setminus T} \{p(j)+1, \dots, j\} \right] \cap \{1, \dots, t\}$. Finally each element in $S \cap \{1, \dots, t\}$ is contained in $\{t_k, \dots, s_k\}$ for some $k \in \{1, \dots, K\}$ and therefore the right-hand side of (4.6) is greater than or equal to $\sum_{i \in \{1, \dots, t\} \cap S} x_{it}$. This gives

$$\sum_{i \in \{1, \dots, t\} \cap T} d_t y_i + \sum_{i \in Q(t)} d_t z_i \geq \sum_{i \in \{1, \dots, t\} \cap S} x_{it} \quad (4.7)$$

Addition of the inequalities (4.7) over all $t \in \{1, \dots, l\}$ gives

$$\sum_{t=1}^l \sum_{i \in \{1, \dots, t\} \cap T} d_t y_i + \sum_{t=1}^l \sum_{i \in Q(t)} d_t z_i \geq \sum_{t=1}^l \sum_{i \in \{1, \dots, t\} \cap S} x_{it} \quad (4.8)$$

The first term in the left-hand side of (4.8) gives

$$\sum_{t=1}^l \sum_{i \in \{1, \dots, t\} \cap T} d_t y_i = \sum_{i \in T} \sum_{t=i}^l d_t y_i = \sum_{i \in T} d_{il} y_i \quad (4.9)$$

For the second part of the left-hand side of (4.8) it holds that summing over i only takes place for t such that $i \in Q(t)$, i.e.

$$\sum_{t=1}^l \sum_{i \in Q(t)} d_t z_i = \sum_{i=1}^l \left[\sum_{t: i \in Q(t)} d_t z_i \right] = \sum_{j \in S \setminus T} d_{jl} (z_{p(j)+1} + \dots + z_j) \quad (4.10)$$

The right-hand side of (4.8) is rewritten as follows, using (4.2):

$$d_{ll} = \sum_{t=1}^l d_t = \sum_{t=1}^l \left[\sum_{i=1}^t x_{it} \right] = \sum_{t=1}^l \left[\sum_{i \in \{1, \dots, t\} \cap S} x_{it} + \sum_{i \in \{1, \dots, t\} \setminus S} x_{it} \right]$$

Moreover

$$\sum_{t=1}^l \sum_{i \in \{1, \dots, t\} \setminus S} x_{it} = \sum_{i \in \{1, \dots, l\} \setminus S} \sum_{t=i}^l x_{it} \leq \sum_{i \in \{1, \dots, l\} \setminus S} \sum_{t=i}^n x_{it} = \sum_{i \in \{1, \dots, l\} \setminus S} x_i$$

and therefore

$$\sum_{t=1}^l \sum_{i \in \{1, \dots, t\} \cap S} x_{it} = d_{1l} - \sum_{t=1}^l \sum_{i \in \{1, \dots, t\} \setminus S} x_{it} \geq d_{1l} - \sum_{i \in \{1, \dots, l\} \setminus S} x_i \quad (4.11)$$

Substituting (4.9)–(4.11) in (4.8) gives the desired (l, S, T) -inequality.

The number of constraints in (PL) together with the (i_1, i_2, t) -inequalities is $O(n^3)$. This number can be reduced to $O(n^2)$ by observing that there always exists an optimal solution to ELSS in which the variables are such that $d_{t+1}x_{it} \geq d_t x_{i, t+1}$ for $i \in \{1, \dots, n\}$ and $t \in \{i, \dots, n\}$.

In fact the only (i_1, i_2, t) -inequalities that are necessary are those with $t = i_2$. By induction it follows that for $i \leq i_2 < t$: $d_t \bullet x_{i, i_2} \geq d_{i_2} \bullet x_{it}$. Multiplying the (i_1, i_2, i_2) -inequality by d_t and using the last inequality gives

$$d_{i_2} \bullet d_t (y_{i_1} + z_{i_1+1} + \dots + z_{i_2}) \geq d_t \bullet \sum_{i=i_1}^{i_2} x_{i, i_2} \geq d_{i_2} \bullet \sum_{i=i_1}^{i_2} x_{it}$$

and thus the (i_1, i_2, t) -inequality is implied, provided that d_{i_2} is positive.

5. Separation for the (l,S,T) -inequalities

Here we show that the separation algorithm for the (l,S,T) -inequalities can be formulated as a shortest path problem.

We fix l . Then we define three nodes for each period $i \in \{1, \dots, l\}$: u_i , v_i and w_i . Moreover, a starting node n_0 and an ending node n_l are defined.

There are two arcs with n_0 as a tail: (n_0, u_0) , and (n_0, v_0) both with zero costs. Moreover there are three arcs with n_l as head: (u_l, n_l) , (v_l, n_l) and (w_l, n_l) , also with zero costs.

To model the (l,S,T) -inequalities in a network we define three types of arcs.

Type 1: arcs (u_{i-1}, u_i) , (v_{i-1}, u_i) , (w_{i-1}, u_i) with cost x_i .

Type 2: arcs (u_{i-1}, v_i) , (v_{i-1}, v_i) , (w_{i-1}, v_i) with cost $d_{il}y_i$.

Type 3: arcs (v_{j-1}, w_k) , (w_{j-1}, w_k) with cost $\sum_{i=p(j)+1}^{j-1} x_i + d_{kl} \sum_{i=p(j)+1}^j z_i$.

FIGURE 1

It is readily checked that each path in the network corresponds to the left-hand side of a unique (l,S,T) -inequality. In particular the nodes $\{v_i\}$ and $\{w_i\}$ define the sets T and $S \setminus T$ respectively. Therefore the shortest path in the network is compared with d_{1l} .

There are $O(l^2)$ arcs in the network, and since it is acyclic the shortest path problem in the network can be solved in $O(l^2)$ time. Doing so for each $l \leq n$ gives an $O(n^3)$ algorithm, to find the most violated (l,S,T) -inequality. This is to be compared with the single max-flow calculation on a graph with $O(n^2)$ nodes derived in Rardin and Wolsey [9].

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