

STOCHASTIC APPROACH TO THE ANALYSIS OF HIGHWAY PAVEMENTS

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ABSTRACT

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Submitted to the Department of Civil Engineering on January 22, 1971 in partial fulfillment of the requirements for the degree of Master of Science.

A probabilistic method of analysis is presented as an integrated part of a rational approach to the analysis and design of highway pavements.

The suggested approach is based on the Monte Carlo simulation procedure. The pavement is represented by a mathematical model based on "Layered Systems Theory". It consists of three different layers with various mechanical properties which are acted upon by vehicular loading and environmental conditions.

The stochastic nature of the model is derived from the changes in the environment and the variability and inhomogeneity of the materials properties. This results in unpredictable behavior of the system associated with probabilities of overloading or inadequate capacity of the system of some components thereof to carry its stipulated functions.

The behavior of the system is characterized by its response to various excitations. This response may be in the form of developed stresses, strains or deflections at any point in the system, or it may, at later stages, take the form of damage manifested by cracks or excessive deformations. Regardless of the nature of response, it is uncertain in nature and should be characterized statistically rather than deterministically.

The stochastic approach for the analysis of pavement systems, therefore, provides realistic and sufficient information about the behavior of the system in operational environment.

This approach seems to be promising and can be pursued further for a comprehensive study of the performance

and failure of pavement systems under realistic operational environment. However, it is suggested that obtaining closed form probabilistic solutions may be more efficient at these subsequent stages of performance evaluation and study, where simulation has proved to be very costly.

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I. INTRODUCTION

In recent years the behavior of materials and structures have been the subject of extensive studies. These studies have emphasized the variability which occurs in the magnitude and distribution of the structural loadings, in the properties of the material, in the surrounding operational environment, and in the response of the structures and other engineering systems to such excitations (1)*.

In highway systems, the increasing use of unconventional road structures emphasizes the need for a better understanding of the contribution which each element of the pavement structure and the surrounding environment (including mechanical loads), makes to the overall behavior of the whole system.

A highway pavement system is a joint product of a complex interaction of the pavement structure, vehicular loads, and environmental conditions operating on the system. The behavior and the performance of the system, therefore is greatly influenced by these parameters. Any variability in one or more of these parameters implies a variability in the response and the overall performance of the system.

This study presents a simulation procedure based on the "Monte Carlo" method for the investigation of variability in the response of the pavement system. The method has been used in a variety of disciplines to study and predict the

* The numbers in the parenthesis refer to the list of references.

behavior of both deterministic and stochastic phenomena. Simplified stochastic models which yield both mean behavior and deviations from the mean can be obtained using the proposed simulation procedure.

A three-layer model representing a highway pavement system is analyzed, taking into consideration the variability of certain parameters in the structure itself as well as in the surrounding environment operating on the system. Cumulative distribution functions of the response of the system under variable loads and environmental conditions are obtained. This study is only a demonstration of the effectiveness of the method, and is not necessarily an exact evaluation of the actual performance of the pavement system under real operational environment.

This study is presented in five chapters. In Chapter II, the principles of computer simulation techniques are discussed, with the relevant justifications for the use of these techniques. Also discussed in this chapter are the Monte Carlo method of analysis as a sampling technique and its application to physical problems. Chapter III presents a methodology for the application of the Monte Carlo method to the analysis of a three-layer model representing a highway pavement system. A numerical example and results are also presented in this chapter.

A summary and conclusions are found in Chapter IV, while Chapter V presents some recommendations for future work.

II. COMPUTER SIMULATION TECHNIQUES

2.1 Definition and Scope

The word "simulation" has been used quite freely to refer to a number of different things. Recognizing the inherent inconsistencies and ambiguities involved in the use of the term, many definitions have emerged for simulation. Churchman has defined "simulation" as follows:

"x simulates y" is true if and only if:

- a) x and y are formal systems
- b) y is taken to be the real system, and
- c) x is taken to be an approximation to the real system (8)

Shubik's definition of simulation, however, appears to be more appropriate because it is typical of more popular definitions (41), it states:

"A simulation of a system or an organism is the operation of a model or simulator which is a representation of the system or organism. The model is amenable to manipulations which would be impossible, too expensive, or impractical to perform on the entity it portrays. The operation of the model can be studied and, from it, properties concerning the behavior of the actual system or its subsystems can be inferred."

For the purpose of this study, however, a narrower defi-

inition of simulation will be used, and it will be restricted to experimentation on mathematical models. Also, our primary interest lies in simulation experiments that are performed on digital computers. In addition, we are concerned with experiments which take place over extended periods of time, under stochastic or dynamic conditions, and which have solutions that are not necessarily deterministic by strictly analytical means.

With these constraints, the following definition, similar to that suggested by Naylor et al. (33), is used in this study:

"Simulation is a numerical technique for conducting experiments on a digital computer, which involves certain types of logical and mathematical models that describe the behavior of a physical system (or some component thereof) over extended periods of real time".

2.2 Rationale for Computer Simulation

It is recognized that in order to study and predict the future behavior of any system, certain steps must be taken in a systematic manner; these include:

1. Observation of the physical system.
2. Formulation of a hypothesis of a mathematical model that attempts to explain the observations of the system.
3. Prediction of the behavior of the system on the basis of the hypothesis by using the mathematical or logical deduction, i.e., by obtaining solutions to the mathematical model.

4. Performance of experiments to test the validity of the hypothesis or the mathematical model.

Generally, it may not be plausible to follow all these steps for any particular problem, and some form of simulation may be a satisfactory substitute.

For example it may be either impossible or very costly to make field observations on the real system. In highway systems for instance, it is almost impossible to perform experiments on the pavement structure where all combinations of the factors affecting its performance, such as temperature, moisture, loads, different combinations of materials properties for each layer can be used in such tests.

Furthermore, the observed system may be so complex that it is impossible to describe it in terms of mathematical equations for which analytic solutions that could be used to predict the behavior of this system are possible to obtain. An example of this is the complex interaction between the environmental factors and the materials' properties in the pavement structure, and the interaction between the environmental factors and the response of the structure itself. These make it virtually impossible to describe the performance of the system in an operational environment in mathematical forms. In such cases, simulation have proved to be an effective tool to describe and predict the future performance of the systems (33).

Although in some cases a mathematical model can be formu-

lated to describe the system, it may not be possible, however, to obtain a solution to it by ordinary analytical techniques. Again, the complexity in the highway systems and other economic systems can well provide examples for this case. In such cases it may be possible to use complicated mathematical models to simulate the systems under consideration. Although this approach does not guarantee precise prediction of the future performance or exact solutions to the model describing the system; it is possible to experiment with a variety of alternative solutions and decision rules to determine which solutions or decision rules are more realistic than others in predicting the behavior of the system. Therefore, computer simulation techniques such as the Monte Carlo method, which has been employed in this study, are used as efficient techniques of numerical analysis for solving complicated stochastic models or systems.

The principal justification for computer simulation is its ability to overcome the aforementioned difficulties in implementing a scientific method to study and analyze physical and other systems. There are, however other reasons for which computer simulation may be necessary. The following are a few of these additional reasons. They are not intended to be mutually exclusive and are closely related to the above discussion.

1. The use of computer simulation permits the study of systems with complex internal interaction

between their different components, by breaking down each system into subsystems, where it may be possible to model these subsystems and analyze them separately.

2. Detailed observations on the system being simulated may lead to a better understanding of the system and to suggestions for improving it, which otherwise would not be possible. This may include the study of the effects of certain informational, environmental, or characteristic changes on the behavior of the system. This is achieved by making alterations in the model of the system and observing the effects of these alterations on the performance of the system.
3. Simulation can be used to foresee the implications of introducing new components into the system. Also, it is very useful with new situations about which little or no knowledge is available. In such cases simulation can serve as a "preservice test" to try out new alternatives for physical and geometric characteristics of a system, before taking the risk of experimenting it on the real system. Economy and safety, the main objectives in engineering design are, hence, satisfied by the implementation of computer simulation.
4. In certain stochastic problems, the sequence of events may be of particular importance, where information about expected values may not be sufficient to describe the process. Monte Carlo methods may be the only satisfactory way of providing the information in such cases. The sequence of occurrence of certain environmental and loading effects has a great importance on the evaluation of the performance of a highway pavement and the degree of damage that exists at any period in the lifetime of the pavement.

2.3 Monte Carlo Methods

The systematic development of the Monte Carlo methods started in the early 1940's, in nuclear physics where attempts

were made to simulate the probabilistic problems concerned with random neutron diffusion in fissile materials (19).

In general, Monte Carlo methods comprise that branch of experimental mathematics which is concerned with experiments on random numbers. The simplest Monte Carlo approach to probabilistic problems is to observe numbers which are randomly chosen in such a manner that they simulate the physical process being studied, and to infer the probable solution for the behavior of the physical system from the behavior of these random numbers.

Problems handled by the Monte Carlo methods can be of two types: probabilistic or deterministic depending on whether or not they are directly concerned with the behavior and the outcome of random processes (19).

The first group consists of those problems which involve some kind of stochastic process. The second group are those deterministic mathematical problems which cannot be solved by strictly deterministic methods. It may however be possible to obtain approximate solutions to the latter group of problems by simulating a stochastic process which has moments, density functions, or cumulative distribution functions that satisfy the functional relationships or the solution requirements of the deterministic problem. Examples of this group are solutions to high order difference equations and multiple integral problems.

The greatest success of the Monte Carlo method has been in those areas where the basic mathematical problem itself consists of the investigation of some random process. Therefore, it seems obvious that this method can serve as a powerful tool to solve a boundary-value problem with random input parameters. This is one of the main reasons why this particular method has been chosen for the analysis of the highway pavement problem.

2.3.1 Monte Carlo Analysis

In order to define the characteristics of the Monte Carlo method, it is suitable to present a simple example on how the method works for solution of mathematical problems.

The development of mathematical statistics played an important role in the computation of integrals. Since "probability" can always be regarded as a measure, the problem of determining the probability of some event or its mathematical expectation can be reduced to a problem of computing some integral, such as the following:

$$\int_0^1 \phi(\xi) d\xi \quad (2.1)$$

Assume that the values of the function $\phi(\xi)$ lie between 0 and 1, i.e., $0 \leq \phi(\xi) \leq 1$ for $a \leq \xi \leq b$. Therefore the problem is to find the area A of the region R, (Figure 1), bounded by the

curve $\eta=\phi(\xi)$, the ξ -axis, and the coordinates $\xi=0$ and $\xi=1$.

Naturally, the restrictions imposed on the function $\phi(\xi)$ are not necessary, since there is a possibility of shifting and scaling.

Now let a point (x,y) fall randomly in the square $0 \leq \xi \leq 1$, $0 \leq \eta \leq 1$, with independent coordinates which are uniformly distributed between 0 and 1. Since $0 \leq x \leq 1$, and $0 \leq y \leq 1$, the probability (p) that the point (x,y) falls within the area under the curve is equal to A , which is the required area.

Using any technique for finding independent uniformly distributed variables as discussed in the following section, say x and y , the following condition should be satisfied:

$$\phi(x) < y \quad (2.2)$$

in order to guarantee that the random point (x,y) lies within the region R under the curve. Therefore, N pairs of sampled random variables are taken and a test is run on each to determine whether they satisfy the inequality (2.2). If this inequality holds for n pairs out of N , the ratio of n/N is approximately equal to the probability that any random point (x,y) falls within the region R , so

$$\frac{n}{N} \approx p = \int_0^1 \phi(\xi) d\xi \quad (2.3)$$

It is clear then that the number of tests N will affect the accuracy of the computation of such integrals and the

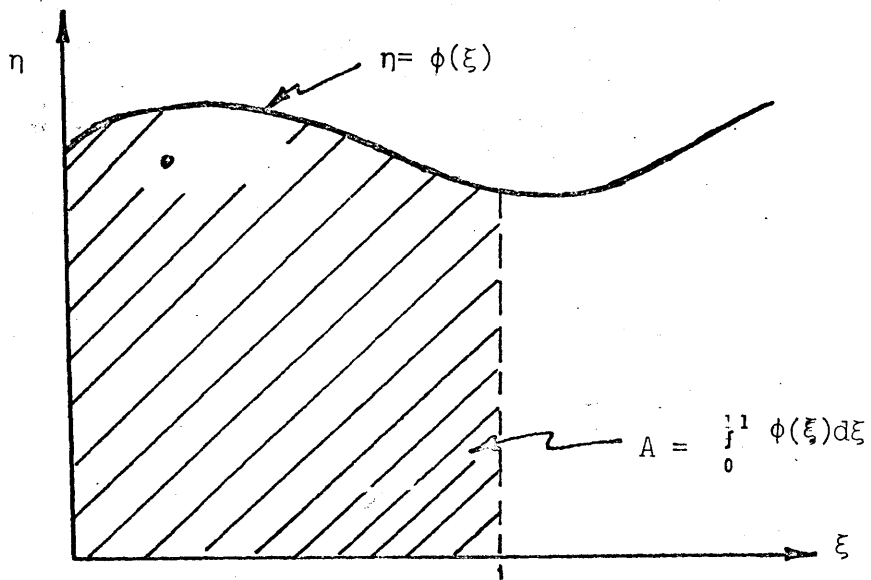


Figure 1. USE OF THE MONTE CARLO METHOD FOR COMPUTING INTEGRALS.

associated error. Also it is interesting to notice that the restrictions that are usually required to evaluate this integral such as the smoothness of the function need not be imposed in this method. All that is required is that the function be bounded and measurable.

A more general case is that in which a modeled process of the type discussed in the previous example, is used for estimating the unknown mathematical expectation of some random variable x . The same example is used here; i.e., it is required to evaluate the integral $\int_0^1 \phi(\xi)d\xi$.

Let (y) be a uniformly distributed variable over the range $(0,1)$. Then the mathematical expectation of the variable $x = \phi(y)$ is

$$M_x = \int_0^1 \phi(\xi)d\xi \quad (2.4)$$

It is necessary, therefore, to sample N independent values of the variable y_1, y_2, \dots, y_N , in order to evaluate the integral. It is also necessary to compute the arithmetic mean:

$$\bar{X} = \frac{1}{N} \sum_{i=1}^N \phi(y_i) \quad (2.5)$$

This value of the arithmetic \bar{X} is approximately equal to the value of the integral. The value of the errors involved in computing some of the values of $\phi(y_i)$ will be

"smoothed out" if the value of N is large. This will guarantee the stability of the method against any disturbances arising from defect of randomness in the machine.

The process, then, involves the estimation of the probability of some event A , or its mathematical expectation by means of a modeled process. The following characteristic features can be inferred from the above discussion:

1. Large number of computations of a uniform type is performed, and
2. The error involved in the computation is "smoothed out" for larger number of samples
3. It is also known that this method needs a comparatively small amount of "memory for storage of intermediate results which is well suited for multi-dimensional problems" (8). This point is extremely significant in very large and complex problems where the storage problem becomes an important issue in the computation process (8,19,28).

The above discussion shows that the Monte Carlo method is a modeling procedure where a random event A , occurring with probability p , is modeled by means of the independent variable.

2.4 Random Numbers

The essential feature common to all Monte Carlo computations is that at some point a random value is substituted for a corresponding set of actual values with similar statistical

properties. This random value is called "random number", on the basis that it could well have been produced by chance by any suitable random process. However, the fact that random numbers are not usually produced in a random way does not influence their effectiveness in this method; the important thing is the distribution of these numbers and not the source they come from.

In order to discuss the techniques for generating random numbers, it is essential to define some terms that are closely related to the properties and the use of these numbers according to the way they are produced.

First, it is important to define what is meant by a "random event" and "probability". A random event is an event which has a chance of happening, and probability is the numerical measure of that chance.

In Monte Carlo work, random numbers are classified into three categories, according to the way they are produced and used, random, pseudorandom, and quasirandom.

Random numbers, y , are the numbers that are produced by chance and follow a standardized rectangular distribution of the type shown in equation (2.6).

$$F(y) = \begin{cases} 0, & y < 0 \\ y, & 0 < y < 1 \\ 1, & y > 1 \end{cases} \quad (2.6)$$

where $F(y)$ is the cumulative frequency distribution of the function y .

However, in practice these so-called "random numbers" are substituted by some other numbers which are convenient to produce and are equally effective from statistical point of view.

For electronic digital computers it is most convenient to calculate a sequence of numbers one at a time as required by a specified rule. These numbers, however, are so devised that usual statistical tests will detect any significant departure from randomness. This sequence is called "pseudo-random". One good advantage of the use of a specified rule in producing random numbers is that the sequence is reproducible for purposes of computational checking.

Pseudorandom numbers are generally used in all classes of problems of the Monte Carlo type. However, in some cases the violation of some statistical tests of randomness may not invalidate the results. In such cases non-random sequences may deliberately be used, provided that this sequence have the particular statistical problem. Such a sequence is called "quasirandom" (19).

Several methods of generating sequences of random numbers are available. Naturally, all the methods embody some

quasirandom physical process that generates sequences of random numbers of a desired length and property (33). One of the principle requirements of these sequences, as in any other random sampling procedures, is statistical independence (18).

Three alternative methods are used to generate sequences of random numbers; they are:

1. Manual methods
2. Library methods
3. Computer methods

Manual methods include such slow procedures as coin flipping, dice rolling, card shuffling, etc., which are the simplest but the least practicable methods.

A number of library tables for random numbers have been published (37). These numbers are generated by one of the aforementioned methods before being tabulated. The one advantage of such tables is that they offer reproducible sequences of random numbers. However, the method lacks the speed and, in some cases, the sufficiency of the numbers contained in the tables where it is not desirable to use the same "random data" for solution of all the problems!

Computer methods include: analog computer methods, and digital computer methods.

Analog computer methods depend on some random physical process (such as the behavior of an electric current), thus they are fast, but the sequences they generate are again non-reproducible.

Three modes for providing random numbers on digital computers have been suggested by Tocher (45): external provision, internal generation by a random physical process, and internal generation of sequences of digits by a recurrence relation.

In examining several methods for generating random numbers, it seems that an acceptable method to be used for such purposes must provide sequences of random numbers having the following properties:

1. They are uniformly distributed,
2. They should be statistically independent,
3. They can be reproducible, and
4. Through a desired length of a sequence, they should be non-repeating.

Furthermore, for this method to be largely acceptable, it must be capable of generating random numbers at high rates of speed and with minimum amount of computer memory capacity (33,39).

2.5 Sampling of Random Events

The generation of simulated statistics (random variates)* is entirely statistical in nature and is carried out by supplying

*The term "variates" means a random variable having a certain mathematical expectation or probability of occurrence.

pseudorandom numbers generated by one of the methods mentioned in the previous section. These numbers are supplied into the process or system under study (where the system is represented by a probabilistic model), and then numbers (random variates) are obtained from it as the required solution. In general, simulation involves replacing an actual statistical span of elements by its theoretical counterpart, i.e., a span described by some assumed standard statistical or probability distribution and then sampling from this theoretical population by means of some type of random number generator (33). However, in some cases it may not be possible to find a standard theoretical distribution that describes a particular stochastic process or some of its components. In such cases, the stochastic process can be reproduced or simulated only by sampling from empirical distributions rather than from theoretical ones (This, naturally, assumes the existence of empirical data.).

In considering stochastic processes involving either continuous or discrete random variables, a function $F(x)$, known as the "cumulative distribution function" of x , denotes the probability that a random variable X takes on the value of x or less. If the random variable is discrete, then x takes on specific values and $F(x)$ is a step function. If $F(x)$ is continuous over the domain of x , then the probability density function is $f(x) = dF(x)/dx$. The cumulative distribution

function can be stated mathematically as

$$F(x) = P(X < x) = \int_{-\infty}^x f(t) dt \quad (2.7)$$

where $F(x)$ is defined over the range $0 \leq F(x) \leq 1$, and $f(t)$ represents the value of the probability density function of the random variable X .

Several methods for generating pseudorandom numbers or uniformly distributed random variates over the interval $(0,1)$ have been developed (33). Uniformly distributed random variates will be denoted by d , when $0 \leq d \leq 1$, and $F(d) = d$.

There are three methods for generating variates from probability distributions -- the "inverse transformation" method, the "rejection" method, and the "composition" method. These methods are discussed in references (33,41); however, a brief description of the first method is presented here because of its relation to the simulation of the highway system under consideration.

Inverse transformation method for generating stochastic variates on a computer is done as follows (see figure 2).

If one wishes to generate random variates x_i 's from some particular statistical population whose density function is given by $f(x)$, the cumulative distribution function $F(x)$ first must be obtained. Since $F(x)$ is defined over the range 0 to 1, one can generate uniformly distributed random numbers over the

same range and set $F(x) = d$. Therefore, for any particular value of d , say d_0 , which has been generated by any of the methods mentioned previously, it is possible to find the corresponding value for x , which is in this case x_0 . This is done by inverting the function F , if it is defined. So

$$d_0 = F_X(x_0) \quad (2.8)$$

$$x_0 = F_X^{-1}(d_0) \quad (2.9)$$

where $F_X^{-1}(d)$ is the inverse transformation of d on the unit interval into the domain of x . This can be summarized mathematically by saying that if random numbers corresponding to a given $F(x)$ are generated (equation 2.10),

$$d = F_X(x) = \int_{-\infty}^x f(t) dt \quad (2.10)$$

then

$$P(X \leq x) = F_X(x) = P[d \leq F(x)] = P[F_X^{-1}(d) \leq x] \quad (2.11)$$

and consequently $F_X^{-1}(d)$ is a variable that has a probability density function $f(x)$. This is equivalent to solving equation for x in terms of d . Figure 2 is an illustration of this method.

2.6 Monte Carlo Simulation Models and Their Properties

The primary concern in this section is that with mathematical models. Mathematical models of systems in general consist of four well-defined elements: components, variables,

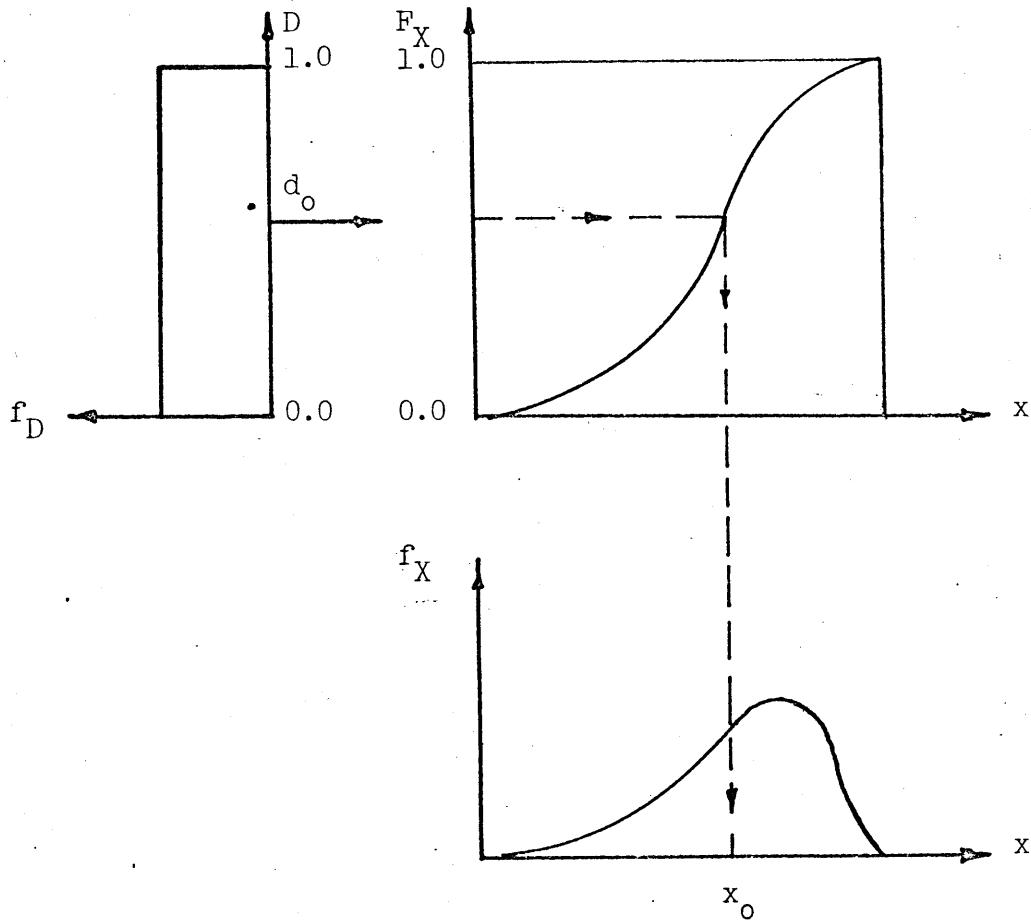


Figure 2. SAMPLING PROCEDURE BY THE INVERSE TRANSFORMATION METHOD.

parameters, and functional relationships (34).

Components of the mathematical models tend to vary widely depending on the nature of the model being simulated and the purpose of simulation. A highway pavement structure can be a component of highway transportation system. While the different layers, the geometry, etc. can well serve as components of a highway pavement structure which is under consideration in this study.

The variables that appear in the model are used to relate one component to another and may be conveniently classified as exogenous variables, status variables, and endogenous variables.

Exogenous variables are the input variables and are assumed to have been predetermined independently of the model being simulated. They may be regarded as acting on the system but not being acted upon or influenced by the system (34,35).

The state of the system over a certain period of time is described by the status variables. These variables interact with both the exogenous and endogenous variables according to an existing functional relationship of the elements of the system.

The output of the system is represented by the endogenous variables. Clearly, these variables are generated from the interaction of the input variables and the status variables according to some existing functional relationships.

Whether a particular variable should be classified as an exogenous variable, a status variable, or an endogenous variable depends on the purpose of the research. For example, vertical deflection may be regarded as an endogenous variable in a study concerned with the pure analysis of load application on a layered system, but may legitimately be treated as an exogenous variable in models concerned with predicting cumulative damage and distress of highway pavements. Exogenous variables may be used in two different ways in simulation experiments. They may either be treated as given parameters (determined by the environment, geometrical, and physical factors associated with the system), which of course have to be estimated first, and read into the computer as input data, or if they are stochastic variables, they may be generated internally by the computer by one of the methods mentioned in Section 2.4.

In the language of experimental design, exogenous variables or parameters are categorized as "factors". In conducting computer simulation experiments on a given system, the main concern is with the effects of the different levels of the various factors on the endogenous variables of the system. This is to say that a computer simulation experiment comprises a series of computer runs in which the effects of alternative factor levels on the endogenous variables are tested empirically (using simulation data) (33).

The functional relationships describing the interaction of the variables and components of a model are two-fold -- identities and operating characteristics. Both identities and operating characteristics are used to generate the behavior of the system. Identities may take the form of either definitions or tautological statements about the components of the model. For a pavement, the vertical deflection may be defined as the difference between the vertical level before a load was applied and that upon load application. An operating characteristic is a hypothesis, usually mathematical equation, relating the system's endogenous and status variables to its exogenous variables (33). Compatibility equations and stress-strain relationships for a layered system are examples of the operating characteristics of the pavement system. Operating characteristics for stochastic processes take the form of probability density functions. Unlike components and variables, which can be directly observed from the real system the parameters of operating characteristics can only be derived on the basis of statistical inference. Naturally, the accuracy of the results of simulation depend on the accuracy of these estimates of the system's parameters.

In this study, the functional relationships describing the interaction between the variables and the components of an engineering model are called "congruity relationships". The

reason behind this is the fact that these equations and definitions relate the different variables and components and describe their inter-compatibility and congruences.

2.7 Representation of the Elements of the Simulation Model in the Monte Carlo Analysis

To illustrate the aforementioned system of classifying elements of mathematical models, and to set forward the problem under consideration, the elements chosen in this section represents a typical example of the problem being faced in the real world for any engineering system.

The behavior of a material in a given operational environment can be represented by a set of responses, R_i where the subscript (i) is a number that varies in unit steps from 1 to the number of responses desired, say N. The choice of the response terms depends on the particular aspects of the material behavior under consideration. The set of response terms R_i 's constitutes the endogenous variables in the simulation model.

The material is characterized by a set of relevant properties P_j , and the environment is described by a set of conditions C_k . The subscripts j and k take the values 1, 2, ..., n and 1, 2, ..., m respectively, where n is the number of pertinent material properties, and m is the number of prevailing environmental conditions considered.

In general, material properties, environmental conditions and response terms are all expected to vary with time.

The three sets of quantities respectively can be regarded as a vector of material properties, an environmental vector, and a response vector.

In a deterministic approach, a functional relationship between each response term and the associated material properties and environmental conditions is usually assumed to exist. Material properties also vary systematically with the environment. These relations are the ones referred to as the Congruity Relationships in the previous section of this Chapter. So:

$$R_i = \psi_i [P_1, P_2, \dots, P_j, \dots, P_n, C_1, C_2, \dots, C_k, \dots, C_m] \quad (2.12)$$

$$P_j = \beta_j [C_1, C_2, \dots, C_k, \dots, C_m] \quad (2.13)$$

However, both material properties and environmental conditions are subject to considerable random variability over fairly wide ranges, even under well-controlled laboratory tests. For brevity the attention is focused in this section on the situation where the environmental factors are not correlated. The modifications which are required to account for the correlation of the environmental factors are discussed in section (2.8) of this chapter. Therefore, the C_k vectors are treated as random variables with probability density functions f_{C_k} and associated cumulative

distributions F_{c_k} . When the environmental factors are correlated*, their joint frequency distributions** yield the necessary statistical data. If the environmental factors are not correlated, their independent frequency distribution sufficiently describe the environment.

Materials properties are inherently variable. Even though the observed variabilities can partly be imputed to the variability in environmental conditions and to experimental and measurement errors, material properties basically can vary under idealized, constant environmental conditions and identical test specimens. Therefore, the terms P_j are also considered to be random variables with probability density functions f_{P_j} and cumulative distributions F_{P_j} .

In as much as the material properties are dependent on the environment conditions, statistical correlation is implied by equation (2.13). However, even under strict conditions of stable environment, material properties can be inherently correlated (24). The joint density function $f_{(P_1, P_2, \dots, P_j, \dots, P_n)}$ rather than the density functions f_{P_j} gives "complete" information

* This correlation exists when there is an interaction between the environmental parameters. An example of this is the interaction between moisture and temperature and the effect of one on the other.

** This may be written as $f(C_1, C_2, \dots, C_m)$.

about the inherently correlated material properties.

Variability in material properties and environmental conditions in any engineering system implies variability in material behavior, i.e., in the response terms R_i . To any system, in general, the basic inputs are the constituent materials characterized by a set of relevant properties, and environmental conditions surrounding the system and affecting its operation. The environment is meant to include loads (mechanical and thermal) as well. So the material properties and the environmental conditions are the basic inputs to the model, i.e., the exogenous variables. The geometry of the layers and of the load enters the model through the congruity relationships, and are also inputs to the model. A set of density functions f_{R_i} or alternatively cumulative distribution functions F_{R_i} represents the variability in material behavior and response, i.e., the endogenous variables.

To evaluate f_{R_i} , prerequisite data should be available for the density functions f_{P_j} and f_{C_k} . Even if these density functions are somehow evaluated, then considerable difficulty can arise in determining f_{R_i} by analytical methods. Such difficulties can be encountered if f_{P_j} and f_{C_k} are not normal and the congruity relationships are not linear. In these cases,

a numerical solution can be obtained by the Monte Carlo method.

The simulation method for the evaluation of the cumulative distribution function F_{R_i} has been proposed in an algorithmic form which is suitable for computer programming. The method is probabilistic in its approach and is based on conditional probability of the form shown below.

Initially, we consider a situation in which the endogenous variables (i.e., the response terms R_i) are related to m non-correlated environmental variables C_k ($k=1,2,\dots,m$) and n material properties P_j ($j=1,2,\dots,n$). The cumulative distribution functions F_{C_k} and F_{P_j} are assumed to have been previously determined and that the congruity relationships of the form of equations (2.12) and (2.13) are at hand. The method comprises the following steps:

1. Draw the first set of values C_{K_1} , $k = 1,2,\dots,m$ of the environmental factors C_K from populations with cumulative distributions F_{C_K} .
2. Obtain the conditional probability distribution function of each material property P_j for the values C_{K_1} available from step 1 above:

$$F_{P_j|C_K} (P_j < p_j | C_K = c_{K_1}), k = 1,2,\dots,m \quad (2.14)$$

Whereas this equation takes the following form if the material properties are not influenced by the environment:

$$F_{P_j|C_k} = F_{P_j} \quad (2.14a)$$

3. Using the distribution functions obtained from step 2, draw the first set of values p_{j_1} , $j = 1, 2, \dots, n$ of the material properties P_j .

4. Compute the first set of endogenous variables, R_{i_1} , using the congruity relationship (2.10).

5. Repeat the previous steps M times to obtain M sample values of the R_i . The summary for the conditional probability used in this procedure is stated in equation (2.15) below:

$$F_{P_j|C_k} (P_j \leq p_j | C_k = c_{k_1}), j = 1, 2, \dots, m \quad (2.15)$$

where c_{j_1} is any set of values of c_j from populations with cumulative distributions F_{C_k} .

In much the same algorithm, i.e.,

$$F_{P_j|C_k} (P_j | C_k) \cdot F_{C_k} (C_k) = F_{P_j} (P_j) \quad (2.16)$$

the inherent interdependence of material properties can be taken into account. When the environmental factors are correlated, some modifications have to be introduced in the above algorithm in a similar way (24).

As a result of M simulations, histograms, means, variances, and percentage points can be obtained. If the number M is sufficiently large, the histograms can accurately represent the continuous distribution of the parent populations.

From the above discussion, it is clear that this method is based upon, and is only reliable as the technique used to obtain a sample value x of a random variable x with a given distribution F_x . Various techniques have been suggested for this purpose; in fact there is a considerable amount of literature devoted to this subject (32,33). Most techniques (22,33,47) are based on the generation of pseudorandom numbers which are uniformly distributed in the region between 0 and 1, which is discussed in Section (2.5) above.

The "inverse transformation" method suggested in section (2.5) has been employed in this study to generate random variates from certain probability distributions. However, care should be taken in the selection of random number generators as some are less efficient than others, depending on the nature of the problem, the parameters involved in the simulation, and the statistical properties of these parameters. An approximate normal deviate generator has been used in case of normally distributed properties in this study. The generator which is part of the IBM/360 system, and can be found under SUBROUTINE GAUSS, in the system's library,

is based on the Central Limit theorem. It uses 12 uniform random numbers to compute each normal deviate, which is done by calling another generator (SUBROUTINE RANDU) twelve times. The latter has been used when the material properties and environmental conditions are assumed to be uniformly distributed. RANDU is based on the "power residue" method to compute sets of randomly distributed numbers (22). Listings of both subroutines (i.e., RANDU and GAUSS) are found in Reference (22).

In the next chapter, application of the above method is presented as applied to a mathematical model representing a highway pavement structure taking into account the effect of the variability in the material properties and environmental conditions on the behavior of the pavement under a static condition of load application.

2.8 Final Procedural Remarks on the Use of the Method

The simulation procedures suggested and discussed above is a simple numerical method giving statistical answers to specific problems which are not amenable to analytical procedures due to their inherent complexity and interacting characteristics. The method is approximate in nature, however, adequate currency can be attained if the number of simulations is "sufficiently" large.*

In this case, the decision as to how many samples are to be drawn

* The "Sufficiency" conditions here depend on the available and the required statistical data.

out should be preceded by sensitivity analysis. The choice of sample size to be used for simulation experiments is one of the most important decisions to be made in planning a simulation study. It is completely inappropriate to select these sample sizes arbitrarily and then assume that the estimates thereby obtained are sufficiently accurate to yield valid conclusions. Instead, it is essential that statistical analysis be conducted to determine the required sample sizes. Hillier and Lieberman (21), Meier et al. (29), Naylor et al. (33), and Wagner (45) suggest various techniques for determining the size of simulation experiments.

Several other techniques have been developed to reduce the number of simulation experiments. They are either regression type of analyses or variance analyses (11,19,21,29,33,45). Variance reducing techniques are aimed to increase the information in the "interesting regions" of the distribution functions F_{R_i} , and consequently to decrease the information in the "non-interesting regions or ranges".* For instance, most structures usually are designed with a very low probability of failure, so that the low probability regions of the distribution functions of the variables

* Information on the entire cumulative distribution function of the variables representing the material behavior is obtained by statistically taking a sufficiently large number of values simulated using the Monte Carlo method. However, only a small portion of the distributions, referred to as the "interesting region", may be of interest in design and safety considerations.

contributing to such failure in the structure will be of prime interest (46). Therefore, it can be concluded that a larger number of simulations over the range of interest would simultaneously yield fairly good estimates of the cumulative frequency distributions over than range, and a reduction of the computer time for simulation. This is achieved by conducting a sensitivity study on the system under consideration to determine the regions of most interest.

The other factors which have an influence on the cumulative frequency distribution of the endogenous variables are the probability density functions of the exogenous parameters (i.e., the environmental variables and the material properties), their interaction and their correlations. In case of interacting parameters, it is suggested that a joint density function of the form shown in equations (2.17) and (2.18) below, be used rather than the single density functions. If these parameters are stochastically independent, then

$$f(C_1, C_2, \dots, C) = \prod_{k=1}^m \quad (2.17)$$

where the probability that the response R falls below a particular value r , will be

$$F_R(r) = P(R < r) = \int_G \dots \int_{k=1}^m \quad (2.18)$$

The restriction $R < r$ defines the region of interest G .

The probability density functions of the different simulated parameters are either assumed or obtained by some statistical tests. Sampling from the actual, statistically determined distribution is superior to that obtained from assumed distribution. However, when the statistical data for the density functions of the parameters under consideration are not available, special care should be taken in assuming such density functions. This can be done by looking into the literature for statistical representation of the same or similar parameters.

III. THREE-LAYER HALF-SPACE VISCOELASTIC SYSTEM

3.1 Model for the Pavement System

A pavement system is represented by a three layer model with two layers of a finite depth and the third layer being infinitely deep. Horizontally, the layers are assumed to extend infinitely. The materials in the layers are linearly elastic or viscoelastic, isotropic, with properties varying in a certain statistical manner. The load is assumed to be a single load uniformly distributed over a circular area at the surface of the top layer. The model is shown schematically in Figure (3). The formulation of the problem for the numerical solution of the stresses, strains, and deflections for the model is that developed in References 13, 14, and 30.

The material properties that are pertinent here are the compliance or the creep function, and the Poisson's ratio. Geometric properties are represented by the heights of the different layers.

The exogenous variables of the model are: material properties, geometric factors, environmental conditions (including mechanical loads).

Poisson's ratio is assumed to be constant and does not vary with the environment. Therefore, the compliance or the creep function is the only property which is assumed to be influenced by the environment, and is also assumed to be statistically distributed in a certain form.

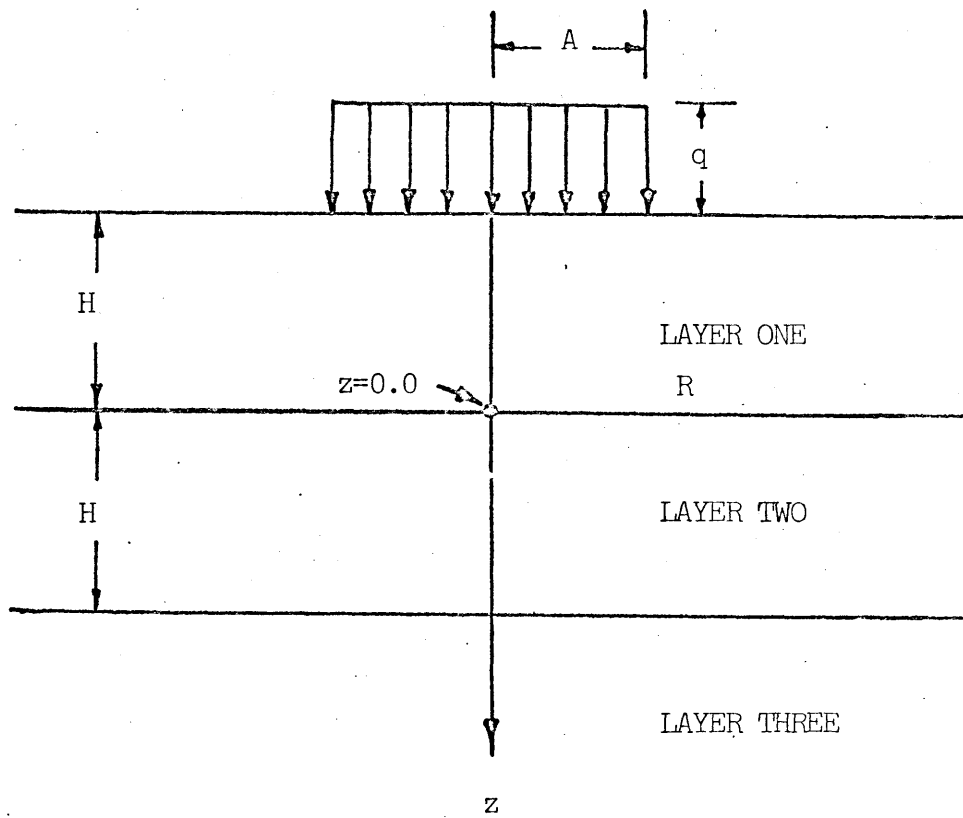


Figure 3. THREE - LAYER SYSTEM

The environmental operating on a highway system is assumed to be composed of three components: the traffic load, the temperature, and the humidity or moisture.

The traffic load is independent of the values of temperature and moisture, but there is an unknown relationship between temperature and moisture. Temperature is assumed to be a random variable having a certain distribution in the range (T_1, T_2) , where T_1 and T_2 are the extreme points of an assumed working range of temperature. The values for an average temperature values over a one year period in the Boston area were obtained from tables of the U.S. Weather Bureau in Boston. Then two distributions for the temperature were assumed, having the extreme values reported by the weather bureau:

- a) Uniform (rectangular) distribution over the range between T_1 and T_2
- b) Normal (Gaussian) distribution over the same range, where a statistical average and mean were calculated from the values of temperature obtained from the above tables.

Several investigations have been conducted to study the effect of temperature on the modulus of the asphalt and soil layers in pavements (7,33,38). Figure (4) has been used in this study to establish an empirical relationship between the temperature and the modulus of the different layers. This has been based on a study conducted by Dormon and Metcalf (7), which is derived from experimental obser-

vations. The moduli of the materials constituting the layered system are assumed to vary with the temperature in the following manner:

$$E_i = A_i e^{-\alpha_i(T+C)}, \quad i = 1,2,3 \quad (3.1)$$

where "A" and " α " are assumed to be constants for the layers, and the subscript (i) refers to the layer of interest (Figure 4). Ideally, the two parameters (A and α) should also be considered random variables with certain statistical distributions. "A" represents the value of the modulus at $T = -C$, where C is the value of the temperature at which the creep function or the modulus have been determined, or some reference temperature. At a given temperature T, which is a random variable distributed in the range (T_1, T_2) , the value of the compliance D can vary between $D_l(T)$ and $D_u(T)$, where the subscripts (l) and (u) refer to the lower and upper bound values of the function. The position of (D) will greatly depend on the moisture. No direct relationship was ascertained to determine the coupled effect of the moisture and temperature in evaluating the modulus or the creep properties of the material in the layers of a pavement system. Further work in this area is necessary.

Two curves are therefore arbitrarily drawn for the representation of the functional relation that has been assumed in equation (3.1) between the material properties and temperature.

The upper-bound curve is for the best condition of moisture, which may be the driest, and the lower-bound curve is for moisture conditions approaching saturation. This is true when the relation of equation (3.1) is for the modulus or the creep function of the materials. In case of the compliance, the inverse of the relation exists, i.e.,

$$D_i = \frac{1}{A_i} e^{\alpha_i (T+C)} \quad (3.2)$$

The upper-bound curves discussed above become the lower-bound curves for the compliance, and vice versa.

From the above discussion, it is clear that the effect of the moisture has been implicitly incorporated in the analysis, although no direct and explicit relation has been established between both the moisture content and the temperature on one hand, and the material properties, on the other hand.

Figure (4) shows the relation between the temperature and the modulus of the material with the assumption that:

- a) The moduli and the temperature are uniformly distributed between upper and lower bounds defining best and worst moisture conditions.*

* Note that both distributions assumed here have been arbitrarily chosen for the sake of demonstration. Any realistic or hypothetical type of probability density function can be used in the model to represent the behavior of the elements of the model.

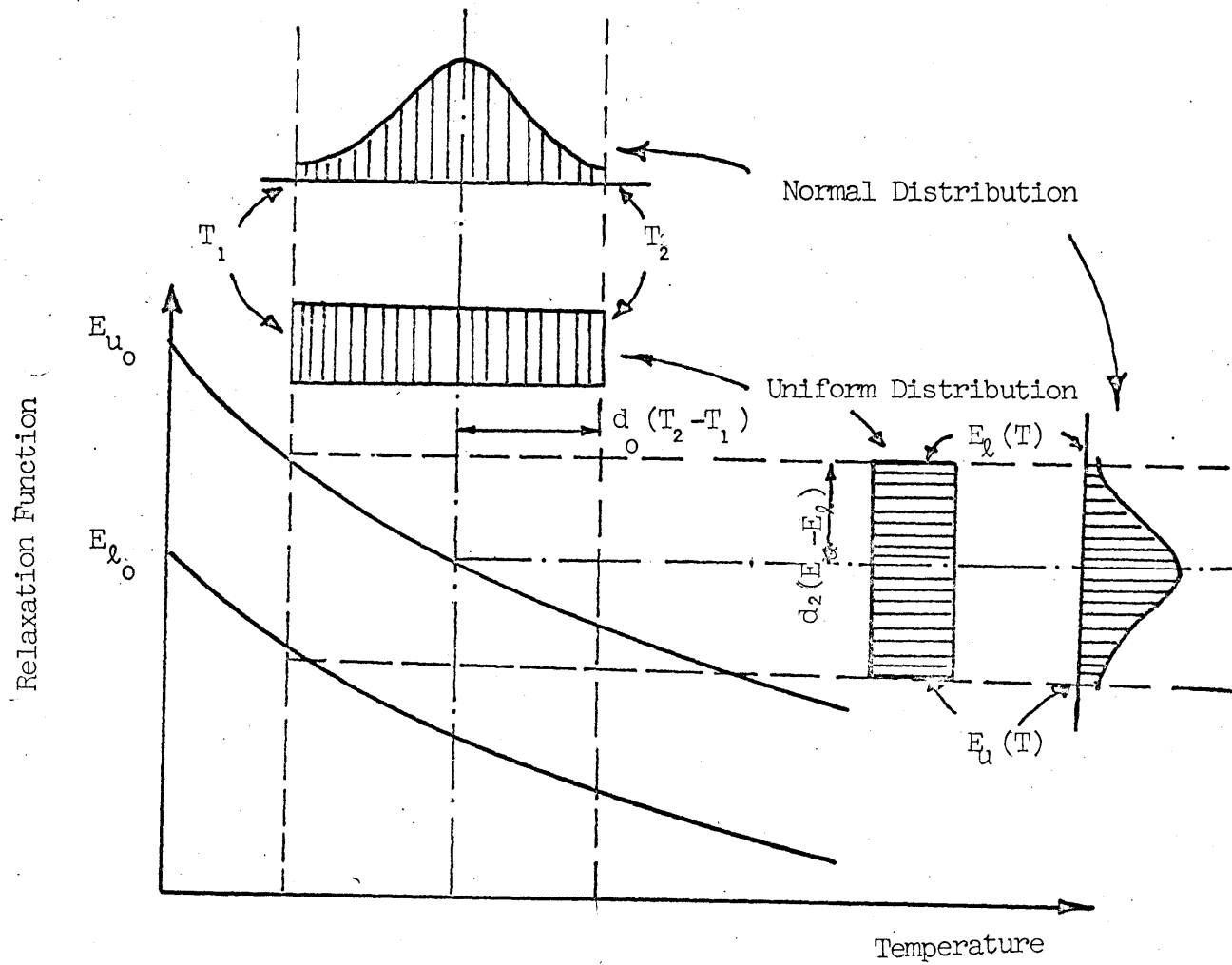


Figure 4. VARIATION OF THE RELAXATION FUNCTION WITH TEMPERATURE AT A FIXED TIME.

- b) The modulus as well as the temperature, are assumed to be normally distributed between the above limits.*

In the first case, the distributions are assumed to be uniform (i.e., rectangular) of the following form:

$$T = T_1 + d_1 (T_2 - T_1) \quad (3.3)$$

$$D = D_1 + d_2 (D_u - D_1) \quad (3.4)$$

where "d₁" and "d₂" are pseudorandom numbers uniformly distributed in the range (0 ≤ (d₁, d₂) ≤ 1). The above distribution is shown in Figure (4).

The coefficient "α" in the assumed exponential relationship between the temperature and the compliance, can also be assumed to be a statistical variable with a certain distribution, as shown in equation (3.5) below:

$$\alpha = \alpha_1 + d_3 (\alpha_2 - \alpha_1) \quad (3.5)$$

where the term α is uniformly distributed** in the range be-

* Note that both distributions assumed here have been arbitrarily chosen for the sake of demonstration. Any realistic or hypothetical type of probability density function can be used in the model to represent the behavior of the elements of the model

** The same discussion in the above footnote is applicable to the variable α.

tween α_1 and α_2 . This assumption is more realistic since it satisfies the modulus-temperature superposition and shifting principles, but it will result in a more complicated situation and will considerably affect the computer time. In the present analysis α is assumed to be a constant for each layer.

In the second case, i.e., when the temperature and the compliance are assumed to be normally distributed in an assumed working range of values (Figure 4), a standard normal (Gaussian) distribution has been chosen using a random number generator for normally distributed variables as illustrated in section (2.7) of Chapter II.

The above techniques and assumptions are employed using a computer program that considers a three-layer system with a static load applied at the top of the surface layer (13, 14, 30). The program is a primary model for the study of the behavior of pavement systems under traffic load. It calculates the stresses, deflections, and strains developed at any point in the system. The program handles linear elastic, linear viscoelastic, or partially viscoelastic system. The formulation of the problem for analytical solution for the stresses, strains, or displacements of the three-layer system is found in References (13, 14, 39).

The effect of the variation in the environment and in the material properties is taken into consideration in the following manner. The compliance or the creep functions used as input to the program are represented in the form of a

series of exponentials, namely the Dirichlet Series, for mathematical conveniences (equation 3.6).

$$D_j = \sum_{i=1}^n G_i e^{-t\delta_i}, j = 1,2,3 \text{ (layer number)} \quad (3.6)$$

Therefore, for a given axle load, a radius "a" is determined as the contact area between the wheels and the pavement with a certain load intensity, and, depending on the temperature of the surroundings, the material property is randomly chosen for each layer, i.e., the value of the compliance or the creep function is selected from a given spectrum between upper and lower bound values D_l and D_u . D_l and D_u are used as input to the program in the following manner:

$$D_j^l = \sum_{i=1}^n G_i^l e^{-t\delta_i}, j = 1,2,3 \quad (3.7)$$

$$D_j^u = \sum_{i=1}^n G_i^u e^{-t\delta_i}, j = 1,2,3 \quad (3.8)$$

where the superscripts "l" and "u" on the coefficient of the exponential series (G_i) denote upper and lower values respectively. The variation in the coefficients (G_i) will eventually yield a variation in the creep or elastic compliances. The assumption made here for simplicity, is that only the instantaneous or the elastic portion of the creep function varies statistically, by fixing the value of the retardation time ($1/\delta_i$). However, the whole curve may vary statistically

in an unknown manner. More statistical tests can contribute to the understanding of such variation. Future changes to accommodate such variation can be made by changing the probability density function which has been arbitrarily assumed in this study.

Selecting the value of the compliance or the creep function in this model fixes a value for the moisture content which can easily be calculated, provided that the functional relationship between the modulus or the compliance and the moisture is given.

The above process is repeated a number of times for a selected sample value of temperature. The number of iterations mainly depends on the sensitivity of the material properties to the variations in the environment and to the statistical characteristics of the material properties as has been discussed in section (2.8) of Chapter II. A flow chart of the computer program describing this process is shown in Appendix I.

In order to make use of the data, the values of the response terms are calculated for a given set of environmental variables by selecting a range of material properties for a given monthly variation in the temperature depending on the surrounding moisture conditions. This requires the determination of the monthly, rather than daily, temperature variation in each layer over a cycle of one year, for example. The magnitude of stresses or strains is calculated on this

basis for several times each month, and a most probable value can be predicted. This procedure is repeated over the whole year period. Assuming that the results are additive, if a critical or intolerable value of stress is reached, then the system is assumed to have partially failed. The analysis is applicable to the case where the value of the deflection is limited and therefore, the value of the compressive stress or strain at the second interface* is limited. However, the values of temperatures in this study were chosen randomly between upper and lower limits over the whole year and are based on a monthly variation because the sequence of their occurrence is not critical when the static load case is studied. When the repeated loading case is studied, it is important to emphasize on the significance of the sequence of occurrence of events to account for accumulation of response over extended periods of time.

In the next section, an illustrative example is presented where numerical values of the inputs and outputs of the model are also listed. Discussion of the results obtained through the computer programs are also presented in this section.

3.2 Numerical Example of Simulation of the Three-Layer System

To illustrate the effectiveness of the techniques dis-

* The interface between the second layer and the subgrade.

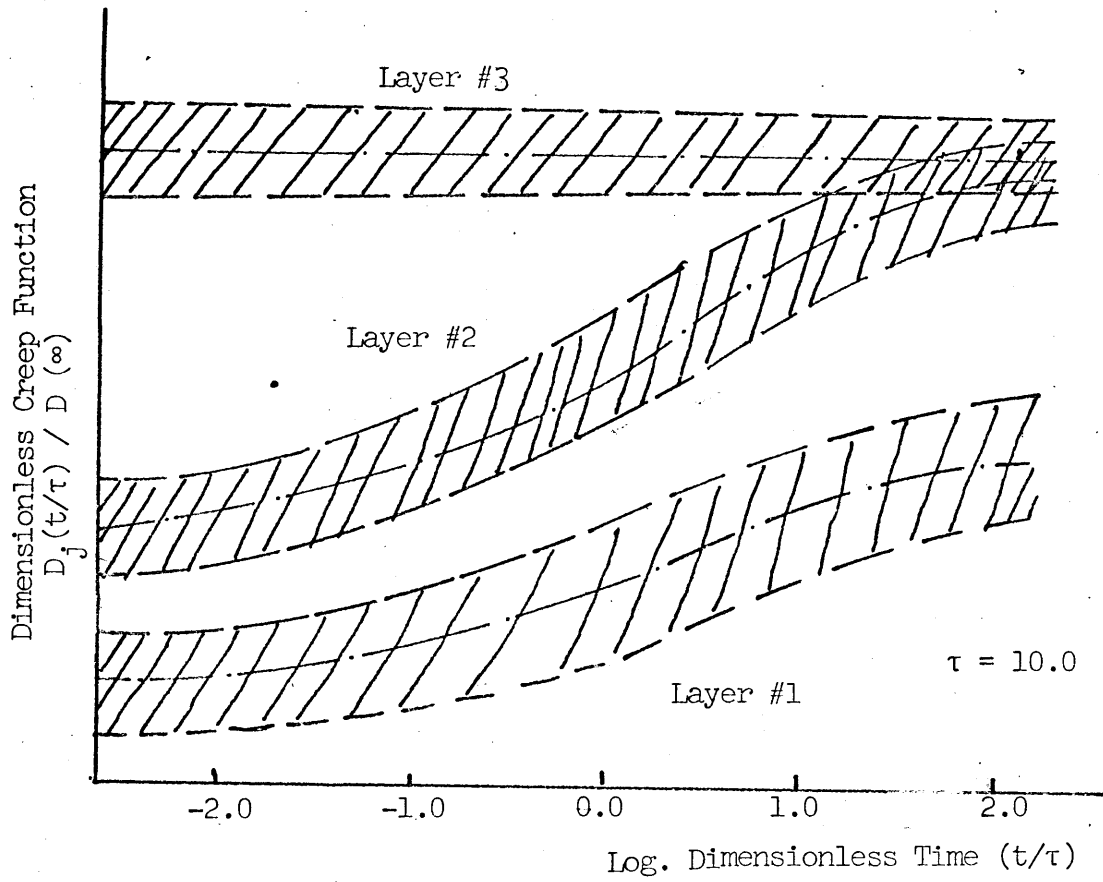


Figure 5. CREEP FUNCTIONS OF THE SIMULATED PAVEMENT SYSTEM

$$D_j(t) = \sum_i G_i e^{-t\delta_i}$$

cussed above, and to give typical results, a three-layer half space viscoelastic system with the following geometry and material properties have been analyzed.

$$\frac{a}{h} = 1.19$$

$$\frac{H}{h} = 2.0$$

$$u_{i,j}^{D_i}(t) = \sum_{j=1}^6 u_{i,j}^{G_i^j} e^{-t\delta_i}$$

$$e_{i,j}^{D_i}(t) = \sum_{j=1}^6 e_{i,j}^{G_i^j} e^{-t\delta_i}$$

where the values of G_i^j , and δ_j are given in Tables 1, and 2, and the compliances are also shown in Figure (5) for materials used in each layer.

3.3 Results and Discussion

Simulation of the data in this example was conducted by drawing 100 sample values of the input variables which were assumed to have two different statistical properties:

- 1) Uniform distribution for the temperature and the viscoelastic creep compliances, and
- 2) Normal (Gaussian) distribution for the variables mentioned above.

Figure (6) shows typical response functions in terms of the vertical strains (ϵ_{zz}) at the first interface under the center of the loaded area, versus time. This figure shows that for a 15%

TABLE 1

Extreme Values for the Coefficients of the Dirichlet Series
Representation of the Creep Compliance $\sum G_i e^{-t\delta_i}$ *

First Layer	Second Layer	Third Layer
Upper Extremes	Upper Extremes	Upper Extremes
$u G_1^1 = -0.5750$ $u G_1^2 = -0.0863$ $u G_1^3 = -0.0575$ $u G_1^4 = -0.0863$ $u G_1^5 = -0.0575$ $u G_1^6 = 0.5750$	$u G_2^1 = -1.1500$ $u G_2^2 = -0.1725$ $u G_2^3 = -0.1150$ $u G_2^4 = -0.1725$ $u G_2^5 = 0.1150$ $u G_2^6 = 1.1500$	$u G_3^1 = 0.0$ $u G_3^2 = 0.0$ $u G_3^3 = 0.0$ $u G_3^4 = 0.0$ $u G_3^5 = 0.0$ $u G_3^6 = 1.1500$
Lower Extremes	Lower Extremes	Lower Extremes
$l G^1 = -0.4250$ $l G^2 = -0.0638$ $l G^3 = -0.0425$ $l G^4 = -0.0638$ $l G^5 = -0.0425$ $l G^6 = 0.4250$	$l G^1 = -0.8500$ $l G^2 = -0.1275$ $l G^3 = -0.0850$ $l G^4 = -0.1275$ $l G^5 = -0.0850$ $l G^6 = 0.8500$	$l G^1 = -0.0$ $l G^2 = 0.0$ $l G^3 = 0.0$ $l G^4 = 0.0$ $l G^5 = 0.0$ $l G^6 = 0.8500$

* All the G_j^i values are multiplied by 10^3 , so the actual value of $u G_1^1$ for example is (-0.0005750).

TABLE 2

Exponents of the Coefficients of the Creep Compliance

in the Dirichlet Series Representation $\sum G_i e^{-t\delta_i}$

$$\delta_1 = 10.00$$

$$\delta_2 = 3.162$$

$$\delta_3 = 1.00$$

$$\delta_4 = 0.316$$

$$\delta_5 = 0.10$$

$$\delta_6 = 0.0$$

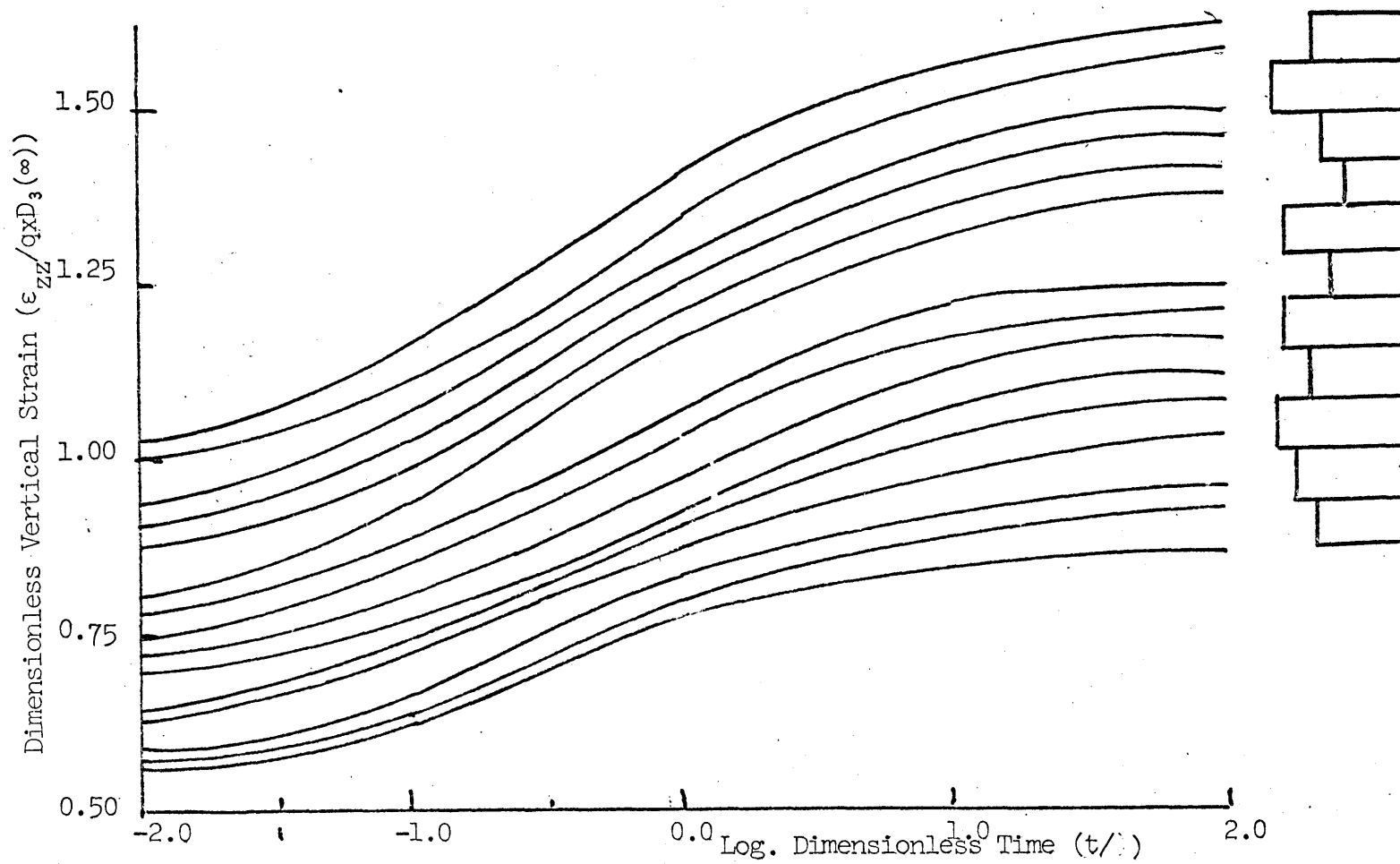


Figure 6. TYPICAL DISTRIBUTION OF THE TIME - DEPENDENT NORMALIZED VERTICAL STRAIN FOR MATERIAL PROPERTIES SHOWN IN FIGURE 5.

variations in the input creep compliances a relatively wide scatter results in the response function. This also shows that for each value of the creep compliances of the different layers, any of the response curves shown in Figure (6) has a chance of occurrence. This variation in the response is substantial enough to justify the use of a probabilistic treatment.

The frequency distributions of the vertical strain at two different points in time are plotted in Figures (7) through (10). Figures (7) and (8) represent frequency distribution for the first case, i.e., when the probability density functions of the input variables are assumed to be uniform. While Figures (9) and (10) represent the corresponding frequency distributions of the vertical strain for normally distributed input variables. Each histogram in Figures (7) through (10) is in fact a cross-section at that particular point in time of Figure (6).

The trend in these histograms appears to be toward that of the corresponding distribution of the input variables, although there are some peaks or irregularities. The reason for these deviations from the assumed distributions may be attributed to the fact that the number of samples drawn for the simulation experiment was not sufficiently large to be representative of the parent populations.

To validate and check these description of the histograms, it is possible to conduct simple statistical tests called "Goodness-of-fit" tests, to estimate the coincidence of the obtained results with

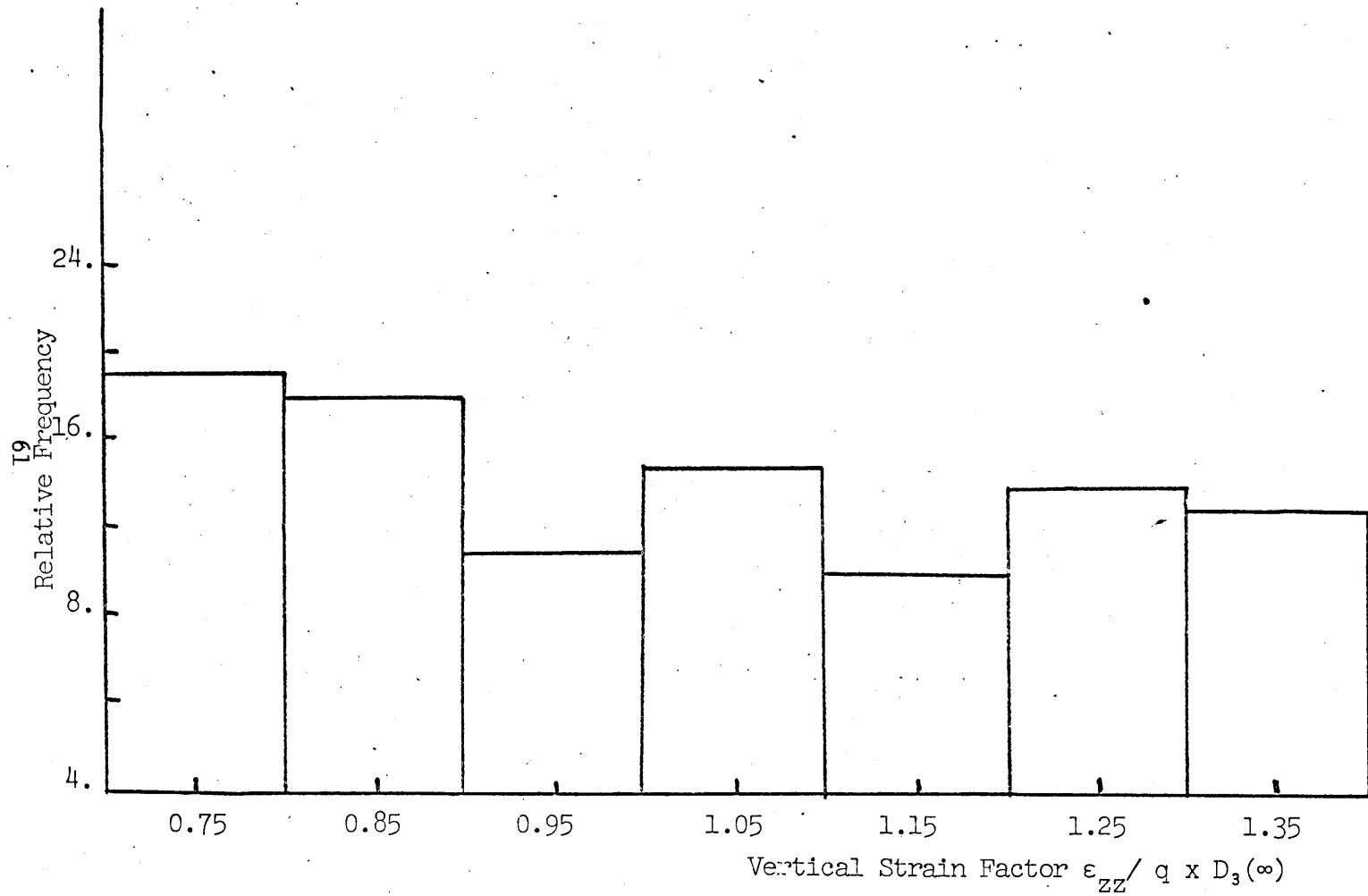


Figure 7. RELATIVE FREQUENCY DISTRIBUTION OF VERTICAL STRAIN AT $t/\tau = 0.0$ FOR UNIFORMLY DISTRIBUTED RANDOM INPUTS.

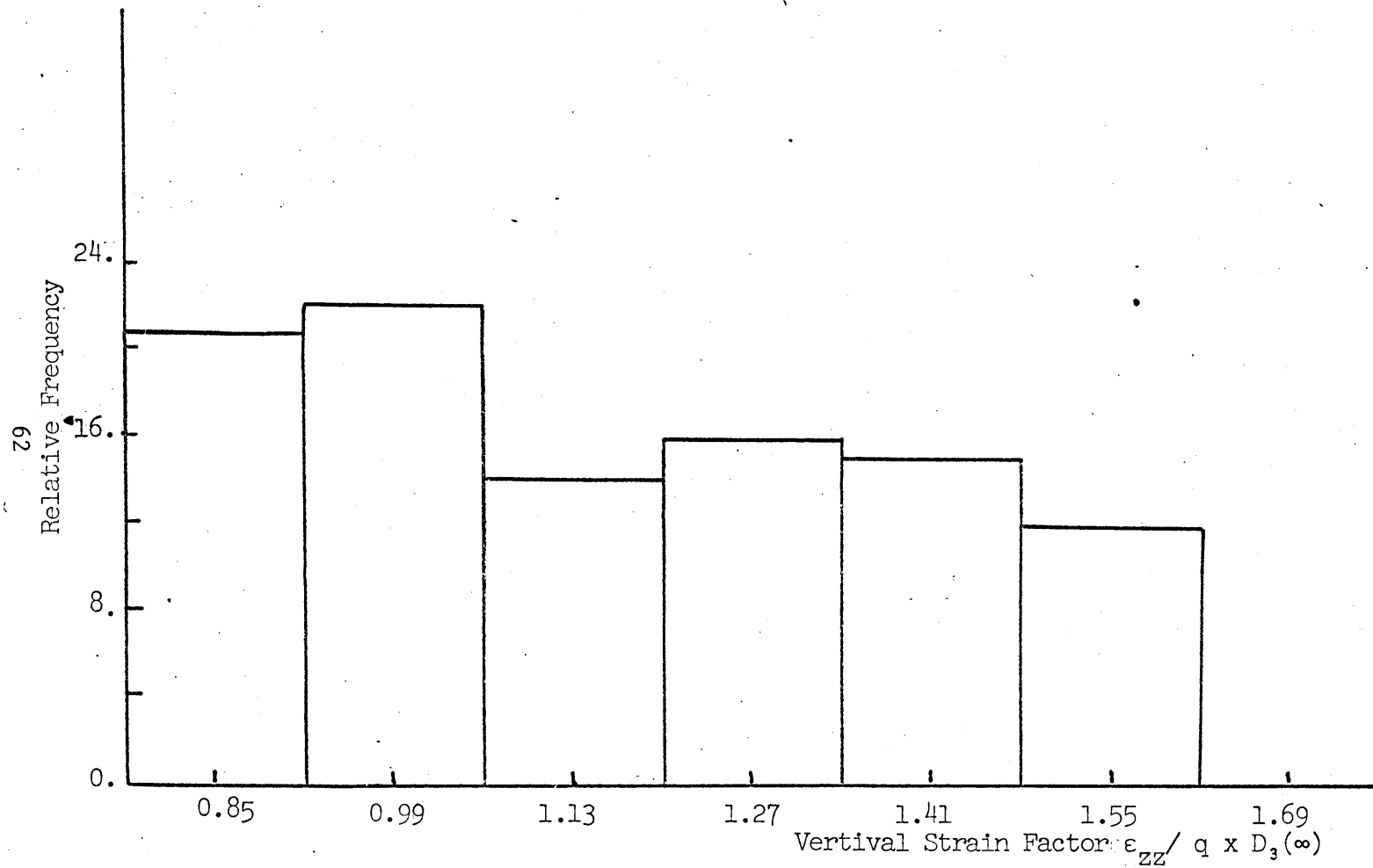


Figure 8. FREQUENCY DISTRIBUTION OF VERTICAL STRAIN AT $t/\tau = 10.0$ FOR NORMALLY DISTRIBUTED RANDOM INPUTS.

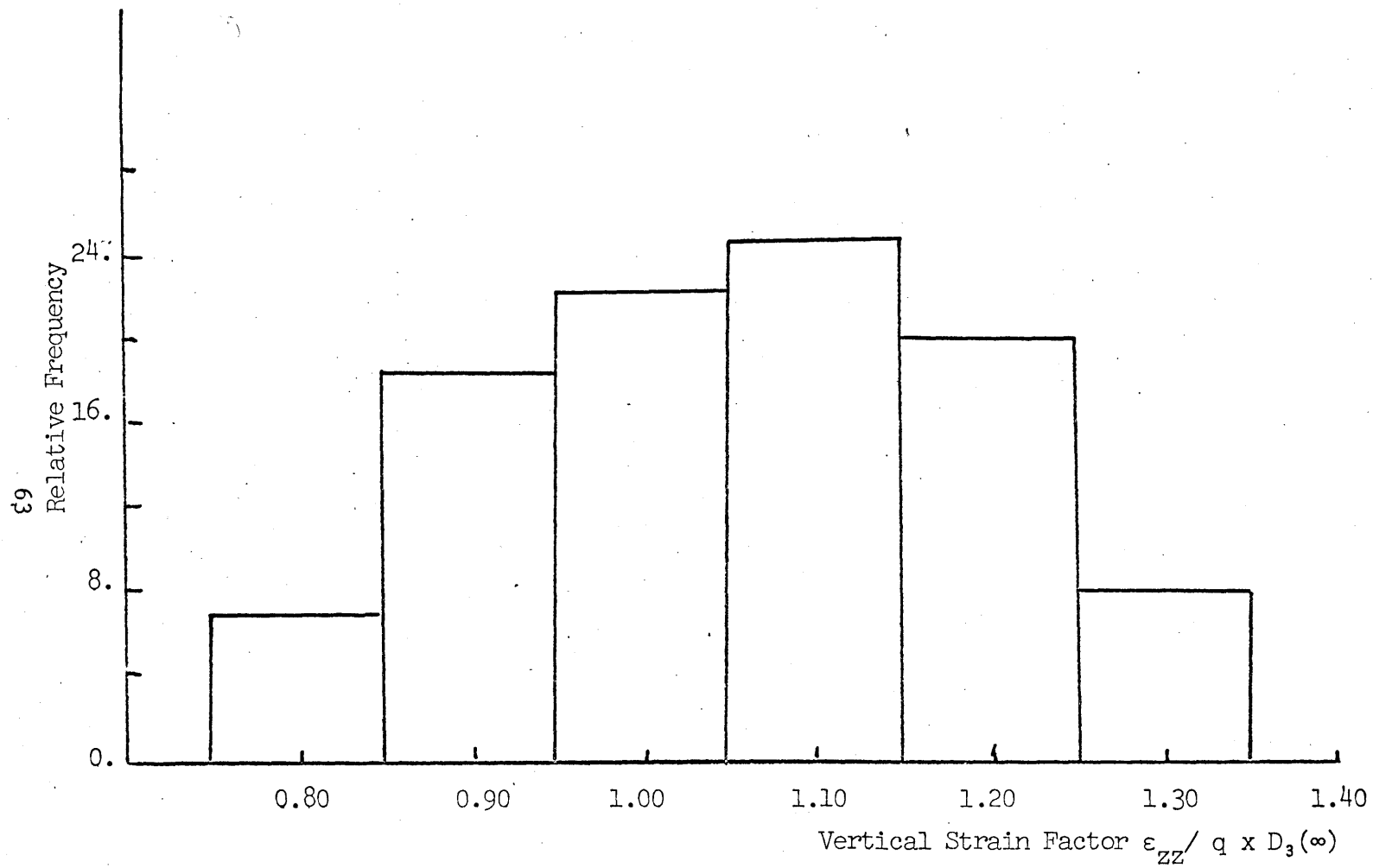


Figure 9. RELATIVE FREQUENCY DISTRIBUTION OF VERTICAL STRAIN AT $t/\tau = 0.0$ FOR NORMALLY DISTRIBUTED RANDOM INPUTS.

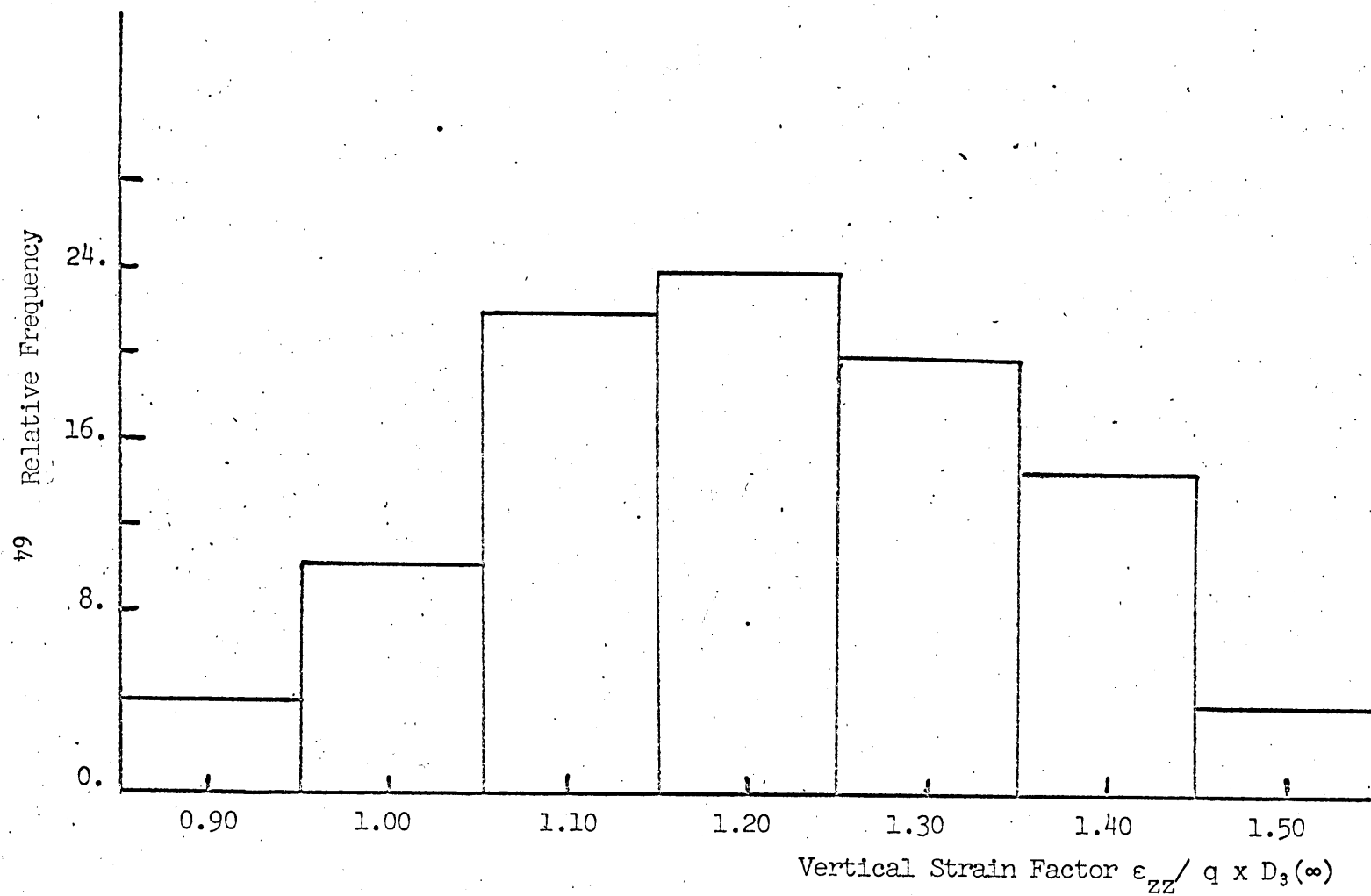


Figure 10. RELATIVE FREQUENCY DISTRIBUTION OF VERTICAL STRAIN AT $t/\tau = 10$. FOR NORMALLY DISTRIBUTED RANDOM INPUTS.

those from theories. A χ^2 - test was, therefore, conducted on the results of the above simulation. Both cases showed that they fit their corresponding theoretical distributions within a reasonable degree of accuracy.

Moreover, to confirm the fact that the lack of adequate number of samples is responsible for the discrepancy between theoretical distributions and those obtained by simulation, the temperature distributions obtained from the IBM System/360 Random Number Generator are plotted in Figures (11) and (12). Figure (11) is a frequency distribution of uniformly distributed temperatures, and Figure (12) is that of normally distributed temperatures. These figures show a trend similar to the corresponding distributions of the response terms shown in Figures (7) through (10).

The cumulative distributions of the above histograms are plotted in Figures (13) through (18).

The above results and their scatter show the importance of the statistical nature of the materials properties and other input variables that will describe the resulting scatter in the response of the pavement to load and environment. In order to use the results of such simulations is the analysis of response of the pavement systems, one may use first and second order movements, i.e., the mean, the variance and the coefficient of variation. From this, a summary of the simulation may be plotted

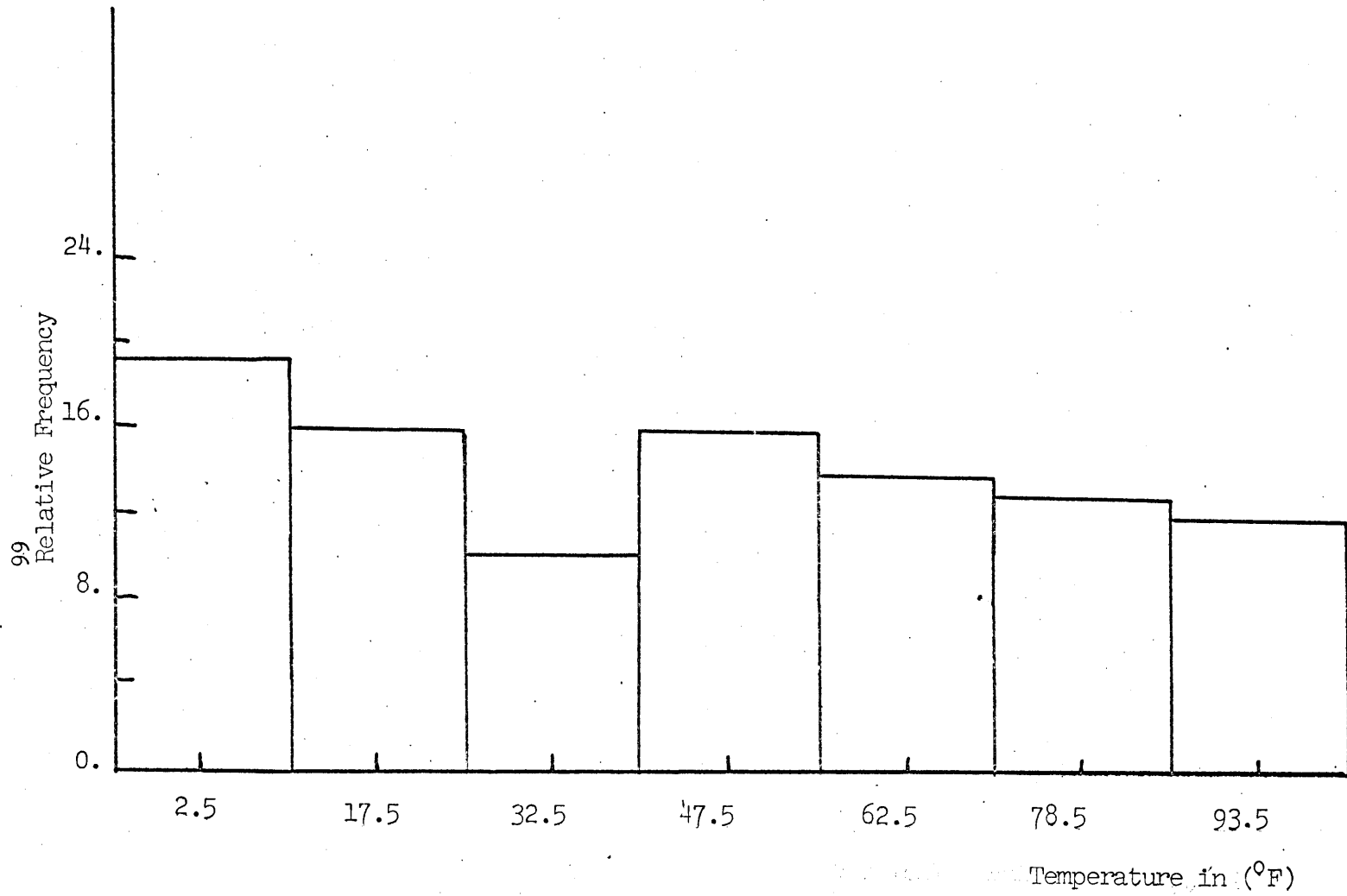


Figure 11. RELATIVE FREQUENCY DISTRIBUTION OF UNIFORMLY DISTRIBUTED TEMPERATURES AS GENERATED BY AN IBM / SYSTEM 360 RANDOM NUMBER GENERATOR.

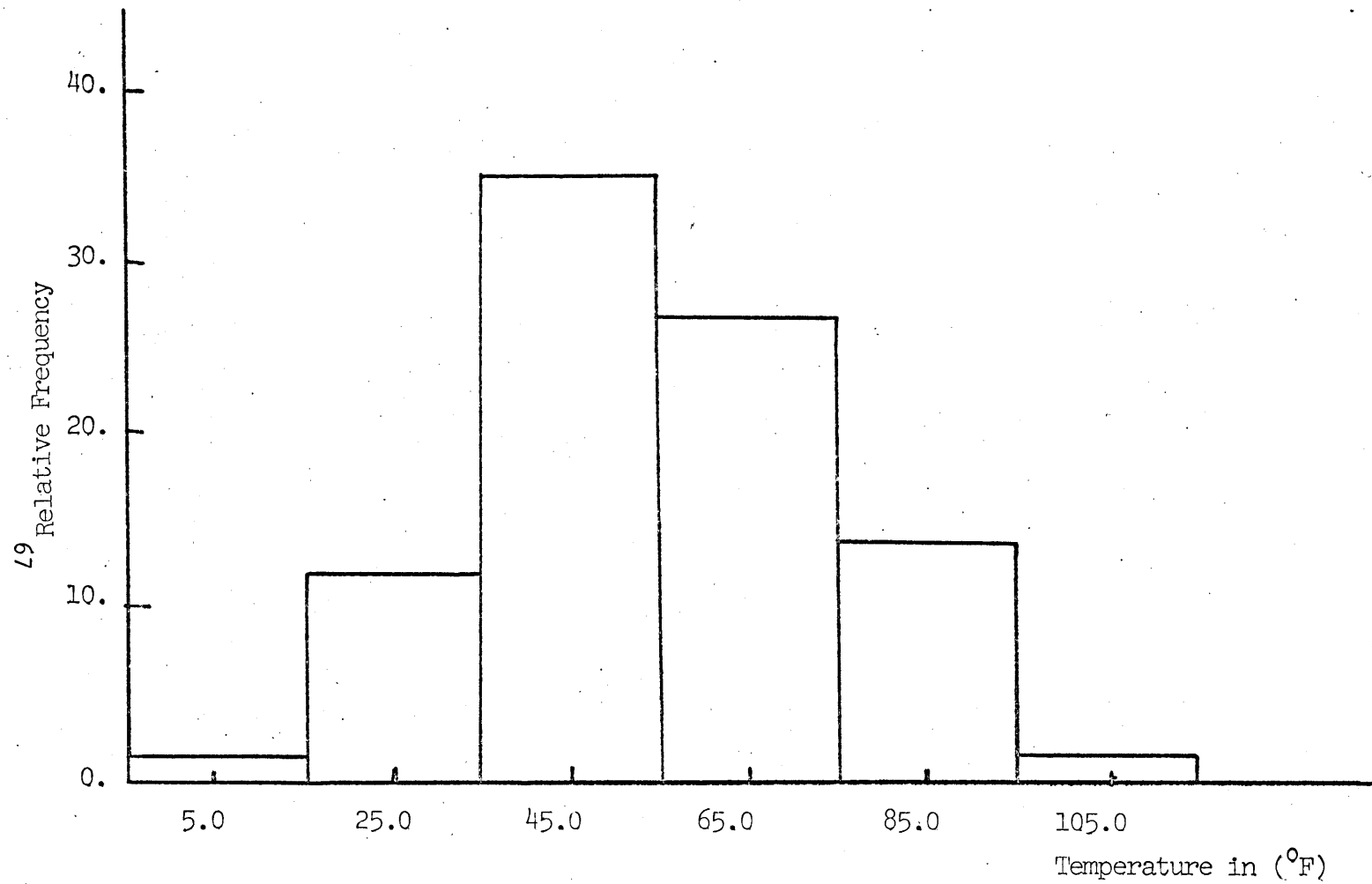


Figure 12. RELATIVE FREQUENCY DISTRIBUTION OF NORMALLY DISTRIBUTED TEMPERATURES AS GENERATED BY AN IBM / SYSTEM 360 RANDOM NUMBER GENERATOR.

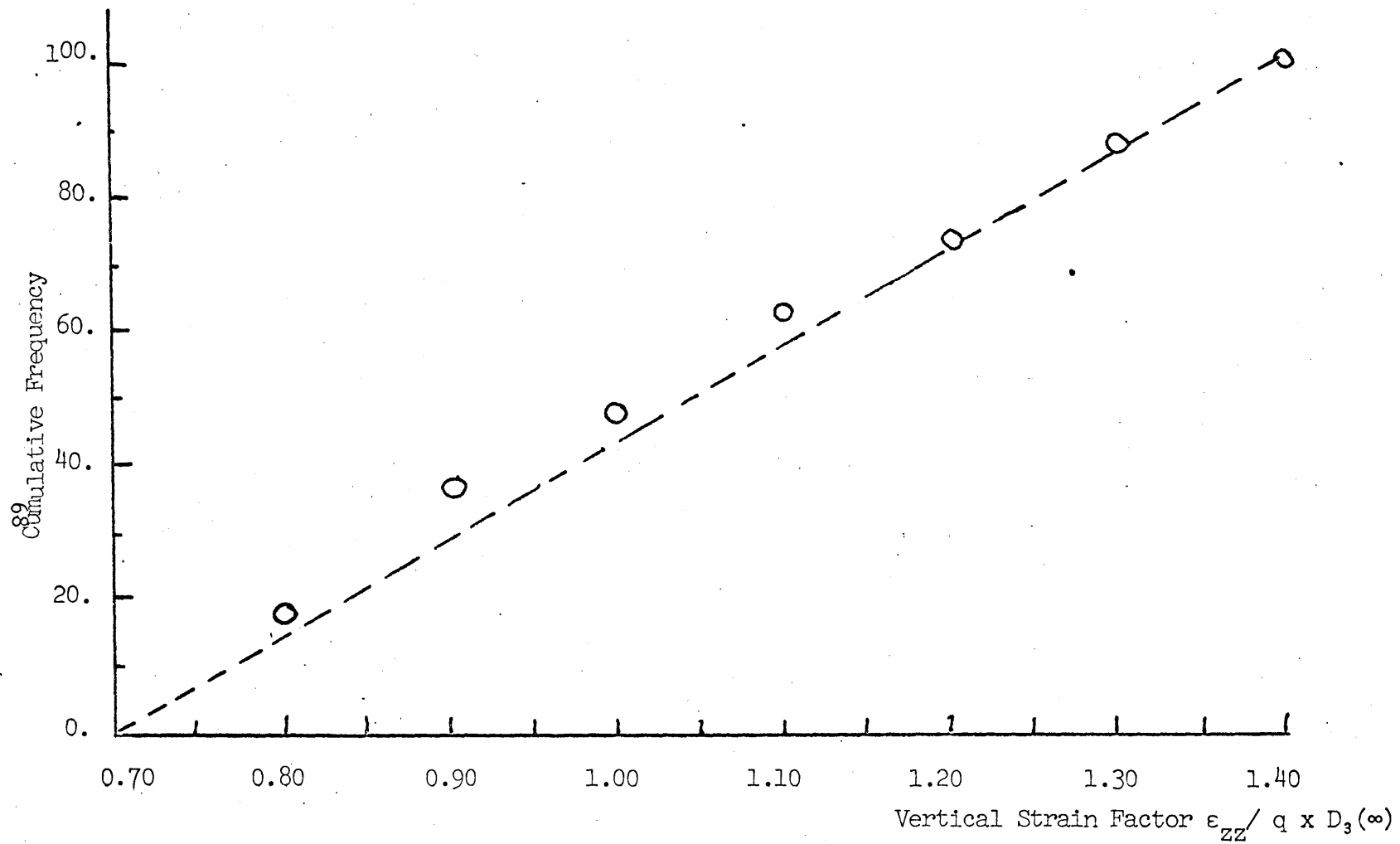


Figure 13. CUMULATIVE DISTRIBUTION OF VERTICAL STRAIN FOR FREQUENCY DISTRIBUTION SHOWN IN FIGURE 7.

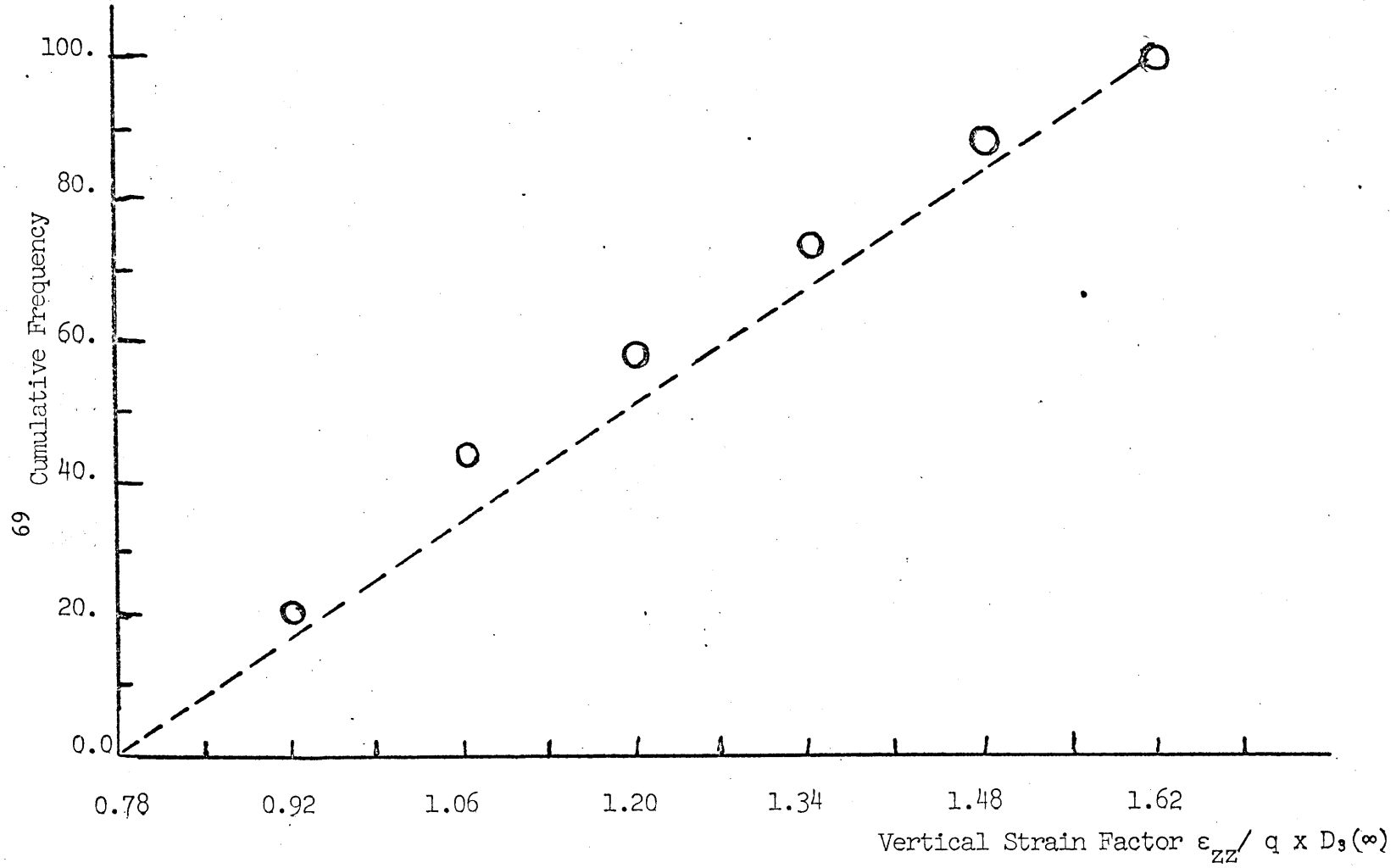


Figure 14. CUMULATIVE DISTRIBUTION OF VERTICAL STRAINS FOR FREQUENCY DISTRIBUTION SHOWN IN FIGURE 8.

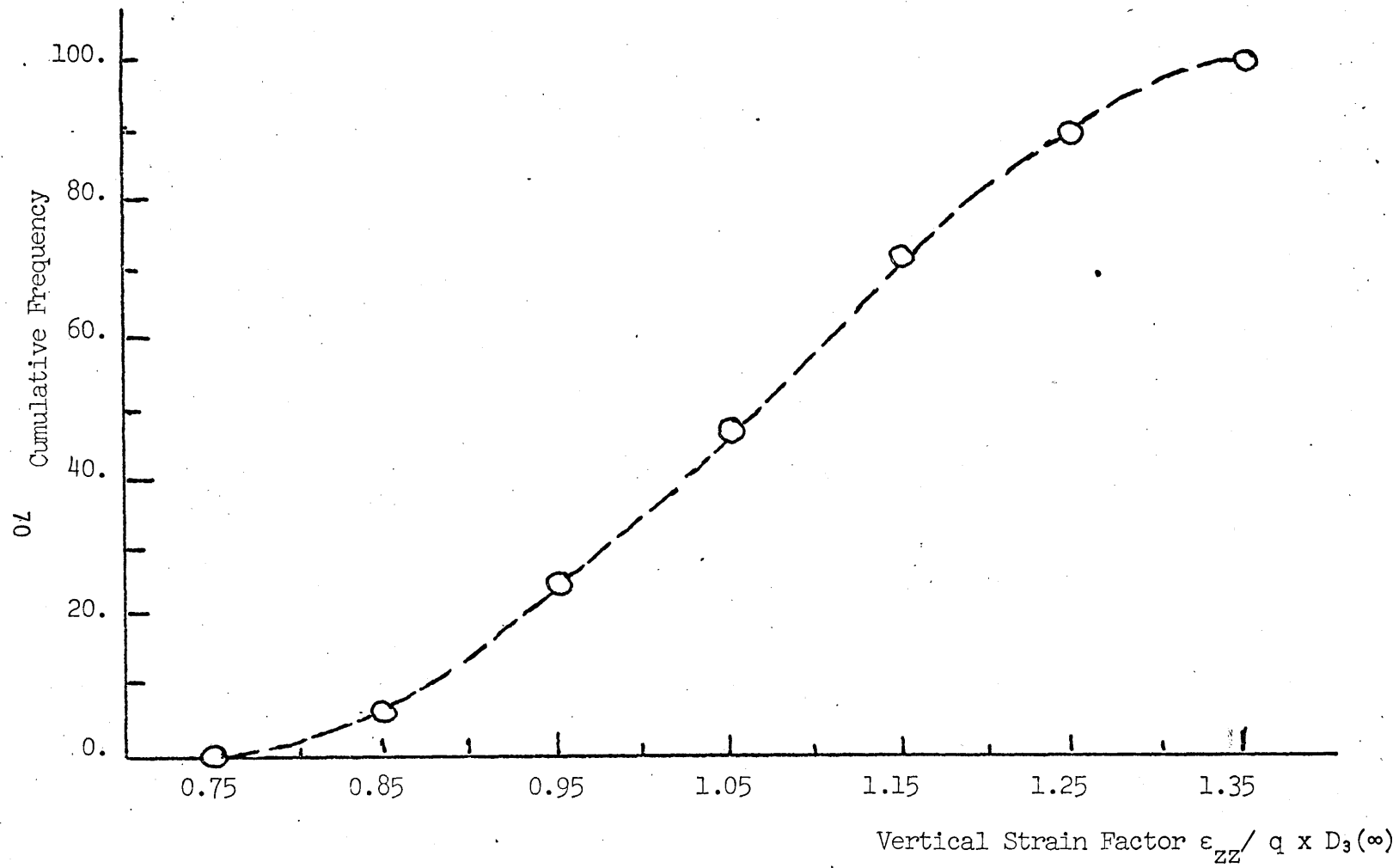


Figure 15. CUMULATIVE DISTRIBUTION OF VERTICAL STRAINS FOR FREQUENCY DISTRIBUTION SHOWN

IN FIGURE 9..

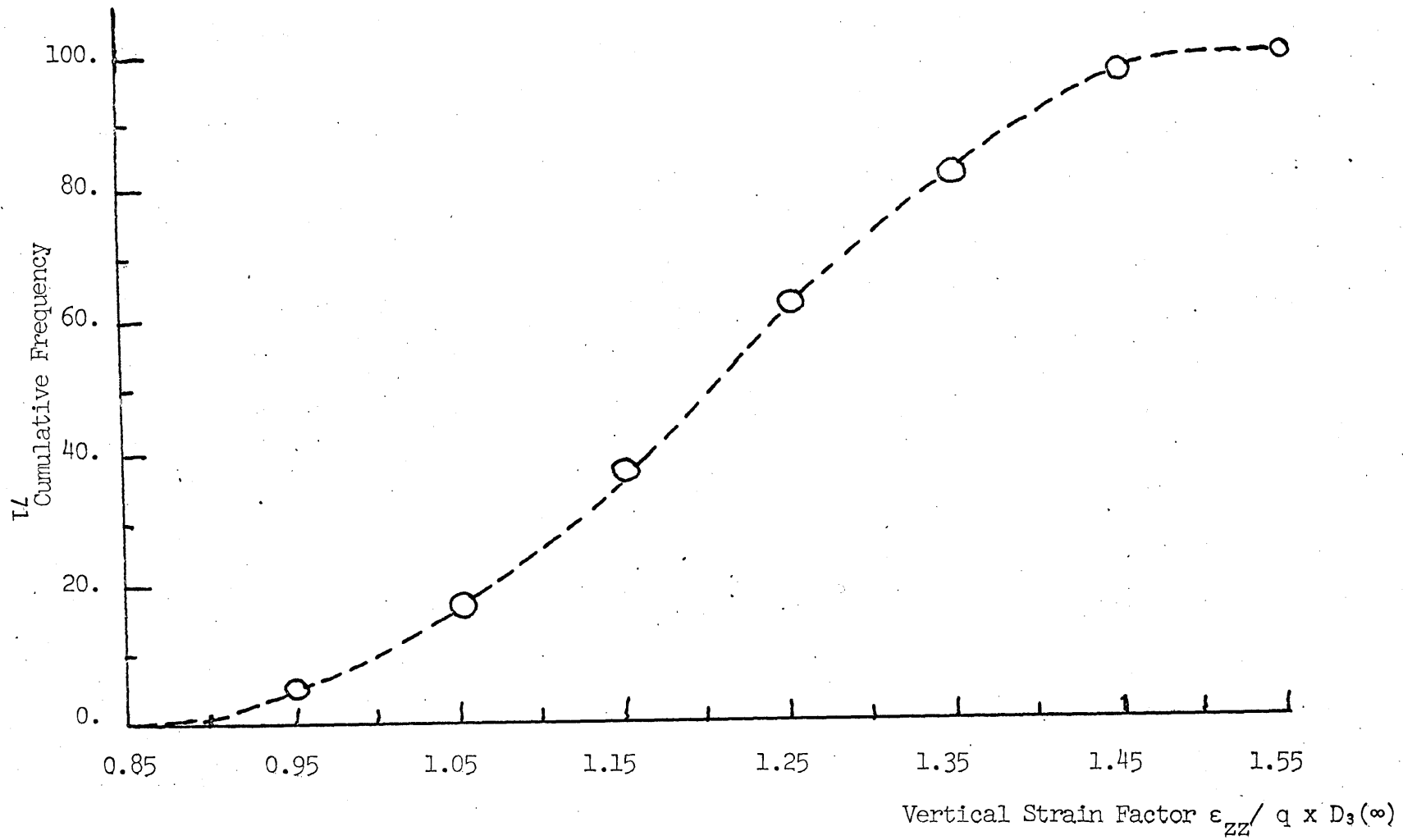


Figure 16. CUMULATIVE DISTRIBUTION OF VERTICAL STRAINS FOR FREQUENCY DISTRIBUTION SHOWN IN

FIGURE 10.

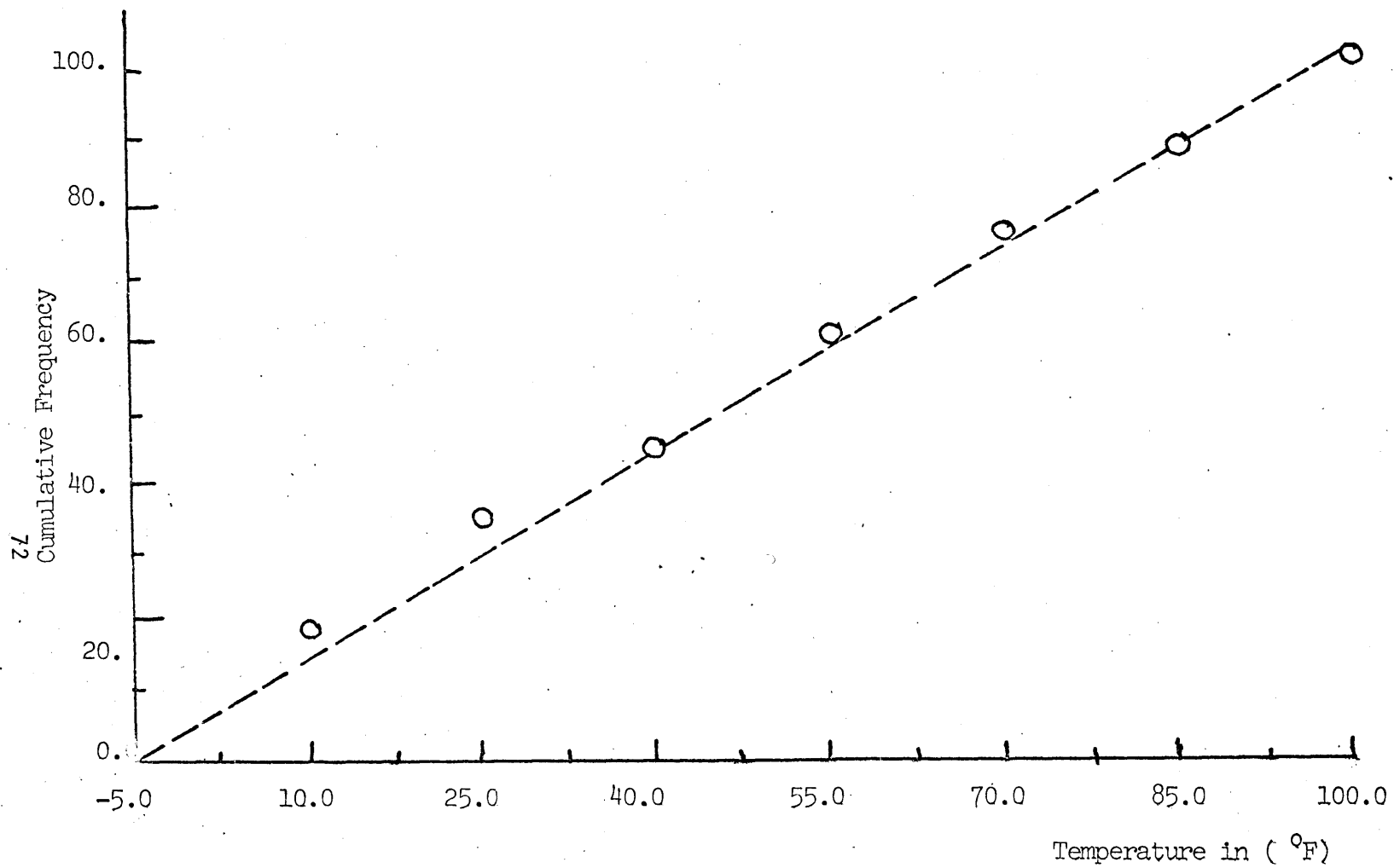


Figure 17. CUMULATIVE DISTRIBUTION OF UNIFORMLY DISTRIBUTED TEMPERATURES.

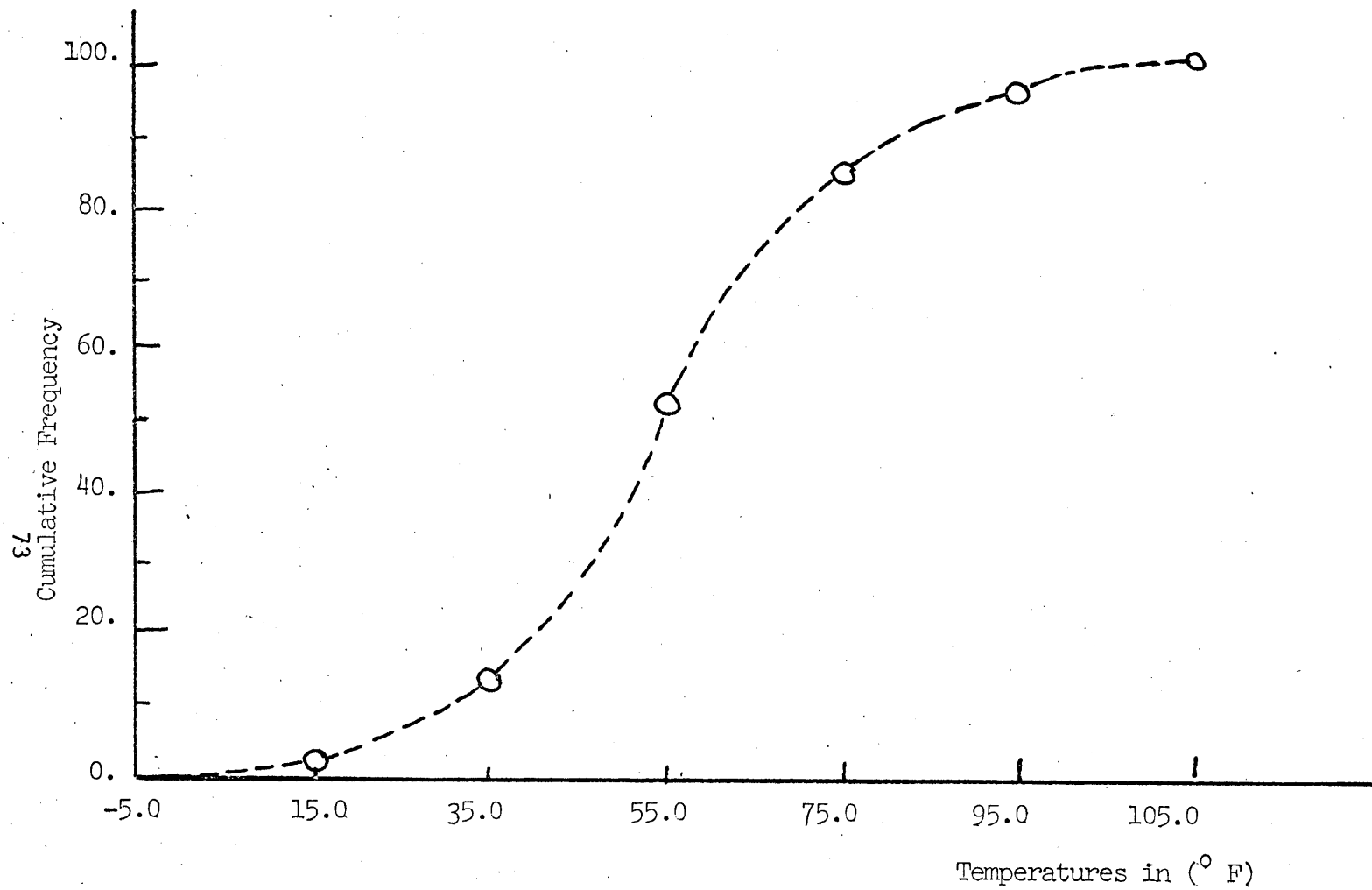


Figure 18. CUMULATIVE DISTRIBUTION OF NORMALLY DISTRIBUTED TEMPERATURES.

as those in Figures (19) and (20). Figure (19) describes the mean, deviations, and extreme values for the time-dependent strain shown in Figure (6), when the input properties were assumed to have a uniform distribution. The corresponding values for the normally distributed input variables are shown in Figure (20).

In the design process, it is more realistic to consider all information similar to that shown in Figures (19) and (20). Using averages and single values for the design may result in a very conservative design, or else failure may be more eminent than that predicted.

Finally, variations in the load function may result in a change of the physical properties of the materials in the pavement that would affect significantly its response. This type of behavior is not being accounted for if classical averaging procedures are followed in the design. While it is obvious that the extreme values of the response shown in Figures (19) and (20) may be due to this type of behavior, and therefore it allows the designer to consider the uncertainty associated with their occurrence to account for these properly in the design.

The next step to be taken in this type of analysis is to use the results obtained from the simulation of the system under a single stationary load and operational environment into a repetitive load mode applied randomly to the system. From this, the effect of load repetitions and varying environment on the

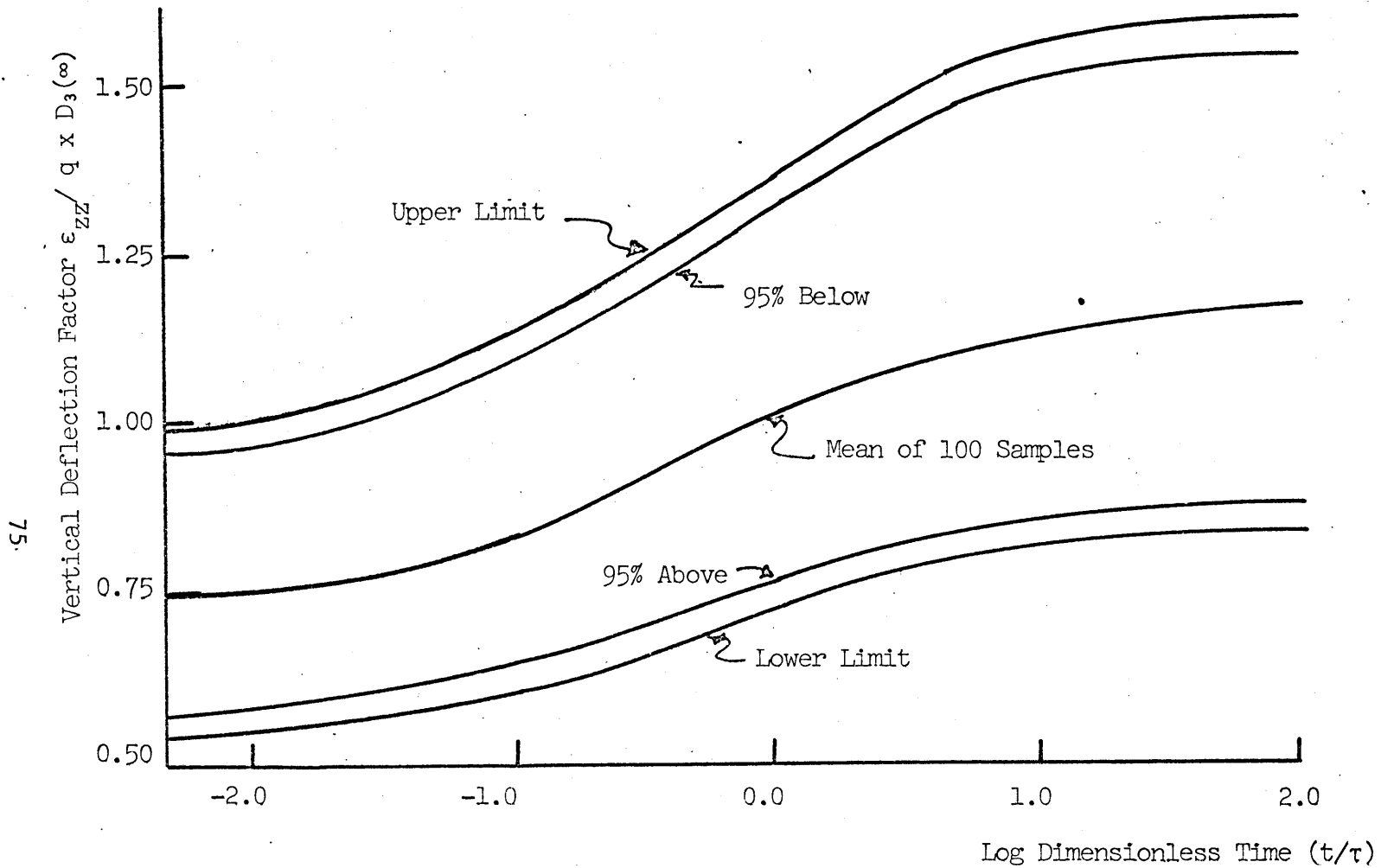


Figure 19. SUMMARY OF SIMULATION OF VERTICAL STRAINS FOR UNIFORMLY DISTRIBUTED RANDOM INPUTS.

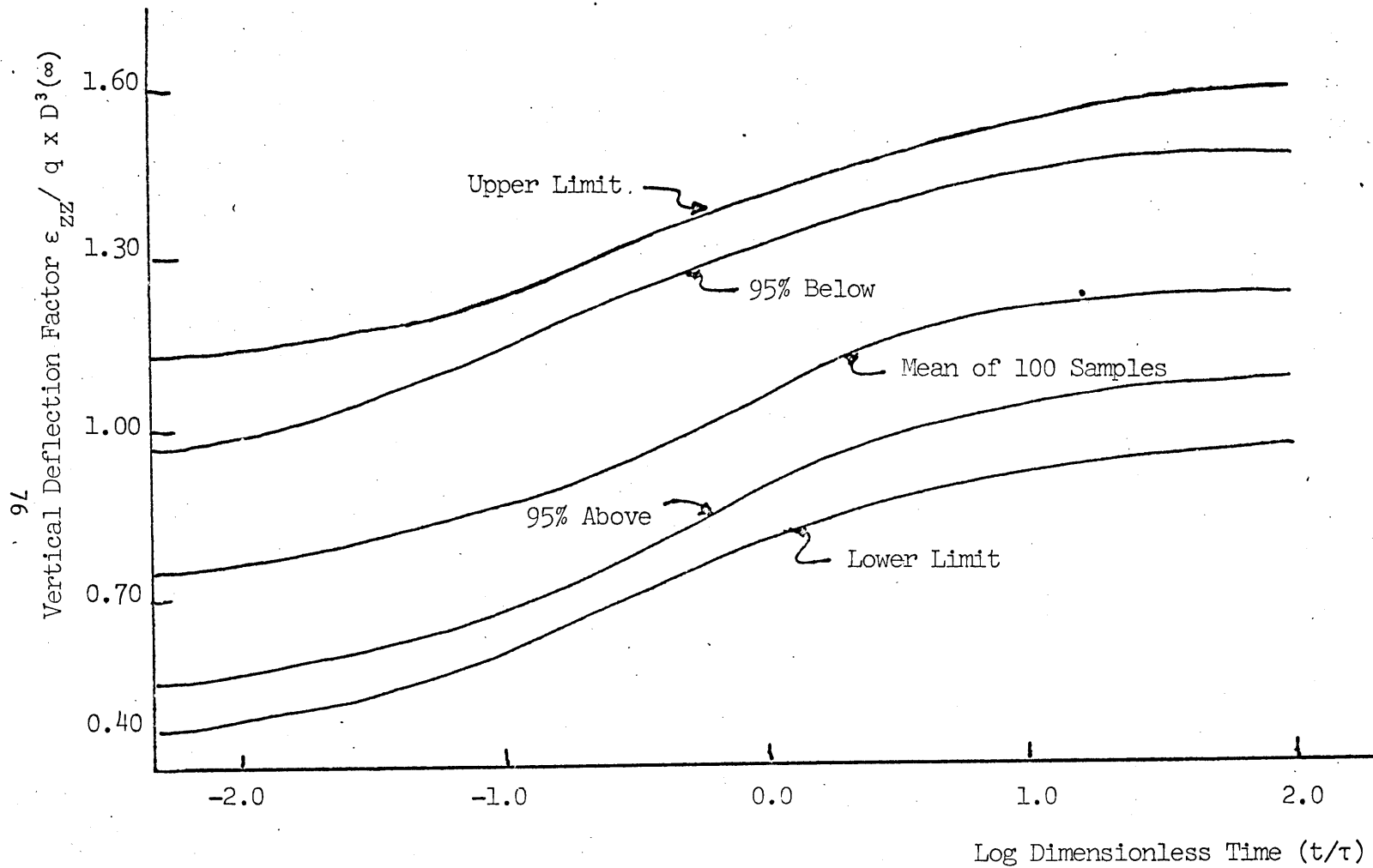


Figure 20. SUMMARY OF SIMULATION OF VERTICAL STRAINS FOR NORMALLY DISTRIBUTED RANDOM INPUTS.

response and behavior of the system can be studied. Consequently the analysis of the so called primary response behavior of the three-layer system under realistic load and environmental excitations would be completed.

The response of the system to a repeated loading mode under constant environment has been studied deterministically, and can be found in references (13) and (14).

It is clear that the study presented in this thesis is essential for the study of damage and failure of pavement systems. This stage of damage progression and failure may be characterized as the secondary response stage as distinguished from the primary response stage presented in this work.

IV. CONCLUSIONS

It has been shown that simulation is a rather promising approach in dealing with problems that involve various degrees of uncertainty due to the variation in certain parameters in them. The highway pavement is a good example of these problems, and simulation provides a systematic approach for developing a meaningful probabilistic input-output relationship. Another advantage associated with the use of the simulation procedure discussed above is that it can handle any irregular shape of probability density function of the input parameters.

However, it is clear that in order to obtain a useful and accurate probabilistic output, the number of simulation experiments to be conducted has to be very large. A sensitivity study is needed to optimize the number of samples required for a specific problem (21). This means that high accuracy requires large computer time, which can be a major setback in the use of the method for simulation of the model.

Therefore, it seems that if the probability density functions of the input parameters have some standardized forms, a more realistic way to attack the problems will be that of using a closed form probabilistic solution. This closed form solution will provide very useful information regarding probabilistic properties of the output, such as the mean, the variance, coefficient of variation, etc., which are needed for purposes of design or further analysis. Since

this approach has proved useful in many applications in engineering, it seems that even if the shapes of the probability density functions are not that of some standardized forms, it can be approximated to fit a standardized theoretical form. The error involved here is hardly significant due to the high degree of uncertainty associated with the problem. Therefore, if the stochastic properties of the problems are known, an alternative formulation and solution of the problems based on the above techniques will be considerably more economical.

However, in the present analysis of response of the pavement system under static load conditions, it is not feasible to use closed form probabilistic solutions. The reason for this is due to the fact that the response terms are expressed numerically as a function of time. This means that the resulting response is not found in a single-valued form. Therefore, simulation is thought to be a reasonable approach in this analysis.

V. RECOMMENDATION FOR FUTURE WORK

5.1 Primary Response Model

An immediate extension for this approach would seem to study the system under realistic operational environment. Assuming that the system is linear, the classical input-output relation for linear systems may be expressed in the following form of a convolution integral:

$$y(t) = \int_{-\infty}^t h(t - \tau) \cdot x(\tau) d\tau \quad (5.1)$$

where $y(t)$ represents the response of the system,

$x(\tau)$ represents a history of the excitation function, and

$h(t - \tau)$ is a characterization function of the system, and

is usually called the "response function".

Equation (5.1) considers a linear system as a black box, characterized by its response function $h(t - \tau)$, and is being acted upon by a history of some excitations described by $x(\tau)$, as shown in figure (21).

This relation is a very useful one, and may be used as described below to study the behavior of the system under realistic operational environment.

Since the vehicular load is applied on the pavement in a repeated mode, it may be represented by some frequency wave such as a half sinusoid, or haversine function, etc., to describe the history of load. The response function $h(t - \tau)$ can be

represented by the response of the system to a unit step load. This response is obtained using the method described in the thesis. However, since the environment is an important factor in the response of the system, equation (5.1) may be modified to include another term $\phi(s)$ describing an arbitrary history of the environment (equation 5.2).

$$y_s(t) = \int_{-\infty}^t x(\tau) h[(t - \tau), \phi(s)] dt \quad (5.2)$$

The excitation function $x(\tau)$ may be treated as a random variable, i.e. with random amplitude and frequency of arrivals of vehicles. The response function can also be treated as a random variable with certain statistical properties and associated means and variances. A simulation study may then be conducted to study the cumulative response under this type of random excitation and environmental history. Damage may then be accumulated according to a certain damage rule, such as that suggested by Miner:

$$\sum_i D_i = \sum_i \frac{n_i}{N_f} = 1 \quad (5.3)$$

Healing and recovery may also be accounted for by some time-dependent process characterizing the system.

Different manifestation of damage can be predicted and accumulated by the suggested model, each by satisfying a certain criterion until an intolerable threshold is reached by one or a combination of more than one type of damage. At this limit,

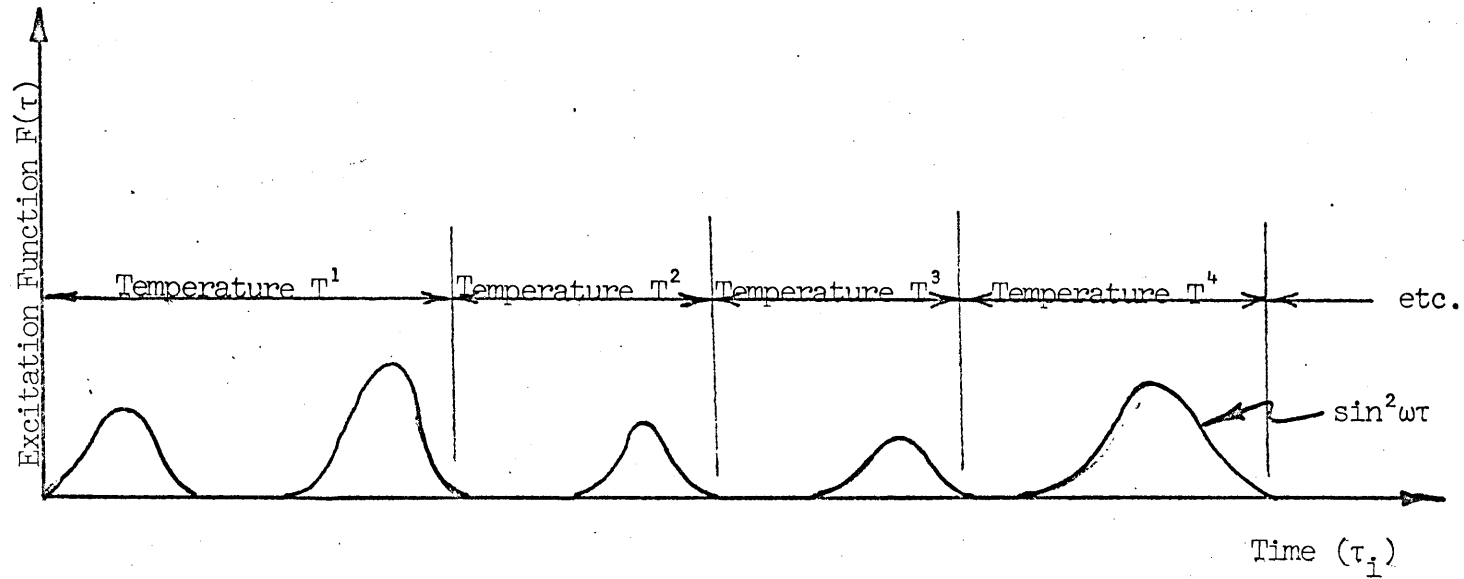


Figure 21. HISTORY OF REPEATED LOAD AND ENVIRONMENT ACTING ON A PAVEMENT SYSTEM.

the system is considered to have failed structurally.

Another alternative for simulation of this process, which has proved to be costly, is that of using a closed form probabilistic solution.

Equation (5.1) in fact describes a deterministic system. A probabilistic description in the time domain of the process may be written as:

$$R_y(\tau) = \int \int h(\alpha) h(\beta) R_x(\tau + \alpha - \beta) d\alpha d\beta \quad (5.4)$$

If the statistical properties of the pattern of load application as well as that of the environment are known, and if the statistical scatter of the material properties is also known, a probabilistic solution may be developed to yield the probabilistic information that will be provided otherwise by simulation.

5.2 Performance Prediction Model

Highway pavements belong to a class of structures which are identified as structure-sensitive systems. Structure-sensitive systems are those engineering systems in which damage or failure of a component results in a loss in the level of performance, rather than the abrupt incidence of total failure. For these systems, internal damage develops within the operational environment over a certain period of time, and failure is viewed as the ultimate conditions which result from the loss of performance.

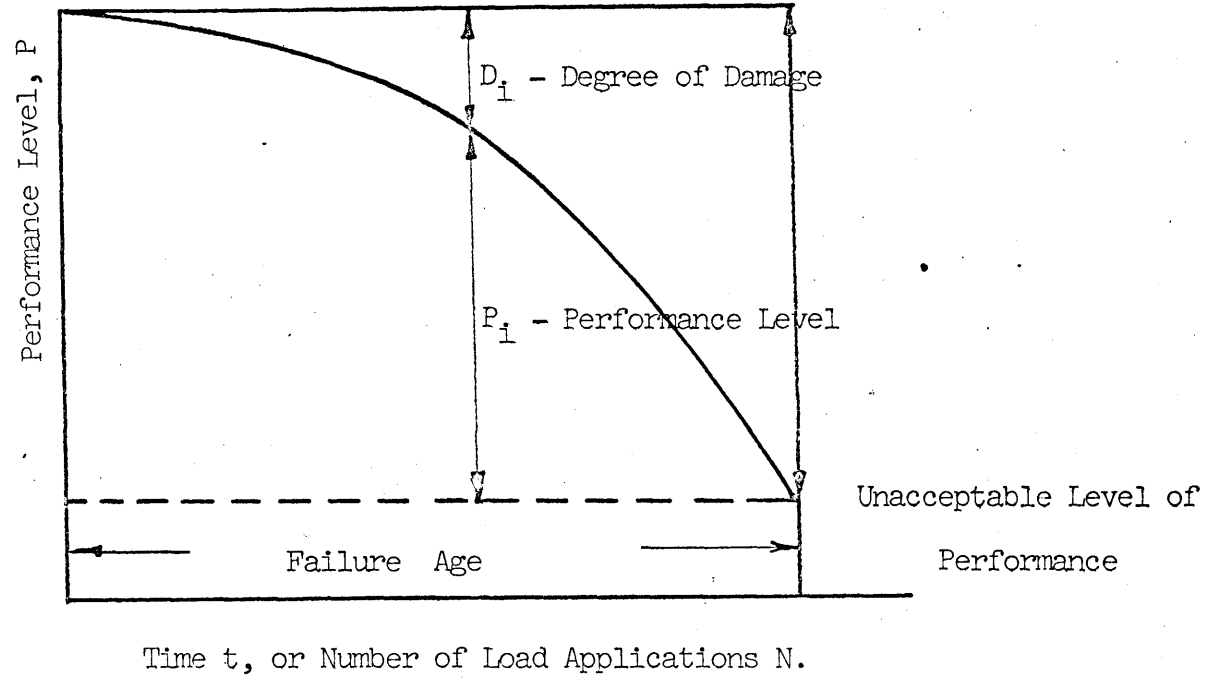


Figure 22. TWO - DIMENSIONAL SIMULATION OF THE PERFORMANCE OF A PAVEMENT STRUCTURE.

Failure, therefore, is the extent of damage which has been accumulated as a consequence of structural deterioration over a range of stress, strain, time, and environmental conditions in an operational environment.

The performance level of pavement system, as a structure-sensitive system, may be defined as the degree to which the stipulated functions of the system are executed within the environment. This level is, therefore, dependent on the history of the applied load and its distribution, on the quality of the construction materials used and their spatial distribution, on the history of the environment, and on the extent to which proper maintenance practices are executed over the entire life of the system.

Finally, damage in the structure may be defined as the extent of structural deterioration resulting in a loss in the performance of the system.

Figure (22) illustrates that the performance of the system diminishes in some way until an unacceptable level is attained. This behavior results from the combined action of the load and the environment during the operational life period of the system.

Therefore, performance, which is in this case the integrity level of the system at any time is one minus the amount of damage accumulated within that time.

$$P_i(t_i) = 1 - D_i(t_i) \quad (5.5)$$

Where D_i is the amount of damage accumulated from $y_s(t)$ in equation (5.2).

Since damage is probabilistic in nature, the performance level will be dependent on the temporal and spatial distribution of damage at any time during the life period of the system. Damage progression in highway pavements can be represented by a Markov process model. A Markov process, is one with the following properties:

$$P[X(n+1) = x_{n+1} \mid (X(1) = x_1) \quad (X(2) = x_2) \quad \dots \quad (X(n) = x_n)] = P[X(n+1) = x_{n+1} \mid X(n) = x_n] \quad (5.6)$$

This simply states that there is only one step dependence. The future state depends only on the current state, and the dependence of the future events on the past is of a particularly simple nature.

The transition of the state of the system may be represented by birth and death processes with the birth representing more damage due to cumulative response and aging effects, and death representing some level of maintenance introduced at that stage.

Each stage in the Markov chain will represent a certain level of damage, or otherwise performance level, accumulating over a period of time, in this case it may be a few months or one year,

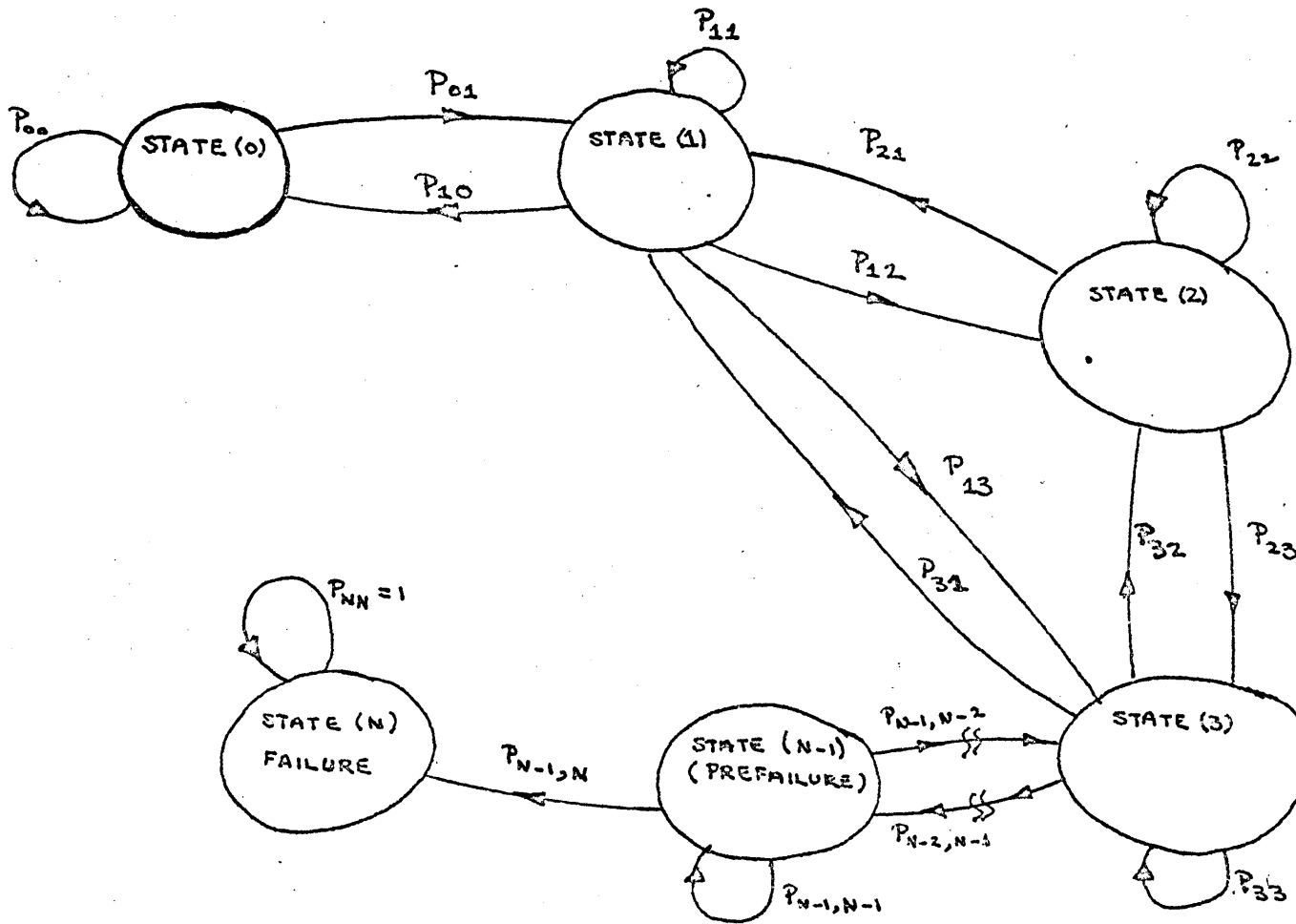


Figure 23. MARKOV PROCESS SIMULATION OF THE DAMAGE PROCESS IN A PAVEMENT SYSTEM

as the case may be. The transition probability matrix can then be established on this basis. The rewarding matrix will express the amount of damage or loss in performance that will be involved in the transition from one state to another. The final stage is one where failure takes place at, and can be reached when the damage reaches some intolerable limit, or when the performance reaches some unacceptable level, at which time the system is considered to have failed.

This analysis provides very useful information that can be used in design practices based on reliability criteria. A distribution of the life time of the system can be obtained by finding the distribution of the time to reach the final state which, in this case is a trapping state since the system is rendered unusable upon entering that state. The amount of maintenance required throughout the life of the system can also be predicted through the model, since maintenance will be responsible for a possible transition from one state to a previous one (filling a crack, a hole, etc.).

Therefore, an important factor is achieved also in this process, which is the introduction of maintenance prediction in the design process based on quantitative and scheduling estimation of the maintenance required throughout the life time of the system. From an economics point of view, this will also be very helpful in estimation of the expected values of construction as

well as operation of the highway system.

A schematic representation of this process is presented in figure (23).

5.3 Summary

Factors contributing to the initiation, propagation, and propagation of damage can be divided into three categories: (a) materials properties and pavement geometry, (b) load variables, and (c) climatic conditions. A substantial variability is associated with the measurement or prediction of each of these factors, thereby resulting in a stochastic nature of the response and behavior of pavement systems. To account for these variabilities, the damage model should be capable of yielding statistical estimates of the temporal and spatial distribution of the different modes of structural deterioration resulting from the action of load and environment throughout the service life of the system. A pavement system is represented by a three-layer Viscoelastic system describing its physical and geometrical properties. The load application can be represented by a Poisson process of random occurrence at a certain rate of arrival. Temperature, moisture, and other environmental variables may be assigned some statistical distribution of a standard type such as normal distribution, uniform distribution, etc.

Damage is accumulated due to repeated load action within the operational environment. A Markov process model can be used to describe the progression of damage over some relatively long periods of time. Each state in the Markov process defines a certain level of damage or performance. A transition matrix will provide the probabilities of the transition of the different states into others. The reward matrix will provide some quantitative measure of damage or loss in the performance level through the transition from one state to another. Maintenance practices will cause the transition of a certain state to a previous one, in other words, it will raise the level of performance or decrease the amount of damage in the system.

Failure is then the state of intolerable extent of damage or unacceptable level of performance.

The above analysis provides a realistic study of the behavior and performance of highway pavement systems based on realistic inputs and outputs of the system. The system is also characterized by a model which is based on a true representation of the physical behavior of the system as well as its geometrical properties.

The following features designate the above method of approach from the viewpoint of design practices:

1. Prediction of the distribution of the performance level of

the system at any period throughout its lifetime.

2. As a consequence of (1) above, maintenance estimates and scheduling will be based on more realistic grounds. This will facilitate incorporating maintenance in the design process as well as the economical analysis of costs of construction and operation of the system.
3. Prediction of the distribution of the lifetime of the pavement. This is very important in any design process as well as economical analysis, since resurfacing is required after this period.
4. All the above analyses are based on a probabilistic approach which accounts for the unpredictable occurrences of events, an approach which is more realistic and more reliable.

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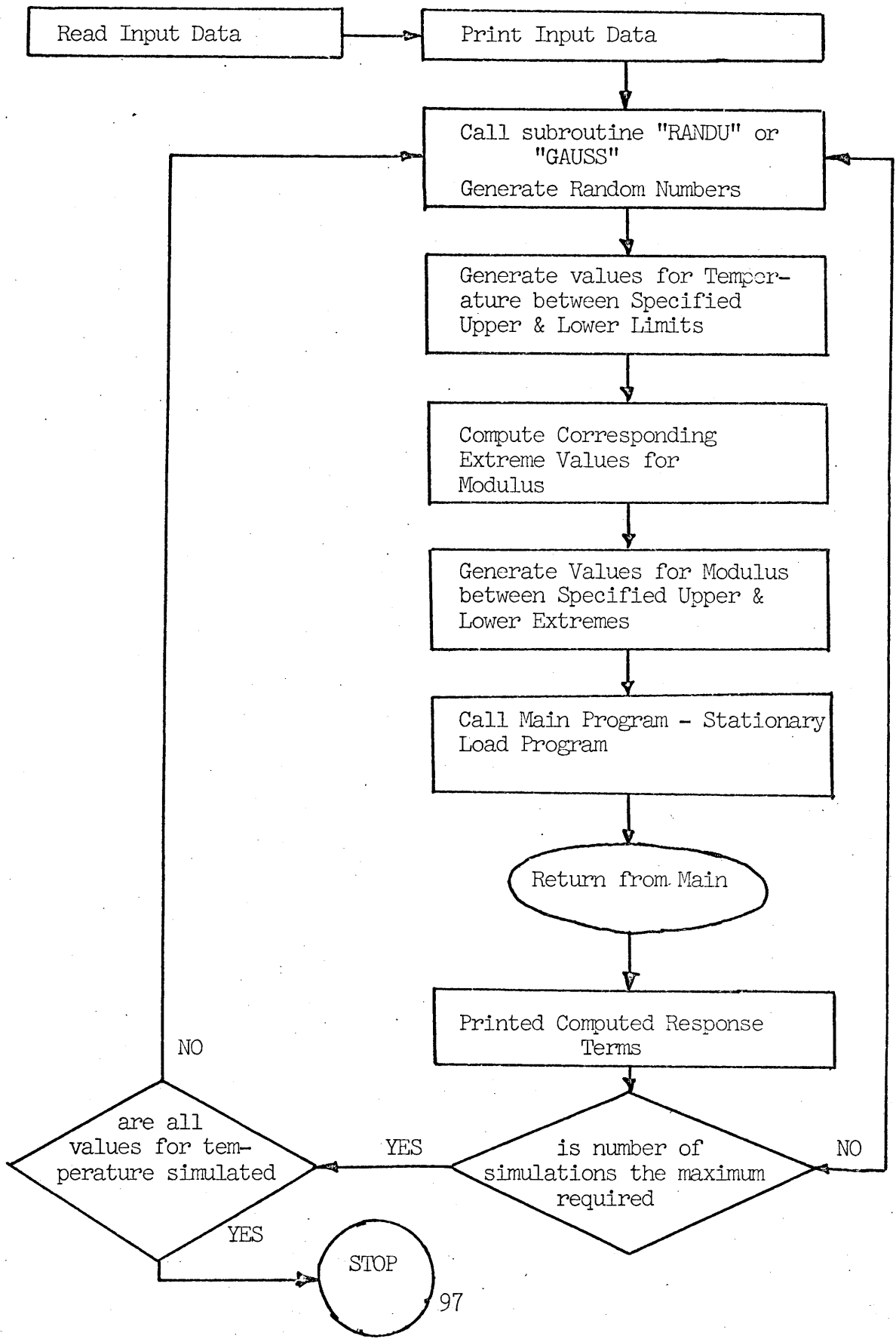
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APPENDIX I

COMPUTER PROGRAM

This Appendix Contains a Flow Chart and a Program Listing
of the Simulation Program for the Three-Layer Viscoelastic System.



C THIS IS THE MAIN PROGRAM FOR THE ANALYSIS OF A LINEAR VISCOELASTICMAIN0001
 C THREE-LAYER HALF-SPACE UNDER A UNIFORM CIRCULAR LOAD, FOR THE CASEMAIN0002
 C THAT THE MULTIPLE CONVOLUTION INTEGRALS ARE EVALUATED EXACTLY. MAIN0003
 C IN ADDITION TO THAT, THE PROGRAM USES THE MONTE CARLO SIMULATION MAIN0004
 C PROCEDURE TO GENERATE RANDOM NUMBERS AS REPRESENTATIVE SAMPLES MAIN0005
 C FOR THE VALUES OF THE RESPONSE TERMS DESIRED. THIS IS A STOCHAS- MAIN0006
 C TIC APPROACH TO PREDICT THE PROBABILITY THAT A DESIRED RESPONSE MAIN0007
 C TERM(BE IT STRESS, STRAIN, OR DEFLECTION) TAKES A CERTAIN VALUE. MAIN0008
 C THE PROGRAM TAKES INTO CONSIDERATION THE EFFECTS OF ONE INHERENT MAIN0009
 C VARIATION IN THE PROPERTIES OF THE MATERIAL(WHICH IS IN THIS CASE MAIN0010
 C THE CREEP COMPLIANCE OF THE DIFFERENT LAYERS CONSTITUTING THE MAIN0011
 C SYSTEM). IT ALSO TAKES INTO ACCOUNT THE EFFECT OF THE CONSTANTLY MAIN0012
 C VARYING ENVIRONMENTAL CONDITIONS(SUCH AS TEMPERATURE AND MOISTURE)MAIN0013
 C ON THE BEHAVIOR OF THE SYSTEM. MAIN0014
 C THE NECESSARY SUBROUTINES ARE VISCC, CNVIT,CNSTNT,SOLVE,AND TERPU.MAIN0015
 C ALSO NECESSARY IS THE REAL FUNCTION SUBPROGRAM JB. MAIN0016
 C THE INPUT IS IDER, ILAYER, IDEFLE, H, A, R, ZZ, ,NJJJ, DELTX, MAIN0017
 C DELXX, N, MNN, NS, NMONTH, IARB, T1(), T2(), THE VECTORS YL1(),MAIN0018
 C YL2(), YL3(), YU1(), YU2(), YU3(),A1, A2, A3, AND DELTA(). MAIN0019
 C YL2(), YL3(), YU1(), YU2(), YU3(),A1, A2, A3, TLIM, DELTA().MAIN0020
 C IDER IS A DUMMY FOR THE STRAINS, IST IS A DUMMY WHICH, TOGETHER MAIN0021
 C WITH IDEFLE DETERMINES WHICH STRESS, STRAIN OR DEFLECTION IS DESIRMAIN0022
 C IST IS 1 FOR NORMAL STRESS, NORMAL STRAIN OR NORMAL DEFLECTION, MAIN0023
 C IS 2 FOR SHEAR STRESS, RADIAL STRAIN OR RADIAL DEFLECTION, AND IS MAIN0024
 C 3 FOR RADIAL STRESS. H IS THE THICKNESS OF THE SECOND LAYER(THE MAIN0025
 C THICKNESS OF THE FIRST LAYER IS ONE). A IS THE RADIUS OF THE LOAD.MAIN0026
 C R IS THE OFF-SET AT WHICH THE RESPONSE IS DESIRED. ZZ IS THE DEPTHMAIN0027
 C AT WHICH THE SOLUTION IS DESIRED. ILAYER IS THE LAYER OF INTERESTMAIN0028
 C (1, 2, OR 3). IDEFLE IS POSITIVE IF THE DEFLECTION IS DESIRED, MAIN0029
 C ZERO FOR THE STRESSES, AND NEGATIVE IF THE STRAINS ARE DESIRED. MAIN0030
 C NJJJ IS AN INPUT TO THE SUBROUTINE SOLVE, AND IS EXPLAINED IN MAIN0031
 C DETAIL THERE. DELTX AND DELXX ARE INPUTS TO THE SUBROUTINE TIME MAIN0032
 C AND ARE EXPLAINED IN DETAIL THERE. N AND MNN ARE ALSO INPUT. N MAIN0033
 C IS THE NUMBER OF TERMS IN THE DIRICHLET SERIES REPRESENTATIONS OFMAIN0034
 C THE INPUT CREEP FUNCTONS. MNN IS THE NUMBER OF POINTS IN TIME AT MAIN0035
 C WHICH THE SOLUTION IS DESIRED. THE VECTORS YL1(), YL2(), YL3()MAIN0036

```

C ARE THE CONSTANTS FOR THE SERIES REPRESENTATIONS OF THE CREEP MAIN0037
C FUNCTIONS OF THE ASSUMED LOWER BOUND FOR THE FIRST, SECOND AND MAIN0038
C THIRD LAYERS RESPECTIVELY. WHILE THE VECTORS YU1( ), YU2( ), AND MAIN0039
C YU3( ), ARE THE SAME CONSTANTS FOR AN ASSUMED UPPER LIMIT(PROBABLY MAIN0040
C AT THE MOST FAVORATE MOISTURE CONDITIONS).. NS IS THE NUMBER OF MAIN0041
C SAMPLES TO BE DRAWN EACH MONTH. NMONTH IS THE NUMBER OF MONTHS MAIN0042
C OVER WHICH THE SIMULATION IS CONDUCTED. IARB IS AN ODD INTEGER NOT MAIN0043
C MORE THAN 7 CHARECTERS TO BE USED AS INPUT TO THE RANDOM NUMBER MAIN0044
C GENERATION SUBROUTINE, IT IS HOWEVER ADVISED TO USE A VALUE OF MAIN0045
C 65549 FOR A BETTER STASTICAL DISTRIBUTION OF THE GENERATED RANDOM MAIN0046
C NUMBERS. T1( ), AND T2( ) ARE VECTORS CONTAINING RESPECTIVELY THE MAIN0047
C LOWER AND UPPER LIMITS OF TEMPERATURS FOR EACH MCNTH OF THE YEAR. MAIN0048
C TLIM IS THE INITIAL VALUE OF TEMPERATURE AT WHICH BOTH THE UPPER MAIN0049
C AND LOWER CREEP CURVES HAVE BEEN MEASURED. NTEMP IS THE NUMBER OF MAIN0050
C TEMPERATURES USED IN COMPUTATIONS. HE RESULTS OF THE PROGRAM ARE MAIN0051
C THE DESIRED, STRAIN OR DISPLACEMENT AT EACH OF THE NNN TIMES. MAIN0052
66 DIMENSION G(20,3) MAIN0053
DIMENSION w(201) MAIN0054
DIMENSION E1(6),E2(6),E3(6),EA(360),EB(360),EC(360),C1(20,12),C2( MAIN0055
120,12),C3(20,12),P(20,12),EL1(6,20,12),EL2(6,20,12),EL3(6,20,12), MAIN0056
2EU1(6,20,12),EU2(6,20,12),EU3(6,20,12),EF(6,20,12),ES(6,20,12),ET( MAIN0057
36,20,12),T1(12),T2(12),YL1(6),YL2(6),YL3(6),YU1(6),YU2(6),YU3(6) MAIN0058
COMMON CC(8,20),DD(8,20),FF(8,20),T(201),DELTA(20) MAIN0059
COMMON/NAME/IDER, ITEM,NNN,IST,W,IDEFLE,G,H,A,R,ZZ,ILAYER, MAIN0060
1 N,NJJJ,DELXX,DELTX MAIN0061
EQUIVALENCE(G(1,1),E1(1)),(G(1,2),E2(1)),(G(1,3),E3(1)) MAIN0062
DO 222 III=1,100 MAIN0063
C THE LOOP THROUGH 726 ALLOWS MULTIPLE SETS OF DATA TO BE RUN. MAIN0064
READ(5,9)IDER MAIN0065
WRITE(6,88)IDER MAIN0066
88 FORMAT(7H IDER = 110) MAIN0067
ITEM=IDER MAIN0068
C ITEM IS A DUMMY AND IS GIVEN THE VALUES OF -1, 0, 1 FOR THE NORMA MAIN0069
C STRAIN, CIRCUMFERENTIAL STRAIN, AND RADIAL STRAIN RESPECTIVELY. MAIN0070
READ(5,20) ILAYER,IDEFLE MAIN0071
READ(5,52)IST,H,A,R,ZZ MAIN0072

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100

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20 FORMAT(5I5) MAIN0073
52 FORMAT(I5/5F10.5) MAIN0074
WRITE(6,210)IST,ILAYER,IDEFLE,H,A,R,ZZ MAIN0075
210 FORMAT(7H IST = I10/10H ILAYER = I10/10H IDEFLE = I10/ MAIN0076
15H H = F10.5/5H A = F10.5/5H R = F10.5/6H ZZ = F10.5) MAIN0077
READ(5,20)NJJJ MAIN0078
C NJ AND NJJ ARE INPUTS TO THE SUBROUTINE SOLVE. THEY HAVE NO SIG- MAIN0079
C NIFICANCE IN THE PRESENT USE OF THAT SUBROUTINE AND ARE GIVEN MAIN0080
C ARBITRARY VALUES. MAIN0081
READ(5,1) DELTX,DELXX MAIN0082
READ(5,20)N,NNN MAIN0083
1 FORMAT(6F10.5) MAIN0084
3344 FORMAT(4I5) MAIN0085
READ(5,3344)NS,NMONTH,NTEMP,IARB MAIN0086
WRITE(6,7998)NS,NMONTH MAIN0087
7998 FORMAT(4X,'NS = ',I10,' NMONTH = ',I5) MAIN0088
READ(5,700)(T1(II),II=1,NTEMP) MAIN0089
READ(5,700)(T2(II),II=1,NTEMP) MAIN0090
WRITE(6,701)(T1(II),II=1,NTEMP) MAIN0091
WRITE(6,702)(T2(II),II=1,NTEMP) MAIN0092
700 FORMAT(12F6.2) MAIN0093
701 FORMAT(4X,'LOWER MONTHLY TEMPERATURES = ',12F7.2) MAIN0094
702 FORMAT(4X,'UPPER MONTHLY TEMPERATURES = ',12F7.2) MAIN0095
DO 797 KL=1,N MAIN0096
E1(KL)=0.0 MAIN0097
E2(KL)=0.0 MAIN0098
797 E3(KL)=0.0 MAIN0099
DO 232 J=1,NS MAIN0100
C1(J,I)=0.0 MAIN0101
C2(J,I)=0.0 MAIN0102
C3(J,I)=0.0 MAIN0103
P(J,I)=0.0 MAIN0104
DO 232 K=1,N MAIN0105
EL1(K,J,I)=0.0 MAIN0106
EL2(K,J,I)=0.0 MAIN0107
EL3(K,J,I)=0.0 MAIN0108
```


	EU1(K,J,I)=0.0	MAIN0109
	EU2(K,J,I)=0.0	MAIN0110
	EU3(K,J,I)=0.0	MAIN0111
	EF(K,J,I)=0.0	MAIN0112
	ES(K,J,I)=0.0	MAIN0113
232	ET(K,J,I)=0.0	MAIN0114
	KIL=NS*NMOUTH*N	MAIN0115
	DO 9001 LLL=1,KIL	MAIN0116
	EA(LLL)=0.0	MAIN0117
	EB(LLL)=0.0	MAIN0118
9001	EC(LLL)=0.0	MAIN0119
	READ(5,703) (YL1(KL),KL=1,N)	MAIN0120
	READ(5,703) (YL2(KL),KL=1,N)	MAIN0121
	READ(5,703) (YL3(KL),KL=1,N)	MAIN0122
	READ(5,703) (YU1(KL),KL=1,N)	MAIN0123
	READ(5,703) (YU2(KL),KL=1,N)	MAIN0124
	READ(5,703) (YU3(KL),KL=1,N)	MAIN0125
101	703 FORMAT(6F10.4)	MAIN0126
	READ(5,706)A1,A2,A3	MAIN0127
	706 FORMAT(3F10.5)	MAIN0128
	WRITE(6,707)A1,A2,A3	MAIN0129
	707 FORMAT(4X,'EXPONENT OF MODULUS VS TEMPERATURE CURVE = ',3F10.5)	MAIN0130
	READ(5,1112)TLIM	MAIN0131
C	TLIM IS THE VALUE OF THE TEMPERATURE AT WHICH THE INPUT CREEP	MAIN0132
C	FUNCTIONS OF THE DIFFERENT LAYERS ARE ORIGINALLY EVALUATED.	MAIN0133
	1112 FORMAT(F7.2)	MAIN0134
	READ(5,50) (DELTA(KL),KL=1,N)	MAIN0135
50	FORMAT(8F10.5)	MAIN0136
C	DELTA(2) IS AN INPUT TO SUBROUTINE TIME, AND IC EXPLAINED THERE.	MAIN0137
	READ(5,9)IDNST	MAIN0138
C	IDNST IS A DUMMY VARIABLE WHICH GIVES THE OPTION FOR THE SHAPE OF	MAIN0139
C	THE DENSITY FUNCTION TO BE USED FOR BOTH THE CREEP FUNCTIONS OF	MAIN0140
C	THE LAYERS AND THE TEMPERATURE OR ENVIRONMENTAL FUNCTION. IF THE	MAIN0141
C	VALUE OF THIS VARIABLE IS ZERO, THEN THE DENSITY FUNCTIONS ARE	MAIN0142
C	NORMALLY DISTRIBUTED, OTHERWISE, THE DENSITY FUNCTIONS ARE UNIFORM-	MAIN0143
C	MLY DISTRIBUTED(I.E. RECTANGULAR DISTRIBUTION).	MAIN0144

	IF(IONST)215,216,215	MAIN0145
216	IZ=IARB	MAIN0146
C	IZ IS AN ODD INTEGER BETWEEN ZERO AND 2**39. IT IS USED AS INPUT	MAIN0147
C	TO LIBRARY SUBROUTINE GAUSS.	MAIN0148
	WRITE(6,1120)	MAIN0149
1120	FORMAT(4X,'THE DENSITY FUNCTIONS ARE NORMALLY DISTRIBUTED IN THE	MAIN0150
	1 FOLLOWING SIMULATION')	MAIN0151
C	THE STEPS THROUGH 9791 ARE TO FIND THE MEAN AND STANDARD DEVIATION	MAIN0152
C	OF NORMALLY DISTRIBUTED TEMPERATURE OBSERVATIONS OVER ONE YEAR.	MAIN0153
	AA=0.0	MAIN0154
	AB=0.0	MAIN0155
	DO 9790 I=1,NTEMP	MAIN0156
	AA=AA+T1(I)+T2(I)	MAIN0157
9790	AB=AB+T1(I)**2+T2(I)**2	MAIN0158
	YY=NTEMP**2.0-1.0	MAIN0159
	AM1=AA/(YY+1.)	MAIN0160
	SS1=AB/YY-AM1*AA/YY	MAIN0161
C	SS1 IS THE VARIANCE OF A NORMALLY DISTRIBUTED XX-OBSERVATION.	MAIN0162
	S1=SS1**0.50	MAIN0163
C	S1 IS THE STANDARD DEVIATION OF THE NORMAL DISTRIBUTION FOR THE	MAIN0164
C	TEMPERATURE, AND AM1 IS THE MEAN FOR THE SAME DISTRIBUTION OVER	MAIN0165
C	A ONE-YEAR PERIOD TAKEN FROM SEVERAL OBSERVATIONS OF THE WEATHER	MAIN0166
C	BUREAU. THOSE VALUES OF THE TEMPERATURE, HOWEVER, SHOULD BE THE	MAIN0167
C	THE REAL TEMPERATURES IN THE LAYERS OF THE SYSTEM, WHERE THE	MAIN0168
C	VALUE AND THE DISTRIBUTION OF THE TEMPERATURE AND OTHER FACTORS	MAIN0169
C	ARE DIFFERENT FOR EACH LAYER.	MAIN0170
	DEV=0.33*AM1	MAIN0171
	IF(S1-DEV)1125,1125,1126	MAIN0172
1126	S1=DEV	MAIN0173
C	THIS RESTRICTION IS IMPOSED BECAUSE OF THE TYPE OF DATA USED HERE	MAIN0174
C	HOWEVER, IF THE DISTRIBUTION IS KNOWN AND THE DATA ARE REAL, THIS	MAIN0175
C	RESTRICTION SHOULD BE REMOVED FROM THE PROGRAM.	MAIN0176
1125	GO TO 899	MAIN0177
215	IY=IARB	MAIN0178
C	IY IS SIMILAR TO IZ. IT IS USED AS AN INPUT TO SUBROUTINE RANDU.	MAIN0179
	WRITE(6,1130)	MAIN0180

1130	FORMAT(4X,'THE DENSITY FUNCTIONS ARE UNIFCRMLY DISTRIBUTED IN THE	MAIN0181
	1FOLLOWING SIMULATION')	MAIN0182
899	DC 222 J=1,NS	MAIN0183
	DO 222 I=1,NMONTM	MAIN0184
	KKJ=1	MAIN0185
	IF(IDNST)708,919,708	MAIN0186
708	IX=IY	MAIN0187
	CALL RANDU(IX,IY,DIN)	MAIN0188
C	RANDU IS A SUBROUTINE WHICH GENERATES A SET OF UNIFCRMLY DISTRIBU-	MAIN0189
C	TED RANDOM NUMBERS.	MAIN0190
	P(J,I)=T1(I)+DIN*(T2(I)-T1(I))	MAIN0191
	GO TO 717	MAIN0192
919	CALL GAUSS(IZ,S1,AM1,P(J,I))	MAIN0193
C	GAUSS IS A SUBROUTINE WHICH GENERATES A SET OF NORMALLY DISTRIB-	MAIN0194
C	UTED RANDOM VARIABLES. THIS SUBROUTINE IS CALLED FROM THE IBM/360	MAIN0195
C	SYSTEM SCIENTIFIC SUBROUTINE PACKAGE.	MAIN0196
717	WRITE(6,713)P(J,I)	MAIN0197
713	FORMAT(4X,'TEMPERATURE GENERATED IS = ',F7.2)	MAIN0198
	C1(J,I)=-A1*(P(J,I)-TLIM)	MAIN0199
	C2(J,I)=-A2*(P(J,I)-TLIM)	MAIN0200
	C3(J,I)=-A3*(P(J,I)-TLIM)	MAIN0201
	DO 722 K=1,N	MAIN0202
755	EL1(K,J,I)=YL1(K)*EXP(C1(J,I))	MAIN0203
	EL2(K,J,I)=YL2(K)*EXP(C2(J,I))	MAIN0204
	EL3(K,J,I)=YL3(K)*EXP(C3(J,I))	MAIN0205
	EU1(K,J,I)=YU1(K)*EXP(C1(J,I))	MAIN0206
	EU2(K,J,I)=YU2(K)*EXP(C2(J,I))	MAIN0207
	EU3(K,J,I)=YU3(K)*EXP(C3(J,I))	MAIN0208
	IF(IDNST)920,921,920	MAIN0209
920	EF(K,J,I)=EL1(K,J,I)+DIN*(EU1(K,J,I)-EL1(K,J,I))	MAIN0210
	ES(K,J,I)=EL2(K,J,I)+DIN*(EU2(K,J,I)-EL2(K,J,I))	MAIN0211
	ET(K,J,I)=EL3(K,J,I)+DIN*(EU3(K,J,I)-EL3(K,J,I))	MAIN0212
	GO TO 795	MAIN0213
921	AM2=0.50*(EL1(K,J,I)+EU1(K,J,I))	MAIN0214
	S2=0.15*AM2	MAIN0215
	CALL GAUSS(IZ,S2,AM2,EF(K,J,I))	MAIN0216

104

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AM3=0.50*(EL2(K,J,I)+EU2(K,J,I))
S3=0.15*AM3
CALL GAUSS(IZ,S3,AM3,ES(K,J,I))
AM4=0.50*(EL3(K,J,I)+EU3(K,J,I))
S4=0.15*AM4
CALL GAUSS(IZ,S4,AM4,ET(K,J,I))
795 II=K+(J-1)*N0+(I-1)*NS*6
EA(II)=EF(K,J,I)
EB(II)=ES(K,J,I)
EC(II)=ET(K,J,I)
IF(KKJ-N)984,984,987
987 KKJ=1
984 E1(KKJ)=EA(II)
E2(KKJ)=EB(II)
E3(KKJ)=EC(II)
KKJ=KKJ+1
722 CCNTINUE
WRITE(6,721)(E1(KKJ),KKJ=1,N)
WRITE(6,721)(E2(KKJ),KKJ=1,N)
WRITE(6,721)(E3(KKJ),KKJ=1,N)
C E1( ), E2( ), E3( ) ARE THE SELECTED SAMPLES FOR THE COEFFICIENTS
C OF THE CREEP FUNCTION FOR THIS PARTICULAR SIMULATION PROCESS.
721 FORMAT(4X'CREEP COEFFICIENTS OF THE LAYERS = ',6F10.5)
725 IF(ITEM)7,7,96
96 CALL VISCO
DO 8865 L=1,NNN
8865 WRITE(6,98)T(L),W(L)
98 FORMAT(8H TIME = E15.8,17H RADIAL STRAIN = E15.8)
GO TO 222
7 CALL VISCO
IF(IDEFL)10,12,11
10 IF(ITEM)13,14,14
13 DO 15 L=1,NNN
15 WRITE(6,97)T(L),W(L)
97 FORMAT(8H TIME = E15.8,17H NORMAL STRAIN = E15.8)
GO TO 222

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MAIN0217
MAIN0218
MAIN0219
MAIN0220
MAIN0221
MAIN0222
MAIN0223
MAIN0224
MAIN0225
MAIN0226
MAIN0227
MAIN0228
MAIN0229
MAIN0230
MAIN0231
MAIN0232
MAIN0233
MAIN0234
MAIN0235
MAIN0236
MAIN0237
MAIN0238
MAIN0239
MAIN0240
MAIN0241
MAIN0242
MAIN0243
MAIN0244
MAIN0245
MAIN0246
MAIN0247
MAIN0248
MAIN0249
MAIN0250
MAIN0251
MAIN0252

```

14 DO 17 L=1,NNN
17 WRITE(6,86)T(L),W(L)
36 FORMAT(9H TIME = E15.8,26H CIRCUMFERENTIAL STRAIN = E15.8)
GO TO 222
11 IF(IST-2)45,46,46
45 DO 120 L=1,NNN
120 WRITE(6,77)T(L),W(L)
77 FORMAT(8H TIME = E15.8,21H NORMAL DEFLECTION = E15.8)
GO TO 222
46 DO 121 L=1,NNN
121 WRITE(6,95)T(L),W(L)
95 FORMAT(8H TIME = E15.8,21H RADIAL DEFLECTION = E15.8)
GO TO 222
12 IF(IST-2)44,43,42
44 DO 122 L=1,NNN
122 WRITE(6,452)T(L),W(L)
105 452 FORMAT(8H TIME = E15.8,17H NORMAL STRESS = E15.8)
GO TO 222
43 DO 231 L=1,NNN
231 WRITE(6,355)T(L),W(L)
355 FORMAT(8H TIME = E15.8,16H SHEAR STRESS = E15.8)
GO TO 222
42 DO 111 L=1,NNN
111 WRITE(6,777)T(L),W(L)
777 FORMAT(8H TIME = E15.8,17H RADIAL STRESS = E15.8)
9 FORMAT(I5)
222 CONTINUE
STOP
END

MAIN0253
MAIN0254
MAIN0255
MAIN0256
MAIN0257
MAIN0258
MAIN0259
MAIN0260
MAIN0261
MAIN0262
MAIN0263
MAIN0264
MAIN0265
MAIN0266
MAIN0266
MAIN0267
MAIN0268
MAIN0269
MAIN0270
MAIN0271
MAIN0272
MAIN0273
MAIN0274
MAIN0275
MAIN0276
MAIN0277
MAIN0278
MAIN0279
MAIN0280
MAIN0281

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SUBROUTINE VISCO
COMMON/MADE/IDER, ITEM, NNN, IST, W, IDEFLE, G, F, A, R, ZZ, ILAYER,
1  N, NJJJ, DELXX, DELTX
COMMON CC(8,20), DC(8,20), FF(8,20), T(201), DELTA(20)
DIMENSION E1(20), E2(20), E3(20), G(20,3), BT1(8,20), BT2(8,20),
1  BT3(8,20), B1(8,20,18), B2(8,20,18), B3(8,20,18), B(8,20,18,3),
2  EM(13), SII(13,201), SIII(13,201), MTX(2,2,3,3), IU(3,3),
3  MTXM(3,3), MTXM(3,3), MXS(3,3), W(201)
EQUIVALENCE(G(1,1), E1(1)), (G(1,2), E2(1)), (G(1,3), E3(1))
EQUIVALENCE(B(1), B1(1)), (B(1,1,1,2), B2(1)), (B(1,1,1,3), B3(1))
DATA MXS/1,2,2,3,3,3,3,3,3/
DATA EM/0.,.2.,.4.,.7,1.,2.,3.,4.,5.,6.,7.,8.,9./
DATA MTX/16,0,7,0,3,1,8,0,8,0,4,2,9,0,6,5/
DATA IU/1,5,3,3,6,6,2,0,6/
DATA MTXM/0,0,0,0,1,0,1,2,2/, MTXM/1,2,2,1,2,1,1,2,2/
REAL JB
DIMENSION BES(91), BESS(91)
ITYP=3
IF(IDEFLE.EQ.0) ITYP=1
IF(IDEFLE.EQ.1) ITYP=2
IDEN=1
IF(IST.EQ.1) GO TO 1
IDEN=2
IF(IST.EQ.2.AND.IDEFLE.EQ.0) IDEN=3
IF(IST.EQ.2.AND.IDEFLE.EQ.-1.AND.ITEM.EQ.0) IDEN=3
1  CONTINUE
C-----CALCULATE TIMES FOR WHICH A SOLUTION IS DESIRED
EX=10.**DELXX
T(1)=0.
T(2)=10.**(DELTX+DELXX)
DO 7 K=3,NNN
7  T(K)=T(K-1)*EX
IOWA=IO(ITYP, IDEN)
C-----CALCULATE CONVOLUTION INTEGRALS
DO 20 I1=1,2
DO 10 I=1,N

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VISCO001
VISCO002
VISCO003
VISCO004
VISCO005
VISCO006
VISCO007
VISCO008
VISCO009
VISCO010
VISCO011
VISCO012
VISCO013
VISCO014
VISCO015
VISCO016
VISCO017
VISCO018
VISCO019
VISCO020
VISCO021
VISCO022
VISCO023
VISCO024
VISCO025
VISCO026
VISCO027
VISCO028
VISCO029
VISCO030
VISCO031
VISCO032
VISCO033
VISCO034
VISCO035
VISCO036

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10      BT1(1,I)=G(I,I1)
      DO 20 I2=I1,2
      CALL CNVINT(BT2,BT1,G(1,I2),N,1)
      DO 20 I3= 2,3
      CALL CNVINT(BT3,BT2,G(1,I3),N,2)
      DO 20 I4=I3,3
20     CALL CNVINT(B1(1,1,MTX(I1,I2,I3,I4)),BT3,G(1,I4),N,3)
      NT=9*MINO(2,ILAYER)
      NP=4
      NPP=4
      IF(ILAYER.EQ.1) GO TO 40
      DO 30 I=1,10,9
      DO 30 J=1,9
30     CALL CNVINT(B2(1,1,J+I-1),B1(1,1,J),G(1,3-MINO(2,I)),N,4)
      NP=5
      NPP=5
40     IF(ITYP.EQ.1) GO TO 60
      DO 50 I=1,NT
50     CALL CNVINT(B3(1,1,I),B(1,1,I,NP-3),G(1,ILAYER),N,NP)
      NPP=NPP+1
60     DO 70 K=1,13
C-----DO FOR 13 VALUES OF M...
      CALL CNSTNT(EM(K),H,ZZ,ILAYER,NT,NP,NPP,N,IOWA,B(1,1,1,NP-3),
1     B(1,1,1,MXS(ILAYER,ITYP)))
      CALL SOLVE(SII,CC,FF,K,N,NP,NPP,NNN,NJJJ)
      IF(ITYP.NE.1.OR.IDEN.NE.2) GO TO 70
      CALL SOLVE(SIII,DD,FF,K,N,NP,NPP,NNN,NJJJ)
70     CONTINUE
C-----CALCULATE BESSEL MULTIPLIERS
      RJ=0.
      DO 75 J=1,91
      BES(J)=JB(MTXN(ITYP,IDEN),RJ,R)*JB(MTXM(ITYP,IDEN),A,RJ)
      IF(IDEN.EQ.2.AND.ITYP.NE.2)BESS(J)=JB(2,RJ,R)*JB(2,A,RJ)
75     RJ=RJ+.1
C-----FOR EACH VALUE OF TIME COMPUTE INTEGRALS W.R.T. M
      IF(IDEN.EQ.2.AND.ITYP.NE.2) GO TO 100

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VISC0037
VISC0038
VISC0039
VISC0040
VISC0041
VISC0042
VISC0043
VISC0044
VISC0045
VISC0046
VISC0047
VISC0048
VISC0049
VISC0050
VISC0051
VISC0052
VISC0053
VISC0054
VISC0055
VISC0056
VISC0057
VISC0058
VISC0059
VISC0060
VISC0061
VISC0062
VISC0063
VISC0064
VISC0065
VISC0066
VISC0067
VISC0068
VISC0069
VISC0070
VISC0071
VISC0072

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      DO 90 I=1,NNN
        DO 80 J=4,13
80      IF(SII(J-1,I)*SII(J,I).LE.0.) SII(J,I)=0.
90      W(I)=TERPO(SII(1,I),BES)*A
        RETURN
100     DO 110 I=1,NNN
        W1=TERPO(SII(1,I),BES)
        IF(ITYP.EQ.1) W2=TERPO(SIII(1,I),BESS)
        IF(ITYP.NE.1) W2=-TERPO(SII(1,I),BESS)
110     W(I)=A*(W1+W2)
        RETURN
        END
```

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VISC0073
VISC0074
VISC0075
VISC0076
VISC0077
VISC0078
VISC0079
VISC0080
VISC0081
VISC0082
VISC0083
VISC0084
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SUBROUTINE CNVINT(A,B,G,N,M)
C-----THIS CALCULATES A CONVOLUTION INTEGRAL EXACTLY
C   A IS THE RESULT
C   B IS THE ARGUMENT
C   G IS THE CREEP FUNCTION
C   N IS THE LENGTH OF THE SERIES
C   M IS (DEGREE OF POLYNOMIALS OF B)+1
DIMENSION A(8,20),B(8,20),G(20)
COMMON CC(8,20),DD(8,20),FF(8,20),T(201),DELTA(20)
C-----EVALUATE CREEP FUNCTION AT ZERO
ZER=0.
DO 5 I=1,N
5   ZER=ZER+G(I)
   MSUC=M+1
   DO 100 L=1,MSUC
   DO 100 J=1,N
109 C-----CALCULATE A(L,J)
   RES=0.
   IF(L.NE.1) GO TO 25
   SUBT=0.
   DO 10 I=1,N
   IF(I.EQ.J) GO TO 10
   DELS=1./((DELTA(I)-DELTA(J)))
   DO 15 K=1,M
   SUBT=SUBT+B(K,I)*DELS
15   DELS=DELS*K/((DELTA(I)-DELTA(J)))
10   CONTINUE
   RES=RES-SUBT*G(J)*DELTA(J)
   GO TO 20
25   RES=RES-B(L-1,J)*DELTA(J)*G(J)/(L-1)
20   IF(L.EQ.MSUC) GO TO 100
   SUBT=0.
   DO 40 I=1,N
   IF(I.EQ.J) GO TO 40
   SSUB=0.
   DELS=1./((DELTA(J)-DELTA(I)))

```

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CNVI0001
CNVI0002
CNVI0003
CNVI0004
CNVI0005
CNVI0006
CNVI0007
CNVI0008
CNVI0009
CNVI0010
CNVI0011
CNVI0012
CNVI0013
CNVI0014
CNVI0015
CNVI0016
CNVI0017
CNVI0018
CNVI0019
CNVI0020
CNVI0021
CNVI0022
CNVI0023
CNVI0024
CNVI0025
CNVI0026
CNVI0027
CNVI0028
CNVI0029
CNVI0030
CNVI0031
CNVI0032
CNVI0033
CNVI0034
CNVI0035
CNVI0036

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```
      DO 30 K=L,M
      SSUB=SSUB+B(K,J)*DELS
30     DELS=DELS*K/(DELTA(J)-DELTA(I))
      SUBT=SUBT+SSUB*G(I)*DELTA(I)
40     CONTINUE
      RES=RES+SUBT+ZER*B(L,J)
100    A(L,J)=RES
      RETURN
      END
```

```
CNVI0037
CNVI0038
CNVI0039
CNVI0040
CNVI0041
CNVI0042
CNVI0043
CNVI0044
CNVI0045
```


G15=.5- G0
 G16=-G15
 G17=.5+ G3
 G18=-G17
 G19=.5+ G4
 G20=.5- G4
 Z4=DEXP(2.*S)
 G27=2.*Z4
 G28=(1.+2.*EM*H)*Z4
 G21=G27*G7-G28*G2+G1
 G22=G27*G8+G28*G2-G1
 G23=G27*G9+G28*G13+G17
 G24=G27*G10+G28*G14+G18
 G25=G27*G11+G28*G15+G19
 G26=G27*G12+G28*G16+G20
 G35=(1.-2.*S)*Z4
 G36=-2.*S*S*Z4
 G29=G35*G7+G7-G36*G2
 G30=G35*G8-G8+G36*G2
 G31=G35*G9+G9+G36*G13
 G32=G35*G10-G10+G36*G14
 G33=G35*G11+G11+G36*G15
 G34=G35*G12-G12+G36*G16
 L=0
 Z5=DEXP(S)
 Z6=DEXP(-S)
 G53=Z5
 G54=-Z6
 G55=S*Z5
 G56=-S*Z6
 G37=G53
 G38=G54
 G39=G55
 G40=G56
 3 G41=G37*G7+G38*G7-G39* G2+G40 *G1
 G42=-(G38*G29+G40)*G21)

CNST0037
 CNST0038
 CNST0039
 CNST0040
 CNST0041
 CNST0042
 CNST0043
 CNST0044
 CNST0045
 CNST0046
 CNST0047
 CNST0048
 CNST0049
 CNST0050
 CNST0051
 CNST0052
 CNST0053
 CNST0054
 CNST0055
 CNST0056
 CNST0057
 CNST0058
 CNST0059
 CNST0060
 CNST0061
 CNST0062
 CNST0063
 CNST0064
 CNST0065
 CNST0066
 CNST0067
 CNST0068
 CNST0069
 CNST0070
 CNST0071
 CNST0072

G43=G37*G8-G38*G8+G39 *G2-G40 *G1
G44=-(G38*G30+G40*G22)
G45=G37*G9+G38*G9+G39*G13+G40*G17
G46=-(G38*G31+G40*G23)
G47=G37*G10-G38*G10+G39*G14+G40*G18
G48=-(G38*G32+G40*G24)
G49=G37*G11+G38*G11+G39*G15+G40*G19
G50=-G38*G33-G40*G25
G51=G37*G12-G38*G12+G39*G16+G40*G20
G52=-G38*G34-G40*G26
IF(L)1,1,2

1 L=5

G57=G41
G58=G42
G59=G43
G60=G44
G61=G45
G62=G46
G63=G47
G64=G48
G65=G49
G66=G50
G67=G51
G68=G52
G38=-G38
G39=(1.+S)*Z5
G40=-(1.-S)*Z6
GO TU 3

2 A1=G45
A2=G46
A3=G47
A4=G48
A5=G65
A6=G66
A7=G67
A8=G68

CNST0073
CNST0074
CNST0075
CNST0076
CNST0077
CNST0078
CNST0079
CNST0080
CNST0081
CNST0082
CNST0083
CNST0084
CNST0085
CNST0086
CNST0087
CNST0088
CNST0089
CNST0090
CNST0091
CNST0092
CNST0093
CNST0094
CNST0095
CNST0096
CNST0097
CNST0098
CNST0099
CNST0100
CNST0101
CNST0102
CNST0103
CNST0104
CNST0105
CNST0106
CNST0107
CNST0108

B1=G49
B2=G50
B3=G51
B4=G52
B5=G61
B6=G62
B7=G63
B8=G64

8 C(1)=A1*A5-B1*B5
C(2)=A2*A5+A1*A6-B2*B5-B1*B6
C(3)=A3*A5+A1*A7-B3*B5-B1*B7
C(4)=A4*A5+A3*A6+A2*A7+A1*A8-B4*B5-B3*B6-B2*B7-B1*B8
C(5)=A2*A6-B2*B6
C(6)=A4*A6+A2*A8-B4*B6-B2*B8
C(7)=A3*A7-B3*B7
C(8)=A4*A7+A3*A8-B4*B7-B3*B8
C(9)=A4*A8-B4*B8

IF(L)4,5,6

6 DO 7 I=1,9

C THE V(I) TERMS ARE THE THETA(I) TERMS OF THE TEXT

7 V(I)=C(I)

A1=G49
A2=G50
A3=G51
A4=G52
A5=G57
A6=G58
A7=G59
A8=G60
B1=G41
B2=G42
B3=G43
B4=G44
B5=G65
B6=G66
B7=G67

CNST0109
CNST0110
CNST0111
CNST0112
CNST0113
CNST0114
CNST0115
CNST0116
CNST0117
CNST0118
CNST0119
CNST0120
CNST0121
CNST0122
CNST0123
CNST0124
CNST0125
CNST0126
CNST0127
CNST0128
CNST0129
CNST0130
CNST0131
CNST0132
CNST0133
CNST0134
CNST0135
CNST0136
CNST0137
CNST0138
CNST0139
CNST0140
CNST0141
CNST0142
CNST0143
CNST0144

```

B8=G68
L=0
GO TO 8
5 L=-5
DO 9 I=1,9
9 Q(3,1,I)=C(I)
A1=G61
A2=G62
A3=G63
A4=G64
A5=G41
A6=G42
A7=G43
A8=G44
B1=G45
B2=G46
B3=G47
B4=G48
B5=G57
B6=G58
B7=G59
B8=G60
GO TO 8
4 DO 10 I=1,9
10 Q(4,1,I)=C(I)
DO 11 I=1,9
Q3=Q(3,1,I)
Q4=Q(4,1,I)
Q(1,1,I)=V(I)*G1+G3*Q3+G4*Q4
Q(2,1,I)=V(I)*G2+G5*Q3+G6*Q4
Q(1,2,I)=V(I)*G7+G9*Q3+G11*Q4
Q(2,2,I)=Q(1,2,I)
Q(3,2,I)=-V(I) *G2+G13*Q3+G15*Q4
Q(4,2,I)= V(I) *G1+G17*Q3+G19*Q4
Q(4,3,I)= V(I)*G21+G23*Q3+G25*Q4
Q(2,3,I)=V(I)*G29+G31*Q3+G33*Q4

```

```

CNST0145
CNST0146
CNST0147
CNST0148
CNST0149
CNST0150
CNST0151
CNST0152
CNST0153
CNST0154
CNST0155
CNST0156
CNST0157
CNST0158
CNST0159
CNST0160
CNST0161
CNST0162
CNST0163
CNST0164
CNST0165
CNST0166
CNST0167
CNST0168
CNST0169
CNST0170
CNST0171
CNST0172
CNST0173
CNST0174
CNST0175
CNST0176
CNST0177
CNST0178
CNST0179
CNST0180

```

```

J=I+9
Q(1,2,J)=V(I)*G8+G10*Q3+G12*Q4
Q(2,2,J)=-Q(1,2,J)
Q(3,2,J)= V(I) *G2+G14*Q3+G16*Q4
Q(4,2,J)=-V(I) *G1+G18*Q3+G20*Q4
Q(4,3,J)= V(I)*G22+G24*Q3+G26*Q4
11 Q(2,3,J)=V(I)*G30+G32*Q3+G34*Q4

```

```

EZ=EM*ZZ
EZ1=DEXP(EZ)
EZ2=DEXP(-EZ)

```

C THE ALAM(I,J) TERMS ARE THE LAMDA(I,J) S OF THE TEXT

```

ALAM(1,1)=-EZ1
ALAM(1,2)=-EZ2
ALAM(1,3)=-EZ*EZ1
ALAM(1,4)=-EZ*EZ2
ALAM(2,1)=-ALAM(1,1)
ALAM(2,2)=ALAM(1,2)
ALAM(2,3)=ALAM(2,1)-ALAM(1,3)
ALAM(2,4)=-ALAM(1,2)+ALAM(1,4)
ALAM(3,1)=ALAM(2,1)
ALAM(3,2)=-ALAM(2,2)
ALAM(3,3)=2.*ALAM(3,1)-ALAM(1,3)
ALAM(3,4)=2.*ALAM(2,2)-ALAM(1,4)
ALAM(4,1)=ALAM(1,1)
ALAM(4,2)=ALAM(1,2)
ALAM(4,3)=-ALAM(2,3)
ALAM(4,4)=ALAM(2,4)
ALAM(5,1)=-1.5*EZ1
ALAM(5,2)=1.5*EZ2
ALAM(5,3)=-1.5*EZ*EZ1
ALAM(5,4)=-1.5*ALAM(1,4)
ALAM(6,1)=1.5*EZ1
ALAM(6,2)=1.5*EZ2
ALAM(6,3)=1.5*ALAM(2,3)
ALAM(6,4)=-1.5*ALAM(2,4)
ALAM(8,1)=-1.5*EM*EZ1

```

```

CNST0181
CNST0182
CNST0183
CNST0184
CNST0185
CNST0186
CNST0187
CNST0188
CNST0189
CNST0190
CNST0191
CNST0192
CNST0193
CNST0194
CNST0195
CNST0196
CNST0197
CNST0198
CNST0199
CNST0200
CNST0201
CNST0202
CNST0203
CNST0204
CNST0205
CNST0206
CNST0207
CNST0208
CNST0209
CNST0210
CNST0211
CNST0212
CNST0213
CNST0214
CNST0215
CNST0216

```



```
ALAM(8,2)=-1.5*EM#EZ2
ALAM(8,3)=ALAM(8,1)*(1.+EZ)
ALAM(8,4)=-ALAM(8,2)*(1.-EZ)
DO 910 L=1,NPP
DO 910 J=1,N
CC(L,J)=0.
DD(L,J)=0.
FF(L,J)=0.
IF(L.GT.NP) GO TO 890
DO 920 I=1,9
920 FF(L,J)=FF(L,J)+V(I)*B(L,J,I)
890 DO 910 I=1,NT
P1=0.
P2=0.
DO 900 M=1,4
P1=P1+Q(M,ILAYER,I)*ALAM(IQWA,M)
900 P2=P2+Q(M,ILAYER,I)*ALAM(4,M)
CC(L,J)=CC(L,J)+BB(L,J,I)*P1
910 DD(L,J)=DD(L,J)+BB(L,J,I)*P2
RETURN
END
```

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```
CNST0217
CNST0218
CNST0219
CNST0220
CNST0221
CNST0222
CNST0223
CNST0224
CNST0225
CNST0226
CNST0227
CNST0228
CNST0229
CNST0230
CNST0231
CNST0232
CNST0233
CNST0234
CNST0235
CNST0236
CNST0237
```

```

SUBROUTINE SOLVE(SI,BB,B,KK,N,M,MM,NNN,NJJJ)
C-----THIS CALCULATES THE SLUTION OF THE INTEGRAL EQUATION
C   SI(KK,1...NNN) IS THE SCLUTION
C   BB(1...MM,1...N),B(1...M,1...N) ARE THE FUNCTIGNS (B THE KERNAL)
C   NJJJ IS THE # OF INTERVALS USED
COMMON CC(8,20),DD(8,20),FF(8,20),T(20),DELTA(20)
DIMENSION SI(13,201),B(8,20),BB(8,20),BET(3)
C-----CALCULATE SOLUTION AT T=0.
  BETA=0.
  ARG=0.
  DO 10 I=1,N
    BETA=BETA+B(1,I)
10   ARG=ARG+BB(1,I)
  SI(KK,1)=ARG/BETA
  DO 70 K=2,NNN
C-----CALCULATE SOLUTION AT T=T(K)
  ISN=1
  ARG=0.
  BET(3)=0.
  DO 30 L=1,N
    SSUMA=0.
    SSUMB=0.
    DO 15 J=1,MM
      SSUMA=BB(MM-J+1,L)+T(K)*SSUMA
15     DO 20 J=1,M
      SSUMB= B(M-J+1,L)+T(K)*SSUMB
20     EX=EXP(-T(K)*DELTA(L))
      IF(EX.LT.1E-10) EX=0.
      ARG=ARG+SSUMA*EX
30     BET(3)=BET(3)+SSUMB*EX
  PSI=0.
  SI(KK,K)=0.
  MIN=MAX0(K-NJJJ,2)
  DO 60 J=MIN,K
    ISN=-ISN
  BET(2+ISN)=0.

```

```

SOLV0001
SOLV0002
SOLV0003
SCLV0004
SOLV0005
SOLV0006
SOLV0007
SOLV0008
SCLV0009
SOLV0010
SOLV0011
SOLV0012
SOLV0013
SCLV0014
SOLV0015
SOLV0016
SOLV0017
SOLV0018
SOLV0019
SOLV0020
SOLV0021
SOLV0022
SOLV0023
SOLV0024
SCLV0025
SOLV0026
SOLV0027
SOLV0028
SOLV0029
SCLV0030
SOLV0031
SOLV0032
SOLV0033
SOLV0034
SOLV0035
SOLV0036

```

	DO 50 L=1,N	SOLV0037
	SSUM=0.	SOLV0038
	DO 40 LL=1,M	SOLV0039
40	SSUM=B(M-LL+1,L)+(T(K)-T(J))*SSUM	SOLV0040
	EX=EXP(-(T(K)-T(J))*DELTA(L))	SOLV0041
	IF(EX.LT.1E-10) EX=C.	SOLV0042
50	BET(2+ISN)=BET(2+ISN)+SSUM*EX	SOLV0043
	JJ=J-1	SOLV0044
	IF(J.EQ.MIN) JJ=1	SOLV0045
60	PSI=PSI-(SI(KK,J)+SI(KK,JJ))*(PET(1)-BET(3))*ISN	SOLV0046
70	SI(KK,K)=(2.*ARG+PSI)/(BET(1)+BET(3))	SOLV0047
	RETURN	SOLV0048
	END	SOLV0049

```

FUNCTION TERPO(S,BES)
C-----THIS COMPUTES THE INTEGRAL W.R.T. M
C   S(1...13) CONTAINS PSI AT DIFFERENT M'S
C   BES(1...91) CONTAINS THE BESSEL MULTIPLIES
      DIMENSION S(13),FUN(91),BES(91)
C-----INTERPOLATE 91 VALUES OF S
      A=(S(1)-2.*S(2)+S(3))*12.5
      V=A*.2+(S(2)-S(1))*5.
      FUN(1)=S(1)
      FUN(2)=A*.01-V*.1+S(2)
      FUN(3)=S(2)
      FUN(4)=A*.01+V*.1+S(2)
      FUN(5)=S(3)
      A=(S(3)-2.*S(4)+S(5))/.18
      V=A*.3+(S(4)-S(3))/.3
      FUN(6)=A*.04-V*.2+S(4)
      FUN(7)=A*.01-V*.1+S(4)
      FUN(8)=S(4)
      FUN(9)=A*.01+V*.1+S(4)
      FUN(10)=A*.04+V*.2+S(4)
      KK=10
      DO 10 K=5,11,2
      A=(S(K)-2.*S(K+1)+S(K+2))*5
      V=A+S(K+1)-S(K)
      DO 10 I=1,20
      KK=KK+1
10      FUN(KK)=A*(I-11)**2*.01+V*(I-11)*.1+S(K+1)
      FUN(91)=S(13)
C-----USE SIMPSON'S RULE FOR THE INTEGRATION
      7 WI=0.
      DO 70 J=2,88,2
70      WI=WI+4.*BES(J)*FUN(J)+2.*BES(J+1)*FUN(J+1)
      WI=WI+BES(1)*FUN(1)+4.*BES(90)*FUN(90)+BES(91)*FUN(91)
      TERPO=WI*.1/3.
      RETURN
      END

```

```

TERP0001
TERP0002
TERP0003
TERP0004
TERP0005
TERP0006
TERP0007
TERP0008
TERP0009
TERP0010
TERP0011
TERP0012
TERP0013
TERP0014
TERP0015
TERP0016
TERP0017
TERP0018
TERP0019
TERP0020
TERP0021
TERP0022
TERP0023
TERP0024
TERP0025
TERP0026
TERP0027
TERP0028
TERP0029
TERP0030
TERP0031
TERP0032
TERP0033
TERP0034
TERP0035
TERP0036

```

```

REAL FUNCTION JB(N,A,B)
C-----JB(J,A,B)=J0(A*B)
C      JB(1,A,B)=J1(A*B)
C      JB(2,A,B)=J1(A*B)/B
C      FOR A*B>12 AN ASYMTOTIC APPROX IS USED
      J=MINO(1,N)
      S=A*B
      IF(S.LE.12.) GO TO 10
      PHI=S-.7854
      IF(J.EQ.1) PHI=S-2.3562
      JB=((2./3.14159/S)**.5)*COS(PHI)
      IF(N.GT.1) JB=JB/B
      RETURN
10     TERM=1.
      IF(N.EQ.2) TERM=A*.5
      IF(N.EQ.1) TERM=S*.5
      JB=TERM
121    DO 20 I=1,22
      TERM=-S*S/(4*I*(I+J))*TERM
      IF(ABS(TERM).LT..0001) RETURN
20     JB=JB+TERM
      RETURN
      END

```

```

JB 0001
JB 0002
JB 0003
JB 0004
JB 0005
JB 0006
JB 0007
JB 0008
JB 0009
JB 0010
JB 0011
JB 0012
JB 0013
JB 0014
JB 0015
JB 0016
JB 0017
JB 0018
JB 0019
JB 0020
JB 0021
JB 0022
JB 0023

```