

VARIATIONAL PROCEDURE FOR
 φ^4 -SCALAR FIELD THEORY

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Submitted to the Department of Physics
in Partial Fulfillment of
the Requirements for the Degree of

DOCTOR OF PHILOSOPHY

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

September 1989

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1

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TABLE OF CONTENTS

	Page
TABLE OF CONTENTS	2
ABSTRACT	3
ACKNOWLEDGEMENTS	4
I. INTRODUCTION	5
II. VARIATIONAL CALCULATION IN QFT	8
II.1 Schrödinger Representation for QFT	8
II.2 Variational Method	13
II.2.1 Time-Independent	13
II.2.2 Time-Dependent	14
II.3 Renormalization Technique	15
III. APPLICATIONS OF VARIATIONAL METHOD	17
A. Non-Linear Oscillator Problem	17
A.1 Time-Independent	17
A.2 Time-Dependent	23
B. φ^4 -Scalar Field Theory	25
B.1 Time-Independent	25
B.1.1 $m^2 = \text{constant}$	31
B.1.2 $m^2 \neq \text{constant}$	40
B.2 Time-Dependent	46
B.2.1 $m^2 = m^2(t)$	51
B.2.2 $m^2 = m^2(\vec{x}, t)$	52
B.3 Formal Solution	54
IV. EXTENSION TO GAUGE FIELD	58
A Yang-Mills Field – ($SU(3)$)	58
CONCLUSIONS	66
APPENDIXES	67
REFERENCES	81

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ABSTRACT

The both time-independent and time-dependent variational procedure described and applied to φ^4 -scalar field theory, and as an extension, the time independent variational procedure also applied to Yang-Mills gauge fields.

Thesis Supervisor: Professor Arthur K. Kerman

ACKNOWLEDGEMENTS

I'd like to express my gratitude to Prof. Kerman not only for what I have learned from him about Physics but also for his patience and understanding. Many times I went his office with a feeling of — but I always left his office with enthusiasm to do more . His tolerance and patience is something I will always remember and appreciate.It changed my attitude. I can say that I have become ,thru his example,more tolerant and more understanding of others.

Whenever I talked to Prof. Polonyi ,his explanations has always made me realize how little I knew of Physics.

I'm pleased to have Prof. Feshbach in my thesis committee. I have certainly enjoyed his lectures.

Finally,I'd like to thank to CTP for making all this is possible.

CHAPTER-I

INTRODUCTION

Since K. Symanzik published his paper [14] on the renormalization there has been a renewed interest in the application of Schrödinger representation to the field theories. Formalism for both boson and fermions has been set up. Publications of Jackiw [28] and Floreanini [21] gives a complete picture of the formalism.

Most of the applications deal with scalar fields and to some extent the gauge fields [1,9]. Many people have studied φ^4 -scalar field theory in the Schrödinger representation [5,6,8,10,11]. To my knowledge no one has ever published any work related to the space dependence of the variational parameters. Space dependence is necessary if we want to include a source term since it couples to the mean value of the field.

Stevenson [4] has approached the problem using plane waves as the basis to expand the field operator. He takes the expectation value of the field operator to be constant and treats the mass term in the frequency as variationally. It is inherent to his approach that there is no way of extending the variational parameters to include space-time dependence. This a restriction on the calculation even though it may initially look covariant, In contrast to our result his calculation does not restrict the bare coupling constant to be negative .

Bardeen [22], in his approach to gauge field ,has formulated the wave functional as an extension of that of the QED ground state wave functional. Even though it

is gauge invariant it does not allow any analytical calculations since it involves an exponential of a quartic function of the fields. Numerical evaluation require a continuum gauge field measure,that is itself a major difficulty in these calculations.

We are motivated to study the singularities of the φ^4 -scalar field theory due to the similarities it has with Yang-Mills gauge field; both of them are quartic in the fields.We tried to find out what we can learn about the structure of the singularities,that will be useful in the study of the Yang-Mills fields.Even though Yang-Mills theory is more complicated due to the gauge invariance and color degrees of freedom,their singularities display similar structure;quadratic and logarithmic.Elimination of the zero mode from the variational parameter corresponding to propogator removes the cubic singularity.Removal of it is essential for the renormalization of the terms appearing in the effective potential.

In the renormalization of φ^4 ground state effective potential the renormalization is introduced in the differential equations for variational parameters.Since differential equations are obtained variationally from the effective potential it is expected that the effective potential will be finite with the same renormalization.This way of introducing renormalization simplifies the algebra a lot.The main idea is to switch to another variational parameter in a way that that defining expression for the new variational parameter will be finite.After renormalizing the effective potential we can verify it variationally. When there is a space dependence in the variational parameter, assuming that the space dependence does not modify the singularities we can generate a formal expansion of the variational parameter around the constant value of the new variational parameter up to an order in which all singular terms are

included. In a scalar theory, singular terms appear as the coefficients of the powers of the new variational parameter. This fact allows us to absorb them in the free parameter of the theory. What makes Yang-Mills theory difficult in the presence of the mean field is the spin and the color dependence of the singular coefficients. Even though they are diagonal in a particular representation since they will be absorbed by the free parameters of the theory, which are just the ordinary c-numbers, it is necessary to kill their spin and color dependence.

The time dependent case is simple enough to tackle if the canonical variables commutes with each other. In that case the kinetic term can be evaluated to see that it is finite if we switch to proper new variable that will renormalize the effective potential.

CHAPTER-II

VARIATIONAL CALCULATIONS IN QFT

II.1 SCHRÖDINGER REPRESENTATION FOR QFT

QFT can be viewed as QM with infinite degrees of freedom. Since the degrees of freedom involved are the values of the fields $\{\phi^a(x)\}$ at each space point \vec{x} , it is necessary to explain what functional, functional differential, and functional integrations, *etc.*, are.

Functional: It is a mapping of the space of sufficiently smooth functions $\{f(\vec{x})\}$ into the Real or Complex numbers

$$F : f \rightarrow f[f] \quad (2.1)$$

e.g.

$$F[f] = \int_V d\vec{x} K_F(\vec{x}) f(\vec{x}) \quad (2.2)$$

$K_F(\vec{x})$ is called a kernel. The value assigned to $F[f]$ depends on the function itself rather than its value at a given point $\{\vec{x}\}$ if we discretize the integral by dividing the space of \vec{x} into N -cells and let the \vec{x}_i lie on each cell and $f_i = f(x_i)$. We can rewrite it as

$$F[f] = \lim_{N \rightarrow \infty} F_N(f_1, f_2, \dots, f_N) = \lim_{N \rightarrow \infty} \sum_{i=1}^N \Delta K_F(i) f_i \quad (2.3)$$

in this sense it is clear that a functional is a generalization of an ordinary function to accommodate the continuous index labeling the variables f_i . Note that $F[f, g, \dots]$ need not be linear in $f(x), g(x), \dots$, *etc.*

A Taylor series expansion of a function may also be generalized to functionals

$$F_2[f] = \sum_{n=0}^{\infty} \frac{1}{n!} \int d\vec{x}_1 \dots d\vec{x}_N K_N(\vec{x}_1, \dots, \vec{x}_N) f(\vec{x}_1) \dots f(\vec{x}_N) \quad (2.4)$$

where $K_N(\dots \vec{x}_i \dots \vec{x}_j \dots) = K(\dots \vec{x}_j \dots \vec{x}_i \dots)$ symmetric under the exchange of two of its arguments.

Functional Derivative: Defined as

$$\frac{\delta F[f]}{\delta f(\vec{y})} \equiv \lim_{\epsilon \rightarrow 0} \frac{F[f + \epsilon \delta_{\vec{y}}] - F[f]}{\epsilon} \quad (2.5)$$

where

$$\delta_{\vec{y}}(\vec{x}) = \delta(\vec{x} - \vec{y}) \quad (2.6)$$

To see that it is a generalization of the ordinary derivative of a function, using discretized version of $f[\vec{y}]$ and $F[f]$, we should write eq.(2.5) as

$$f_i = f(\vec{y}_i)$$

$$\frac{\delta F[f_1 \dots f_i, \dots]}{\delta f_i} = \lim_{\epsilon \rightarrow 0} \frac{F(f_1 \dots f_i + \epsilon, \dots) - F(f_1 \dots f_i \dots)}{\epsilon} \quad (2.7)$$

e.g.

$$F[f] = \int d\vec{x} K_F(\vec{x}) f(\vec{x}) \quad (2.8)$$

Using

$$\frac{\delta f(\vec{x})}{\delta f(\vec{y})} \equiv \delta(\vec{x} - \vec{y})$$

we obtain

$$\frac{\delta F[f]}{\delta f(\vec{y})} = \int d\vec{x} K_F(\vec{x}) \delta(\vec{x} - \vec{y})$$

$$\frac{\delta F[f]}{\delta(\vec{y})} = K_F(\vec{y}) \quad (2.9)$$

Functional derivative of expressions that do not involve integrals are found by re-expressing them as integrals,

e.g.

$$\begin{aligned} \frac{\delta}{\delta B^a(\vec{x})} \left(\frac{d}{dy_\alpha} B^b(\vec{y}) \right) &= \frac{\delta}{\delta B^a(\vec{x})} \sum_c \int d\vec{y} \frac{d}{dy_\alpha} \delta(\vec{y} - \vec{z}) B(\vec{z}) \delta^{bc} \\ &= \sum_c \int d\vec{z} \frac{d}{dy_\alpha} \delta(\vec{y} - \vec{z}) \delta^{bc} \frac{\delta B^c(\vec{z})}{\delta B^a(\vec{x})} \end{aligned} \quad (2.10)$$

using

$$\frac{\delta B^c(\vec{z})}{\delta B^a(\vec{x})} = \delta_a^c \delta(\vec{y} - \vec{z})$$

we obtain

$$\frac{\delta}{\delta B^a(\vec{x})} \left(\frac{d}{dy_\alpha} B^b(\vec{y}) \right) = \delta_a^b \frac{d}{dy_\alpha} \delta(\vec{x} - \vec{y}) \quad . \quad (2.11)$$

Functional Integration : Defined as

$$\begin{aligned} \int D(f) F[f] &\equiv \int \prod_{\vec{x}} df(\vec{x}) F[f] \\ &\equiv \lim_{N \rightarrow \infty} \int_{-\infty}^{+\infty} df_1 \dots df_N F_N(f_1 \dots f_N) \quad . \end{aligned} \quad (2.12)$$

and it is nothing more than infinite dimensional integration. Note that the range of integration for the variables, f_i , is all the allowed values.

Functionality: It is a rule that assigns a (Real or Complex) number to a (Real or Complex) functional

$$\psi[\varphi] \longrightarrow E$$

e.g. energy functionality defined as

$$E = E\{\psi\} \equiv \frac{\int \psi^*[\varphi] \hat{H} \psi[\varphi] D(\varphi)}{\int |\psi[\varphi]|^2 D(\varphi)} \quad . \quad (2.13)$$

\hat{H} is some operator acting on the functional $\psi[\varphi]$. Note that in our notation brackets $()$, $[]$, and $\{ \}$ are used around the argument to indicate a function , functional, and functionality, respectively.

e.g

() indicates a function , $f(\vec{x})$

[] indicates a functional , $F[f]$

{ } indicates a functionality , $E\{\psi\}$.

Generalization of the Functional Differentiation to Functionality:

A functionality $E = E\{\psi\}$ is differentiable at the functional $\psi[\varphi]$ if

$$\frac{\partial}{\partial z} E\{\psi + z\Omega\}|_{z=0} = \int (DE/D\psi)\Omega D(\varphi) \quad (2.14)$$

exists as a linear functionality in Ω for certain suitable dense of functionals, $\Omega = \Omega[\varphi]$. As a variational principle $DE/D\psi = 0$ leads to a Schrödinger functional differential equation

$$\hat{H}\psi[\varphi] = E\psi[\varphi] \quad (2.15)$$

where $\psi[\varphi]$ are the stationary functional states. If $\psi[\varphi; f]$ is prescribed in a functional form but with a function $f(\mathbf{x})$ as a free “parameter” then when the energy functionality (2.13) is evaluated we are left with a functional $E[f]$.

$f(\vec{x})$ can be determined by solving the relatively simple Rayleigh–Ritz partial-differential equation

$$\frac{\delta E[f]}{\delta f(\vec{x})} = 0 \quad (2.16)$$

to find stationary value of $E[f]$ with respect to the function $f(\vec{x})$.Ref[29,39]

Application to Scalar Field Theory

Let $\phi(x)$ be a real scalar field and $L(t) = \int d\vec{x} \left\{ \frac{1}{2} (\partial_\mu \phi)^2 - V(\phi) \right\}$ be the associated Lagrangian of the theory and the canonical field to $\phi(x)$ defines as

$$\begin{aligned} \pi(x) &\equiv \frac{\delta L}{\delta \dot{\phi}(x)} & \dot{\phi}(x) &\equiv \frac{\partial}{\partial t} \phi(x) \\ &= \dot{\phi}(x) \end{aligned} \quad (2.17)$$

and the Hamiltonian form

$$\begin{aligned} L &= \int d\vec{x} \pi(x) \dot{\phi}(x) - H[\pi, \phi] \\ H(\pi, \phi) &= \int d\vec{x} \left\{ \frac{1}{2} \pi^2(x) + \frac{1}{2} (\vec{\nabla} \phi)^2 + V(\phi) \right\} \end{aligned} \quad (2.18)$$

by promoting the fields $\pi(x)$, $\phi(x)$ into operators we introduce the quantization as

$$\left[\hat{\pi}(\vec{x}, t), \hat{\phi}(t, \vec{y}) \right] = -i\delta(\vec{x} - \vec{y}) \quad (2.19)$$

Now the problem is to find eigenstates and eigenvalues of the Hamiltonian operator \hat{H}

$$\begin{aligned} \hat{H} &= \int d\vec{x} \left\{ \frac{1}{2} \hat{\pi}(\vec{x}, t)^2 + \frac{1}{2} (\vec{\nabla} \hat{\phi})^2 + V(\hat{\phi}) \right\} \\ \hat{H}|\psi\rangle &= E|\psi\rangle . \end{aligned} \quad (2.20)$$

In the functional Schrödinger representation we have the following correspondence

$$\begin{aligned} |\psi\rangle &\longleftrightarrow \psi[\phi(\vec{x})] \\ \hat{\phi}(t, \vec{x})|\psi\rangle &\longleftrightarrow \phi(\vec{x})\psi[\phi] \\ \hat{\pi}(t, \vec{x})|\psi\rangle &\longleftrightarrow \frac{1}{i} \frac{\delta}{\delta \phi(\vec{x})} \psi[\phi] \\ \hat{H}(\hat{\pi}, \hat{\phi}) &\longleftrightarrow \hat{H} \left(\frac{\delta}{i\delta \phi}, \phi \right) \end{aligned} \quad (2.21)$$

and

$$\hat{H} \left(\frac{\delta}{i\delta \phi(x)}, \phi(x) \right) \psi[\varphi] = E\psi[\varphi]$$

that is

$$\int d\vec{x} \left\{ -\frac{\delta^2}{\delta\phi(\vec{x})^2} + \frac{1}{2} \left(\vec{\nabla}\phi(\vec{x}) \right)^2 + V(\phi(\vec{x})) \right\} \psi[\phi] = E\psi[\phi] \quad (2.22)$$

with the norm $\langle\psi|\psi\rangle$ defined as $\int D(\phi)\psi^*[\phi]\psi[\phi]$ Ref.[3]

Since the exact solution of this problem is not possible, in the next section we will develop a variational approach to obtain some results for the ground state energy.

II.2 VARIATIONAL METHOD

II.2.1 Time-Independent (Rayleigh–Ritz)

If we multiply Eq. (2.22) from the left with $\psi^*[\phi]$ and integrate over the $\phi(\vec{x})$ field we obtain an energy functionality

$$E\{\psi\} = \frac{\int D(\phi)\psi^*[\phi]\hat{H}\left[\frac{\delta}{i\delta\phi(x)}, \phi(x)\right]\psi[\phi]}{\int D[\phi]|\psi[\phi]|^2} . \quad (2.23)$$

It is known from QM that any trial wave function that approximates ground state wave function gives an upper bound for the ground state energy. Similarly, any trial wave functional $\psi[\phi]$ that approximates the ground state wave functional gives an upper bound for the ground state energy of the theory.

If some undetermined parameters are introduced in the wave function, it is possible to improve the upper bound value by extremizing ground state energy with respect to the parameters. Similarly, in the following we choose a trial wave functional with a function $f(\vec{x})$ as a free “parameter” $\psi[\varphi; f]$. It allows us to obtain energy functionality as a functional $E[f]$. Extremizing it leads to

$$\delta E[f]/\delta f(\vec{x}) = 0 \quad (2.24)$$

a relatively simple “Rayleigh–Ritz partial differential equation for $f(\vec{x})$; solutions minimize the energy.

The above procedure describes the time-independent Rayleigh-Ritz variational method for functionals. Ref.[16]

II.2.2 Time-Dependent Variational Method for Functionals

It is a generalization of Dirac’s less known time-dependent variational procedure. We start by defining functionality as

$$L\{\psi\} = \int dt \left\langle \psi(t) \left| \left(i \frac{\partial}{\partial t} - \hat{H} \right) \right| \psi(t) \right\rangle / \langle \psi | \psi \rangle \quad (2.25)$$

a variation of $L\{\psi\}$ with respect to the functional $\psi[\varphi; t]$ leads to

$$i \frac{\partial}{\partial t} |\psi(t)\rangle = \hat{H} |\psi(t)\rangle \quad (2.26)$$

time-dependent functional Schrödinger equation. If the operator \hat{H} is time-independent then the time-dependence of the state functional will be nothing more than a phase factor e^{-itE} multiplying the state (E constant). Ref.[23,24,30,41]

Let us choose a trial wave functional with an even number of parameters that depends on time. That is, the time-dependence of the wave functional is through that of parameters $\psi[\varphi(\vec{x}); f(t, \vec{x}), g(t, \vec{x}), \dots]$.

After evaluating the functionality with respect to the field $\varphi(\vec{x})$ we are left with a functional of the parameters. If the parameters are introduced in the wave functional in a way that they can be read off as canonical in $\alpha[f, g, \dots]$ then it is

possible to identify the effective Hamiltonian functional from now on what we will call an effective Lagrange functional.

$$L[f, g] = \int dt d\vec{x} f \dot{g} - \mathcal{H}[f, g] \quad (2.27)$$

Equations that determine the parameters are the Hamilton equations

$$\begin{aligned} -\dot{f}(\vec{x}, t) &= \frac{\delta \mathcal{H}[f, g]}{\delta g(\vec{x}, t)} \\ \dot{g}(\vec{x}, t) &= \frac{\delta \mathcal{H}[f, g]}{\delta f(\vec{x}, t)} \end{aligned} \quad (2.28)$$

which are time-dependent partial differential equations. Note that when the parameters are time-independent they are reduced to Rayleigh–Ritz variational equations for the functionals

$$\frac{\delta \mathcal{H}[f, g]}{\delta g(\vec{x})} = 0 \quad \frac{\delta \mathcal{H}[f, g]}{\delta f(\vec{x})} = 0 \quad . \quad (2.29)$$

In this way it is possible, by keeping all quantum effects, to reduce quantum mechanics (QM) into classical mechanics (CM) and quantum field theory (QFT) into classical field theory (CFT).

The above time-dependent partial-differential equations should be supplied with some boundary conditions at a given time t .

II.3 RENORMALIZATION TECHNIQUE

Since QFT is plagued with infinities it is necessary, even after reducing it to CFT over parameters, to introduce a technique to render the physical quantities finite.

As will be seen in Chapter II, Section B, when the parameter is a function of more than one argument, $G(\vec{x}, \vec{y})$, the diagonal elements, $G(\vec{x}, \vec{x})$ turn out to be

infinite and partial differential equations that contain diagonal elements are not well defined, too. We introduce a cut-off, Λ , in the upper-bound of the Fourier integral of the parameter. $G(\vec{x}, \vec{x})$ is obtained as

$$G(\vec{x}, \vec{x}) = \lim_{\Lambda \rightarrow \infty} G_{\Lambda}(\vec{x}, \vec{x}) = \text{finite terms} \tag{2.30}$$

$$+ \text{divergent terms} + \text{vanishing terms} .$$

Finite terms are the ones independent of a cut-off, Λ divergent terms are the ones which go to infinity as $\Lambda \rightarrow \infty$ and vanishing terms are the ones which go to zero as the cut-off, Λ , goes to infinity.

The vanishing terms are ignored and the divergent terms are combined with the bare mass and coupling constants of the theory to define finite (or renormalized) values of the mass and coupling constant. In other words, the bare mass and coupling constant is chosen as the cut-off, Λ , dependent in a way they will cancel the divergent terms. In this way, the differential equations are well defined. Since the differential equations are variationally obtained from the effective Hamiltonian. We expect that it is finite up to a constant with respect to variational parameters. That constant could be infinite in the limit $\Lambda \rightarrow \infty$ (indeed it is). Since the Hamiltonian (or energy) is determined up to a constant, it has no relevance. In fact we will choose to scale the energy with respect to that of the free field theory (renormalized coupling constant = 0). (In the application of variational procedure to φ^4 -scalar field theory the technique described above will become more clear.)

CHAPTER-III

APPLICATIONS OF VARIATIONAL METHOD

As an application of the variational method we start with the non-linear oscillator problem to make our way out from simple to more complicated ones.

A. Non-Linear Oscillator Problem

Non-linear oscillator problem can be regarded zero dimensional field theory. It serves an introduction to the φ^4 -scalar field theory.

A.1 Time-Independent

Let

$$L(q, \dot{q}) = \frac{1}{2}\dot{q}^2 - V(q) \quad (3.1)$$

where

$$V(q) = \frac{1}{2}aq^2 + \frac{b}{24}q^4 \quad (3.2)$$

be the Lagrangian of a particle, of unit mass, moving in a potential $V(q)$. It also corresponds to zero dimensional field theory. Defining the Hamiltonian as

$$H(p, q) = \frac{1}{2}p^2(t) + V(q(t)) \quad (3.3)$$

where

$$p(t) = \frac{\delta^2 L}{\delta \dot{q}(t)} \quad \text{and} \quad H(p, q) = p\dot{q} - L(\dot{q}, q) \quad (3.4)$$

and introducing quantization

$$[\hat{p}(t), \hat{q}(t)] = -i\hat{I}\hbar \quad (3.5)$$
$$\hat{H}(\hat{p}, \hat{q}) = \frac{1}{2}\hat{p}^2(t) + V(\hat{q}(t))$$

we write the eigenvalue problem as

$$\hat{H}(\hat{p}, \hat{q})|\psi\rangle = E|\psi\rangle \quad (3.6)$$

or, in the coordinate representation

$$\begin{aligned} |\psi\rangle &\longrightarrow \psi(q) \\ \hat{p}(t)|\psi\rangle &\longrightarrow \frac{\hbar}{i} \frac{d}{dq}\psi(q) \\ \hat{q}(t)|\psi\rangle &\longrightarrow q\psi(q) \\ \left[\frac{-\hbar^2}{2} \frac{d^2}{dq^2} + V(q) \right] \psi(q) &= E\psi(q) \end{aligned} \quad (3.7)$$

multiplying (3.7) from the left with $\psi^*(q)$ and integrating over q we obtain

$$E = \frac{\int_{-\infty}^{+\infty} dq \psi^*(q) \left[-\frac{\hbar^2}{2} \frac{d^2}{dq^2} + V(q) \right] \psi(q)}{\int dq |\psi(q)|^2} \quad (3.8)$$

Let us choose

$$\psi_0(q) = N \exp \left\{ -\frac{1}{4}(q - q_0)g^{-1}(g - g_0) \right\} \quad (3.9)$$

Where $g > 0$, as a trial wave function with undetermined parameters (q_0, g) to have an approximate value of the ground state energy.

After evaluating the integration (Appendix A) we are left with the ground state energy as a function of the parameters

$$E(q_0, g) = V(q_0) + \frac{\hbar^2}{8}g^{-1} + \frac{1}{2} \left(a + \frac{b}{2}q_0^2 \right) g + \frac{b}{8}g^2 \quad (3.10)$$

The first term in eq.(3.10) is the classical contribution and the rest are the quantum contributions. Rayleigh-ritz variation of $E(q_0, g)$

$$\frac{\partial E}{\partial g} \equiv 0 \quad \text{and} \quad \frac{\partial E}{\partial q_0} \equiv 0$$

gives the following two equations for the the values of (q_0, g) extremizing eq.(3.10)

$$(a + \frac{b}{2}g + \frac{b}{8}q_0^2)q_0 = 0 \quad (3.11a)$$

and

$$g^3 + \left(\frac{2a}{b} + q_0^2\right)g^2 - \frac{\hbar^2}{b} = 0 \quad (3.11b)$$

to be the sure that the possible solutions will give the minimum of the energy we demand that they should satisfy the stability conditions

$$\frac{\partial^2 E}{\partial q_0^2} \geq 0 \quad \text{and} \quad \frac{\partial^2 E}{\partial g^2} \geq 0 \quad (3.12)$$

and they are

$$a + \frac{3b}{8}q_0^2 + \frac{b}{2}g \geq 0 \quad (3.13a)$$

and

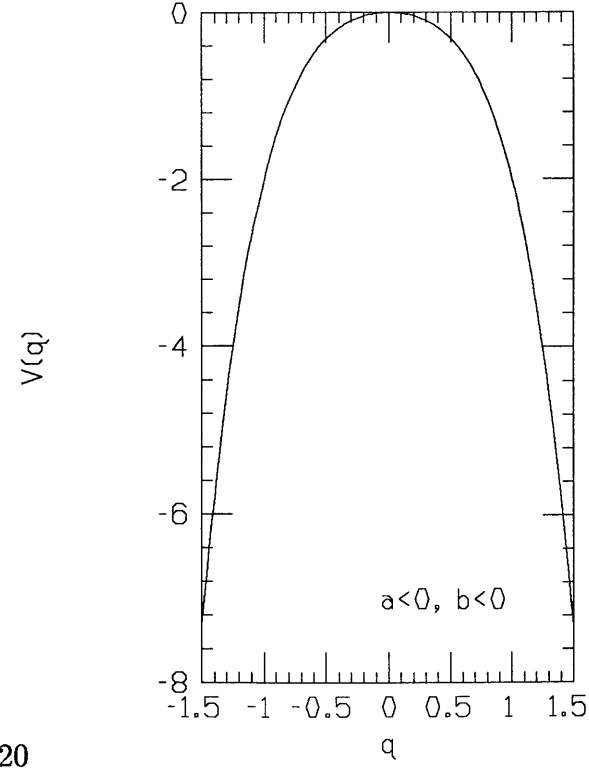
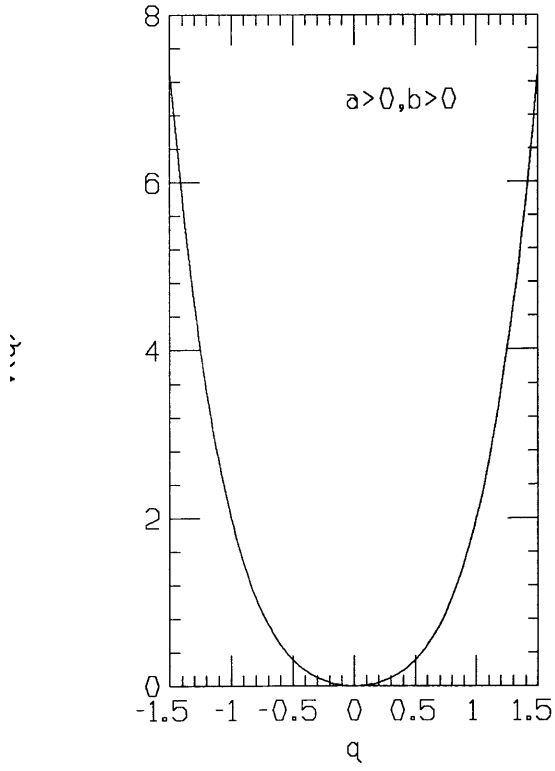
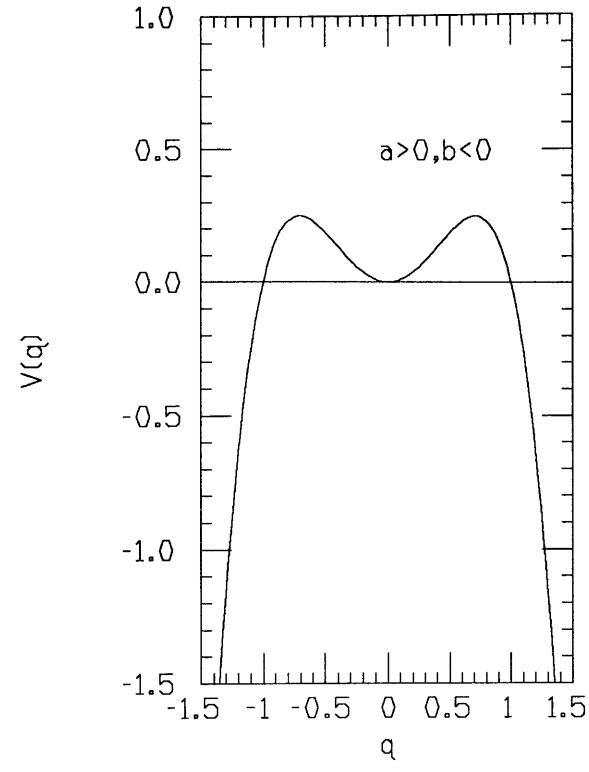
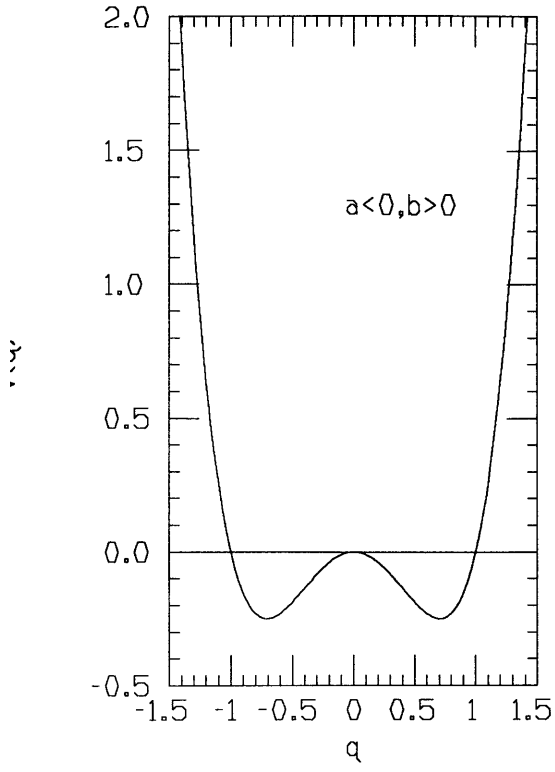
$$\frac{\hbar^2}{4}g^{-3} + \frac{b}{4} \geq 0 \quad (3.13b)$$

We have one more condition on g that is, $g > 0$. Let us plot the potential, $V(q)$, before trying to solve these algebraic equations.

$$V(q) = \frac{1}{2}aq^2 + \frac{b}{24}q^4 \quad (3.14)$$

The potentials in Figs. 1 and 2, classically, will lead to stable periodic solutions and quantum mechanically to stable ground states. In Fig. 3, classically may lead to stable periodic solutions and quantum mechanically to unstable ground states. Figure 4 does not support any classical and quantum solutions.

Figures Fig.1-4



Now, lets investigate the possible solutions for g and q , from eq.(3.11a-b) we find one set of solutions as

I. When $q_0 = 0$, and $g = g_k^*$ where

$$g_k^* = \left| \frac{2a}{3b} \right| \left[-1 + 2 \cos \left(\frac{\phi}{3} + 120^\circ k \right) \right] \quad (3.15a)$$

and ϕ is given by

$$\cos \phi = \eta \left| 1 - \frac{D}{a^3} \right| \quad (3.15b)$$

Where

$$D = \frac{1}{2} \left(\frac{3}{2} \right)^3 \hbar^2 b^2 \geq 0 \quad (3.16)$$

and

$$\eta = \begin{cases} -1 & \text{if } \frac{a^3}{Db} > \frac{1}{b} \\ +1 & \text{if } \frac{a^3}{Db} < \frac{1}{b} \end{cases}$$

The requirement $g_k^* > 0$ implies the following conditions

$$\begin{aligned} -1 + 2 \cos \left(\frac{\phi}{3} + 120^\circ k \right) &> 0 \\ \cos \left(\frac{\phi}{3} + 120^\circ k \right) &> \frac{1}{2} = \cos 60^\circ \\ 0 \leq \frac{\phi}{3} + 120^\circ k &< 60 \quad k = 0, 1, 2 \\ \implies k = 0 &\quad \text{and} \quad 0 \leq \phi < 180 \quad . \end{aligned}$$

$$g_0^* = g^* = \left| \frac{2a}{3b} \right| \left(-1 + 2 \cos \frac{\phi}{3} \right)$$

and its range is $0 < g \leq \left| \frac{2a}{3b} \right|$

The fact that $|\cos \phi| < 1$ implies $a > 0$ and $\frac{a^3}{D} > \frac{1}{2}$. Therefore this solution can be valid only for figure 1 and 2. It makes sense since the wave function will be centered around the minimum of the potential to minimize its energy.

The stability conditions (3.13a-b), when $q = 0$,

$$bg > -2a$$

and

$$1 > -g^3 \frac{b}{2\hbar}$$

restricts the possible values of (a, b) for a minimum. For $b > 0$ all values of $a > 0$ and $b > 0$ are allowed this corresponds to figure 1. As for the figure 2, since $b < 0$ the first stability condition is fulfilled by $b < 0$ and $a > 0$ but the second condition puts the following restriction on the values of a and b

$$2 \left(\frac{3}{2} \right)^3 \hbar^2 > \frac{a^3}{b^2}$$

this condition is the outcome of the fact that the potential well should be large enough to support a quantum solution.

II. $q_0 = \tilde{q}_0^*$, $g = \tilde{g}^*$

$$\tilde{g}^{*3} + \left(\frac{2a}{b} + \tilde{q}_0^{*2} \right) \tilde{g}^{*2} - \frac{\hbar^2}{b} = 0 \quad (3.17a)$$

and

$$a + \frac{b}{2} \tilde{g}^* + \frac{b}{8} \tilde{q}_0^{*2} = 0 \quad (3.17b)$$

eliminating q_0^* we easily obtain

$$\tilde{g}^{*3} + \frac{2a}{b} \tilde{g}^{*2} + \frac{\hbar^2}{3b} = 0 \quad (3.18)$$

The solution of this equation can be obtained from g_k by substituting

$$\tilde{g}_k^* = g_k^* \left(\hbar^2 \longrightarrow -\frac{\hbar^2}{3} \right)$$

The effect of the substitution is to change the sign in eq.(3.18)

$$\tilde{g}^* = \left| \frac{2a}{b} \right| \frac{1}{3} \left(-1 + 2 \cos \frac{\tilde{\phi}}{3} \right) \quad (3.19a)$$

where

$$\cos \tilde{\phi} = \eta \left| 1 + \frac{D}{a^3} \right| \quad (3.19b)$$

The change of sign in the $\cos \tilde{\phi}$ implies that $a < 0$. Therefore this solution is valid only for the figure 3.

Note that

$$\begin{aligned} \tilde{q}_0^* &= \mp \sqrt{-\frac{8a}{b} + 4g^*} \\ &= \mp \sqrt{-\frac{6a}{b} + 4g^* - \frac{2a}{b}} \end{aligned} \quad (3.20)$$

the wave function centered $4g^* + \left| \frac{2a}{b} \right|$ amount away from the minima of the potential to the right or left.

A.2 Time-Dependent

The effective Lagrangian

$$\begin{aligned} I &= \int dt L(t) \\ L(t) &= \left\langle \psi(t) \left| i \frac{\partial}{\partial t} - \hat{\mathcal{H}} \right| \psi(t) \right\rangle / \langle \psi | \psi \rangle \end{aligned} \quad (3.21)$$

The trial wave function we choose in a way that

$$\begin{aligned} \psi_0(q, t) &= \exp(i[q(t) - q_0(t)] \pi_0) \\ &\times \exp \left\{ -\frac{1}{4} (q - q_0) [Q^{-2}(t) + 2iQ^{-1}(t)P(t)] (q - q_0)(t) \right\} \end{aligned} \quad (3.22)$$

(q_0, π_0) and (Q, P) will be pairs of canonical parameters in the effective lagrangian

$$\begin{aligned} \psi_0(q, t) = & \exp(i(q - q_0(t))\pi_0(t)) \\ & \times \left\{ -\frac{1}{4}(1 - q_0(t))^2 [Q^{-2}(t) - 2iQ^{-1}(t)P(t)] \right\} \end{aligned} \quad (3.23)$$

where

$$\begin{aligned} \hat{\mathcal{H}}(p, q) = & -\frac{1}{2} \frac{\partial^2}{\partial q^2} + V(q) \\ L(t) = & \left\langle \psi_0(t) \left| i \frac{\partial}{\partial t} \right| \psi_0(t) \right\rangle / \langle \psi_0 | \psi_0 \rangle - \left\langle \psi_0(t) | \hat{\mathcal{H}} | \psi_0(t) \right\rangle / \langle \psi_0 | \psi_0 \rangle \end{aligned} \quad (3.24)$$

After integrating over variable q we obtain the effective lagrangian as (Appendix-C)

$$\begin{aligned} L(t) = & \left\{ \pi_0 \dot{q}_0 - \frac{i}{4} Q^2 \frac{\partial}{\partial t} (Q^{-2} - 2iQ^{-1}P) \right. \\ & \left. - \left[\frac{1}{2} \pi_0^2 + \left(\frac{1}{8} Q^{-2} + P^2 \right) + V(q_0) + \frac{1}{2} \left(a + \frac{b}{a} q_0^2 \right) Q^2 + \frac{b}{8} Q^4 \right] \right\} \end{aligned} \quad (3.25)$$

we rewrite the above expression as

$$L(t) = \left\{ \pi_0 \dot{q}_0 + P \dot{Q} + \frac{d}{dt} \left(\frac{i}{2} \ln Q - \frac{1}{2} QP \right) - \mathcal{H}(\pi_0, q_0, P, Q) \right\} \quad (3.26a)$$

where

$$\mathcal{H} = \frac{1}{2} \pi_0^2 + \frac{1}{2} P^2 + \frac{1}{8} Q^{-2} + V(q_0) + \frac{1}{2} \left(a + \frac{b}{2} q_0^2 \right) Q^2 + \frac{b}{8} Q^4 \quad (3.26b)$$

Ignoring the total time derivative we infer that (π_0, q_0) and $P, Q)$ are canonical pairs

of variables. Therefore, $\hat{\mathcal{H}}$ can be identified as Hamiltonian and the dynamics will

be determined from Hamilton-Jocabi equations. They are

$$\begin{aligned} -\dot{\pi}_0(t) = & \frac{\partial \mathcal{H}}{\partial q_0(t)} & \dot{q}_0 = & \frac{\partial \mathcal{H}}{\partial \pi_0(t)} \\ -\dot{P}(t) = & \frac{\partial \mathcal{H}}{\partial Q(t)} & \dot{Q}(t) = & \frac{\partial \mathcal{H}}{\partial P(t)} \end{aligned}$$

The first two eq. gives

$$\dot{\pi}_0 = \frac{\partial V}{\partial q_0} + \frac{b}{2}q_0Q^2 = aq_0 + \frac{b}{16}q_0^3 + \frac{b}{2}q_0Q^2$$

$$\dot{q}_0 = \pi_0$$

$$\implies -\ddot{q}_0 = \left[a + \frac{b}{2}Q^2(t) \right] q_0(t) + \frac{b}{6}q_0^3(t)$$

and the others

$$-\dot{P}(t) = -\frac{1}{4}Q^{-3} + \left(a + \frac{b}{2}q_0^2 \right) Q + \frac{b}{2}Q^3$$

$$\dot{Q} = P$$

$$-\ddot{Q} = \left[a + \frac{b}{2}Q^2(t) + \frac{b}{2}q_0^2(t) \right] Q - \frac{1}{4}Q^{-3}$$

define $\omega^3(t) = a + \frac{b}{2}(q_0^2 + Q^2(t))$

$$\ddot{q}_0 = -\omega^2(t)q_0(t) + \frac{b}{3}q_0^3(t)$$

$$\ddot{Q}(t) = -\omega^2(t)Q(t) + \frac{1}{4}Q^{-3}(t)$$

Ref.[4]

B. φ^4 SCALAR FIELD THEORY

B.1 Time Independent

As a next step, we choose φ^4 -Field Theory to apply the variational procedure described since it is a generalization of the non-linear oscillator problem to 4-dimensional field theory and serves an introduction to the application of the variational procedure to the gauge field theories. Particularly, Yang-Mills SU(3) gauge fields. Both of them are quartic in fields.

Now, the problem is to calculate the expectation value of the φ^4 -scalar field theory hamiltonian for a given trial wave functional and make sense out of it by the renormalization technique described. Ref.[5,22,26,27]

The hamiltonian of concern is given by (1.20) where the potential is

$$V(\varphi) = \frac{1}{2}a\varphi^2(\vec{x}) + \frac{b}{4!}\varphi^4(\vec{x}) \quad (3.27)$$

a and b are the bare mass and the bare coupling constants, respectively. Due to renormalization they will be chosen as cut-off dependent

As for the trial wave functional it will be the generalization of the one we used in the non-linear oscillator problem. Quadratic exponentials are the only ones with which we can, analytically, evaluate functional integrals, therefore we choose the following form as our trial wave functional,

$$\psi\{\varphi\} = N \exp \left\{ \frac{-1}{4} \int_{-\infty}^{+\infty} d\vec{x} d\vec{y} [\varphi(\vec{x}) - \varphi_0(\vec{x})] G^{-1}(\vec{x}, \vec{y}) [\varphi(\vec{y}) - \varphi_0(\vec{y})] \right\} \quad (3.28)$$

If we calculate the mean-value of the field operator for a given \vec{x} we obtain

$$\langle \psi | \hat{\varphi}(\vec{x}, t) | \psi \rangle = \varphi_0(\vec{x}) \langle \psi | \psi \rangle \quad (3.29)$$

Where $\varphi_0(\vec{x})$ can be identified as the mean-field. It is a measure of the localization of the field operator for a given \vec{x} . As for the two point function

$$\langle \psi | \hat{\varphi}(\vec{x}) \hat{\varphi}(\vec{y}) | \psi \rangle = [\varphi_0(\vec{x})\varphi_0(\vec{y}) + G(\vec{x}, \vec{y})] \langle \psi | \psi \rangle \quad (3.30)$$

$G(\vec{x}, \vec{y})$ can be identified as the propagator in the presence of the mean-field. Its determinant has to be positive for the convergence of the wave functional as $\varphi(\vec{x}) \rightarrow \infty$ and it is symmetric under exchange of its arguments. $G(\vec{x}, \vec{y}) = G(\vec{y}, \vec{x})$

From QM point of view $\varphi_0(\vec{x})$ indicates the localization of the wave functional for a given variable and $G(\vec{x}, \vec{y})$ stands for the width of the gaussian. Both $\varphi_0(\vec{x})$ and $G(\vec{x}, \vec{y})$ are undetermined parameters whose values will be found variationally from the energy functionality.

Calculation of the energy functionality: Expectation value of the hamilton operator is given as

$$E[\varphi_0, G] = \langle \psi | \hat{H} | \psi \rangle / \langle \psi | \psi \rangle \quad (3.31)$$

Where

$$\langle \psi | \hat{H} | \psi \rangle = \int d\vec{x} \int D(\varphi) \psi^*[\varphi] \left[\frac{-1}{2} \frac{\delta^2}{\delta\varphi^2} + \frac{1}{2} (\vec{\nabla}\varphi)^2 + V(\varphi) \right] \psi[\varphi] \quad (3.32)$$

and $\psi[\varphi]$ is given by eq(3.28) to simplify notation lets define $\tilde{\varphi}(\vec{x}) = \varphi(\vec{x}) - \varphi_0(\vec{x})$ and $(\tilde{\varphi}, G^{-1}\tilde{\varphi})$ as the exponent appearing in the wave functional. If we rewrite $\langle \psi | \hat{H} | \psi \rangle$ after we shift the field variable we obtain

$$\begin{aligned} \langle \psi | \hat{H} | \psi \rangle = & \int d\vec{x} \int D(\tilde{\varphi}) \exp \left[\frac{-1}{4} (\tilde{\varphi}, G^{-1}\tilde{\varphi}) \right] \times \\ & \left\{ \frac{-1}{2} \frac{\delta^2}{\delta\tilde{\varphi}^2(\vec{x})} + \frac{1}{2} (\vec{\nabla}\tilde{\varphi} + \vec{\nabla}\varphi_0)^2 + V(\tilde{\varphi} + \varphi_0) \right\} \exp \left[\frac{-1}{4} (\tilde{\varphi}, G^{-1}\tilde{\varphi}) \right] \end{aligned} \quad (3.33)$$

where

$$V(\tilde{\varphi} + \varphi_0) = V(\varphi_0) + \left(\frac{1}{2}a + \frac{b}{4}\varphi_0^2 \right) \tilde{\varphi}^2 + \frac{b}{24}\tilde{\varphi}^4 + \text{terms linear and quadratic in } \tilde{\varphi}$$

Since the exponent is quadratic in $\tilde{\varphi}$ we will ignore the odd powers of $\tilde{\varphi}$ multiplying the exponential term in the integrand. The result is

$$\begin{aligned} \langle \psi | \hat{H} | \psi \rangle = & \langle \psi | \psi \rangle E_{cl} + \int d\vec{x} \int D(\tilde{\varphi}) \exp \left[\frac{-1}{4} (\tilde{\varphi}, G^{-1}\tilde{\varphi}) \right] \times \\ & \left\{ \frac{-1}{2} \frac{\delta^2}{\delta\tilde{\varphi}^2(\vec{x})} + \frac{1}{2} (\vec{\nabla}\tilde{\varphi})^2 + \frac{1}{2} \left(a + \frac{b}{2}\varphi_0^2 \right) \tilde{\varphi}^2 + \frac{b}{24}\tilde{\varphi}^4 \right\} \exp \left[\frac{-1}{4} (\tilde{\varphi}, G^{-1}\tilde{\varphi}) \right] \end{aligned} \quad (3.34)$$

where

$$E_{cl} = \int d\vec{x} \left\{ \frac{1}{2} (\vec{\nabla} \varphi_0)^2 + V(\varphi_0) \right\}$$

E_{cl} is the classical energy. Let's have a close look to each term separately. The action of the functional derivative on our wave functional brings down $\frac{-1}{2} (G^{-1}(\vec{x}, \cdot), \tilde{\varphi}_0)$ term in front of the wave functional. Second application generates the following two terms $[\frac{1}{4} G^{-1}(\vec{x}, \vec{x}) + \frac{-1}{8} (G^{-1}(\vec{x}, \cdot) \tilde{\varphi}, \tilde{\varphi} G^{-1}(\cdot, \vec{x}))]$ in front of the wave functional. Functional integration over $\tilde{\varphi}(\vec{x})$ converts the two $\tilde{\varphi}$ into G therefore the over all contribution from the functional derivative is $\frac{1}{8} TR(G^{-1}) < \psi | \psi >$. Similarly, the second and third terms generate $\frac{1}{2} TR(KG) < \psi | \psi >$. The fourth term requires special attention (details will be given in appendix E). It is $\frac{b}{8} TR TR(GG) < \psi | \psi >$. The over all result is

$$E[\varphi_0, G] = E_{cl} + \frac{1}{8} TR(G^{-1}) + \frac{1}{2} TR(KG) + \frac{b}{8} TR TR(GG) \quad (3.35)$$

where

$$K(\vec{x}, \vec{x}) = \left\{ -\vec{\nabla}^2 + [a + \frac{1}{2} b \varphi^2(\vec{x})] \right\} \delta(\vec{x} - \vec{x}) \quad (3.36)$$

and $TR TR$ stands for double trace. Note the following relation

$$K(\vec{x}, \vec{y}) = \frac{\delta^2 E_{cl}}{\delta \varphi_0(\vec{x}) \delta \varphi_0(\vec{y})} \quad (3.36a)$$

The values of $\varphi_0(\vec{x})$ and $G(\vec{x}, \vec{y})$ are found by

$$\frac{\delta E[\varphi_0, G]}{\delta \varphi_0(\vec{x})} = 0 \quad \text{and} \quad \frac{\delta E[\varphi_0, G]}{\delta G(\vec{x}, \vec{y})} = 0 \quad (3.37)$$

and they are

$$-\nabla_x^2 \varphi_0(\vec{x}) + a \varphi_0(\vec{x}) + \frac{b}{6} \varphi_0^3(\vec{x}) + \frac{b}{2} \varphi_0(\vec{x}) G(\vec{x}, \vec{x}) = 0 \quad (3.37a)$$

and

$$\frac{1}{4}G^{-2}(\vec{x}, \vec{y}) = \left\{ -\nabla_{\vec{x}}^2 + a + \frac{b}{2}\varphi_0^2(\vec{x}) + \frac{b}{2}G(\vec{x}, \vec{x}) \right\} \delta(\vec{x} - \vec{y}) \quad (3.37b)$$

(see Appendix F)

Define

$$m^2(\vec{x}) \equiv a + \frac{b}{2}(\varphi_0^2(\vec{x}) + G(\vec{x}, \vec{x})) \quad (3.38)$$

and

$$\hat{p}^2 \equiv \left(\frac{1}{i} \vec{\nabla} \right)^2 = -\vec{\nabla}^2$$

by rewriting the equations we obtain

$$\left[\hat{\nabla} + m^2(\vec{x}) - \frac{b}{3}\varphi_0(\vec{x}) \right] \varphi_0(\vec{x}) = 0 \quad (3.39a)$$

and

$$\frac{1}{4}G^{-2}(\vec{x}, \vec{y}) = \left[\hat{p}^2 + \hat{m}^2(\vec{x}) \right] \delta(\vec{x} - \vec{y}) \quad (3.39b)$$

we also cast the second equation into operator form by defining

$$G(\vec{x}, \vec{y}) \equiv \langle \vec{x} | \hat{G} | \vec{y} \rangle$$

and

$$m^2(\vec{x})\delta(\vec{x} - \vec{y}) \equiv \langle \vec{x} | \hat{m}^2 | \vec{y} \rangle$$

$$\frac{1}{4}\hat{G}^{-2} = \hat{p}^2 + \hat{m}^2$$

or

$$\hat{G} = \frac{1}{2\sqrt{\hat{p}^2 + \hat{m}^2}} \quad (3.40)$$

in explicit form

$$G(\vec{x}, \vec{y}) = \langle \vec{x} | \hat{G} | \vec{y} \rangle$$

$$G(\vec{x}, \vec{y}) = \left\langle \vec{x} \left| \frac{1}{2\sqrt{\hat{p}^2 + \hat{m}^2}} \right| \vec{y} \right\rangle$$

$G(\vec{x}, \vec{y})$ is finite if $\vec{x} \neq \vec{y}$ and infinite when $\vec{x} = \vec{y}$. It can be easily seen by taking $m^2(\vec{x})$ to be constant and going into momentum representation for the operator \hat{p}^2 :

$$G(\vec{x}, \vec{x}) = \left\langle \vec{x} \left| \frac{1}{2\sqrt{\hat{p}^2 + \hat{m}^2}} \right| \vec{x} \right\rangle$$

$$= \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{e^{i0}}{2\sqrt{\vec{p}^2 + m^2}}$$

$$= \int_{-\infty}^{+\infty} \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{2\sqrt{\vec{p}^2 + m^2}} = \frac{4\pi}{(2\pi)^3} \int_0^\infty d^2 p \frac{1}{2\sqrt{p^2 + m^2}}$$

$$\xrightarrow{p \rightarrow \infty} \int_0^\infty \frac{p^2 dp}{p} \sim \int_0^\infty p dp \sim p^2 \Big|_0^\infty$$

$G(\vec{x}, \vec{x})$ has at the first order quadratic and to the second order a logarithmic divergence.

Since $G(\vec{x}, \vec{x})$ appears in $m^2(\vec{x})$ and its trace in the energy expression a renormalization scheme is necessary to make sense out of all these terms plagued with infinity. The renormalization method is applied, as explained earlier, by introducing a cut-off Λ in the range of the momentum integral and to choose the mass, $a = a(\Lambda)$, and the coupling constant, $b = b(\Lambda)$ to be cut-off dependent in a way that they will absorb the infinity that appear in the $m^2(\vec{x})$ term. Through this procedure the variationally obtained equations will be well-defined and in turn, the energy expression will be finite up to a infinite constant. This infinite constant will be eliminated by scaling energy with respect to that of the non-interacting free field theory.

B.1.1 $m^2=\text{constant}$ For a homogeneous and isotropic vacuum

$$G(\vec{x}, \vec{y}) = G(\vec{x} - \vec{y})$$

$$\text{and} \tag{3.41}$$

$$\varphi_0(\vec{x}) = \varphi_0 \text{ constant} .$$

It implies $m^2(\vec{x}) = m^2$ constant (independent of \vec{x}). Therefore

$$\begin{aligned} G(\vec{x}, \vec{x}) &= \left\langle \vec{x} \left| \frac{1}{1\sqrt{\hat{\vec{p}}^2 + m^2}} \right| \vec{x} \right\rangle \\ &= \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{e^{i0}}{2\sqrt{\vec{p}^2 + m^2}} \end{aligned} \tag{3.42}$$

$$G(\vec{x}, \vec{x}) = \frac{4\pi}{2(2\pi)^3} \int_0^\Lambda p^2 dp \frac{1}{\sqrt{p^2 + m^2}} \quad (\Lambda \rightarrow \infty)$$

Λ is our regulator (cut-off) that will be taken to infinity later. (See appendix G).

The result of integration is

$$\begin{aligned} G(\vec{x}, \vec{x}) &= \frac{4\pi}{2(2\pi)^3} \left\{ \frac{1}{2} p \sqrt{p^2 + m^2} - \frac{1}{2} m^2 \ln \left(p + \sqrt{p^2 + m^2} \right) \right\} \Big|_0^\Lambda \\ &= \frac{4\pi}{2(2\pi)^3} \left\{ \frac{1}{2} \Lambda \sqrt{\Lambda^2 + m^2} - \frac{1}{2} m^2 \ln \left(\Lambda + \sqrt{\Lambda^2 + m^2} \right) \right. \\ &\quad \left. + \frac{1}{2} m^2 \ln \sqrt{m^2} \right\} \end{aligned} \tag{3.43}$$

we expand it into powers of m/Λ and keep only the constant and the divergent terms, using the following expansion

$$\sqrt{\Lambda^2 + m^2} = \Lambda \sqrt{1 + m^2/\Lambda^2} = \Lambda \left(1 + \frac{m^2}{2\Lambda^2} - \frac{m^4}{8\Lambda^4} + \dots \right)$$

we obtain

$$G(\vec{x}, \vec{x}) = \frac{1}{8\pi^2} \left\{ \Lambda^2 + \frac{1}{2} m^2 \ln \frac{m^2}{\Lambda^2} \right\} + \mathcal{O} \left(\frac{1}{\Lambda} \right) \tag{3.44}$$

where $\ln \alpha = 2 \ln 2 - 1$ $\alpha > 0$. As expected $G(\vec{x}, \vec{x})$ has quadratic and logarithmic divergences as Λ goes to ∞ .

In a similar way $G_{-1}(\vec{x}, \vec{x})$ is found as

$$G^{-1}(\vec{x}, \vec{x}) = \frac{1}{4\pi^2} \left\{ \Lambda^4 + \Lambda^2 m^2 + \frac{1}{4} m^4 \ln \frac{m^2}{\beta \Lambda^2} \right\} \quad (3.45)$$

where $\ln \beta = 2 \ln 2 - \frac{1}{2}$. Now, we can evaluate the algebraic equation for m^2 explicitly

$$m^2 \equiv a + \frac{b}{2} (\varphi_0^2 + G(\vec{x}, \vec{x}))$$

or

$$m^2 = a + \frac{b}{2} \varphi_0^2 + \frac{b/2}{8\pi^2} \left(\Lambda^2 + \frac{1}{2} m^2 \ln \frac{m^2}{\alpha \Lambda^2} \right)$$

by rewriting

$$\ln \frac{m^2}{\alpha \Lambda^2} \equiv \ln \frac{m^2}{\mu_R^2} + \ln \frac{\mu_R^2}{\alpha \Lambda^2}$$

we are allowed to regroup Λ^2 dependent terms as

$$m^2 = \mu_R^2 + \frac{g_R}{2} \left(\varphi_0^2 + \frac{1}{16\pi^2} m^2 \ln \frac{m^2}{\mu_R^2} \right) \quad (3.46)$$

where

$$\mu_R^2 \equiv \frac{a + \frac{b\Lambda^2}{16\pi^2}}{1 - \frac{b}{32\pi^2} \ln \frac{\mu_R^2}{\alpha \Lambda^2}} \quad (3.47a)$$

and

$$g_R \equiv \frac{b}{1 - \frac{b}{32\pi^2} \ln \frac{\mu_R^2}{\alpha \Lambda^2}} \quad (3.47b)$$

μ_R^2 and g_R can be regarded as renormalized mass and coupling constants if μ_R^2 and g_R have a fixed value as Λ goes to infinity then we have well-defined algebraic equations for m^2 which can be regarded as new variational parameters. Conversely,

we will have a well-defined ground state energy up to an infinite constant. At this point we introduce renormalization. Instead of choosing bare mass, a , and bare coupling, b , as constants we choose them to be cut-off dependent for a fixed μ_R^2 and g_R . Their form can be obtained by inverting the above equations, that is

$$b(\Lambda) = \frac{g_R}{1 - \frac{g_R}{32\pi^2} \ln \frac{\alpha\Lambda^2}{\mu_R^2}} \quad (3.48a)$$

$$a(\Lambda) = \frac{\mu_R^2 - \frac{g_R\Lambda^2}{16\pi^2}}{1 - \frac{g_R}{32\pi^2} \ln \frac{\alpha\Lambda^2}{\mu_R^2}} \quad (3.48b)$$

Now we can let the Λ go to infinity for a fixed g_R and μ_R^2

$$b(\Lambda) \xrightarrow{\Lambda \rightarrow \infty} 0^- \quad \text{for a finite } g_R$$

(for further analysis look at K. Huang's paper about triviality of φ^4 theory).Ref.[7]

Now, we can turn our attention to calculating ground state energy explicitly

$$E[\varphi_c, G] = E_{cl} + \frac{1}{8} \text{Tr } G^{-1} + \frac{1}{2} \text{TR}(GK) + \frac{b}{8} \text{TR TR}(GG)$$

where

$$\frac{1}{4} \hat{G}^{-2} = \hat{K} + \frac{b}{2} G \hat{I} \quad (3.49)$$

if I substitute \hat{K} in the above equation

$$\begin{aligned} E[\phi, G] &= E_{cl} + \frac{1}{8} \text{TR } G^{-1} + \frac{1}{2} \text{TR} \left[G \left(\frac{1}{4} G^{-2} - \frac{b}{2} G \hat{I} \right) \right] + \frac{b}{8} \text{TR TR}(GG) \\ &= E_{cl} + \frac{1}{4} \text{TR}(G^{-1}) - \frac{b}{8} \text{TR TR}(GG) \end{aligned} \quad (3.50)$$

$G(\vec{x}, \vec{x})$ can also be eliminated in favor of $m^2(\vec{x})$ using

$$m^2(\vec{x}) = a + \frac{b}{2} \varphi_0^2 + \frac{b}{2} G(\vec{x}, \vec{x})$$

or

$$G(\vec{x}, \vec{x}) = \frac{2}{b} (m^2 - a) - \varphi_0^2 \quad (3.51)$$

substituting it in $E[\phi_i, G]$ gives, for the time being ignoring the trace

$$\begin{aligned} E &= E_{\text{cl}} + \frac{1}{4}G^{-1} - \frac{b}{8} \left[\frac{2}{b}(m^2 - a) - \varphi_0^2 \right]^2 \\ &= \frac{1}{2}a\varphi_0^2 + \frac{b}{24}\varphi_0^4 + \frac{1}{4}G^{-1} - \frac{b}{8}\varphi_0^4 - \frac{(m^2 - a)^2}{2b} + \frac{1}{2}\varphi_0^2(m^2 - a) \\ &= +\frac{1}{2}m^2\varphi_0^2 - \frac{b}{12}\varphi_0^4 + \frac{1}{4}G^{-1} - \frac{(m^2 - a)^2}{2b} \end{aligned} \quad (3.52)$$

when we remove the cut-off ($\lambda \rightarrow \infty$) we have seen that $b \rightarrow 0^-$ therefore $-\frac{b}{12}\varphi_0^4$ term goes to zero. Henceforth, we will ignore it.

The overall result

$$E = \frac{1}{2}m^2\varphi_0^2 + \frac{1}{4}G^{-1} - \frac{(m^2 - a)^2}{2b} \quad \text{traced over} \quad (3.53)$$

The term $\frac{(m^2 - a)^2}{b}$ implies that ground state energy can not be obtained perturbatively.

Let's define

$$A \equiv 1 - \frac{g_R}{32\pi^2} \ln \frac{\alpha\Lambda^2}{\mu_R^2} \quad (3.54)$$

Then

$$\begin{aligned} b &= g_R/A \\ a &= \left(\mu_R^2 - \frac{g_R\Lambda^2}{16\pi^2} \right) / A \end{aligned}$$

Now if we substitute G^{-1} , a and b in E

$$\begin{aligned} E &= \frac{1}{2}m^2\varphi_0^2 + \frac{1}{(4\pi)^2} \left(\Lambda^4 + \Lambda^2m^2 + \frac{1}{4}m^4 \ln \frac{m^2}{\beta\Lambda^2} \right) \\ &\quad - \frac{A}{2} \left(\frac{m^4 - 2m^2 \left(\mu_R^2 - \frac{g_R\Lambda^2}{16\pi^2} \right) / A + \left(\mu_R^2 - \frac{g_R\Lambda^2}{16\pi^2} \right)^2 / A^2}{g_R} \right) \end{aligned}$$

ignoring any term which does not have either φ_0 or m^2 we are allowed to do this since all other terms are constant with respect to variation

$$\begin{aligned}
E &= \frac{1}{2}m^2\varphi_0^2 + \frac{1}{(4\pi)^2}\Lambda^2m^2 + \frac{1}{4(4\pi)^2}m^4 \ln \frac{m^2}{\beta\Lambda^2} \\
&\quad - \frac{A}{2} \frac{m^4}{g_R} + \frac{m^2 \left(\mu_R^2 - \frac{g_R\Lambda^2}{16\pi^2} \right)}{g_R} + \frac{\mu_R^2}{2g_R} \\
&\quad + \text{ignored terms} .
\end{aligned}$$

or

$$\begin{aligned}
&= \frac{1}{2}m^2\varphi_0^2 + \frac{1}{(4\pi)^2}\Lambda^2m^2 + \frac{1}{4(4\pi)^2}m^4 \ln \frac{m^2}{\beta\Lambda^2} \\
&\quad - \frac{m^4}{2g_R} + \frac{m^4}{64\pi^2} \ln \frac{\alpha\lambda^2}{\mu_R^2} + \frac{2m^2\mu_R^2}{2g_R} - m^2\Lambda^2/16\pi^2 + \frac{\mu_R^2}{2g_R} \\
&= \frac{1}{2}m^2\varphi_0^2 + \frac{1}{64\pi^2}m^4 \left[\ln \frac{m^2}{\beta\Lambda^2} + \ln \frac{\alpha\Lambda^2}{\mu_R^2} \right] - \frac{(m^2 - \mu_R^2)^2}{2g_R} \\
E &= \frac{1}{2}m^2\varphi_0^2 + \frac{1}{64\pi^2}m^4 \left[\ln \frac{m^2}{\mu_R^2} - \frac{1}{2} \right] \\
&\quad - \frac{(m^2 - \mu_R^2)^2}{2g_R} + (\text{infinite constant})
\end{aligned}$$

Since φ_0 and m^2 are constants, the trace reduces to a volume integral. Defining $V = \int d^3\vec{x}$ we get the overall result as

$$\begin{aligned}
E/V = \epsilon(m^2, \varphi_0) &= \frac{1}{2}m^2\varphi_0^2 + \frac{1}{64\pi^2}m^4 \left[\ln \frac{m^2}{\mu_R^2} - \frac{1}{2} \right] \\
&\quad - \frac{(m^2 - \mu_R^2)^2}{2g_R} + \frac{\mu_R^4}{128\pi^2}
\end{aligned} \tag{3.55}$$

we added a constant $\mu_R^4/128\pi^2$ term to scale the energy with respect to that of non-interacting free scalar field theory, that is

$$\epsilon(m^2 = \mu_R^2, \varphi_0 = 0) = 0 \tag{3.56}$$

chosen to be zero where

$$m^2 = \mu_R^2 + \frac{g_R}{2} \left(\varphi_0^2 + \frac{1}{16\pi^2}m^2 \ln \frac{m^2}{\mu_R^2} \right) \tag{3.57}$$

when $g_R = 0$ $m^2 = \mu_R^2$ is a solution.

To check the consistency of our solution we take the variation of energy density $\epsilon(m^2, \varphi_0^2)$ with respect to m^2

$$\frac{\delta \epsilon}{\delta m^2} = 0$$

$$\frac{1}{2} \varphi_0^2 + \frac{1}{32\pi^2} m^2 \left(\ln \frac{m^2}{\mu_R^2} - \frac{1}{2} \right) + \frac{1}{64\pi^2} m^2 - \frac{(m^2 - \mu_R^2)}{g_R} = 0$$

$$m^2 = \mu_R^2 + \frac{g_R}{2} \left(\varphi_0^2 + \frac{1}{16\pi^2} m^2 \ln \frac{m^2}{\mu_R^2} \right)$$

consistent with the defining equation for m^2 .

Analyses of m^2 :

Let us set $\varphi_0^2 = 0$ and define $x \equiv m^2/\mu_R^2$, $0 < x < \infty$, then the transcendental equation (3.46) is

$$x = 1 + \frac{g_R}{2(4\pi)^2} x \ln x \quad (3.58)$$

One of the solutions is $x = 1$ for a given g_R ($-\infty > g_R < \infty$). $x = 1 \implies m^2(g_R) = \mu_R^2$ for all values of g_R . To find out if it leads to a minimum for energy density we investigate the stability condition note $\epsilon(m^2, \mu_R^2) = \mu_R^4 \epsilon(x)$

$$\frac{\partial^2 \epsilon}{\partial x^2} \geq 0 \quad (3.59)$$

where

$$\epsilon(x) = \frac{1}{64\pi^2} x^2 \left(\ln x - \frac{1}{2} \right) - \frac{(x-1)^2}{2g_R} + \frac{1}{128\pi^2} \quad (3.60)$$

$$\frac{\partial^2 \epsilon}{\partial x^2} = \frac{1}{32\pi^2} \left(\ln x - \frac{1}{2} \right) + \frac{1}{32\pi^2} + \frac{1}{64\pi^2} - \frac{1}{g_R}$$

the stability condition is

$$\begin{aligned} \frac{1}{32\pi^2} \ln x + \frac{1}{32\pi^2} - \frac{1}{g_R} &\geq 0 \\ \ln x &\geq \frac{2(4\pi)^2}{g_R} - 1 \end{aligned}$$

The $x = 1$ solution puts the following constraint on g_R to be a the minimum of the energy density

$$\begin{aligned} 0 &\geq \frac{2(4\pi)^2}{g_R} - 1 \\ \text{if } g_R > 0 &\text{ then } g_R \geq 2(4\pi)^2 \\ \text{if } g_R < 0 &\text{ then } g_R \leq 2(4\pi)^2 \end{aligned} \tag{3.61}$$

So we can conclude that the values of g_R between $0 < g_R < 2(4\pi)^2$ do not lead to a stable minimum. The corresponding energy density is

$$\epsilon(x = 1) = 0 \tag{3.62}$$

If we try to expand $x(g_R)$ into powers of g_R we realize that the transcendental equation leads to

$$\left. \frac{d^n x(g_R)}{dg_R^n} \right|_{g_R=0} = 0 \tag{3.63}$$

for every n ($n = 0, 1, 2, \dots$). Similarly there is no way of expanding the energy density $\epsilon(x)$ to the powers of g_R . Simply, it is non-perturbative. If we closely examine Eq.(3.58) we see that it has another solution. By writing it in the following way

$$\begin{aligned} x &= 1 + gx \ln x \quad \text{where } g = g_R/32\pi^2 \\ 0 &= 1 + gx \ln \left(e^{-1/g} x \right) \end{aligned} \tag{3.64}$$

and defining $v(g) = x(g)e^{-1/g}$ we obtain

$$v \ln v = -\frac{e^{-1/g}}{g} \equiv -\eta(g) \tag{3.65}$$

Since $\eta(g)$ ($0 \leq \eta(g) \leq e^{-1}$) for $0 \leq g < \infty$ we can use it as an expansion parameter to expand $v(g) = v(\eta(g))$ around the $\eta = 0$ using eq.(3.65) The result is

$$v(g) = 1 - \frac{e^{-1/g}}{g} + \frac{1}{2} \frac{e^{-2/g}}{g^2} + \frac{4}{3!} \frac{e^{-3/g}}{g^3} - \frac{9}{4!} \frac{e^{-4/g}}{g^4} + \dots \quad (3.66)$$

and in terms of $x(g)$

$$x(g) = e^{1/g} \left(1 - \frac{e^{-1/g}}{g} + \frac{e^{-2/g}}{2g^2} + \frac{2}{3} \frac{e^{-3/g}}{g^2} - \frac{3}{8} \frac{e^{-4/g}}{g^4} + \dots \right) \quad (3.67)$$

where $g = g_R/32\pi^2$. The solution $x(g)$ is singular at $g = 0$ as $g \rightarrow \infty$, and $x(g \rightarrow \infty) \rightarrow 1$. To see the variation of x as an a function of g we plot

$$x = 1 + gx \ln x \quad (3.68)$$

g dependence of x is trivial

Graphically, the dependence of $\epsilon(x)$ on \sqrt{x} for a given values of g is shown in fig-6.

$$\epsilon(x) = x^2 \left(\ln x - \frac{1}{2} \right) - \frac{(x-1)^2}{g} + \frac{1}{2} \quad (3.69)$$

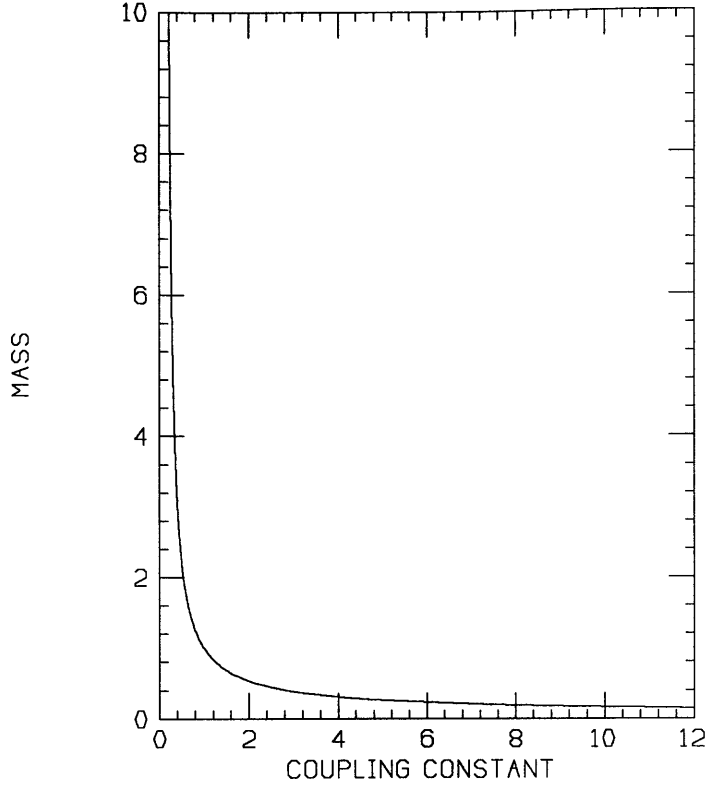
Looking at fig-6a ,we clearly,see that $\epsilon(x)$ has two stationary values ,at $x = 0$ and $x = 1$ the true minimum is at the $x = 1$.As g goes to minus infinity the true minimum still is at $x = 1$.

Looking at fig-6b , we see that it has two minima depending on the value of the g one of them becomes the true minimum of the system.

Since the presence of z shifts the position of the singularities we investigate x as a function of z and g .

$$x = 1 + gz + gx \ln x \quad (3.70)$$

fig-5



If z is greater than e^{-1} there is no singularity but for $0 < z < e^{-1}$ at the value of $x_0, (x_0 \ln x_0 + z = 0)$ a double singularity develops.

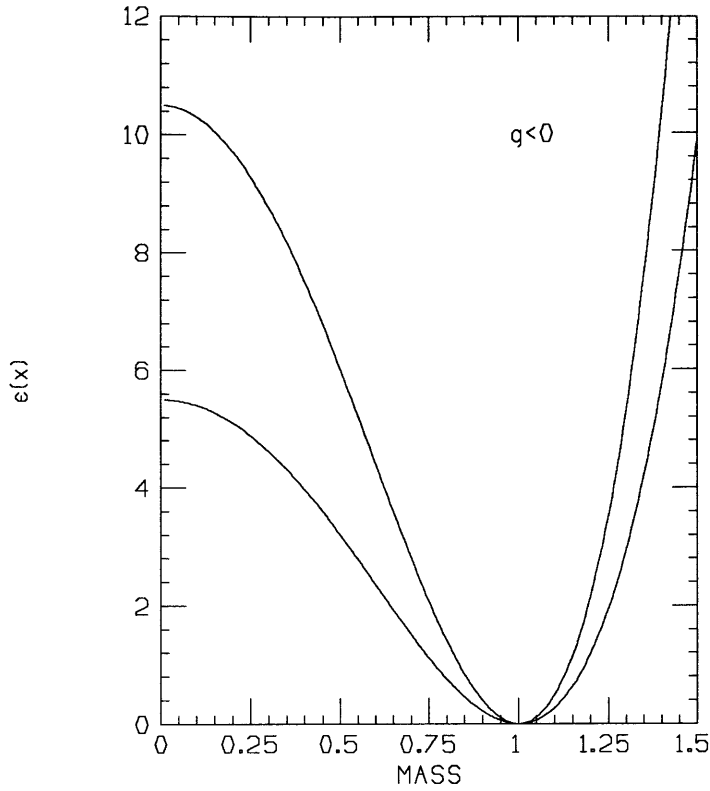
Finally, we write $\epsilon(x, z)$ in terms of x and z

$$\epsilon(z, x) = zx + x^2 \left(\ln x - \frac{1}{2} \right) - \frac{(x-1)^2}{g} + \frac{1}{2} \quad (3.71)$$

Where

$$z = 16\pi^2 \varphi_0^2 / \mu_R^2 \quad g = g_R / 32\pi^2 \quad (3.72)$$

fig-6a



As for the stability of the second solution ,stability condition (3.59) requires

$$\ln x \geq \frac{1}{g} - 1 \quad (3.73)$$

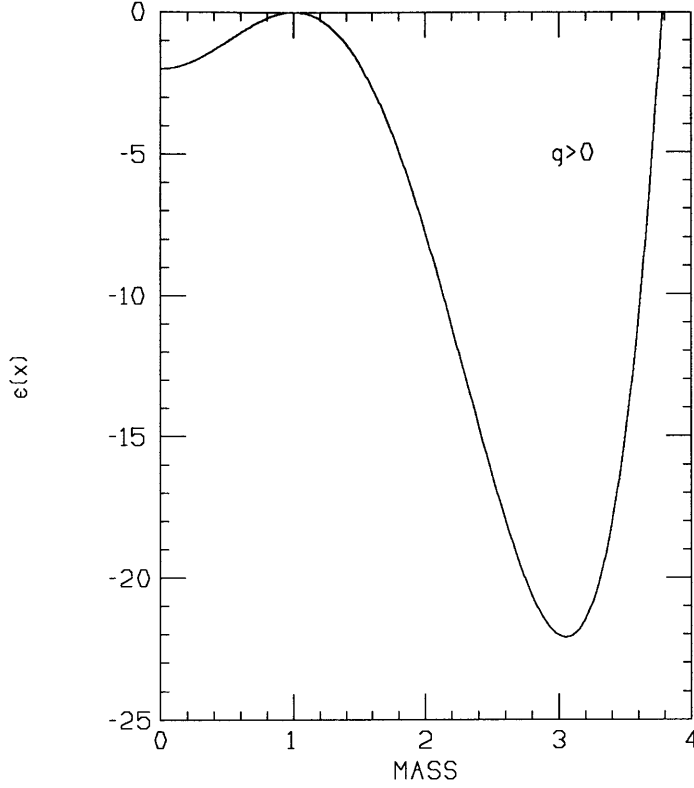
Eliminating x in favor of g gives

$$2 - g \leq \frac{1}{x(g)} \quad (3.74)$$

this equation determines the range of g in which the $\epsilon(x)$ has a stable minimum.

B.1.2 $m^2 \neq \text{constant}$

fig-6b



After gaining insight into the calculation by assuming m^2 and φ_0 are constant.

Now, we want to find out how the result will be modified if $m^2(\vec{x})$ and $\varphi_0(\vec{x})$ are \vec{x} -dependent.

Our calculations up to equations (3.40) were general

$$m^2(\vec{x}) = a + \frac{b}{2} [\varphi_0^2(\vec{x}) + G(\vec{x}, \vec{x})] \quad (3.74a)$$

and

$$G(\vec{x}, \vec{x}) = \left\langle \vec{x} \left| \frac{1}{2\sqrt{\hat{p}^2 + \hat{m}^2}} \right| \vec{x} \right\rangle \quad (3.74b)$$

Since the \hat{p}^2 operator does not commute \hat{m}^2 , we cannot work in momentum space to evaluate the matrix element.

Our goal is to identify the finite and divergent part of the operator. To achieve it we make a formal expansion of the operator around $\hat{m}^2 = \mu_R^2 \hat{I}$ up to an order in which the terms of expansion is divergent. We separate these divergent terms from the rest of the finite ones. In this way we obtain the finite part as the matrix element of a finite operator and divergent parts explicitly.

Let

$$G^{-1}(\vec{x}, \vec{x}) = \left\langle \vec{x} \left| 2\sqrt{\hat{p}^2 + \hat{m}^2} \right| \vec{x} \right\rangle$$

we will first obtain finite and divergent parts of $G^{-1}(\vec{x}, \vec{x})$ and afterwards determine $G(\vec{x}, \vec{x})$ from the following relationship

$$G(\vec{x}, \vec{x}) = \frac{1}{2} \frac{\delta}{\delta \hat{m}^2} G^{-1}(\vec{x}, \vec{x}) \quad (3.75)$$

Expand $\sqrt{\hat{p}^2 + \hat{m}^2}$ formally around $\hat{m}^2 = \mu_R^2 \hat{I}$ up to second order.

$$\begin{aligned} \sqrt{\hat{p}^2 + \hat{m}^2} &= \sqrt{\hat{p}^2 + \mu_R^2} + \frac{1}{2}(\hat{m}^2 - \mu_R^2) [\hat{p}^2 + \mu_R^2]^{-1/2} \\ &\quad - \frac{1}{8} (\hat{m}^2 - \mu_R^2)^2 [\hat{p}^2 + \mu_R^2]^{-3/2} + \frac{1}{2} \hat{G}_F^{-1} \end{aligned} \quad (3.76)$$

by taking the matrix elements we obtain the following expression

$$\begin{aligned} G^{-1}(\vec{x}, \vec{x}) &= G_F^{-1}(\vec{x}, \vec{x}) + \left\langle \vec{x} \left| 2\sqrt{\hat{p}^2 + \mu_R^2} \right| \vec{x} \right\rangle \\ &\quad + (m^2(\vec{x}) - \mu_R^2) \left\langle \vec{x} \left| [\hat{p}^2 + \mu_R^2]^{-1/2} \right| \vec{x} \right\rangle \\ &\quad - \frac{1}{4} (m^2(\vec{x}) - \mu_R^2)^2 \left\langle \vec{x} \left| [\hat{p}^2 + \mu_R^2]^{-3/2} \right| \vec{x} \right\rangle \end{aligned} \quad (3.77)$$

where

$$G_F^{-1}(\vec{x}, \vec{x}) = \left\langle \vec{x} \left| \left\{ 2\sqrt{\hat{p}^2 + \hat{m}} - 2\sqrt{\hat{p}^2 + \mu_R^2} - (\hat{m}^2 - \mu_R^2) [\hat{p}^2 + \mu_R^2]^{1/2} + \frac{1}{4} (\hat{m}^2 - \mu_R^2)^2 [\hat{p}^2 + \mu_R^2]^{-3/2} \right\} \right| \vec{x} \right\rangle \quad (3.78)$$

As you may have noticed in the expansion we have chosen a particular ordering of the operators in divergent terms. Later we will see in a formal solution of the problem that the divergent terms are linear and quadratic in $m^2(\vec{x})$.

We ignore $\left\langle \vec{x} \left| 2\sqrt{\hat{p}^2 + \mu_R^2} \right| \vec{x} \right\rangle$ in $G^{-1}(\vec{x}, \vec{x})$ since its contribution to energy will be nothing more than an infinite constant. We evaluate $\left\langle \vec{x} \left| [\hat{p}^2 + \mu_R^2]^{-1/2} \right| \vec{x} \right\rangle$ using the result of Eq. (3.44)

$$\left\langle \vec{x} \left| [\hat{p}^2 + \mu_R^2]^{-1/2} \right| \vec{x} \right\rangle = \frac{1}{4\pi^2} \left\{ \Lambda^2 + \frac{1}{2} \mu_R^2 \ln \frac{\mu_R^2}{\alpha \Lambda^2} \right\} \quad (3.79)$$

and

$$\begin{aligned} \left\langle \vec{x} \left| [\hat{p}^2 + \mu_R^2]^{-3/2} \right| \vec{x} \right\rangle &= -2 \frac{\delta}{\delta \mu_R^2} \left\langle \vec{x} \left| [\hat{p}^2 + \mu_R^2]^{-1/2} \right| \vec{x} \right\rangle \quad (3.79a) \\ &= -\frac{1}{2\pi^2} \left\{ \frac{1}{2} \ln \frac{\mu_R^2}{\alpha \Lambda^2} + \frac{1}{2} \right\} \\ &= -\frac{1}{4\pi^2} \left\{ 1 + \ln \frac{\mu_R^2}{\alpha \Lambda^2} \right\} \end{aligned}$$

The overall result is

$$\begin{aligned} G^{-1}(\vec{x}, \vec{x}) &= G_F^{-1}(\vec{x}, \vec{x}) + \frac{1}{4\pi^2} (m^2(\vec{x}) - \mu_R^2) \left\{ \Lambda^2 + \frac{1}{2} \mu_R^2 \ln \frac{\mu_R^2}{\alpha \Lambda^2} \right\} \quad (3.80a) \\ &\quad + \frac{1}{(4\pi)^2} (m^2(\vec{x}) - \mu_R^2)^2 \left\{ 1 + \ln \frac{\mu_R^2}{\alpha \Lambda^2} \right\} \end{aligned}$$

and

$$\begin{aligned} G(\vec{x}, \vec{x}) &= G_F(\vec{x}, \vec{x}) + \frac{1}{8\pi^2} \left(\Lambda^2 + \frac{1}{2} \mu_R^2 \ln \frac{\mu_R^2}{\alpha \Lambda^2} \right) \quad (3.80b) \\ &\quad + \frac{1}{(4\pi)^2} (m^2(\vec{x}) - \mu_R^2) \left(1 + \ln \frac{\mu_R^2}{\alpha \Lambda^2} \right) \end{aligned}$$

where

$$G_F(\vec{x}, \vec{x}) = \left\langle \vec{x} \left| \left\{ \frac{1}{2} [\hat{p}^2 + m^2]^{-1/2} - \frac{1}{2} [\hat{p}^2 + \mu_R^2]^{-1/2} + \frac{1}{4} (m^2 - \mu_R^2) [\hat{p}^2 + \mu_R^2]^{-3/2} \right\} \right| \vec{x} \right\rangle \quad (3.81a)$$

$$G(\vec{x}, \vec{x}) = G_F(\vec{x}, \vec{x}) - \frac{1}{(4\pi^2)} \mu_R^2 + \frac{\Lambda^2}{8\pi^2} + \frac{1}{(4\pi)^2} m^2(\vec{x}) \left(1 + \ln \frac{\mu_R^2}{\alpha\Lambda} \right) \quad (3.81b)$$

To renormalize the defining relation for $m^2(x)$ we insert $G(\vec{x}, \vec{x})$ into Eq. (3.74a)

and group divergences by inserting (3.47) for μ_R^2 and g_R we obtain

$$m^2(\vec{x}) = \mu_R^2 + \frac{1}{2} g_R \left[\varphi_0^2(\vec{x}) + \tilde{G}_F(\vec{x}, \vec{x}) \right] \quad (3.82)$$

where

$$\tilde{G}_F(\vec{x}, \vec{x}) = G_F(\vec{x}, \vec{x}) + \frac{1}{16\pi^2} (m^2(\vec{x}) - \mu_R^2) \quad (3.83)$$

Ground state energy from Eq. (3.49) is

$$E[\varphi_0, G] = \int d^3\vec{x} \left\{ \frac{1}{2} (\nabla\varphi_0)^2 + \frac{1}{2} a\varphi_0^2 + \frac{b}{24}\varphi_0^4 \right\} + \frac{1}{8} \text{TR}(G^{-1}) + \frac{1}{2} \text{TR}(GK) + \frac{b}{8} \text{TR} \text{TR}(GG) \quad (3.84)$$

where

$$\frac{1}{4} G^{-2} = K + \frac{b}{2} G\hat{I}$$

We rewrite the energy density $\epsilon(\vec{x})$ as

$$\epsilon(\vec{x}) = \frac{1}{2} (\nabla\varphi_0)^2 + \frac{1}{2} a\varphi_0^2 + \frac{b}{24}\varphi_0^4 + \frac{1}{4} G^{-1} - \frac{b}{8} GG \quad (3.85)$$

using

$$m^2(\vec{x}) = a + \frac{b}{2} (\varphi_0^2 + G(x, x))$$

$$G(x, x) = \varphi_0^2 + \frac{2}{b} (m^2(\vec{x}) - a)$$

$$\begin{aligned}
\epsilon(\vec{x}) &= \frac{1}{2}(\nabla\varphi_0)^2 + \frac{1}{2}a\varphi_0^2 + \frac{b}{24}\varphi_0^4 + \frac{1}{4}G^{-1} - \frac{b}{8}\left[\frac{2}{b}(m^2 - a) - \varphi_0^2\right]^2 \\
&= \frac{1}{2}(\nabla\varphi_0)^2 + \frac{1}{2}a\varphi_0^2 + \frac{b}{24}\varphi_0^4 + \frac{1}{4}G^{-1} - \frac{(m^2 - a)^2}{2b} - \frac{b}{8}\varphi_0^4 \\
&\quad + \frac{1}{2}(m^2 - a)\varphi_0^2 \\
\epsilon(x) &= \frac{1}{2}(\nabla\varphi_0)^2 + \frac{1}{2}m^2\varphi_0^2 - \frac{b}{12}\varphi_0^4 + \frac{1}{4}G^{-1} - \frac{(m^2 - a)^2}{2b} \tag{3.86}
\end{aligned}$$

We can ignore $\frac{b}{12}\varphi_0^4$, b will go to zero as we remove the cut-off, over all terms which have the divergences are

$$\frac{1}{4}G^{-1} - \frac{(m^2 - a)^2}{2b} \tag{3.87}$$

$$G^{-1}(\vec{x}, \vec{x}) = \tilde{G}_F^{-1}(\vec{x}, \vec{x}) + \frac{1}{4\pi^2} \left\{ m^2\Lambda^2 + \frac{1}{4}m^4 \ln \frac{\mu_R^2}{\alpha\Lambda^2} \right\} \tag{3.88}$$

where

$$\tilde{G}_F^{-1}(\vec{x}, \vec{x}) = G_F^{-1}(\vec{x}, \vec{x}) + \frac{1}{(4\pi)^2} (m^2(\vec{x}) - \mu_R^2)^2 \tag{3.88a}$$

if we substitute a , b and G^{-1} into $\epsilon(x)$

$$\begin{aligned}
\epsilon(x) &= \frac{1}{2}(\nabla\varphi_0)^2 + \frac{1}{2}m^2\varphi_0^2 + \frac{1}{3}\tilde{G}_F^{-1}(\vec{x}, \vec{x}) + \frac{1}{(4\pi)^2}m^2\Lambda^2 \\
&\quad + \frac{1}{4(4\pi)^2}m^4 \ln \frac{\mu_R^2}{\alpha\Lambda^2} - \frac{\left[m^2 - \left[\mu_R^2 - \frac{g_R\Lambda^2}{16\pi^2} \right] / A \right]^2}{2g_R/A} \tag{3.89}
\end{aligned}$$

where

$$A = 1 + \frac{g_R}{32\pi^2} \ln \frac{\mu_R^2}{\alpha\Lambda^2}$$

After some calculations we obtain finite ground state energy density as

$$\epsilon(x) = \frac{1}{2}(\nabla\varphi_0(x))^2 + \frac{1}{2}m^2(\vec{x})\varphi_0^2(\vec{x}) + \frac{1}{4}\tilde{G}_F^{-1}(\vec{x}, \vec{x}) - \frac{1}{2g_R} (m^2(\vec{x}) - \mu_R^2)^2 \tag{3.90}$$

$$E[\varphi_0, m^2(\vec{x})] = \int d^3\vec{x} \epsilon(\vec{x})$$

where $m^2(\vec{x})$ is giving by (3.88) in case $m^2(\vec{x}) = \text{constant}$ \tilde{G}_F and \tilde{G}_F^{-1} are equivalent to the relations (3.43)–(3.45), respectively.

B.2 Time-Dependent Variational Calculation

Using Dirac's procedure we define an effective Lagrangian and effective action

$$\begin{aligned}\mathcal{L}(t) &= \left\langle \psi \left| \left(i \frac{\partial}{\partial t} - \hat{\mathcal{H}} \right) \right| \psi \right\rangle / \langle \psi | \psi \rangle \\ I &= \int dt \mathcal{L}(t)\end{aligned}\tag{3.91}$$

where $\hat{\mathcal{H}}$ Hamiltonian of the theory and $|\psi\rangle$ is our trial wave functional with undetermined parameters.

The overall idea is to introduce the parameters in pairs in a way that, after we evaluate the effective Lagrangian, they will turn out to be canonical to each other. This procedure reduces the Quantum Field Theory to the Classical Field Theory over parameters by retaining quantum corrections.

Equations governing time-development of parameters are obtained by setting the variation of the effective action with respect to parameters to zero. Setting

$$\delta I = 0 \quad \text{end-points are fixed}$$

These are nothing more than classical Hamilton–Jacobi equations. The canonical nature of the parameters allows us to identify the effective Hamiltonian.

Time-dependence of the state wave functional is introduced through the parameters and the trial wave functional is parameterized by two pairs of conical parameters $(\varphi_0(\vec{x}, t), \Pi_0(\vec{x}, t))$ and $(Q(\vec{x}, \vec{y}, t), P(\vec{x}, \vec{y}, t))$.

By construction $\varphi_0(\vec{x}, t)$ is the expectation of the field operator $\hat{\varphi}(\vec{x}, t)$

$$\langle \psi | \hat{\varphi}(\vec{x}, t) | \psi \rangle = \varphi_0(\vec{x}, t) \langle \psi | \psi \rangle\tag{3.92}$$

and $Q^2(\vec{x}, \vec{y}, t)$ is the propagator related to the expectation value of $\hat{\varphi}(\vec{x}, t)\hat{\varphi}(\vec{y}, t)$ operator in the presence of $\varphi_0(\vec{x}, t)$

$$\langle \psi | \hat{\varphi}(\vec{x}, t)\hat{\varphi}(\vec{y}, t) | \psi \rangle = [\varphi_0(\vec{x}, t)\varphi_0(\vec{y}, t) + G(\vec{x}, \vec{y}, t)] \langle \psi | \psi \rangle \quad (3.93)$$

where

$$G(\vec{x}, \vec{y}; t) = \int d\vec{z} Q(\vec{x}, \vec{z}; t)Q(\vec{z}, \vec{y}; t)$$

Our trial wave functional is

$$\begin{aligned} |\psi\rangle \rightarrow \psi[\Pi_0, \varphi_0, P, Q] = N \exp \left\{ i \int d\vec{x} \Pi_0(\vec{x}, t) [\varphi(\vec{x}) - \varphi_0(\vec{x}, t)] \right. \\ \left. - \frac{1}{4} \int d\vec{x} d\vec{y} (\varphi(\vec{x}) - \varphi_0(\vec{x}, t)) \left[Q^{-2}(\vec{x}, \vec{y}; t) - 2i \int d\vec{z} Q^{-1}(\vec{x}, \vec{z}, t)P(\vec{z}, \vec{y}, t) \right] \right. \\ \left. (\varphi(\vec{y}) - \varphi_0(\vec{y}, t)) \right\} \end{aligned} \quad (3.94)$$

and the Hamiltonian is given by (1.20)

The canonical momenta for φ_0 and Q is introduced through the phase factor of the wave functional so that when we evaluate $\langle \psi | \frac{\partial}{\partial t} | \psi \rangle$ term in effective Lagrangian will reveal the canonical nature of the parameters.

$$\begin{aligned} \left\langle \psi \left| i \frac{\partial}{\partial t} \right| \psi \right\rangle &= \int D(\varphi) |\psi|^2 i \frac{\partial}{\partial t} \quad (3.95) \\ &\times \left\{ \int i \Pi_c(\varphi - \varphi_0) - \frac{1}{4} \int \int (\varphi - \varphi_c) [Q^{-2} - 2iQ^{-1}P] (\varphi - \varphi_c) \right\} \\ &= \langle \psi | \psi \rangle \int dx \Pi_c(\vec{x}, t) \dot{\varphi}_0(\vec{x}, t) + -\frac{i}{4} \int d\vec{x} d\vec{y} \left\langle \vec{x} \left| \frac{d}{dt} \right. \right. \\ &\times \left((Q^{-2} - 2iQ^{-1}P) \int D(\varphi) |\psi|^2 (\varphi(\vec{x}) - \varphi_0(\vec{x}, t)) (\varphi(\vec{y}) - \varphi_0(\vec{y}, t)) \right) \\ &= \langle \psi | \psi \rangle \left\{ \int d\vec{x} \Pi_c(\vec{x}, t) \dot{\varphi}_c(\vec{x}, t) + -\frac{i}{4} \int d\vec{x} d\vec{y} \left\langle \vec{x} \left| \frac{d}{dt} (Q^{-2} - 2iQ^{-1}P) \right| \vec{y} \right\rangle \right. \\ &\times \left. \langle \vec{x} | Q^2 | \vec{y} \rangle \right\} \end{aligned}$$

The last term in Eq. (3.95)

$$-\frac{i}{4} \int d\vec{x} d\vec{y} \left\langle \vec{x} \left| \frac{d}{dt} (-2iQ^{-1}P) \right| \vec{y} \right\rangle \langle \vec{x} | Q^2 | \vec{y} \rangle$$

can be written as

$$-\frac{1}{2} \text{TR} \left(Q^2 \frac{d}{dt} (Q^{-1}P) \right)$$

using

$$\frac{d}{dt} Q^{-1}(t) = -Q^{-1}(t) \dot{Q}(t) Q^{-1}(t)$$

$$-\frac{1}{2} \text{TR} \left(Q \dot{P} - Q \dot{Q} Q^{-1} P \right)$$

ignoring the total time derivative

$$\text{TR}(P \dot{Q}) + \frac{1}{2} \text{TR} \left(Q \dot{Q} Q^{-1} P - \dot{Q} P \right) \quad (3.92)$$

in general $\hat{Q}(t)$ does not commute with $\dot{\hat{Q}}$; therefore, the second term of (3.99) is a contribution to the Hamiltonian. In the case $\hat{Q}(t)$ and $\dot{\hat{Q}}(t)$ commute it is zero in what follows that is what we will assume.

Ignoring the total time derivatives of the terms final result is

$$\left\langle \psi \left| i \frac{\partial}{\partial t} \right| \psi \right\rangle = \left\{ \int d\vec{x} \Pi_c(\vec{x}, t) \dot{\phi}_c(\vec{x}, t) + \int d\vec{x} \left\langle \vec{x} \left| \hat{P} \hat{Q} \right| \vec{x} \right\rangle \right\} \langle \psi | \psi \rangle \quad (3.97)$$

we used the following properties

$$Q(\vec{x}, \vec{y}; t) = Q(\vec{y}, \vec{x}; t)$$

and

$$\text{TR} \left(Q^{-1} \frac{d}{dt} Q \right) = \frac{d}{dt} \text{TR}(\ln Q)$$

After similar calculations done in section B.1

$$\langle \psi | \hat{\mathcal{H}} | \psi \rangle / \langle \psi | \psi \rangle = E_{\text{cl}} + \text{TR} \left(\frac{1}{2} P^2 + \frac{1}{8} G^{-1} + \frac{1}{2} \hat{G} K + \frac{b}{8} \hat{G} \hat{G} \right) \quad (3.98)$$

where

$$E_{\text{cl}} = \int d\vec{x} \left\{ \frac{1}{2} \Pi_0^2(\vec{x}, t) + \frac{1}{2} \left[\vec{\nabla} \varphi_x(\vec{x}, t) \right]^2 + \frac{a}{2} \varphi_0^2(\vec{x}, t) + \frac{b}{24} \varphi_0^4(\vec{x}, t) \right\} \quad (3.98a)$$

and

$$\begin{aligned} \hat{G} &= \hat{Q}^2 \\ \hat{K} &= \hat{p}^2 + a + \frac{b}{2} \left(\varphi_c^2 + \hat{G}_d \right) \quad \hat{G}_d = \langle x | G | x \rangle \hat{I} \end{aligned}$$

The effective Lagrangian is

$$\mathcal{L}(t) = \int d\vec{x} \left(\Pi_0 \dot{\varphi}_0 + \langle \vec{x} | P \dot{Q} | \vec{x} \rangle \right) - \mathcal{H}(\Pi_0, \varphi_0, R, Q) \quad (3.99)$$

where the effective Hamiltonian is

$$\mathcal{H}(\Pi_0, \varphi_0, P, Q) = E_{\text{cl}} + \text{TR} \left(\frac{1}{2} P^2 + \frac{1}{8} G^{-1} + \frac{1}{2} G K + \frac{b}{8} G G \right) \quad (3.99a)$$

Using the Hamilton–Jacobi equations we do arrive at the following equations of motion for the mean field φ_0

$$\dot{\varphi}_0(\vec{x}, t) = \frac{\delta \mathcal{H}}{\delta \Pi_0(\vec{x}, t)}$$

i.e.

$$\begin{aligned} \dot{\varphi}_0(\vec{x}, t) &= \Pi_0(\vec{x}, t) \\ -\dot{\Pi}_0(\vec{x}, t) &= \frac{\delta \mathcal{H}}{\delta \varphi_0(\vec{x}, t)} \end{aligned}$$

i.e.

$$\begin{aligned} -\dot{\Pi}_0(\vec{x}, t) &= + \frac{\delta E_{\text{cl}}}{\delta \varphi_0(\vec{x}, t)} + \frac{b}{2} (\vec{x}, \vec{x}; t) \varphi_0(\vec{x}, t) \\ -\ddot{\varphi}_0(\vec{x}, t) &= -\nabla^2 \varphi_0(\vec{x}, t) + a \varphi_0(\vec{x}, t) + \frac{b}{8} \varphi_0^3(\vec{x}, t) + \frac{b}{2} G(\vec{x}, \vec{x}, t) \varphi_0(\vec{x}, t) \end{aligned}$$

defining

$$m^2(\vec{x}, t) = a + \frac{b}{2} [\varphi_0^2(\vec{x}, t) + G(\vec{x}, \vec{x}; t)]$$

$$\left\{ \frac{\partial^2}{\partial t^2} - \nabla^2 + \left[m^2(\vec{x}, t) - \frac{b}{2} \varphi_0^2(\vec{x}, t) \right] \right\} \varphi_0(\vec{x}, t) = 0 \quad (3.100)$$

This differential equation governs time development of the mean field $\varphi_0(\vec{x}, t)$ from given a initial $\varphi_i(\vec{x}, t_i)$ field at given time.

The other equation for $Q(\vec{x}, \vec{y}; t)$ is obtained from

$$\dot{Q}(\vec{x}, \vec{y}; t) = \frac{\delta \mathcal{H}}{\delta P(\vec{x}, \vec{y}; t)}$$

$$-\dot{P}(\vec{x}, \vec{y}; t) = \frac{\delta \mathcal{H}}{\delta Q(\vec{x}, \vec{y}; t)}$$

as

$$\hat{\dot{Q}}(t) = \left[\hat{p}^2 + \hat{m}^2(t) \right] \hat{Q}(t) + \frac{1}{4} \hat{Q}^{-3}(t) . \quad (3.101)$$

Equation (3.101) does not give us any insight into the form of the solution. Therefore, we cannot map $G(\vec{x}, \vec{y}; t)$ into another variable in a way that we can isolate the divergences of \hat{G} .

Physical considerations regarding to the form of the propagator, G , suggest that it should have the following form

$$\hat{G}(t) = \frac{1}{\sqrt{\hat{p}^2 + \hat{m}^2(t)}} \quad (3.102)$$

That is, the time-dependence of the propagator, \hat{G} , comes solely from the mass term, $\hat{m}^2(t)$ while it preserves its form as indicated above.

This expected form for G leads to inconsistencies with what is expected from variationally obtained Eq. (3.101) if we put $\hat{G} = \hat{Q}^2$ into Eq. (3.101) it requires that

$$\hat{\dot{Q}}(t) = 0$$

which is not acceptable therefore we treat $\hat{m}(t)$ as out variational parameter in what follows. Ref.[20,25]

B.2.1. $m^2(\vec{x}, t) = m^2(t)$ (Independent of \vec{x})

Let us insert $\hat{G}(t)$ given by Eq. (3.102) with a $m^2(t)$ term independent of \vec{x} into the Lagrangian given by Eq. (3.99) where $\hat{P}(t)$ is equal to $\hat{Q}(t)$

$$\begin{aligned} \mathcal{L}(t) = & \frac{1}{2} \text{TR} \left(\dot{\hat{Q}}^2(t) \right) + \int d\vec{x} \pi_c \dot{\phi}_c \\ & - \left\{ E_{\text{cl}} + \frac{1}{8} G^{-1} + \frac{1}{2} GK + \frac{b}{8} GG \right\} \end{aligned} \quad (3.103)$$

using the results of Section B.1.1 we can re-express the terms within curly brackets as in Eq. (3.55) where m^2 is replaced with $m^2(t)$. In respect to terms in curly brackets t -dependence of m^2 is parametric and it does not effect the manipulations.

The first term of the Lagrangian is

$$\begin{aligned} \frac{1}{2} \text{TR} \left[\hat{Q}^2(t) \right] &= -\frac{1}{2} \int d\vec{x} \left\langle x \left| \frac{1}{32} [\hat{p}^2 + m^2(t)]^{-5/2} [\dot{m}^2(t)]^2 \right| \vec{x} \right\rangle \\ &= \left(\frac{1}{64} \cdot \frac{V}{2\pi^2} \int_0^\infty dp \frac{p^2}{[p^2 + m^2(t)]^{5/2}} \right) [\dot{m}^2(t)]^2 \end{aligned} \quad (3.104)$$

the integral over p converges as p goes to infinity so the term is finite

$$\frac{1}{2} \text{TR} \left(\dot{Q}^2(t) \right) = \frac{1}{64} \frac{1}{2\pi^2} \frac{1}{3} \frac{V}{m^2(t)} [\dot{m}^2(t)]^2$$

to eliminate $1/m^2(t)$ in front of $(\dot{m}^2)^2$ we will switch to $m(t)$ from $m^2(t)$ as a variational parameter. Then we obtain the effective Lagrangian with respect to $m(t)$ as

$$\frac{\mathcal{L}(t)}{V} = \frac{a}{2} [\dot{m}(t)]^2 - \epsilon(m^2(t)) \quad (3.105)$$

where $\epsilon(m^2(t))$ is given by Eq. (3.55) and

$$a = \frac{1}{32\pi^2 \times 3}$$

Euler's equation for the dynamics of $m(t)$ is

$$\frac{d}{dt} \left(\frac{\delta(\mathcal{L}/V)}{\delta \dot{m}(t)} \right) = \frac{\delta(\mathcal{L}/V)}{\delta m(t)} = 2m(t) \frac{\delta(\mathcal{L}/V)}{\delta m^2(t)}$$

it is

$$\begin{aligned} a\ddot{m}(t) &= -2m(t) \frac{\partial \epsilon}{\partial m^2(t)} \\ -a\ddot{m}(t) &= 2m(t) \left\{ \frac{1}{32\pi^2} m^2(t) \ln \frac{m^2(t)}{\mu_R^2} + \frac{1}{2} \varphi_0^2(t) - \frac{(m^2(t) - \mu_R^2)}{g_R} \right\} . \end{aligned} \quad (3.106)$$

in the case $\varphi_0^2(t) = 0$ by scaling g_R as $g = \frac{1}{32\pi^2} g_R$ we can rewrite Eq. (3.106) as

$$-\frac{1}{3} \ddot{m}(t) = 2m(t) \left\{ m^2(t) \ln \frac{m^2(t)}{\mu_R^2} - \frac{(m^2(t) - \mu_R^2)}{g} \right\} .$$

futhermore, by defining $x(t) = m(t)/\mu_R > 0$

$$-\frac{1}{3} \ddot{x} = 2x \left\{ x^2 \ln x^2 - \frac{x^2 - 1}{g} \right\} \quad (3.107)$$

since it is a second order differential equation it requires not only $x(t_0)$ but also $\dot{x}(t_0)$ as initial conditions.

B.3.2 $m^2 = m^2(\vec{x}, t)$ (dependent on \vec{x})

With the purpose of avoiding the ordering problem in the wave functional we choose our variational parameters in the following way

$$\psi_0 \sim \exp \left\{ \frac{-1}{4} \int \phi (G^{-1} - 4i\Sigma) \phi \right\} \quad (3.108)$$

The effective lagrangian is obtained as

$$\mathcal{L}(t) = \text{TR}(\Sigma \dot{G}) - \mathcal{H}(\Sigma, G) \quad (3.109)$$

Where

$$\mathcal{H} = 2 \text{TR} (\Sigma G \Sigma) + \text{TR} (\mathcal{V}(G)) \quad (3.110)$$

$\mathcal{V}(G)$ is given by eq.(3.35). We have ignored the φ dependence of the wave functional for the time being since it has no effect on the present calculations and it can be added later on. The cononical nature of the Σ and G is very clear from the formalism therefore application of the Hamilton equations gives us

$$\begin{aligned} \dot{G}(\vec{x}, \vec{y}; t) &= \frac{\delta \mathcal{H}}{\delta \Sigma(\vec{x}, \vec{y}; t)} \\ \dot{\hat{G}} &= 2 \left(\Sigma \hat{G} + \hat{G} \Sigma \right) \end{aligned} \quad (3.111)$$

and

$$\begin{aligned} -\dot{\Sigma}(\vec{x}, \vec{y}; t) &= \frac{\delta \mathcal{H}}{\delta G(\vec{x}, \vec{y}; t)} \\ -\dot{\Sigma} &= 2\Sigma^2 + \frac{\delta \mathcal{H}}{\delta G(\vec{x}, \vec{y}; t)} \end{aligned} \quad (3.112)$$

Using eq.(3.112) we can show that

$$\text{TR}(\Sigma \dot{\hat{G}}) = 4 \text{TR}(\Sigma G \Sigma) \quad (3.113)$$

With the help of eq.(3.113) we can write the effective lagrangian as

$$\mathcal{L}(t) = \frac{1}{2} \text{TR}(\Sigma \dot{\hat{G}}) - \text{TR} (\mathcal{V}(G)) \quad (3.114)$$

Where $\Sigma = \Sigma(G, \dot{\hat{G}})$

The solution of the operator eq.(3.111) is

$$2\Sigma = \int_0^\infty d\beta \exp\{-\beta G\} \dot{\hat{G}} \exp\{-\beta G\} \quad (3.114a)$$

We can eliminate the Σ term from the lagrangian using eq.(3.114a) and write it in terms of G and \dot{G} as

$$\mathcal{L}(t) = \frac{1}{2} \int_0^\infty d\beta \text{TR}(\exp\{-\beta G\} \dot{G} \exp\{-\beta G\} \dot{G}) - \text{TR}(\mathcal{V}(G)) \quad (3.115)$$

If we pick up the form of the G as

$$\frac{1}{4} \hat{G}^2(t) = \hat{p}^2 + \hat{m}^2(t)$$

The form of the \dot{G} is given by

$$\dot{G} = -4 \int_0^\infty d\alpha \exp\{-\alpha G\} G^2 \dot{m}^2 G^2 \exp\{-\alpha G\} \quad (3.116)$$

Eq.(3.116) allows us to write the lagrangian in terms of \dot{m} and m The over all result is

$$\mathcal{L}(t) = \frac{1}{2} \int dx dy \dot{m}(\vec{x}, t) \mathcal{M}(\vec{x}, \vec{y}; m) \dot{m}(\vec{y}, t) - \text{TR}(\mathcal{V}) \quad (3.117)$$

Where

$$\mathcal{M}(\vec{x}, \vec{y}; m) = 32 \int_0^\infty d\beta \beta^2 |\langle \vec{x} | G^2 \exp\{-\beta G\} G^2 | \vec{y} \rangle|^2 m(x, t) m(y, t) \quad (3.117a)$$

details of this formal manipulations will be left as a future study.

B.3 Formal Solution of the Problem

Let's write the effective Hamiltonian as

$$E = \frac{1}{8} \text{TR}(G^{-1}) + \frac{1}{2} \text{TR}(GK) + \frac{1}{4} g^2 \text{TR}(GG) \quad (3.118)$$

where

$$\hat{K} = \vec{p}^2 + \mu^2 \hat{I}$$

[Band the last trace is a double trace. Varying E with respect to G gives

$$\frac{\delta E}{\delta G} = 0 \implies \frac{1}{4}\hat{G}^{-2} = \hat{K} + g^2\hat{G}_d \quad (3.119)$$

define

$$\hat{m}^2 \equiv \mu^2\hat{I} + g^2\hat{G}_d \quad (3.120)$$

and eliminate \hat{K} from E in favor of \hat{m}^2

$$\begin{aligned} E &= \frac{1}{8} \text{TR}(G^{-1}) + \frac{1}{2} \text{TR} \left(G \left(\frac{1}{4}G^{-2} - (\hat{m}^2 - \mu^2) \right) \right) + \frac{1}{4}g^2D \text{TR}(GG) \\ &= \frac{1}{4} \text{TR}(G^{-1}) - \frac{1}{2} \text{TR} \left(\hat{G} (\hat{m}^2 - \mu^2\hat{I}) \right) + \frac{1}{4} \text{TR} \left(G (m^2 - \mu^2\hat{I}) \right) \end{aligned}$$

and eliminating $g^2\hat{G}_d$ in favor of \hat{m}^2 we obtain the following result

$$E = \frac{1}{4} \text{TR} \left[G^{-1} - G (m^2 - \mu^2\hat{I}) \right] \quad (3.121)$$

where

$$\begin{aligned} \hat{G} &= \frac{1}{2\sqrt{\vec{p}^2 + \hat{m}^2}} \\ \hat{m}^2 &= \mu^2\hat{I} + g^2\hat{G}_d \end{aligned}$$

Further we can eliminate \hat{G} in favor of \hat{m}^2

$$\hat{G}_d = \frac{(\hat{m}^2 - \mu^2)}{g^2}$$

Substituting this form of G in Eq. (3.121) we can rewrite it as

$$E = \frac{1}{4} \text{TR} \left(G^{-1} - \frac{(m^2 - \mu^2)^2}{g^2} \right) \quad (3.122)$$

Let us assume the divergent part of $G^{-1}(\vec{x}, \vec{x})$ is at most quadratic in \hat{m}^2 .

That is

$$G^{-1}(\vec{x}, \vec{x}) = G_F^{-1}(\vec{x}, \vec{x}) + G_2(\Lambda)m^2(\vec{x})m^2(\vec{x}) + 2G_1(\Lambda)m^2(\vec{x}) + G_0(\Lambda) \quad (3.123)$$

and $G(\vec{x}, \vec{x})$ is obtained by

$$G = \frac{1}{2} \frac{\delta}{\delta \hat{m}^2} G^{-1} \quad (3.124)$$

$$G(\vec{x}, \vec{x}) = G_F(\vec{x}, \vec{x}) + G_2(\Lambda)m(\vec{x}) + G_1(\Lambda)$$

$G_0(\Lambda)$, $G_1(\Lambda)$ and $G_2(\Lambda)$ singular coefficients as cut-off Λ goes to infinity and G_F is the finite part of G . To renormalize the \hat{m}^2 and E we do not need to know the forms of $G(\Lambda)$. It is enough to know that the divergent part of $G(\vec{x}, \vec{x})$ is linear in $m^2(\vec{x})$. In fact the form of G admits divergence terms up to the first order in $m^2(\vec{x})$

Substituting $G(\vec{x}, \vec{x})$ in $m^2(\vec{x})$

$$m^2(\vec{x}) = \mu^2 + g^2 (G_F(\vec{x}, \vec{x}) + G_1(\Lambda) + G_2(\Lambda)m^2(\vec{x}))$$

and combining $m^2(\vec{x})$ terms

$$m^2(\vec{x}) = \frac{\mu^2 + g^2 G_1(\Lambda)}{1 - g^2 G_2(\Lambda)} + \frac{g^2}{1 - g^2 G_2(\Lambda)} G_F(\vec{x}, \vec{x})$$

and defining

$$\begin{aligned} \mu_R^2 &= \frac{\mu^2 + g^2 G_1(\Lambda)}{1 - g^2 G_2(\Lambda)} \\ g_R^2 &= \frac{g^2}{1 - g^2 G_2(\Lambda)} \end{aligned} \quad (3.125a)$$

we can rewrite $m^2(\vec{x})$ as

$$m^2(\vec{x}) = \mu_R^2 + g_R^2 G_F(\vec{x}, \vec{x}) \quad (3, 125)$$

we also obtain $\mu^2(\Lambda)$ and $g^2(\Lambda)$ for a fixed μ_R^2 and g_R^2 by inverting Eq. (3.125) as

$$\begin{aligned} g^2 &= \frac{g_R^2}{1 + g_R^2 G_2(\Lambda)} \\ \mu^2 &= \frac{\mu_R^2 - g_R^2 G_1(\Lambda)}{1 + g_R^2 G_2(\Lambda)} \end{aligned} \quad (3.126)$$

note as $\Lambda \rightarrow \infty$, $g^2 \rightarrow 0$.

Now to renormalize the energy, E we insert G , $\mu^2(\Lambda)$ and $g^2(\Lambda)$ into Eq. (3.122)

$$E = \frac{1}{4} \text{TR} \left(G_F^{-1} + G_0(\Lambda) + 2G_1(\Lambda)\hat{m}^2 + G_2(\Lambda)\hat{m}^2\hat{m}^2 - \frac{(\hat{m}^2 - [\mu_R^2 - g_R^2 G_1(\Lambda)]/A)^2}{g_R^2} A \right)$$

where $A \equiv 1 + g_R^2 G_2(\Lambda)$.

Singular terms linear and quadratic in \hat{m}^2 cancels and ignoring singular terms constant with respect to variation we obtain the energy, E as

$$E = \frac{1}{4} \text{TR} \left(G_F^{-1} - \frac{(\hat{m}^2 - \mu_R^2)^2}{g_R^2} \right) \quad (3.127)$$

or

$$E(m^2) = \frac{1}{4} \int d^2\vec{x} \left\{ G_F^{-1}(\vec{x}, \vec{x}) - \frac{(m^2(\vec{x}) - \mu_R^2)^2}{g_R^2} \right\} . \quad (3.128)$$

Note $E(m^2(\vec{x}) \equiv \mu_R^2) = 0$ due to $G_F(\vec{x}, \vec{x}) \Big|_{m^2(\vec{x})=\mu_R^2} = 0$. Since $G(\vec{x}, \vec{x})$ is expanded to the powers of $m^2(\vec{x})$ around μ_R^2 .

In case $G(\vec{x}, \vec{x})$ has extra discrete variables such as color and spin indices, $G_{ij}^{ab}(\vec{x}, \vec{x})$, due to appearance of a matrix in $\frac{1}{4}g^2 \text{TR}(\mathcal{M}GG)$ as in Yang–Mills gauge field theory, renormalization of m_{ij}^{2ab} becomes more complicated unless we choose spin and color dependences of G in a way that the \mathcal{M} matrix contracted with GG can be made a number. We will follow this procedure when we are dealing with Yang–Mills field theory in Chapter IV.

CHAPTER IV

EXTENSION TO GAUGE FIELDS

A. Yang–Mills Field – SU(3)

The ideas developed in the preceding chapters need to be applied on a realistic field theoretic model. There is a resemblance between φ^4 -scalar field theory and Yang–Mills $SU(3)$ gauge field theory. Both of them are quartic in fields therefore we expect that the application of what we have developed so far into gauge fields will be straightforward up to a point.

We do prefer to work in temporal gauge ($\mathcal{A}_0(\vec{x}) = 0$) even though it doesn't fix the gauge arbitrariness completely. Our strategy is to quantize the gauge Hamiltonian

$$\mathcal{H} = \int d^3\vec{x} \frac{1}{2} \left\{ \vec{E}^a(\vec{x})\vec{E}^a(\vec{x}) + \vec{B}^a(\vec{x})\vec{B}^a(\vec{x}) + \mu^2 \mathcal{A}^a(\vec{x})\mathcal{A}^a(\vec{x}) \right\} \quad (4.1)$$

where the color-electric field $E_i^a(\vec{x})$ is canonical to $\mathcal{A}_i^a(\vec{x})$ and the color magnetic field is

$$B_i^a(\vec{x}) = \epsilon^{ijk} \partial_j \mathcal{A}_k^a(\vec{x}) + \frac{1}{2} \epsilon^{ijk} f^{abc} \mathcal{A}_j^b(\vec{x}) \mathcal{A}_k^c(\vec{x}) \quad (4.2)$$

as if there is no extra-degrees of freedom and impose the gauge constraint on the trial wave functional.

In functional Schrödinger representation, the canonical variable $-E_i^a(\vec{x})$ is $i \frac{\delta}{\delta \mathcal{A}_i^a(\vec{x})}$.

A more detailed introduction to what follows can be found in Ref. [2]. We choose our trial wave functional as

$$|\psi\rangle \rightarrow \psi \{ \mathcal{A}_i^a(\vec{x}) \} \simeq \exp \left\{ -\frac{1}{4} \int (\mathcal{A} - \bar{\mathcal{A}}) G (\mathcal{A} - \bar{\mathcal{A}}) \right\} \quad (4.3)$$

\int stands for sum over discrete and integral over continuous variables. G is $G_{ij}^{ab}(\vec{x}, \vec{y})$, $\mathcal{A}_i^a(\vec{x})$ is dynamical variable and $\bar{\mathcal{A}}_i^a(\vec{x})$ is a parameter-function and it corresponds to

$$\langle \psi | \mathcal{A}_i^a(\vec{x}) | \psi \rangle / \langle \psi | \psi \rangle = \bar{\mathcal{A}}_i^a(\vec{x}) \quad (4.4)$$

mean field value of $\mathcal{A}_i^a(\vec{x})$ when the system described by the state $|\psi\rangle$.

Our variational parameters are $G(x)$ and $\bar{\mathcal{A}}(x)$. (Note I use x to stand for (\vec{x}, a, i) .) We form the effective potential or the ground state energy as

$$E\{\psi\} = \langle \psi | \hat{H} | \psi \rangle / \langle \psi | \psi \rangle \quad (4.5)$$

where \hat{H} is given by Eq. (4.1), $|\psi\rangle$ is given by Eq. (4.3) and $E(x)$ is the functional derivative with respect to $\mathcal{A}(x)$. After calculations similar to Chapter II we obtain the following result for the effective potential

$$\begin{aligned} E[G, \bar{\mathcal{A}}] &= E_{\text{cl}} + \frac{1}{8} \text{TR}(G^{-1}) + \frac{1}{2} \text{TR}(GK) \\ &+ \frac{1}{4} g^2 \text{TR}(GMG) \end{aligned} \quad (4.6)$$

where

$$\begin{aligned} \bar{B}_i^a(\vec{x}) &= \epsilon^{ijk} \partial_j \bar{\mathcal{A}}_k^a(\vec{x}) + \frac{1}{2} g f^{abc} \bar{\mathcal{A}}_j^b(\vec{x}) \bar{\mathcal{A}}_k^c(\vec{x}) \\ E_{\text{cl}} &= \int d^3 \vec{x} \frac{1}{2} \{ \bar{B}_i^a(\vec{x}) \bar{B}_i^a(\vec{x}) + \mu^2 \bar{\mathcal{A}}_i^a(\vec{x}) \bar{\mathcal{A}}_i^a(\vec{x}) \} \\ K_{ij}^{ab}(\vec{x}, \vec{y}) &= \frac{\delta^2 E_{\text{cl}}}{\delta \bar{\mathcal{A}}_i^a(\vec{x}) \delta \bar{\mathcal{A}}_j^b(\vec{y})} \\ \hat{K} &\equiv (S \cdot D)^2 - g \vec{S} \cdot \vec{B} + \mu^2 I \\ (S^k)_{ij} &= \hat{i} \epsilon_{ijk} \\ \lambda^{abc} &= -\hat{i} f^{abc} \end{aligned} \quad (4.7)$$

$$\hat{D}_i = \hat{I}_c \hat{P}_i - g \hat{\mathcal{A}}_i \quad (4.6)$$

$$\bar{\mathcal{A}}_i(\vec{x}) = \lambda^a \bar{\mathcal{A}}_i^a(\vec{x})$$

$$\bar{B}_i(\vec{x}) = \lambda^a \bar{B}_i^a(\vec{x})$$

$$[S_i, S_j] = \hat{i} \epsilon_{ijk} S_k$$

$$[\lambda^a, \lambda^b] = \hat{i} f^{abc} \lambda^c$$

and

$$\mathcal{M}_{k;\ell m}^{bc;de} = \epsilon_{ijk} \epsilon_{ilm} f^{abc} f^{ade} + 2 \epsilon_{ijl} \epsilon_{imk} f^{abd} f^{aec} \quad (4.9)$$

The way the indices are contracted with G is

$$G_{jk}^{bc}(\vec{x}) \mathcal{M}_{k;\ell m}^{bc;de} G_{\ell m}^{de}(\vec{x}) \quad (4.10)$$

and it also can be written in terms of S_i, λ^a as

$$\text{TR}(S^i \lambda^a G) \text{TR}(S^i \lambda^a G) + 2 \text{TR}(G S^i \lambda^a G S^i \lambda^a) \quad (4.11)$$

Trace is over all spin and color indices and over the continuous index \vec{x} is a double trace.

The gauge constraint is $\vec{D} \cdot \vec{E} = 0$. After quantization it becomes an operator that commutes with the Hamiltonian $[\hat{\mathcal{H}}, \hat{\vec{D}} \cdot \hat{\vec{E}}] = 0$. We incorporate it into the formalism by its action on the state

$$\hat{\vec{D}} \cdot \hat{\vec{E}} |\psi\rangle = 0 \quad (4.12)$$

Since the state is a trial one when $\bar{\mathcal{A}}_i^a(\vec{x})$ is different than zero it is satisfied only on the average

$$\langle \psi | \hat{\vec{D}} \cdot \hat{\vec{E}} | \psi \rangle = 0 \quad (4.13)$$

when $\bar{\mathcal{A}}_i^a(\vec{x})$ is zero, the condition (4.12) can be satisfied by a special form $G_{ij}^{ab}(\vec{x}, \vec{y})$.

The variational principle for G

$$\frac{\delta E[G, \bar{\mathcal{A}}]}{\delta G_{ij}^{ab}(\vec{x}, \vec{y})} \equiv 0$$

leads to

$$\frac{1}{4}G^{-2} = K + g^2 \left[S^i \lambda^a \text{TR}(S^i \lambda^a \hat{G}_d) + S^i \lambda^a \hat{G}_d S^i \lambda^a \right] \quad (4.14)$$

The trace is only over color and spin induced. By defining

$$\hat{m}^2 = \mu^2 I - g(\vec{S} \cdot \vec{B}) + g^2 S^i \lambda^a \text{TR}(S^i \lambda^a \hat{G}_d) + 2g^2 (S^i \lambda^a \hat{G}_d S^i \lambda^a) \quad (4.15)$$

it can be written as

$$\begin{aligned} \frac{1}{4}\hat{G}^{-2} &= (S \cdot D)^2 + \hat{m}^2 \\ \hat{G} &= \frac{1}{\sqrt{(S \cdot D)^2 + \hat{m}^2}} \end{aligned} \quad (4.16)$$

The variational principle for $\bar{\mathcal{A}}$

$$\frac{\delta E}{\delta \bar{\mathcal{A}}_i^a(\vec{x})} \equiv 0$$

leads to

$$D_j^{ab}(\vec{x}) \bar{F}_{ij}^b(\vec{x}) + \mu^2 \bar{\mathcal{A}}_i^a(\vec{x}) + \frac{1}{2} \frac{\delta}{\delta \bar{\mathcal{A}}_i^a(\vec{x})} \text{TR}(GK) = 0 \quad (4.17)$$

where

$$\bar{F}_{ij}^b(\vec{x}) = \epsilon_{ijk} \bar{B}_k^a(\vec{x}) \quad (4.18)$$

These equations (4.16) – (4.17) are intractable in their most general form since it is not possible to isolate and eliminate the divergences. Even though the non-zero value of $\bar{\mathcal{A}}_i^a(\vec{x})$ is necessary for the inclusion of the external source terms. Leaving

it for future study, we will focus our attention in the case of $\bar{\mathcal{A}}_i^a(\vec{x})$ is zero. If we let $\bar{\mathcal{A}}_i^a(\vec{x}) \equiv 0$ then

$$E = \frac{1}{8} \text{TR}(G^{-1}) + \frac{1}{2} \text{TR}(GK) + \frac{1}{4} g^2 \text{TR}(GMG) \quad (4.19)$$

where

$$K = \left(\hat{S} \cdot \hat{p} \right) + \mu^2 \hat{I} = \hat{I}_c \hat{Q}_{ij} \cdot \hat{p}^2 + \mu^2 \hat{I} \quad (4.20)$$

To fulfill the gauge condition we pick up the form of G as

$$G_{ij}^{ab}(\vec{x}, \vec{y}) = \delta^{ab} \left[\hat{Q}_{ij}(\vec{x}) + \frac{1}{\epsilon} \hat{P}_{ij}(\vec{x}) \right] G_0(\vec{x}, \vec{y}) \quad (4.21)$$

where \hat{Q}_{ij} and \hat{P}_{ij} are transverse orthogonal operators defined as

$$\begin{aligned} \hat{Q}_{ij} &= \hat{I} \delta_{ij} - \frac{\hat{p}_i \hat{p}_j}{\hat{p}^2} \\ \hat{P}_{ij} &= \frac{\hat{p}_i \hat{p}_j}{\hat{p}^2} \end{aligned} \quad (4.22)$$

They obey the following relations

$$\begin{aligned} \sum_i \hat{Q}_{ii} &= 2\hat{I} & \sum_i P_{ii} &= \hat{I} \\ \hat{Q}_{ij} \hat{Q}_{jk} &= \hat{Q}_{ik} & \hat{P}_{ij} \hat{P}_{jk} &= \hat{P}_{ik} \\ \hat{Q}_{ij} + \hat{P}_{ij} &= \delta_{ij} \hat{I} & Q_{ij} P_{jk} &= P_{ij} Q_{jk} = 0 \end{aligned} \quad (4.23)$$

We also introduced a parameter ϵ that will be taken to zero at the end of the calculations. Since without ϵ , $G_{ij}^{ab}(\vec{x}, \vec{y})$ is not invertible in spin indices and it has a zero eigenvalue in the representation in which it is diagonal, that causes the Gaussian integrals to diverge. This trick allows us to go on with our calculations without any inconsistency. With this form for G our variational parameter is $G_0(\vec{x}, \vec{y})$ since we can integrate out over color and spin indices.

The inverse of G can be written in operator form as

$$G^{-1} = \hat{G}_0^{-1} (\hat{Q} + \epsilon \hat{P}) \hat{I}_e \quad (4.24)$$

The first two terms of Eq. (4.11) can be evaluated easily to yield

$$\begin{aligned} \frac{1}{8} \text{TR}(G^{-1}) &= \frac{1}{8} N(2 + \epsilon) \text{TR}(\hat{G}_0^{-1}) \\ \frac{1}{2} \text{TR}(GK) &= N \text{TR}(\hat{G}_0 \hat{K}_0) \end{aligned} \quad (4.25)$$

where

$$\hat{K}_0 = \hat{p}^2 + \left(\frac{1 + 2\epsilon}{2\epsilon} \right) \mu^2 \hat{I}$$

$N = \sum_a \delta^{aa}$ color degrees of freedom which is $N = 8$ for $SU(3)$. The last term

$$\frac{1}{4} g^2 \text{TR}(GMG) = \frac{1}{2} g^2 \text{TR}(GS^i \lambda^a GS^i \lambda^a)$$

the first term of (4.11) drops since $\text{TR}(\lambda^a) = 0$

$$= \frac{1}{2} g^2 NC_1 \text{TR}(GS^i GS^i)$$

where $NC_1 = \text{TR}(\lambda^a \lambda^a) = f^{abc} f^{abc}$

$$\begin{aligned} &= \frac{1}{2} g^2 NC_1 \left\{ \text{TR}(\vec{S}^2) \text{TR}(\hat{G}_0 \hat{G}_0) + 2 \left(\frac{1}{\epsilon} - 1 \right) \text{TR}(G_0(S^i P S^i G_0)) \right. \\ &\quad \left. + \left(\frac{1}{\epsilon} - 1 \right)^2 \text{TR}((PG_0)S^i(PG_0)S^i) \right\} \end{aligned}$$

$\text{TR}((PG_0)S^i(PG_0)S^i) = \text{TR}(G_0 S^i P S^i G_0) + \text{surface terms (ignored)}$

$$= \frac{1}{2} g^2 NC_1 \left\{ \text{TR}(\vec{S}^2) \text{TR}(G_0 G_0) + \left(\frac{1}{\epsilon^2} - 1 \right) \text{TR}(G_0(S^i P S^i G_0)) \right\}$$

$\text{TR}(S^i P S^i) = 2\hat{I}$ traced over only spin indices $\text{TR}(\vec{S}^2) = 6$

$$= g^2 NC_1 \left(2 + \frac{1}{\epsilon^2} \right) \text{TR}(G_0 G_0) \quad (4.26)$$

the overall result is

$$\frac{E}{N} = \frac{1}{8}(2 + \epsilon) \text{TR}(G_0^{-1}) + \text{TR}(G_0 K_0) + g^2 C_1 \left(2 + \frac{1}{\epsilon^2}\right) \text{TR}(G_0 G_0) \quad (4.27)$$

the variational principle for G_0

$$\frac{\delta E}{\delta G_0(\vec{x}, \vec{y})} \equiv 0$$

leads to

$$\frac{1}{4}(2 + \epsilon)G_0^{-2} = 2\hat{K}_0 + 4 \left(2 + \frac{1}{\epsilon^2}\right) g^2 C_1 \hat{G}_{0d}$$

or

$$\frac{1}{4}G_0^{-2} = \frac{2}{2 + \epsilon} \vec{p}^2 + \frac{1 + 2\epsilon}{(2 + \epsilon)2\epsilon} \mu^2 + 4 \frac{1 + 2\epsilon^2}{(2 + \epsilon)\epsilon^2} g^2 C_1 \hat{G}_{0d} \quad (4.28)$$

defining

$$m^2 = \frac{1 + 2\epsilon}{(2 + \epsilon)2\epsilon} \mu^2 + 4C_1 \frac{1 + 2\epsilon^2}{(2 + \epsilon)\epsilon^2} g^2 \hat{G}_{0d} \quad (4.29)$$

write

$$\frac{1}{4}\hat{G}_0^{-2} = \hat{p}^2 + \hat{m}^2$$

or

$$\hat{G}_0 = \frac{1}{2\sqrt{\hat{p}^2 + \hat{m}^2}} \quad (4.30)$$

Note in the limit ϵ goes to zero $\frac{2}{2+\epsilon}$ goes to 1 if we scale g and μ^2 as

$$g_0^2 = 2g^2 C_1 \left(2 + \frac{1}{\epsilon^2}\right) \quad (4.31)$$

$$\mu_0^2 = \frac{1 + 2\epsilon}{4\epsilon} \mu^2$$

we can rewrite Eq. (4.27) – (4.29) in a familiar form

$$\frac{E}{N} = \frac{1}{4} \hat{G}_0^{-1} + \hat{G}_0 \hat{K}_0 + \frac{1}{2} g_0^2 G_0 G_0 \quad (4.32)$$

$$\hat{K}_0 = \hat{p}^2 + \mu_0^2 \hat{I}$$

$$m^2 = \frac{2}{2 + \epsilon} \left(\mu_0^2 + g_0^2 \hat{G}_{0d} \right)$$

$$\hat{m}^2 = \mu_0^2 + g_0^2 \hat{G}_{0d} \quad (4.33)$$

by eliminating K_0 in favor of m^2

$$\frac{E}{N} = \frac{1}{2} \text{TR} \left(\hat{G}_0^{-1} - \hat{G}_0 (\hat{m}^2 - \mu_0^2) \right)$$

or

$$\frac{E}{N} = \frac{1}{2} \text{TR} \left(G_0^{-1} - \frac{(\hat{m}^2 - \mu_0^2)^2}{g_0^2} \right) \quad (4.34)$$

This relation is equivalent to the one given in Eq. (3.122). Therefore it can be written in a finite form as

$$\frac{E}{N} = \frac{1}{2} \text{TR} \left(G_{0F}^{-1} - \frac{(\hat{m}^2 - \mu_R^2)^2}{g_R^2} \right) \quad (4.35)$$

and

$$m^2(\vec{x}) = \mu_R^2 + g_R^2 G_{0F}(\vec{x}, \vec{x}) \quad (4.36)$$

Its structure is similar to that of φ^4 -scalar field theory up to some constants and its non-perturbative nature is very clear. For a further reading on the subject check Ref.[12,15,17,18,19,31–37].

CONCLUSIONS

So far we have succeeded in obtaining a finite expression for the ground state energy of φ^4 -scalar field theory; both time-independent and time-dependent. We have seen that the relevant singular terms have both quadratic and logarithmic divergence, and they have the same form when we deal with non-abelian gauge fields if we define G_{ij} to be transverse in spin indices. In case of zero mean field we achieved this with the help of Q_{ij} , Eq.(4.22). We have seen that for the renormalization purposes, the elimination of the longitudinal part of the G_{ij} is essential to remove the cubic singularity.

The results of the chapter III and IV are very promising for the application of variational method to the gauge fields in the presence of the mean gauge field, even though we have some difficulties in the isolation and the elimination of the zero and the negative modes of the propagator in the presence of the mean gauge field (background gauge field). We have seen that negative modes can be made to cancel by guessing the form of the mass parameter in (4.15). We are hoping that we would be able to eliminate the zero mode by defining the proper transverse operator. The generalization of the (4.22), that is, \hat{p}_i is replaced by \hat{D}_i , does not obey the first two relations of (4.23). Therefore we need to define another operator, when it is contracted with K of (4.4), will eliminate the zero mode. We will continue the research on these lines in the future to prove the confinement with the inclusion of the source terms thru (4.12)

APPENDIX A

$$\Gamma(x) = \int_0^{\infty} dt t^{x-1} e^{-t} \quad \text{Gamma function}$$

$$\Gamma(x+1) = x\Gamma(x) \tag{A.1}$$

$$\Gamma(n+1) = n!$$

$$\Gamma(x) = 2\alpha^x \int_0^{\infty} dy y^{2x-1} e^{-\alpha y^2} \tag{A.2}$$

Calculation of $E(q_0, g)$

$$E(q_0, g) = \int_{-\infty}^{+\infty} dq \psi^*(q) \left[-\frac{\hbar}{2} \frac{d^2}{dq^2} + V(q) \right] \psi(q) \tag{A.3}$$

where

$$\psi(q) = N \exp \left\{ -\frac{1}{4}(q - q_0)^2/g \right\} \tag{A.4}$$

and $|N|^2$ is given as

$$|N|^2 = \frac{1}{\Gamma(1/2)\sqrt{2g}}$$

to have the unit normalization

$$\int_{-\infty}^{+\infty} dq |\psi|^2 = 1$$

$$\begin{aligned} E(q_0, g) &= \frac{1}{\Gamma(1/2)\sqrt{2g}} \int_{-\infty}^{+\infty} dq e^{-\frac{1}{4}(q-q_0)^2/g} \left[-\frac{\hbar^2}{2} \frac{d^2}{dq^2} + V(q) \right] \\ &= -\frac{\hbar^2}{2\Gamma(1/2)\sqrt{2g}} \int_{-\infty}^{\infty} dq e^{-\frac{1}{4}q^2/g} \frac{d^2}{dq^2} e^{-\frac{1}{4}q^2/g} \\ &\quad + \frac{1}{\Gamma(1/2)\sqrt{2g}} \int_{-\infty}^{+\infty} dq V(q) e^{-(q-q_0)^2/2g} \\ \frac{d}{dq} e^{-q^2/4g} &= e^{-q^2/4g} (-q/2g) \\ \frac{d^2}{dq^2} e^{-q^2/4g} &= e^{-q^2/4g} (-q/2g)^2 - \frac{1}{2g} e^{-q^2/4g} \end{aligned}$$

Inserting it in $E(q_0, g)$

$$\begin{aligned}
&= -\frac{\hbar^2}{2\Gamma(1/2)\sqrt{2g}} 2 \int_0^\infty dq \left(\frac{q^2}{4g^2} - \frac{1}{2g} \right) e^{-q^2/2g} + \text{II} \\
&= -\frac{\hbar^2}{2\Gamma(1/2)\sqrt{2g}} \left\{ \frac{1}{4g^2} \frac{\Gamma(3/2)}{(1/2g)^{3/2}} - \frac{1}{2g} \frac{\Gamma(1/2)}{(1/2g)^{1/2}} \right\} + \text{II} \\
&= -\hbar^2 \left\{ \frac{1/2\Gamma(1/2)}{2\Gamma(1/2)4g^2} - \frac{1}{4g} \right\} + \text{II} \\
&= -\hbar^2 \left\{ \frac{1}{8} - \frac{1}{4} \right\} g^{-1} + \text{II} \\
E(q_0, g) &= \frac{\hbar^2}{8g} + \text{II}
\end{aligned}$$

Where

$$\begin{aligned}
\text{II} &= \frac{1}{\Gamma(1/2)\sqrt{2g}} \int_{-\infty}^{+\infty} dq V(q + q_0) e^{-q^2/2g} \\
V(q + q_0) &= V(q_0) + q \left(aq_0 + \frac{b}{8}q_0^3 \right) + q^2 \left(\frac{1}{2}a + \frac{b}{4}q_0^2 \right) \\
&\quad + q^3 \left(\frac{b}{8}q_0 \right) + \frac{b}{24}q^4 \\
V(q + q_0) &= V(q_0) + q \left(aq_0 + \frac{b}{8}q_0^3 \right) + q^2 \left(\frac{1}{2}a + \frac{b}{4}q_0^2 \right) + q^3 \left(\frac{b}{8}q_0 \right) + \frac{b}{24}q^4 .
\end{aligned}$$

the odd powers of q do not give any contributions due to symmetry of the the integrand.

$$\begin{aligned}
\text{II} &= V(q_0) + \frac{1}{\Gamma(1/2)\sqrt{2g}} \left(\frac{1}{2}a + \frac{b}{4}q_0^2 \right) \frac{\Gamma(3/2)}{(2g)^{3/2}} + \frac{1}{\Gamma(1/2)\sqrt{2g}} \frac{b}{24} \frac{\Gamma(5/2)}{(1/2g)^{5/2}} \\
\text{II} &= V(q_0) + \frac{1}{2} \left(a + \frac{b}{2}q_0^2 \right) g + \frac{b}{8}g^2
\end{aligned}$$

over all result is

$$E(q_0, g) = \frac{\hbar}{8}g^{-1} + \frac{1}{2} \left(a + \frac{b}{2}q_0^2 \right) g + \frac{b}{8}g^2 + V(q_0) \tag{A.5}$$

Ref.[41]

APPENDIX B

The calculation of the roots of

$$g^3 + \left(\frac{2a}{b} + q_0^2\right)g^2 - \frac{\hbar^2}{b} = 0 \quad (B.1)$$

Let

$$p = \frac{2a}{b} + q_0^2, \quad r = -\frac{\hbar^2}{b}$$

then

$$g^3 + pg^2 + r = 0$$

Define $x = g + \frac{p}{3}$

$$x^3 + cx + d = 0, \quad c = -p^2/3, \quad d = \frac{1}{27}(2p^3 + 27r)$$

if $\frac{d^2}{4} + \frac{c^3}{27} < 0$. The real roots are

$$x_k = 2\sqrt{-\frac{c}{3}} \cos\left(\frac{\phi}{3} + 120^\circ k\right) \quad k = 0, 1, 2 \quad (B.2)$$

and

$$\cos \phi = \begin{cases} \mp \sqrt{\frac{d^2/4}{-c^3/27}} & - \quad d > 0 \\ \mp \sqrt{\frac{d^2/4}{-c^3/27}} & + \quad d < 0 \end{cases}$$

if $\frac{d^2}{4} + \frac{c^3}{27} > 0$ and ($c > 0$) the real root is

$$x = 2\sqrt{\frac{c}{2}} \cot 2\phi$$

where ϕ and ψ

$$\cot 2\psi = \mp \sqrt{\frac{d^2/4}{c^3/27}} \text{ and } \tan \phi = [\tan \psi]^{1/3} \quad \begin{array}{l} + \quad b > 0 \\ - \quad b < 0 \end{array}$$

if $\frac{d^2}{4} + \frac{c^3}{27} = 0$ the roots are

$$x = \mp 2\sqrt{\frac{-c}{3}} \pm \sqrt{\frac{-c}{3}} \pm \sqrt{\frac{-c}{3}} \quad \begin{array}{l} - \quad d > 0 \\ + \quad d < 0 \end{array}$$

$$c = -\frac{1}{3}p^2 = -\frac{1}{3}\left(\frac{2a}{b} + q_0^2\right)^2 < 0$$

$$d = \frac{1}{27}(2p^3 + 27r) = \frac{2}{3^3}\left(\frac{2a}{b} + q_0^2\right)^3 - \frac{\hbar^2}{b}$$

let us calculate

$$\begin{aligned} \frac{d^2}{4} + \frac{c^3}{27} &= \frac{1}{3^6}\left(\frac{2a}{b} + q_0^2\right)^6 - \frac{1}{3^3}\frac{\hbar^2}{b}\left(\frac{2a}{b} + q_0^2\right)^2 + \frac{\hbar^4}{4b^2} + -\frac{1}{3^6}\left(\frac{2a}{b} + q_0^2\right)^6 \\ \frac{d^2}{4} + \frac{c^3}{27} &= \frac{\hbar^4}{4b^2} - \frac{\hbar^2}{3^3b}\left(\frac{2a}{b} + q_0^2\right)^3 \neq 0 \\ \frac{\hbar^2}{4b} &< \frac{1}{27}\left(\frac{2a}{b} + q_0^2\right)^3 \end{aligned}$$

calculate

$$\frac{d^2/4}{-c^3/27} = 1 - \frac{27\hbar^2/b}{\left(\frac{2a}{b} + q_0^2\right)^3} + \frac{3^6\hbar^4/4b^2}{\left(\frac{2a}{b} + q_0^2\right)^6} > 0$$

since $c < 0$ for all values of a and b the possible roots are

$$x_k = \frac{2}{3}\left|\left(\frac{2a}{b} + q_0^2\right)\right| \cos\left(\frac{\phi}{3} + 120^\circ k\right) \quad k = 0, 1, 2 \quad (B.3)$$

and

$$\begin{aligned} \cos \phi &= \mp \sqrt{1 - \theta(\hbar^2) + \theta(\hbar^4)} \quad \begin{array}{l} +d > 0 \\ -d < 0 \end{array} \\ g_k &= x_k - \frac{p}{3} = \frac{1}{3}\left(\frac{2a}{b} + q_0^2\right) \left[-1 + 2 \cos\left(\frac{\phi}{3} + 120^\circ k\right)\right] \\ g_k &= \frac{1}{3}\left|\left(\frac{2a}{b} + q_0^2\right)\right| \left[-1 + 2 \cos\left(\frac{\phi}{3} + 120^\circ k\right)\right] \quad k = 0, 1, 2, \quad (B.4) \end{aligned}$$

The limit $\hbar \rightarrow 0$ corresponds to $\phi = 180$, $k = 1$

$$g_k(\hbar \rightarrow 0) = -\left|\left(\frac{2a}{b} + q_0^2\right)\right|$$

Ref.[43]

APPENDIX C

Evaluation of $\mathcal{L}(t)$

$$\mathcal{L}(t) = \left\langle \psi_0(t) \left| i \frac{\partial}{\partial t} - \hat{\mathcal{H}} \left(i \frac{\partial}{\partial q}, q^0 \right) \right| \psi_0(t) \right\rangle / \langle \psi_0 | \psi_0 \rangle \quad (C.1)$$

where ψ is given by

$$\psi_0(q, t) = \exp [i(q - q_0(t)) \pi_0(t)] \exp \left\{ -\frac{1}{4} (q - q_0)^2 (Q^{-2} - 2iQP) \right\} \quad (C.2)$$

$$\begin{aligned} \left\langle \psi_0(t) \left| i \frac{\partial}{\partial t} \right| \psi_0(t) \right\rangle &= \int dq |\psi(q, t)|^2 \left(-i \frac{\partial}{\partial t} \right) \\ &\times \left\{ i(q - q_0) \pi_0 - \frac{1}{4} (q - q_0)^2 (Q^{-2} - 2iQ^{-1}P) \right\} \end{aligned}$$

using

$$\int dq (q - q_0) |\psi(q, t)|^2 = 0$$

we can evaluate it as

$$\begin{aligned} &= \int dq |\psi(q, t)|^2 \left\{ \pi_0 \dot{q}_0 - \frac{i}{4} (q - q_0)^2 |\psi(q, t)|^2 \frac{\partial}{\partial t} (Q^{-2} - 2iQ^{-1}P) \right\} \\ &\left\langle \psi_0(t) \left| i \frac{\partial}{\partial t} \right| (t) \right\rangle / \langle \psi_0 | \psi_0 \rangle = \pi_0 \dot{q}_0 - \frac{i}{4} \frac{\partial}{\partial t} (Q^{-2} - 2iQ^{-1}P) \\ &\quad \times \frac{\int dq (q - q_0)^2 |\psi(q, t)|^2}{\int dq |\psi(q, t)|^2} \end{aligned}$$

and

$$\begin{aligned} \frac{\int dq (q - q_0^2) |\psi(q, t)|^2}{\int dq |\psi(q, t)|^2} &= \frac{\Gamma(3/2) (Q^{-1}/2)^{-3/2}}{\sqrt{2} \Gamma(1/2) Q} = Q^2 \\ \left\langle \psi_0(t) \left| i \frac{\partial}{\partial t} \right| \psi_0(t) \right\rangle / \langle \psi_0 | \psi_0 \rangle &= \\ &= \pi_0 \dot{q}_0 - Q^2 \frac{i}{4} \frac{\partial}{\partial t} (Q^{-2} - 2iQ^{-1}P) \end{aligned}$$

and

$$\begin{aligned} \left\langle \psi_0(t) \left| \hat{\mathcal{H}} \left(i \frac{\partial}{\partial q}, \hat{q} \right) \right| \psi_0(t) \right\rangle / \langle \psi_0 | \psi_0 \rangle &= \left\langle \psi_0(t) \left| -\frac{\partial^2}{2\partial q^2} \right| \psi_0(t) \right\rangle / \langle \psi_0 | \psi_0 \rangle \\ &+ \langle \psi_0(t) | V(q) | \psi_0(t) \rangle / \langle \psi_0 | \psi_0 \rangle \end{aligned}$$

or

$$\langle \psi_0(t) | V(q) | \psi_0(t) \rangle / \langle \psi_0 | \psi_0 \rangle = V(q_0) + \frac{1}{2} \left(a + \frac{b}{2} q_0^2 \right) Q^2 + \frac{b}{8} Q^4$$

(see Appendix B)

$$\begin{aligned} & \left\langle \psi_0(t) \left| -\frac{1}{2} \frac{\partial^2}{\partial q^2} \right| \psi_0(t) \right\rangle = \\ & = \int dq \psi_0^*(q, t) \frac{-\partial}{2\partial q} \left\{ i\pi - \frac{1}{2}(q - q_0) \left(Q^{-2} - 2i\frac{P}{Q} \right) \right\} \psi_0(q, t) \\ & = \int dq |\psi^*(q, t)|^2 \left(\frac{-1}{2} \right) \left\{ \left[i\pi_0 - \frac{1}{2}(q - q_0) \left(Q^{-2} - \frac{2i}{Q}P \right) \right]^2 - \frac{1}{2} \left(Q^{-2} - \frac{i2}{Q}P \right) \right\} \\ & = \langle \psi_0 | \psi_0 \rangle \left(\frac{-1}{2} \right) \left\{ -\pi_0^2 - \frac{1}{2} \left(Q^{-2} - \frac{2i}{Q}P \right) + \frac{1}{4} \left(Q^{-2} - \frac{2i}{Q}P \right)^2 Q^2 \right\} \\ & \left\langle \psi_0(t) \left| -\frac{1}{2} \frac{\partial^2}{\partial q^2} \right| \psi_0(t) \right\rangle / \langle \psi_0 | \psi_0 \rangle = \frac{1}{2} \pi_0^2 + \frac{1}{8} Q^{-2} + \frac{1}{2} P^2 \end{aligned}$$

APPENDIX D

Show that

$$\langle \psi | \hat{\phi}(\vec{x}) | \psi \rangle = \phi_0(\vec{x}) \langle \psi | \psi \rangle \quad (D.1)$$

where

$$|\psi\rangle \leftrightarrow \exp \left\{ -\frac{1}{4} \int d\vec{x} d\vec{y} [\phi(\vec{x}) - \phi_0(\vec{x})] G^{-1}(\vec{x} - \vec{y}) [\phi(\vec{y}) - \phi_0(\vec{y})] \right\}$$

Proof:

$$\begin{aligned} \langle \psi | \hat{\phi}(\vec{x}) | \psi \rangle &= \int D(\phi) \phi(\vec{x}) \\ &\times \exp \left\{ -\frac{1}{2} \int d\vec{z} d\vec{y} [\phi(\vec{z}) - \phi_0(\vec{z})] G^{-1}(\vec{z}, \vec{y}) [\phi(\vec{y}) - \phi_0(\vec{y})] \right\} \end{aligned}$$

The best way to deal with it is to discretize the variable \vec{x} and take the limit at the end of the calculation.

Let $\vec{x} = \alpha\Delta$ where $\alpha = \{\alpha_1, \alpha_2, \alpha_3\}$ stand for 3-tuple integers.

Define $\phi_\alpha = \phi(\vec{x} = \alpha\Delta)$ and $G_{\alpha\beta} = G(\vec{x} = \alpha\Delta, \vec{y} = \beta\Delta)$

$$\langle \psi | \hat{\phi}(\vec{x}) | \psi \rangle = \lim_{\Delta \rightarrow 0} \int_{-\infty}^{+\infty} \prod_{\gamma} d\phi_{\gamma} \phi_{\nu} e^{-\frac{1}{2} \Delta^2 (\phi_{\alpha} - \phi_{\alpha}^0) G_{\alpha\beta} (\phi_{\beta} - \phi_{\beta}^0)}$$

shift the variable $\phi_{\nu} \rightarrow \phi_{\nu} + \phi_{\nu}^0$ (ϕ_{ν}^0 : finite constant).

$$\langle \psi | \hat{\phi}(\vec{x}) | \psi \rangle = \phi_0(\vec{x}) \langle \psi | \psi \rangle + \lim_{\Delta \rightarrow 0} \int_{-\infty}^{+\infty} \prod_{\gamma} d\phi_{\gamma} \phi_{\nu} e^{-\frac{\Delta^2}{2} \phi_{\alpha} G_{\alpha\beta}^{-1} \phi_{\beta}}$$

Now, we only have to show that the second term is zero.

$$\lim_{\Delta \rightarrow 0} \int \prod_{\gamma} d\phi_{\gamma} \phi_{\nu} e^{-\frac{\Delta^2}{2} \phi_{\alpha} G_{\alpha\beta}^{-1} \phi_{\beta}}$$

If we work in the representation in which $G_{\alpha\beta}$ is diagonal

$$\begin{aligned} G_{\alpha\beta} &= g_\alpha \delta_{\alpha\beta} \quad \text{where } g_\alpha > 0 \\ &= \lim_{\Delta \rightarrow 0} \int_{-\infty}^{+\infty} \prod_\gamma d\phi_\gamma \phi_\nu e^{-\frac{\Delta^2}{2} \phi_\alpha^2 g_\alpha^{-1}} \\ &= 0 \end{aligned}$$

since the integrand is odd for ν -variable.

Show that

$$\begin{aligned} \langle \psi | \hat{\phi}(\vec{x}) \hat{\phi}(\vec{y}) | \psi \rangle &= \lim_{\Delta \rightarrow 0} \int \prod_\gamma d\phi_\gamma \phi_\alpha \phi_\beta e^{-\frac{\Delta^2}{2} (\phi_{\nu_1} - \phi_{\nu_1}^0) G_{\nu_1 \nu_2}^{-1} (\phi_{\nu_2} - \phi_{\nu_2}^0)} \quad (D.2) \\ &\quad \text{shift } \phi_\nu \longrightarrow \phi_\nu + \phi_\nu^0 \\ &= \lim_{\Delta \rightarrow 0} \int \prod_\gamma d\phi_\gamma (\phi_\alpha + \phi_\alpha^0) (\phi_\beta + \phi_\beta^0) e^{-\frac{\Delta^2}{2} \phi_{\nu_1} G_{\nu_1 \nu_2}^{-1} \phi_{\nu_2}} \\ &= \phi_0(\vec{x}) \phi_0(\vec{y}) \langle \psi | \psi \rangle + \lim_{\Delta \rightarrow 0} \int \prod_\gamma d\phi_\gamma [(\phi_\alpha^0 \phi_\beta + \phi_\alpha \phi_\beta^0) + \phi_\alpha \phi_\beta] \\ &\quad \times \exp \left\{ -\frac{\Delta^2}{2} \phi_{\nu_1} G_{\nu_1 \nu_2}^{-1} \phi_{\nu_2} \right\} \end{aligned}$$

Since we have shown above that the integral of odd powers of ϕ_ν multiplying exponential is zero we are left only with a quadratic term.

$$= \phi_0(\vec{x}) \phi_0(\vec{y}) \langle \psi | \psi \rangle + \lim_{\Delta \rightarrow 0} \int \prod_\gamma d\phi_\gamma \phi_\alpha \phi_\beta e^{-\frac{\Delta^2}{2} \phi_{\nu_1} G_{\nu_1 \nu_2}^{-1} \phi_{\nu_2}}$$

Let $A_{\nu_1 \nu_2}$ diagonalize $G_{\nu_1 \nu_2}$

$$G_{\nu_1 \nu_2} = A_{\nu_1 \rho}^T g_\rho A_{\rho \nu_2} \quad (A^T A = I)$$

or

$$\begin{aligned} G_{\nu_1 \nu_2}^{-1} &= A_{\nu_1 \rho}^T g_\rho^{-1} A_{\rho \nu_2} \\ \phi_{\nu_1} G_{\nu_1 \nu_2}^{-1} \phi_{\nu_2} &= (A_{\nu_1 \rho}^T \phi_{\nu_1})^T g_\rho^{-1} (A_{\rho \nu_2} \phi_{\nu_2}) \end{aligned}$$

define

$$\phi'_\rho = A_{\rho\nu_2} \phi_{\nu_2}$$

or

$$\phi_{\nu_2} = A_{\nu_2\rho}^T \phi'_\rho$$

Note

$$\prod_\gamma d\phi_\gamma = J\left(\frac{\phi^0}{\phi'}\right) \prod_\gamma d\phi'_\gamma$$

where

$$J\left(\frac{\phi^0}{\phi'}\right) = \det(A) = 1$$

since $A^T A = I$. Inserting all these definitions in the integrand

$$\lim_{\Delta \rightarrow 0} \prod_\gamma d\phi'_\gamma A_{\alpha\rho_1}^T A_{\beta\rho_2}^T \phi'_{\rho_1} \phi'_{\rho_2} e^{-\frac{\Delta^2}{2} \phi'^2 g_\rho^{-1}}$$

after isolating the integrals over ϕ_{ρ_1} and ϕ_{ρ_2} we are left with

$$+ A_{\alpha\rho_1}^T A_{\beta\rho_2}^T \delta_{\rho_1\rho_2} \lim_{\Delta \rightarrow 0} \left(\frac{\int d\phi_{\rho_1} \phi_{\rho_1}^2 e^{-\frac{\Delta^2}{2} \phi_{\rho_1}^2 g_{\rho_1}^{-1}}}{\int d\phi_{\rho_1} e^{-\frac{\Delta^2}{2} \phi_{\rho_1}^2 g_{\rho_1}^{-1}}} \right) \langle \psi | \psi \rangle$$

using the formula in Appendix A.

$$\begin{aligned} & + \delta_{\rho_1\rho_2} A_{\alpha\rho_1}^T A_{\beta\rho_2}^T \lim_{\Delta \rightarrow 0} \left(\frac{\Gamma(3/2) / (\Delta^2 g_{\rho_1}^{-1} / 2)^{3/2}}{\Gamma(1/2) / (\Delta^2 g_{\rho_1}^{-1} / 2)^{1/2}} \right) \langle \phi | \psi \rangle \\ & + \delta_{\rho_1\rho_2} \frac{g_{\rho_1}}{\Delta^2} A_{\beta\rho_2}^T \langle \psi | \psi \rangle \\ & + A_{\alpha\rho_1}^T \left(\frac{\delta_{\rho_1\rho}}{\Delta} \right) g_\rho \left(\frac{\delta_{\rho\rho_2}}{\Delta} \right) A_{\rho_2\beta} \langle \psi | \psi \rangle \end{aligned}$$

using the continuum version of $\delta_{\alpha\beta} = \Delta \delta(\vec{x} - \vec{y})$ over all result is

$$\langle \psi | \hat{\phi}(\vec{x}) \hat{\phi}(\vec{y}) | \psi \rangle = [\phi_0(\vec{x}) \phi_0(\vec{y}) + G(\vec{x}, \vec{y})] \langle \psi | \psi \rangle \quad (D.3)$$

APPENDIX E

Show that

$$\begin{aligned} & \int D(\phi) \phi(\vec{x}_1) \phi(\vec{x}_2) \phi(\vec{x}_3) \phi(\vec{x}_4) e^{-\frac{1}{2}(\phi, G^{-1} \phi)} \\ &= \{G(x_1, x_2)G(x_3, x_3) + G(x_1, x_3)G(x_2, x_4) + G(x_1, x_4)G(x_2, x_3)\} \langle \psi | \psi \rangle \quad (E.1) \end{aligned}$$

Let do it by discretizing the \vec{x} variable

$$= \lim_{\Delta \rightarrow 0} \int \prod_{\gamma} d\phi_{\gamma} \phi_{\alpha_1} \phi_{\alpha_2} \phi_{\alpha_3} \phi_{\gamma_4} e^{-\frac{\Delta^2}{2} \phi_{\nu_1} G_{\nu_1 \nu_2}^{-1} \phi_{\nu_2}}$$

do a coordinate transformation that will diagonalize G^{-1}

$$\begin{aligned} \phi'_{\rho} &= A_{\rho \nu_2} \phi_{\nu_2} \quad \text{or} \quad \phi_{\nu_2} = A_{\nu_2 \rho}^T \phi'_{\rho} \\ &= A_{\alpha_1 \beta_1}^T A_{\alpha_2 \beta_2}^T A_{\alpha_3 \beta_3}^T A_{\alpha_4 \beta_4}^T \lim_{\Delta \rightarrow 0} \int \prod_{\gamma} d\phi'_{\gamma} \phi'_{\beta_1} \phi'_{\beta_2} \phi'_{\beta_3} \phi'_{\beta_4} \\ &\quad \times \exp \left\{ -\frac{\Delta^2}{2} \phi'_{\rho_2} A_{\nu_1 \rho_2}^T G_{\nu_1 \nu_2}^{-1} A_{\nu_2 \rho_1}^T \phi'_{\rho_1} \right\} \end{aligned}$$

where

$$\begin{aligned} A_{\nu_1 \rho_2}^T G_{\nu_1 \nu_2}^{-1} A_{\nu_2 \rho_1}^T &= g_{\rho_1}^{-1} \delta_{\rho_1 \rho_2} \\ &= A_{\alpha_1 \beta_2}^T A_{\alpha_2 \beta_2}^T A_{\alpha_3 \beta_3}^T A_{\alpha_4 \beta_4}^T \lim_{\Delta \rightarrow 0} \int \prod_{\gamma} d\phi_{\gamma} \phi_{\beta_1} \phi_{\beta_2} \phi_{\beta_3} \phi_{\beta_4} e^{-\frac{\Delta^2}{2} g_{\rho}^{-1} \phi_{\rho}^2} \end{aligned}$$

The only combinations of $(\beta_1, \beta_2, \beta_3, \beta_4)$ yielding non-zero integral values are

$$\beta_1 = \beta_2 \quad , \quad \beta_3 = \beta_4$$

$$\beta_1 = \beta_3 \quad , \quad \beta_2 = \beta_4$$

$$\beta_1 = \beta_4 \quad , \quad \beta_2 = \beta_3$$

all the other values they can take yield odd integrand (note $\beta_1 = \beta_2 = \beta_3 = \beta_4$ is

included in the sum over the above combinations.)

$$\begin{aligned} &= A_{\alpha_1 \beta_2}^T A_{\alpha_2 \beta_2}^T A_{\alpha_3 \beta_3}^T A_{\alpha_4 \beta_4}^T \lim_{\Delta \rightarrow 0} \int \prod_{\gamma} d\phi_{\gamma} \\ &\quad \times \left(\delta_{\beta_1 \beta_2} \delta_{\beta_3 \beta_4} \phi_{\beta_2}^2 \phi_{\beta_3}^2 + \delta_{\beta_1 \beta_3} \delta_{\beta_2} \delta_{\beta_4} \phi_{\beta_1}^2 \phi_{\beta_2}^2 + \delta_{\beta_1 \beta_4} \delta_{\beta_3 \beta_3} \phi_{\beta_1}^2 \phi_{\beta_2}^2 \right) e^{-\frac{\Delta^2}{2} g_{\rho}^{-1} \phi_{\rho}^2} \end{aligned}$$

utilizing the results of Appendix D we obtain

$$= \{G(\vec{x}_1, \vec{x}_2)G(\vec{x}_3, \vec{x}_4) + G(\vec{x}_1, \vec{x}_3)G(\vec{x}_2, \vec{x}_4) + G(\vec{x}_1, \vec{x}_4)G(\vec{x}_2, \vec{x}_3)\} \langle \psi | \psi \rangle \quad (E.2)$$

APPENDIX F

Show that

$$\frac{\delta E[\phi_0, G]}{\delta \phi_0(\vec{x})} = \left[-\nabla^2 + a + \frac{b}{8}\phi_0^2(\vec{x}) + \frac{b}{2}G(\vec{x}, \vec{x}) \right] \phi_0(\vec{x}) \quad (F.1)$$

and

$$\frac{\delta E[\phi_0, G]}{\delta G(\vec{x}, \vec{y})} = -\frac{1}{4}G^{-2}(\vec{x}, \vec{y}) + \left(-\nabla^2 + a + \frac{b}{2}\phi_0^2(\vec{x}) + \frac{b}{2}G(\vec{x}, \vec{x}) \right) \delta(\vec{x} - \vec{y}) \quad (F.2)$$

where

$$E[\phi_0, G] = E_{cl} + \frac{1}{8} \text{TR}(G^{-1}) + \frac{1}{2} \text{TR}(KG) + \frac{b}{8} \text{DTR}(GG)$$

where

$$E_{cl} = \int d^3\vec{x} \left\{ \frac{1}{2} (\nabla\phi_0)^2 + \frac{a}{2}\phi_0^2 + \frac{b}{24}\phi_0^4 \right\}$$

$$K(\vec{x}, \vec{y}) = \left\{ -\nabla^2 + a + \frac{b}{2}\phi_0^2(\vec{x}) \right\} \delta(\vec{x} - \vec{y})$$

a)

$$\begin{aligned} \frac{\delta E}{\delta \phi_0(\vec{x})} &= \frac{\delta}{\delta \phi_0(\vec{x})} \left\{ \int d\vec{y} \left\{ \frac{1}{2} [\nabla\phi_0(\vec{y})]^2 + \frac{a}{2}\phi_0^2(\vec{y}) + \frac{b}{24}\phi_0^4(\vec{y}) \right\} \right. \\ &\quad \left. + \frac{1}{2} \int d\vec{y} d\vec{z} K(\vec{y}, \vec{z})G(\vec{z}, \vec{y}) \right\} \\ &= \int d\vec{y} \left\{ \left[\vec{\nabla}_y \phi_0(\vec{y}) \right] \vec{\nabla}_y \delta(\vec{x} - \vec{y}) + a\phi_0(\vec{y})\delta(\vec{x} - \vec{y}) + \frac{b}{8}\phi_0^3(\vec{y})\delta(\vec{x} - \vec{y}) \right\} \\ &\quad \left. + \frac{b}{2} \int d\vec{y} d\vec{z} \left\{ \phi_0(\vec{y})\delta(\vec{y} - \vec{x})\delta(\vec{y} - \vec{z}) \right\} G(\vec{z}, \vec{y}) \right\} \end{aligned}$$

by dropping the surface term which results from the first term of integrand and integrating out over \vec{y} and \vec{z} variables we obtain

$$\begin{aligned}\frac{\delta E}{\delta \phi_0(\vec{x})} &= \left\{ -\nabla^2 \phi_0(\vec{x}) + a\phi_0(\vec{x}) + \frac{b}{8}\phi_0^3(\vec{x}) + \frac{b}{2}\phi_0(\vec{x})G(\vec{x}, \vec{x}) \right\} \\ &= \left(-\nabla^2 + a + \frac{b}{8}\phi_0^2(\vec{x}) + \frac{b}{2}G(\vec{x}, \vec{x}) \right) \phi_0(\vec{x})\end{aligned}\quad (F.3)$$

b)

$$\begin{aligned}\frac{\delta E}{\delta G(\vec{x}, \vec{y})} &= \left\{ \frac{1}{8} \int d\vec{z} G^{-1}(\vec{z}, \vec{z}) + \frac{1}{2} \int d\vec{z}_1 d\vec{z}_2 K(z_1, z_2) G(\vec{z}_2, \vec{z}_1) + \right. \\ &\left. + \frac{b}{8} \int d\vec{z} G(\vec{z}, \vec{z}) G(\vec{z}, \vec{z}) \right\}\end{aligned}$$

using

$$\frac{\delta G(z_1, z_2)}{\delta G(\vec{x}, \vec{y})} = \delta(\vec{x}_1 - \vec{z}_1) \delta(\vec{y} - \vec{z}_2) + \delta(\vec{x}_1 - \vec{z}_2) \delta(\vec{x}_2 - \vec{x}_2)$$

and integrating over all delta functions, we obtain

$$\frac{\delta E}{\delta G(\vec{x}, \vec{y})} = \frac{\delta}{\delta G(\vec{x}, \vec{y})} \left[\frac{1}{8} \text{Tr}(G^{-1}) \right] + K(\vec{x}, \vec{y}) + \frac{b}{2} G(\vec{x}, \vec{x}) \delta(\vec{x} - \vec{y})$$

To calculate the derivative of G^{-1} respect to G first we work in a discrete representation then differentiate the identity. That is,

$$\frac{\delta}{\delta G(\vec{x}, \vec{y})} G^{-1}(\vec{z}_1, \vec{z}_2) \longrightarrow \frac{\delta}{\delta G_{\alpha\beta}} G_{\nu_1\nu_2}^{-1}$$

$$G_{\nu_1\nu_2}^{-1} G_{\nu_2\nu_3} = \delta_{\nu_1\nu_3}$$

Differentiating both sides with respect to $G_{\alpha\beta}$ gives

$$\begin{aligned}\frac{\delta}{\delta G_{\alpha\beta}} (G^{-1})_{\nu_1\nu_2} G_{\nu_2\nu_3} &= - (G^{-1})_{\nu_1\nu_2} \frac{\delta}{\delta G_{\alpha\beta}} G_{\nu_2\nu_3} \\ &= - (G^{-1})_{\nu_1\nu_2} (\delta_{\alpha\nu_2} \delta_{\beta\nu_3} + \delta_{\alpha\nu_3} \delta_{\beta\nu_2})\end{aligned}$$

multiply with G^{-1} from the right

$$\begin{aligned}\frac{\delta}{\delta G_{\alpha\beta}} (G^{-1})_{\nu_1\nu_2} G_{\nu_2\nu_3} G_{\nu_3\nu_4}^{-1} &= - (G^{-1})_{\nu_1\nu_2} (\delta_{\alpha\nu_2} \delta_{\beta\nu_3} + \delta_{\alpha\nu_3} \delta_{\beta\nu_2}) \\ \frac{\delta}{\delta G_{\alpha\beta}} (G^{-1})_{\nu_1\nu_4} &= -G_{\nu_1\alpha}^{-1} G_{\beta\nu_4}^{-1} G_{\nu_1\beta}^{-1} G_{\alpha\nu_4}^{-1}\end{aligned}$$

Now if we take the trace and use $G_{\nu_1\nu_2} = G_{\nu_2\nu_1}$ we obtain

$$\frac{\delta}{\delta G_{\alpha\beta}} \text{TR}(G^{-1}) = -2 (G^{-1})_{\alpha\beta}$$

or

$$\frac{\delta}{\delta G(\vec{x}, \vec{y})} \text{TR}(G^{-1}) = -2G^{-2}(\vec{x}, \vec{y})$$

The over all result is

$$\begin{aligned}\frac{\delta E}{\delta G(\vec{x}, \vec{y})} &= -\frac{1}{4}G^{-2}(\vec{x}, \vec{y}) + K(\vec{x}, \vec{y}) + \frac{b}{2}G(\vec{x}, \vec{x})\delta(\vec{x} - \vec{y}) \\ &= -\frac{1}{4}G^{-2}(\vec{x}, \vec{y}) + \left\{ -\nabla^2 + a + \frac{b}{2}\phi_0^2(\vec{x}) + \frac{b}{2}G(\vec{x}, \vec{x}) \right\} \delta(\vec{x} - \vec{y})\end{aligned}\tag{F.4}$$

APPENDIX G

$$\int \frac{x^2}{\sqrt{a+(x^2)}} dx = \frac{xu}{2c} - \frac{a}{2c\sqrt{c}} \ln(x\sqrt{c} + u) \quad (\text{G.1})$$

where

$$c > 0 \quad u = \sqrt{a + cx^2}$$

$$\int dx x^2 \sqrt{x^2 + a} = \frac{1}{4}xu^3 - \frac{1}{8}axu - \frac{1}{8}a^2 \ln(x + u) \quad (\text{G.2})$$

where

$$u = \sqrt{x^2 + a}$$

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