STRESSES AND DISPLACEMENTS IN SEMI-INFINITE MEDIA

by

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1966

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Department of Civil Engineering

Certified by . . . .
Thesis Supervisor

Accepted by . . . .
Chairman, Departmental Committee on Graduate Students

ABSTRACT

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This thesis presents a general computer oriented method for the determination of the components of the stress tensor and the displacement vector, the principal stresses, and the principal directions of stress at any point of a semi-infinite elastic medium subjected to static normal and shearing surface loads.

This method has been programmed in FORTRAN IV for an IBM / 360 digital computer and the program, with slight improvements, will also provide the solution for the homogeneous linear-viscoelastic half-space and static loadings.

Thesis Supervisor: Fred Moavenzadeh

Title: Associate Professor

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CHAPTER 1

INTRODUCTION

Analysis of stresses and displacements in semi-infinite elastic media is of special interest in several problems of soil mechanics, such as the design of foundations for structures, highways and machinery. In such cases, a load is distributed over a relatively small area of a body which is only limited in one direction by a plane surface, it is generally assumed that this body is weightless, homogeneous, isotropic and its behavior is linear elastic, Figs. 1, 2, and 3.

This is a restricted case of the more general problem of; analysis of stresses and displacements in layered, non-homogeneous, non-isotropic semi-infinite media with time-dependent properties, subjected to arbitrarily distributed time-varying moving normal and shearing surface loads and subjected to different interphase conditions between the layers, Fig. 4.

Most methods of analysis (see references pages 123 and 124) have attempted to present the solutions of the homogeneous half-space in the closed form, and due to the mathematical complications in obtaining this type of solution for arbitrary load distributions, most of them have been limited to axisymmetric cases whereby the load is distributed over a circle. Closed form solutions for distributed loads over asymmetrical shapes generally involve elliptic integral expressions which have limited their application.

The solution of the basic problem of a normal point load on an elastic homogeneous half-space was obtained by Boussinesq in 1885. Terazawa in 1916 developed the solution for the stresses and displacements at any point in a semi-infinite elastic body under distributed normal loads. This solution is in the form of infinite integrals involving Bessel-Fourier expansions and is applicable to any distributed axi-symmetric loading.

Love solved the same problem through the use of potential functions in 1929. His work was summarized and extended by Fergus and Miner in 1955. Love's work was used to solve the problem of a uniform load distributed over an elliptical area by Deresiewicz in 1959.

A fairly extensive tabulation of stresses, strains, and deflections has been given, for arbitrary Poisson's ratio by Ahlvin and Ulery in 1962.

The most practical method of solution up to date is based on the influence charts developed by Newmark (10 and 11) which are general and have sufficient accuracy for most of the engineering applications but are in disadvantage of requiring excessive time for each point of the semi-infinite medium. Design is thus somewhat limited by the lack of a flexible and accurate method that enables the design engineer to describe the characteristics of the problem (geometry of the loaded area, load distribution, properties of the materials), to obtain in real time the desired distributions of stresses and displacements at any point of the body, to modify in turn his original conception, and to introduce new data and thus follow the steps of successive approximations of the design process. These can be summarized by stating that the most desired characteristics of any method of solution for an engineering problem are:

a. Clear, accurate and simple description of the data.
b. Great flexibility of the method itself to accept and produce different types of information corresponding to different situations.
c. Known and adjustable limits of accuracy.

The purpose of this study is to present a general computer oriented method for the determination of the components of the stress tensor and the displacement vector, the principal stresses and the principal directions of stress at any point of a semi-infinite elastic medium subjected to static normal and shearing surface loads.

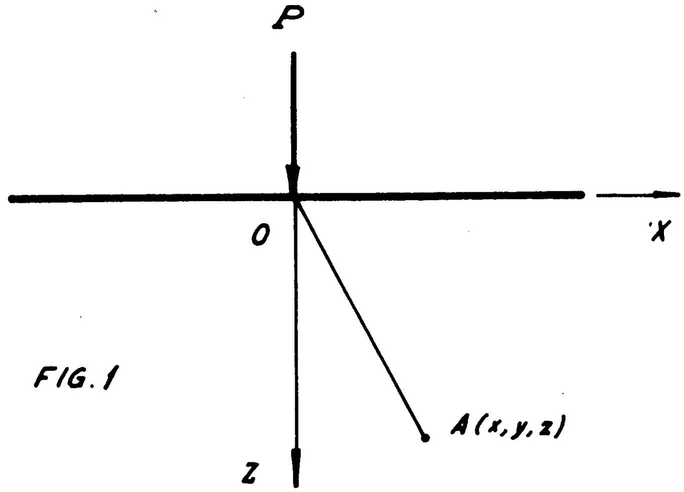


FIG. 1

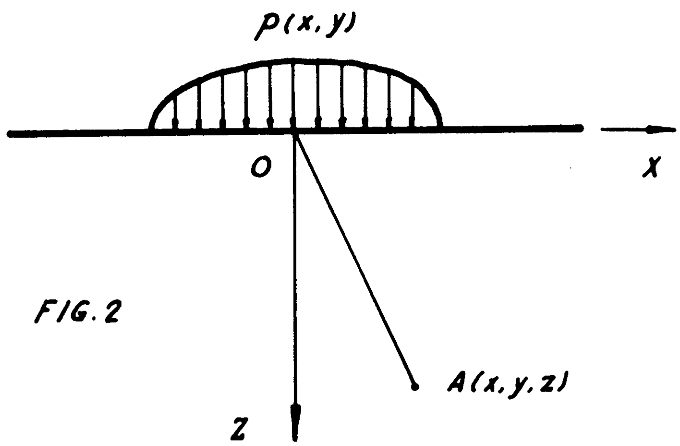


FIG. 2

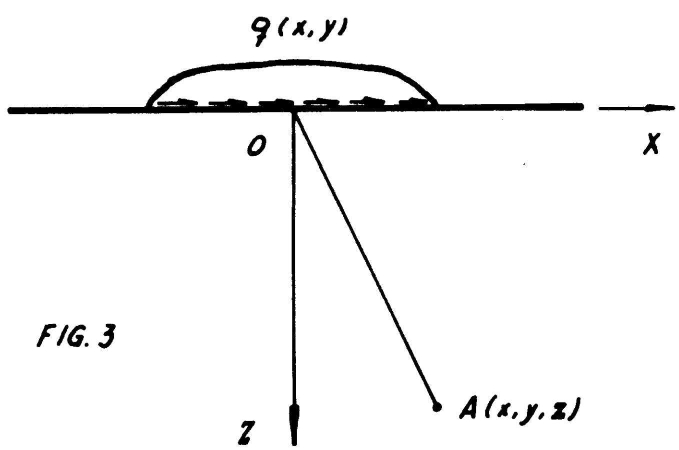


FIG. 3

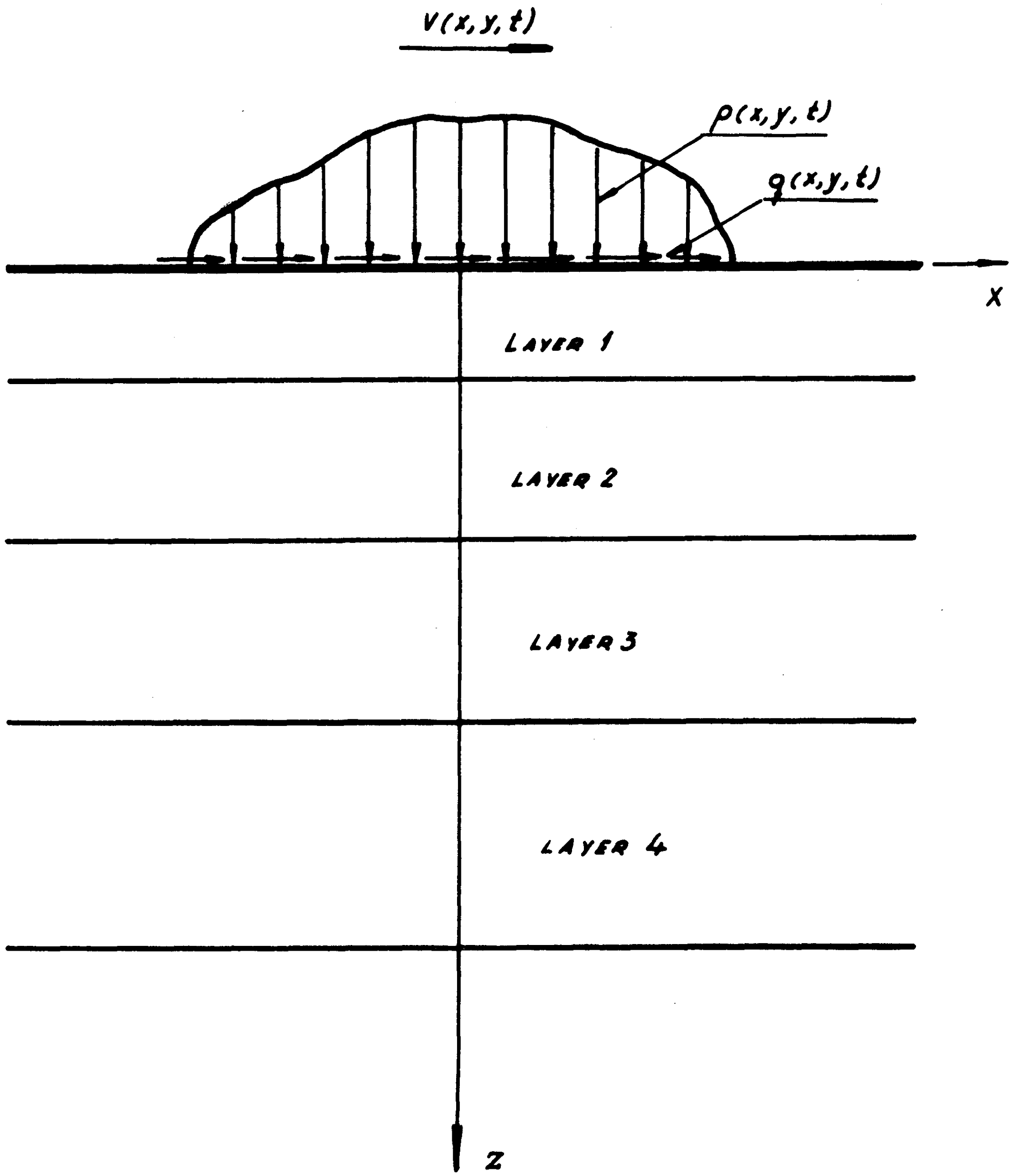
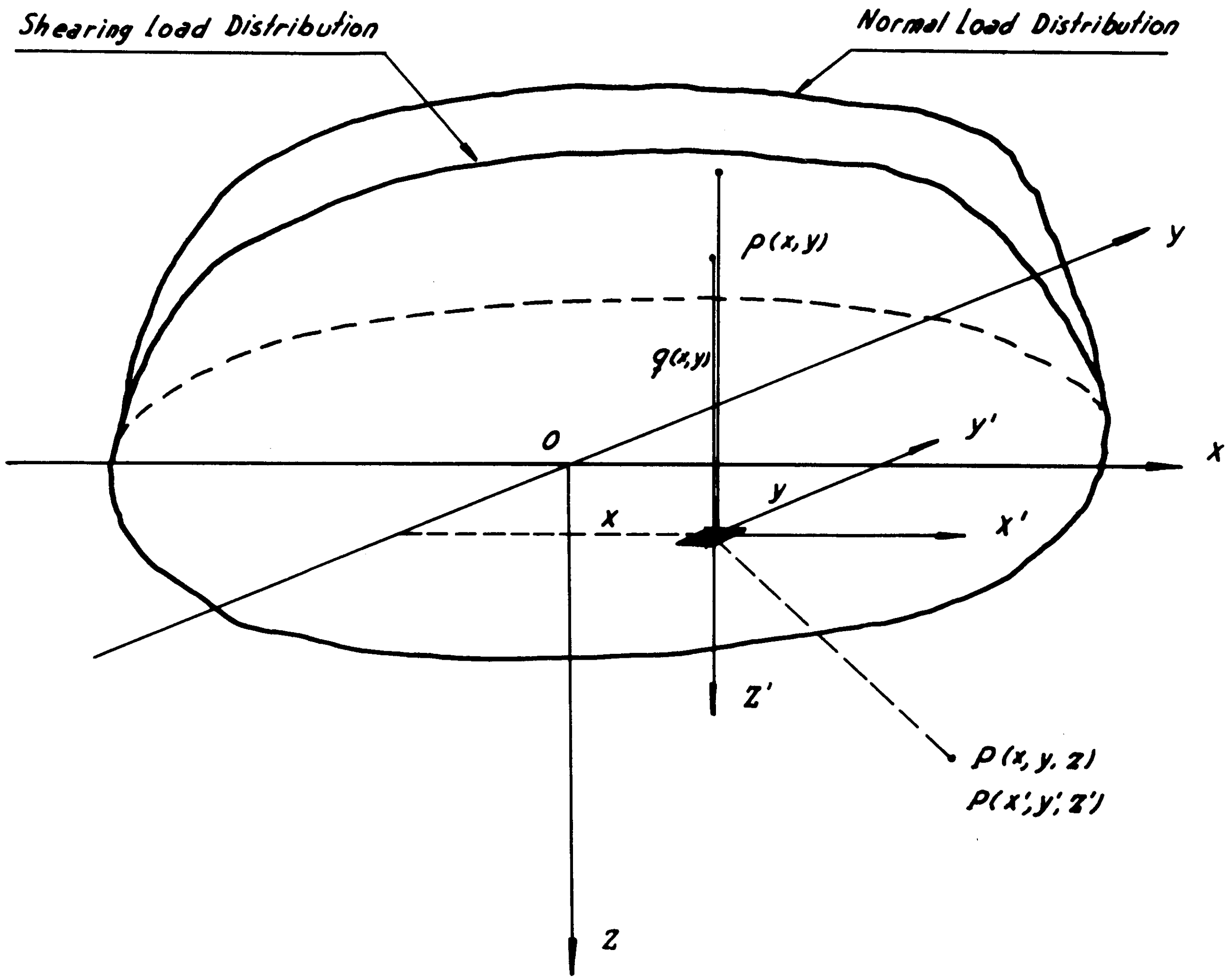


FIG. 4.



UNKNOWNS

$$\begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix} \qquad \begin{bmatrix} u \\ v \\ w \end{bmatrix} \qquad \begin{matrix} \sigma_I & l_1, m_1, n_1 \\ \sigma_{II} & l_2, m_2, n_2 \\ \sigma_{III} & l_3, m_3, n_3 \end{matrix}$$

FIG. 5

CHAPTER 2

LOAD AND LOADED AREA

To provide the method with a fair amount of generality, two basic types of load distributions are considered and superimposed:

I. Distribution of normal surface stresses $p(x,y)$
II. Distribution of shearing surface stresses $q(x,y)$

In the case of the shearing surface stresses it is assumed that they are parallel to the X axis. This assumption does not imply any limitation for the following reasons:

1. The X axis can be selected arbitrarily.
2. The expressions of the stress components can be modified if it is desired to have the shearing stresses parallel to the Y axis.
3. In case of stresses parallel to another axis in the plane X,Y they can always be decomposed in X and Y components and their total effect can be calculated using the principle of superposition.

2a. LOADED AREA

The loaded area shall be enclosed within a grid of M x M square elements, as shown in Figure 6. Each grid element will be characterized by two numbers:

$$i = \text{row number}$$

$$j = \text{column number}$$

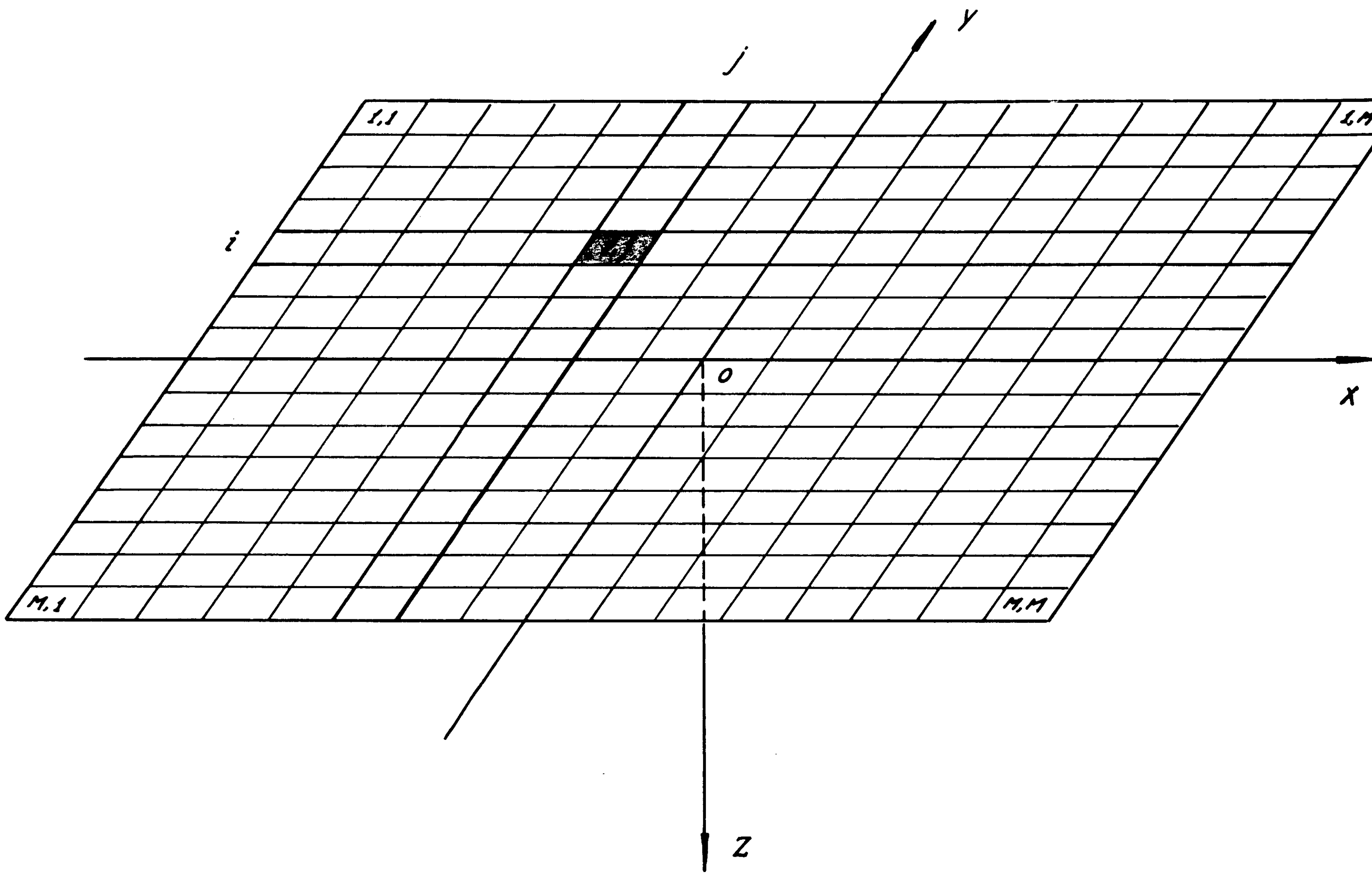


FIG. 6.

2b. LOAD FUNCTION

The load function is defined by two square matrices of order M, P(I,J) and Q(I,J), associated with the grid.

A. Normal load function

Each element ij of the first matrix will contain a number p_{ij} equal to the average intensity of the normal surface stresses applied on the element of area A_{ij} of the grid (Fig. 7).

B. Shearing load function

Similarly, each element ij of the second matrix will contain a number q_{ij} equal to the average intensity of the shearing surface stresses applied on the element of area A_{ij} of the grid (Fig. 8).

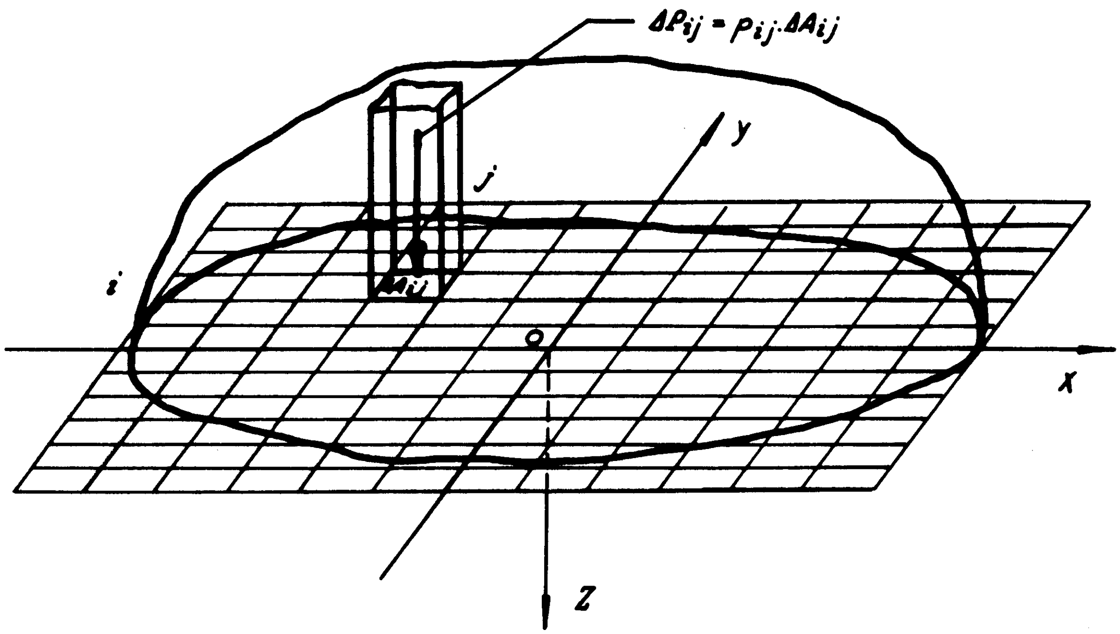
The elements of normal and shearing loads acting on the grid shall be treated in two different ways:

a. As elements of distributed loads of intensities p_{ij} and q_{ij} respectively.

b. As equivalent point loads applied at the center of each element ij, of intensities

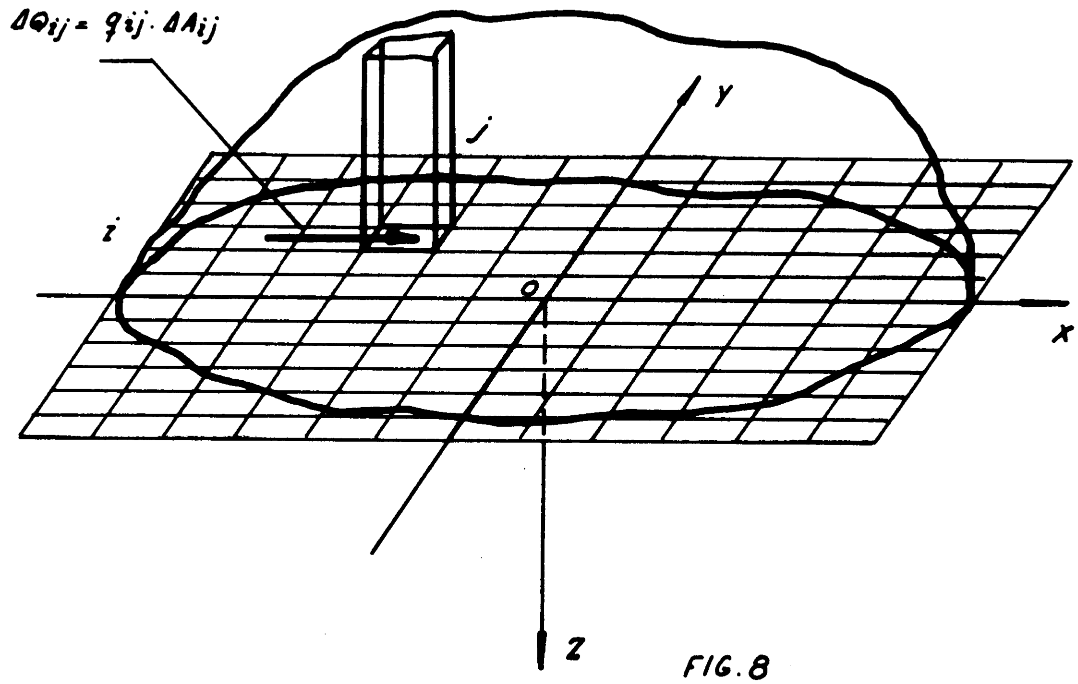
$$P_{ij} = p_{ij} \times A_{ij} \tag{1}$$

$$Q_{ij} = q_{ij} \times A_{ij} \tag{2}$$



$\Delta P_{ij} = p_{ij} \cdot \Delta A_{ij}$
Y
j
i
ΔA_{ij}
O
X
Z

FIG. 7



$\Delta Q_{ij} = q_{ij} \cdot \Delta A_{ij}$
Y
j
i
O
X
Z

FIG. 8

This discrimination shall be done according to the distance from the point $P(x,y,z)$ of the semi-infinite medium to the center of the element ij of the grid. This distance is a measure of the error incurred in considering distributed loads as point loads (see article 4e).

CHAPTER 3

COMPUTATION OF STRESSES AND DISPLACEMENTS

Due to the assumption of linear elastic behavior it is possible to use the following principle:

3a. PRINCIPLE OF SUPERPOSITION

The stresses and displacements that occur at a given point P(x,y,z) of the semi-infinite medium are equal to the sum of the stresses and displacements produced by each individual element of the load matrices P(I,J) and Q(I,J).

Let as before:

p_{ij} = average normal surface stress on element ij

q_{ij} = average shearing surface stress on element ij

σ_{kl} = component of stress tensor at point P(x,y,z) due to total load.

where $k = x,y,z$

$l = x,y,z$

$\Delta\sigma_{kl}$ = component of stress tensor at point P(x,y,z) due to load element ij

Δu_k = component of displacement vector at point P(x,y,z) due to load element ij.

Then by the principle of superposition:

$$\sigma_{kl} = \left[\sum_{\substack{i=1,M \\ j=1,M}} \Delta\sigma_{kl} \right]_P + \left[\sum_{\substack{i=1,M \\ j=1,M}} \Delta\sigma_{kl} \right]_Q \tag{1}$$

$$u_k = \left[\sum_{\substack{i=1,M \\ j=1,M}} \Delta u_k \right]_P + \left[\sum_{\substack{i=1,M \\ j=1,M}} \Delta u_k \right]_Q \tag{2}$$

M = order of load matrices P(I,J) and Q(I,J).

3b. MATHEMATICAL PROCEDURE

For the computation of the components σ_{kl} of the stress tensor and u_k of the displacement vector, the elements of distributed load p_{ij} and q_{ij} are substituted by the equivalent point loads given by expressions (1) and (2) page 13, and the numerical values of $\Delta\sigma_{kl}$ and Δu_k are then obtained by Boussinesq's expressions for normal and shearing point loads (Appendix D, formulas (1) to (12)).

This representation of the load function by a finite number of point loads has inherent the following sources of inaccuracies:

a. Point loads produce infinite discontinuities in the stress and displacement distributions at the point of application.

b. The distributions of the stresses and displacements for the distributed loads and the equivalent point

loads vary significantly in the vicinity of the point of application of the load; this variation gradually reduces as the distance to the point of application is increased. Fig. 9 shows a comparison of the distributions of σ_z for a point load of radius R = 1 at a relative depth Z/R = 1.
The rate of decrease of this difference varies with the intensity and the radius assigned to the equivalent distributed load.

c. As a consequence of b, the degree of accuracy will vary with the size of the grid elements or equivalently, with the number M of subdivisions of the loaded area and with the distance from the point of the semi-infinite medium. This point is discussed further in article 4e of the next chapter.

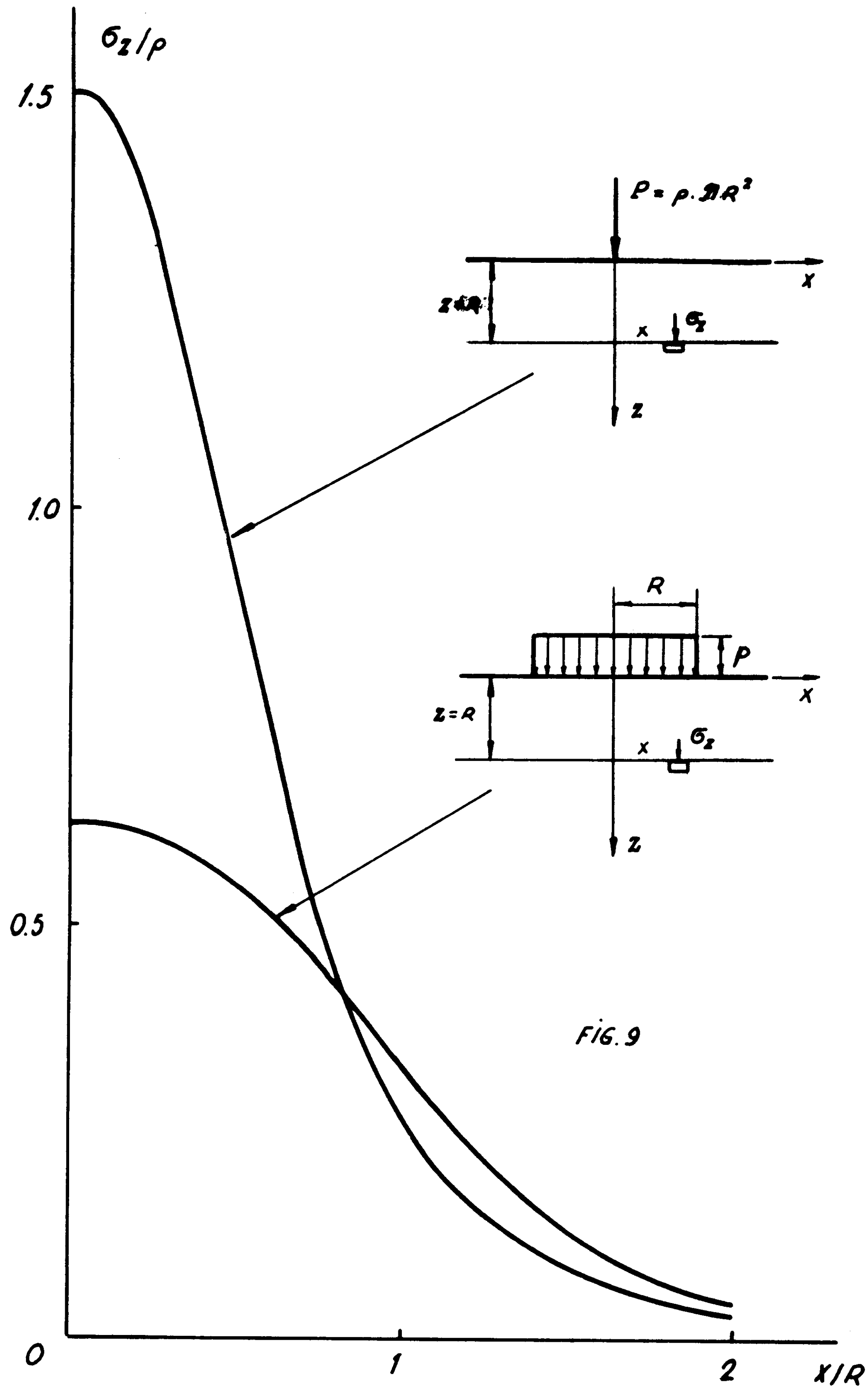


FIG. 9

CHAPTER 4

IMPROVEMENT OF ACCURACY

The evaluation and the improvement of the accuracy of this method requires further analysis of points (a), (b), and (c) of article 3b.

4a. ERROR VOLUME ASSOCIATED TO A LOAD ELEMENT

It was mentioned in section 2b, page 15, that a measure of the difference in the distributions of stresses and displacements corresponding to point and distributed loads is the distance from the point of the semi-infinite medium that is being considered to the load element ij. Assuming that the magnitude if the accepted error is $\pm\,\varepsilon_{max}\%$, it is possible to define for each load element ij a volume of the half-space outside of which the error is less than $|\pm\,\varepsilon_{max}\%|$. This volume shall be designed as the "error volume" associated to the load element ij. Where the point P(x,y,z) of the half space is within this error volume, the element ij is treated as a distributed load. The actual shape and dimensions of this volume will depend on items (a), (b), and (c) of the previous section. After the computation of the preliminary results of article 4e, it was found convenient and simple to adopt as an error volume a square prism of side 2ρ and depth $ZLIM$.

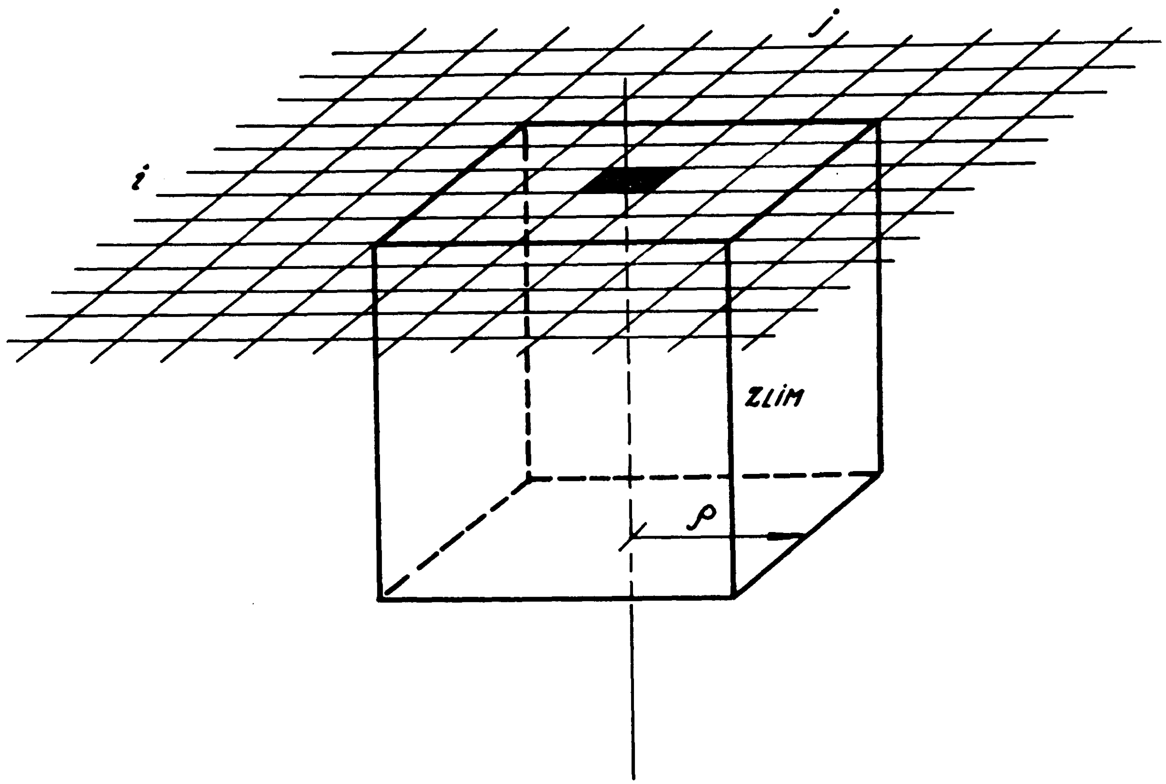


FIG. 10

Error Volume of Load Element ij

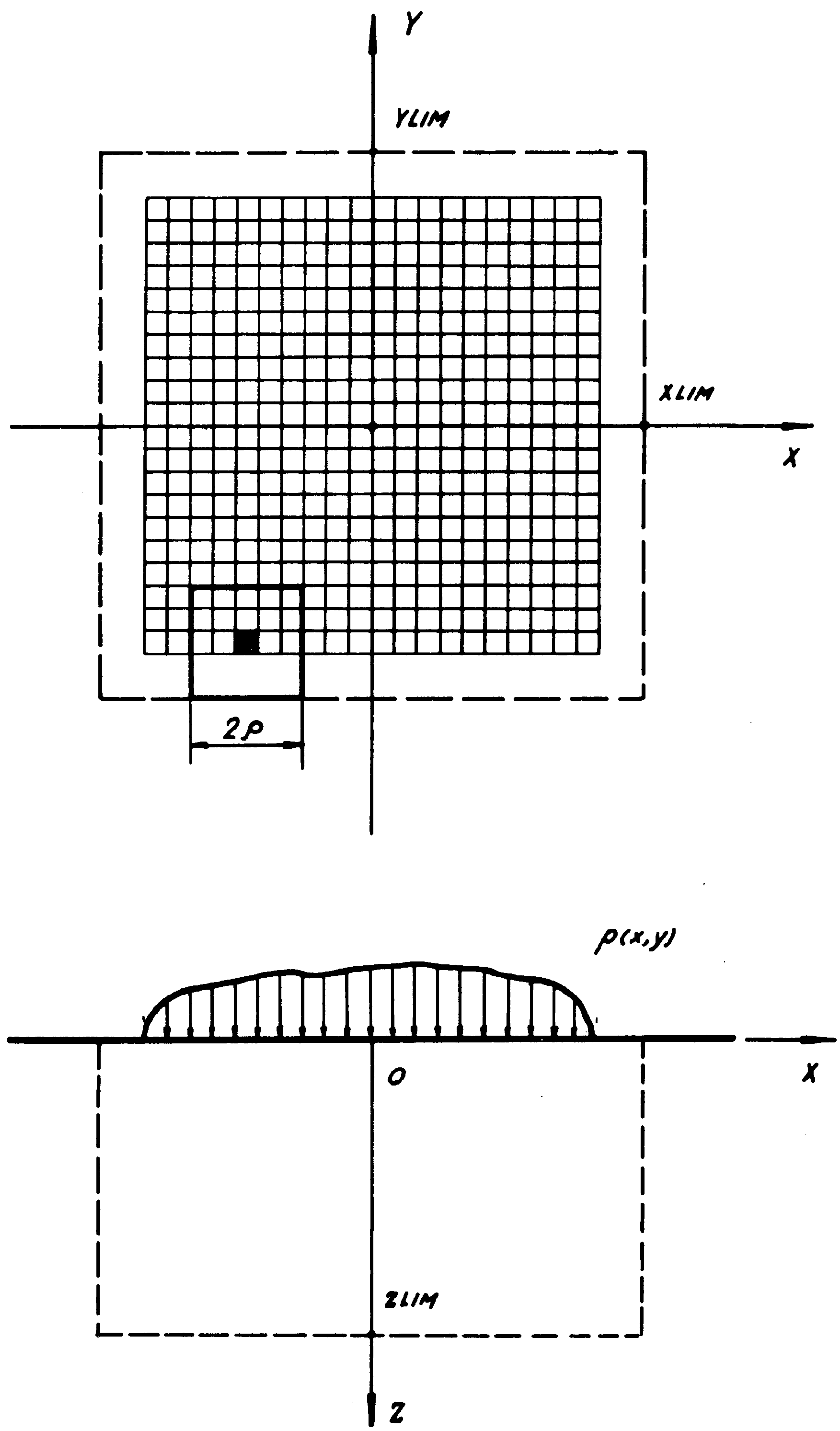


FIG. 11. Error volume of total load.

4b. ERROR VOLUME ASSOCIATED TO THE TOTAL LOAD

The error volume as corresponding to the total load is defined as the integral of the error volumes of all the load elements ij as shown in Fig. 11, and its dimensions are 2XLIM, 2YLIM and ZLIM.

4c. NUMBER OF DISTRIBUTED LOAD ELEMENTS

The procedure to determine these elements is very simple, and is used only when the point P(x,y,z) of the semi-infinite medium is within the error volume associated with the total load.

Drawing the error volume of side 2ρ and depth ZLIM with its longitudinal axis passing through P(x,y,z) as shown in Fig. 12, the intersection with the grid encloses the elements to be counted (only those whose centers are within the square of side 2ρ).

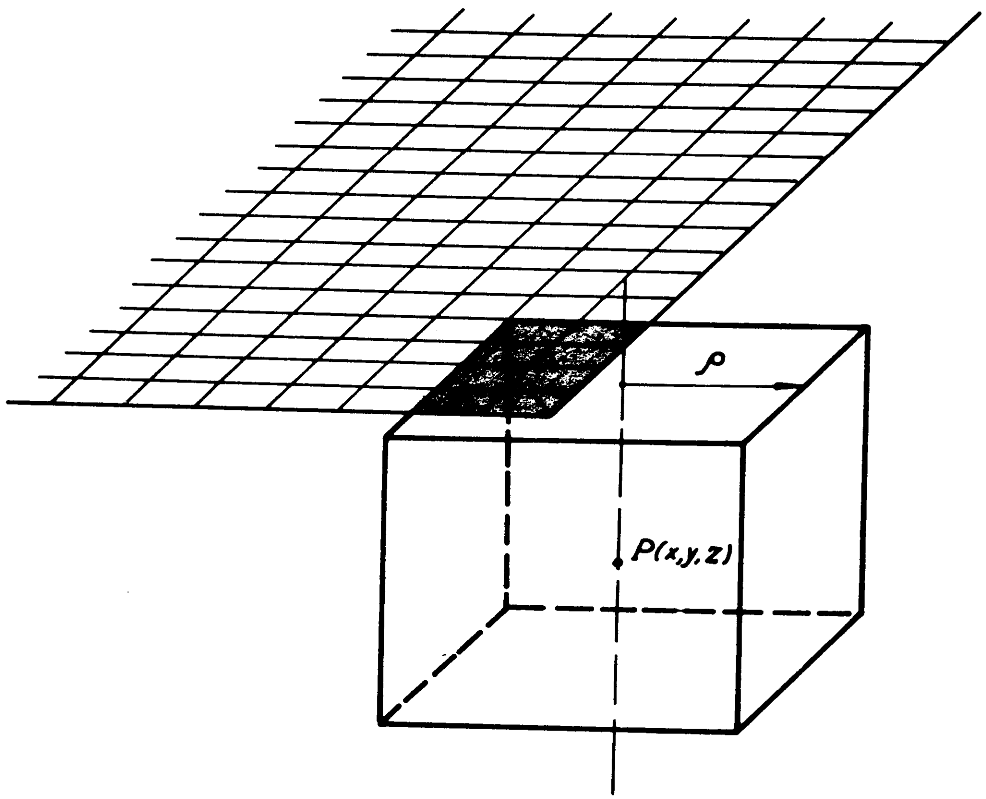


FIG. 12

4d. PROCEDURE TO HANDLE THE DISTRIBUTED LOAD ELEMENTS

Assuming that the load elements shown in Fig. 12 are those to be treated as distributed loads, when considering them one by one with their actual shape, the stress and displacement components $\Delta\sigma_{ij}$ and Δu_k have, for arbitrary points of the semi-infinite medium, complicated expressions generally involving elliptic integrals. Even the expressions for uniform circular loads and arbitrary points of the half space involve elliptic integrals, but the latter can be reduced to rather simple forms for points on the vertical axis passing through the center of the circle. These expressions have been obtained by integration of expressions (1) to (12) in Appendix D.

This point suggests the following approximation, which has been programmed and used successfully in this study.

Considering the results obtained in article 3c, in order to maintain the error to $\varepsilon_{max} = \pm 1\%$, the length of the side of the error volume must be:

$$2\rho = 0.4 \times R$$

R = maximum diameter of the loaded area.

The size of the grid divisions must be:

$$\Delta\rho = 0.5 \times \rho$$

The number of grid elements contained in the side of a square of side length 2ρ is:

$$N = \frac{2\rho}{\Delta\rho} = \frac{0.4 \times R}{0.5 \times \rho} = \frac{0.4 \times R \times 2}{0.5 \times 0.4 \times R} = \frac{0.8}{0.2} = 4$$

These results are illustrated in Fig. 13.

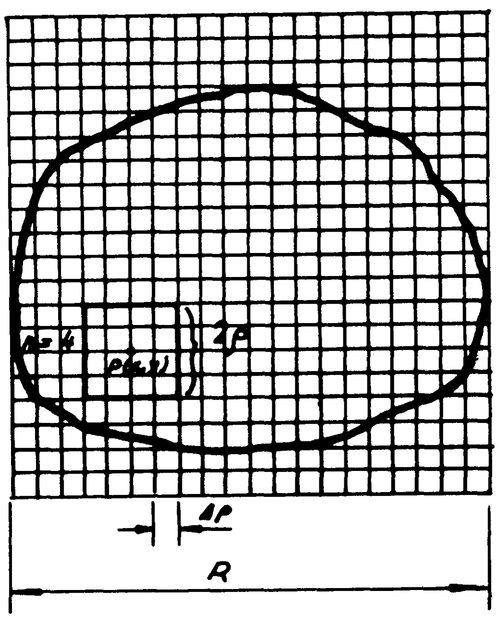
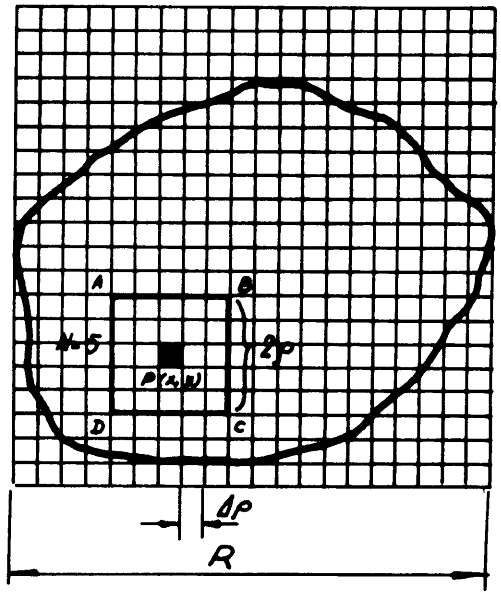


FIG. 13.

As it will be explained in Chapter 5, the center of coordinates will be located at the center of the grid of 20 x 20 elements and position of each grid element is specified by the coordinates of its center, it was found more convenient to assign to the side of the error volume a length equal to 5 grid elements as shown in Fig. 14.



Grid Order M=20

FIG. 14.

Square ABCD = error volume associated with element ij. This square is divided into three concentric square rings as shown in Fig. 15 such that:

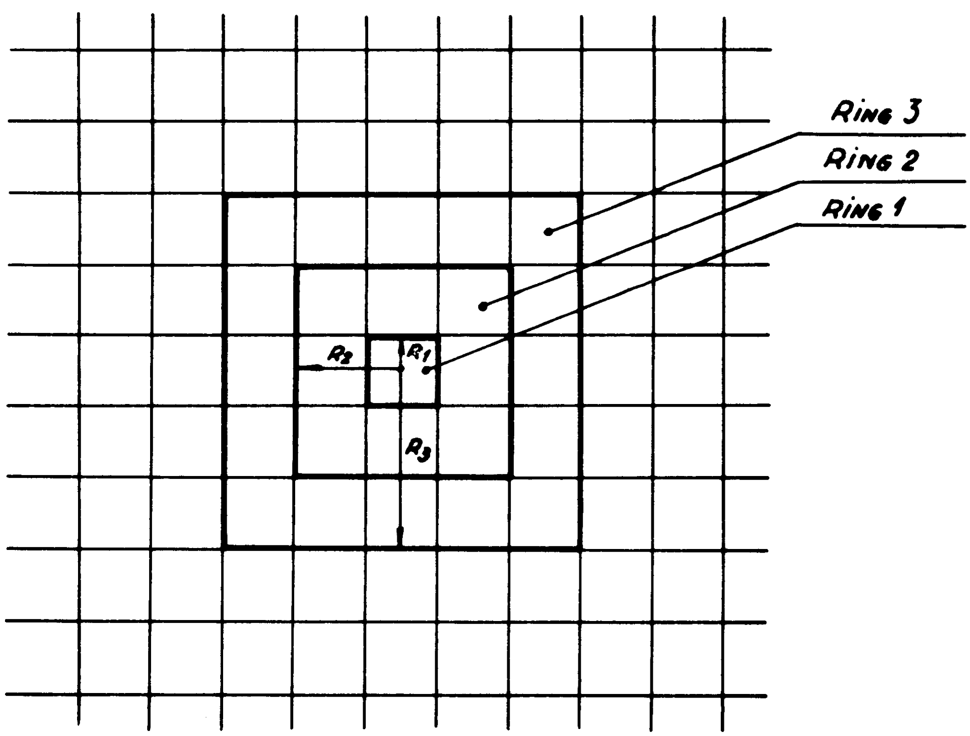


FIG. 15.

R1 $= \Delta\rho/2$ = radius of ring 1

R2 $= 3\Delta\rho/2$ = radius of ring 2

R3 $= 5\Delta\rho/2 = \rho$ = radius of ring 3

K1MAX = 1 = number of elements in ring 1

K2MAX = 8 = number of elements in ring 2

K3MAX = 16 = number of elements in ring 3

K1 = number of elements enclosed by ring 1
in a particular case ($0 \leq$ K1 $\leq$ K1MAX)

K2 = number of elements enclosed by ring 2
in a particular case ($0 \leq$ K2 $\leq$ K2MAX)

K3 = number of elements enclosed by ring 3
in a particular case ($0 \leq$ K3 $\leq$ K3MAX)

The approximation can be described as follows:

1. Check if point P(x,y,z) is within error volume corresponding to the whole load function.
If NO use expressions for point loads (expressions (3) to (20)).
If YES use algorithm 2.

2. Draw prism of radius R1 = ρ with longitudinal axis passing through P(x,y,z).

3. Count the numbers K1, K2 and K3 of load elements that fall within rings R1, R2 and R3 respectively.

4. Calculate the averages

$$p_{av1} = \frac{\sum_{i=1}^{K1} p_{ij}}{K1} \qquad q_{av1} = \frac{\sum_{i=1}^{K1} q_{ij}}{K1}$$

$$p_{av2} = \frac{\sum_{i=1}^{K2} p_{ij}}{K2} \qquad q_{av2} = \frac{\sum_{i=1}^{K2} q_{ij}}{K2}$$

$$p_{av3} = \frac{\sum_{i=1}^{K3} p_{ij}}{K3} \qquad q_{av3} = \frac{\sum_{i=1}^{K3} q_{ij}}{K3}$$

5. Calculate the fractions K1/K1MAX, K2/K2MAX, K3/K3MAX.

6. Place equivalent uniform normal and shearing circular ring loads of intensities:

$$p_{av1} \qquad p_{av2} \qquad p_{av3}$$

$$q_{av1} \qquad q_{av2} \qquad q_{av3}$$

on top of the square rings 1, 2, and 3 respectively.

7. Compute $\Delta\sigma_{kl}$ and Δu_k using formulas for points on the vertical axis of circular loads (Appendix D).

8. Multiply the results by the fractions obtained in (5) respectively.

9. Add these results to those corresponding to the other elements of the grid considered as point loads.

This method of handling the elements within the error volume involves two types of approximations:

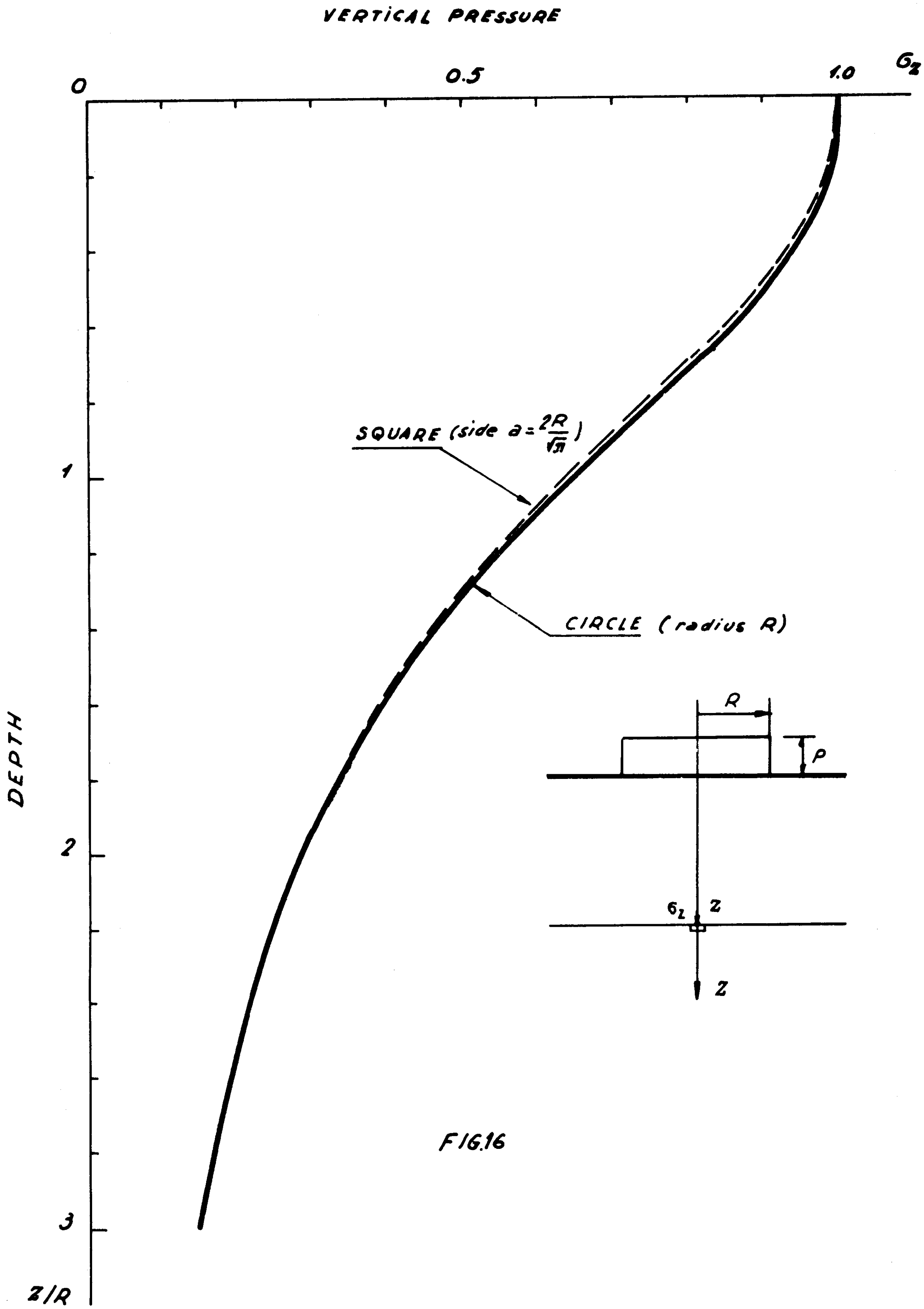
a. It considers a number K of square loads as a fraction K/KMAX of a circular load.

b. It reduces the actual distribution of surface stresses p_{ij} and q_{ij} within the three square rings to the six average intensities shown in page 29.

The determination of the errors introduced by these two approximations leads to the following considerations:

With respect to approximation (1), the diagram, Fig. 16 (see also reference 12, page 124), shows the distribution of vertical stresses σ_z for equivalent circular and square loads (for the same loaded area and the same load intensity) as a function of the depth Z. For X=0 and Z=0 they are coincident and for depths Z/R between 0 and 1 the percentage error is less than 0.01%.

With respect to approximation (2), the results obtained by this method have been compared to those obtained with Newmark's influence charts for the same points of the error volume of the total load, and an accuracy of the order of 99% is obtained for well behaved load functions.



VERTICAL PRESSURE
0
0.5
1.0
σ_z
SQUARE (side $a = \frac{2R}{\sqrt{\pi}}$)
CIRCLE (radius R)
R
p
σ_z
z
z
DEPTH
1
2
3
z/R

FIG.16

4e. SIZE OF THE GRID ELEMENTS AND RADIUS OF THE ERROR VOLUME.

In order to determine the influence of the size of the grid elements on the accuracy of the procedure and in order to determine the radius of the error volume corresponding to one grid element, the following procedure was developed:

a. Consider, for the purpose of this discussion, the loaded area as a circle of radius R, with a uniform circular load of intensity $p = 1$.

b. Calculate the stress and displacement components for points on the Z axis (vertical axis passing through the center) using formulas obtained in Appendix D.

c. Draw a circle of radius ρ concentric to the first, and consider the load on top of this circle as uniformly distributed.

d. Divide the anular ring of width $(R-\rho)$ into m concentric rings of width $\Delta\rho$, where $\Delta\rho$ will be equivalent in this case to the length of the side of one square element of the grid in a real case.

e. Divide the anular ring $(R-\rho)$ into n equal sectors, n being calculated as follows:

$$2\pi\rho = n \times \Delta\rho \qquad \therefore \qquad n = \frac{2\pi\rho}{\Delta\rho}$$

f. Place on the centers of each of the m x n elements of area A_{ij} an equivalent point load of intensity

$$P_{ij} = p_{ij} \times A_{ij} = 1 \times A_{ij}$$

g. Calculate the components of the stress tensor and the displacement vector for the same points of the semi-infinite medium as was done in (b) above.

h. Calculate the difference between results of h and b, and calculate the percentage of error based on values in part (b) above.

i. Repeat this process for different values of $\Delta\rho/\rho$ and ρ/R.

j. Define a value of the accepted error, $|\varepsilon_{max}\,\%|$ and select the number m that produces this error. Take this value of m one half of the order of the load matrices or equivalently as one half of the number of divisions of the grid sides.

This process was carried out only for the stress component σ_z and the following data was used:

$R = 1$

$\rho/R = 0.05$ to 0.20 with increments of 0.025

$\Delta\rho/\rho = 0.1$ to 0.5 with increments of 0.1

Then for any value of ρ :

$$\rho_i / R \quad \left\{ \begin{array}{l} \Delta\rho_1/\rho_i = 0.1 \\ \Delta\rho_2/\rho_i = 0.2 \\ \quad \vdots \qquad - \quad \vdots \\ \Delta\rho_5/\rho_i = 0.5 \end{array} \right.$$

For each case Z/R was varied from 0 to 4 with intervals

$$\Delta Z/R = 0.1$$

This process was programmed for an IBM 360 computer, the flow chart and the program are shown in pages 36 and 37 repectively.

Table II (page 39) shows in a compact form the final selection of M (order of the load matrices).

Adopting a magnitude of the error

$$|\pm \varepsilon_{máx}\%| = 1\%$$

the value of M that produces the closest error interval is (see Table II):

$$M = 2m = 20$$

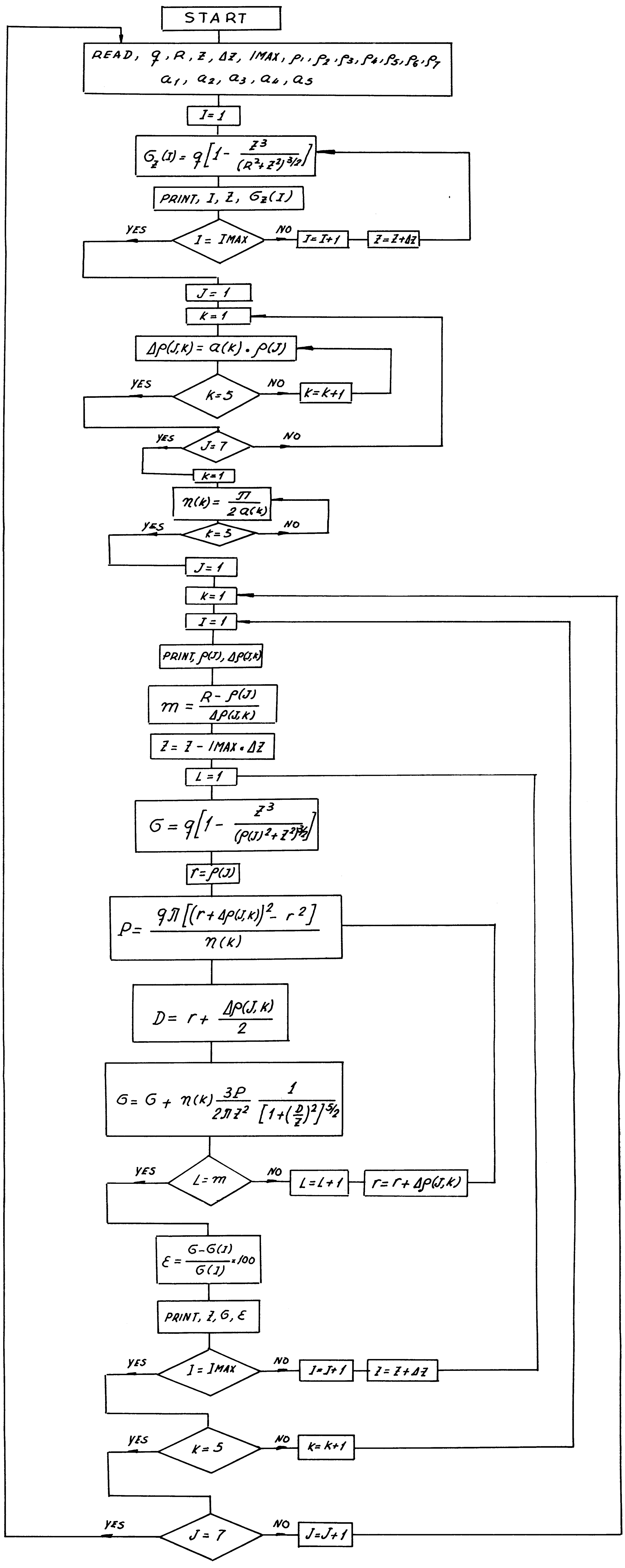
and for M = 20

ρ = 0.2 x R = Radius of error volume

$\Delta\rho$ = 0.5 x = Magnitude of a grid element.

<u>TABLE I</u>

DESCRIPTION	SYMBOL	SYMBOL IN PROGRAM
Intensity of distributed load	q	Q
Radius of loaded area	R	R
Radius of central element	ρ_j	RHO(J)
Width of the concentric rings	$\Delta\rho_{jk}$	DRHO(JK)
Coefficient of width of the rings	a_j	A(J)
Number of divisions in each ring	n_j	FN(J)
Number of rings	m	M
Depth	z	Z
Vertical component of stress for each depth, corresponding to uniform load distribution	σ_{zi}	SIG(I)
Idem for mixed distribution	σ	S
Error	ε	EPS.
Depth subscript	i	I
Center element radius subscript	j	J
Ring width and width coeff.subscript	k	K
Ring subscript	l	L
Distance from point loads to center of circle	d	D
Int. radius of the rings	r	RI



START
READ, q, R, Z, ΔZ, IMAX, ρ1, ρ2, ρ3, ρ4, ρ5, ρ6, ρ7
a1, a2, a3, a4, a5
I = 1
$\sigma_z(I) = q\left[1 - \frac{Z^3}{(R^2+Z^2)^{3/2}}\right]$
PRINT, I, Z, $\sigma_z(I)$
I = IMAX
YES
NO
I = I+1
Z = Z+ΔZ
J = 1
K = 1
$\Delta\rho(J,K) = a(K) \cdot \rho(J)$
K = 5
K = K+1
J = 7
K = 1
$\eta(K) = \frac{\pi}{2a(K)}$
K = 5
J = 1
K = 1
I = 1
PRINT, ρ(J), Δρ(J,K)
$m = \frac{R - \rho(J)}{\Delta\rho(J,K)}$
Z = Z - IMAX · ΔZ
L = 1
$\sigma = q\left[1 - \frac{Z^3}{(\rho(J)^2+Z^2)^{3/2}}\right]$
r = ρ(J)
$P = \frac{q\pi\left[(r+\Delta\rho(J,K))^2 - r^2\right]}{\eta(K)}$
$D = r + \frac{\Delta\rho(J,K)}{2}$
$\sigma = \sigma + \eta(K)\frac{3P}{2\pi Z^2}\frac{1}{\left[1+\left(\frac{D}{Z}\right)^2\right]^{5/2}}$
L = m
L = L+1
r = r + Δρ(J,K)
$\varepsilon = \frac{\sigma - \sigma(I)}{\sigma(I)} \times 100$
PRINT, Z, σ, ε
I = IMAX
I = I+1
Z = Z+ΔZ
K = 5
K = K+1
J = 7
J = J+1

```
/JOB      GO                           DOMINGUEZ A B   6032   BPR
/FTC
C       STRESS DISTRIBUTION STUDY.
        DIMENSION SIG(200),DRHO(10,10),RHO(10),A(10),FN(10)
   10 READ (5,1) Q,R,IMAX,Z,DZ
    1 FORMAT(F2.0,F2.0,I4,F5.1,F5.1)
      READ (5,2) (RHO(I),I=1,7)
    2 FORMAT(7F5.3)
      READ(5,16) (A(K),K=1,5)
   16 FORMAT(5F5.3)
      WRITE(6,30)
   30 FORMAT(11X,1HI,10X,3HZ/R,13X,6HSIG(I),//)
      DO 4 I=1,IMAX
      SIG(I)=Q*(1.-(Z**3.)/(SQRT((R**2.+Z**2.)**3.)))
      WRITE(6,3) I,Z,SIG(I)
    3 FORMAT(I12,8X,F5.1,F15.8)
    4 Z=Z+DZ
      WRITE(6,20)
   20 FORMAT(1H )
      DO 5 J=1,7
      DO 5 K=1,5
    5 DRHO(J,K)=A(K)*RHO(J)
      DO 6 K=1,5
    6 FN(K)=3.14/(2.0*A(K))
      DO 9 J=1,7
      WRITE(6,25)
   25 FORMAT(1H )
      DO 9 K=1,5
      WRITE(6,26)
   26 FORMAT(1H )
      WRITE(6,31)
   31 FORMAT(11X,1HJ,3X,1HK,10X,6HRHO(J),10X,9HDRHO(J,K),/)
      WRITE(6,15) J,K,RHO(J),DRHO(J,K)
   15 FORMAT(10X,I2,2X,I2,10X,F6.3,4X,F15.8,//)
      WRITE(6,32)
   32 FORMAT(11X,1HI,11X,1HZ,15X,1HS,19X,3HDIF,17X,3HEPS,/)
      M=(R-RHO(J))/DRHO(J,K)
      AIMAX=IMAX
      Z=Z-AIMAX*DZ
      DO 9 I=1,IMAX
      RI=RHO(J)
      S=Q*(1.-(Z**3.)/(SQRT((RHO(J)**2.+Z**2.)**3.)))
      DO 7 L=1,M
      P=Q*3.14*((RI+DRHO(J,K))**2.-RI**2.)/FN(K)
      FL=L
      D=RI+DRHO(J,K)/2.
      S=S+FN(K)*3.*P/(2.*3.14*(Z**2.)*(SQRT(1.+(D/Z)**2.)**5.))
    7 RI=RI+DRHO(J,K)
      DIF=S-SIG(I)
      EPS=DIF*100/SIG(I)
      WRITE(6,8) I,Z,S,DIF,EPS
    8 FORMAT(I12,8X,F5.1,5X,F15.8,5X,F15.8,5X,F15.8)
    9 Z=Z+DZ
      GO TO 10
      END
```

```
/DATA
1.1.  40  0.1  0.1
0.0500.0750.1000.1250.1500.1750.200
0.1000.2000.3000.4000.500
/END OF FILE
```

TABLE II

ρ/R	$\Delta\rho/\rho$	$\Delta\rho/R$	M	$\rho/\Delta\rho$	ε máx %	
					−	+
0.050	0.1	0.0050	400	10	0.0017	0.0457
	0.2	0.0100	200	5	0.0002	0.0194
	0.3	0.0150	134	3.33	0.9237	0.0000
	0.4	0.0200	100	2.5	1.8441	0.0553
	0.5	0.0250	80	2	0.0100	0.1209
0.075	0.1	0.0075	267	10.	0.4603	0.0003
	0.2	0.0150	134	5	1.8427	0.0000
	0.3	0.0225	89	3.33	0.4568	0.0398
	0.4	0.0300	67	2.5	4.5847	0.0220
	0.5	0.03750	54	2	4.5823	0.0651
0.100	0.1	0.0100	200	10	0.0328	0.0167
	0.2	0.0200	100	5	0.1329	0.0277
	0.3	0.0300	67	3.33	0.3012	0.0547
	0.4	0.0400	50	2.5	3.6661	0.0000
	0.5	0.0500	40	2	0.8478	0.1441
0.125	0.1	0.0125	160	10	0.0374	0.02479
	0.2	0.0250	80	5	0.1514	0.0370
	0.3	0.0375	54	3.33	2.2900	0.0000
	0.4	0.0500	40	2.5	4.5718	0.0000
	0.5	0.0625	32	2	0.9360	0.1603
0.150	0.1	0.0150	134	10	1.8323	0.0000
	0.2	0.0300	67	5	1.8291	0.0000
	0.3	0.0450	45	3.33	7.2870	0.0000
	0.4	0.0600	33	2.5	1.8182	0.0000
	0.5	0.0750	27	2	4.5573	0.0000
0.175	0.1	0.0175	114	10	0.4435	0.0000
	0.2	0.0350	57	5	3.6585	0.0000
	0.3	0.0525	38	3.33	6.8305	0.0000
	0.4	0.0700	29	2.5	9.9595	0.0000
	0.5	0.0[illegible]	23	2	6.8132	0.0000
0.200	0.1	0.0200	100	10	0.0279	0.0495
	0.2	0.0400	50	5	0.1284	0.0733
	0.3	0.0600	33	3.33	3.6470	0.0000
	0.4	0.0800	25	2.5	0.5355	0.1773
	0.5	0.1000	20	2	0.8450	0.2495

4f. DEPTH OF THE ERROR VOLUME

In order to determine the value of ZLIM, the circle of radius R is divided into concentric rings of width $\Delta\rho/\rho$ = 0.5 and n sectors, n being as before:

$$n = \frac{2\pi R}{\Delta\rho}$$

Equivalent point loads $P_{ij} = p_{ij} \times A_{ij}$ are placed at the center of each element and the values of the vertical stress σ_z are calculated for points on the Z axis and compared with those obtained in item (b) of the previous section.

The program corresponding to this process is shown in page 41 .

The substitution of distributed loads by equivalent point loads can be started, as shown in Table III, at a depth Z/R = 0.4 in order to maintain an error or $\pm$ 1%. But for purposes of practical convenience it is an accepted value:

$$Z/R = 1$$

$$\therefore \quad ZLIM = R$$

```
/JOB      GO                         DOMINGUEZ A B    6032    BPR
/FTC
C       STRESS DISTRIBUTION STUDY
   10 READ(5,1) Q,R,IMAX,Z,DZ,RHO,A,DRHO
    1 FORMAT(2F3.0,I5,2F5.1,F12.8,F5.1,F5.1)
      WRITE(6,2)
    2 FORMAT(11X,1HI,10X,3HZ/R,12X,3HSIG,20X,1HS,19X,3HDIF,17X,3HEPS,/)
      DO 9 I=1,IMAX
      SIG=Q*(1.-(Z**3.)/(SQRT((R**2.+Z**2.)**3.)))
      FN=3.14/(2.*A)
      M=(R-RHO)/DRHO
      RI=RHO
      S=0.
      DO 7 L=1,M
      P=Q*3.14*((RI+DRHO)**2.-RI**2.)/FN
      D=RI+DRHO/2.
      S=S+FN*3.*P/(2.*3.14*(Z**2.)*(SQRT(1.+(D/Z)**2.)**5.))
    7 RI=RI+DRHO
      DIF=S-SIG
      EPS=DIF*100./SIG
      WRITE(6,8) I,Z,SIG,S,DIF,EPS
    8 FORMAT(I12,8X,F5.1,5X,F15.8,5X,F15.8,5X,F15.8,5X,F15.8)
    9 Z=Z+DZ
      GO TO 10
      END
/DATA
 1. 1.  100  0.1  0.1  0.00000001  0.5  .1
/END OF FILE
```

TABLE III

Z/R	DISTRIBUTED LOAD	POINT LOADS	ERROR %
0.1	0.9990	1.1773	17.84
0.2	0.9924	1.0284	3.63
0.3	0.9762	0.9910	1.51
0.4	0.9487	0.9569	0.86
0.5	0.9105	0.9159	0.58
0.6	0.8638	0.8676	0.44
0.7	0.8114	0.8143	0.36
0.8	0.7562	0.7585	0.31
0.9	0.7006	0.7025	0.27
1.0	0.6464	0.6480	0.24

CHAPTER 5

COORDINATE SYSTEMS

This method basically utilizes two systems of coordinates:

1. A main cartesian system of fixed axes X,Y,Z. Its origin O is located at the center of the square grid of 20 x 20 elements. The X and Y axes are parallel to the sides of the grid and the Z axis is perpendicular to the grid plane, their positive directions are shown in Fig. 17.

2. An auxiliary cartesian system of movable axes X',Y',Z'. Its origin O' is, at each step ij of the process, located at the center of the corresponding ij element of the grid (Fig. 17).

These axes are parallel to the X,Y,Z axes respectively and have the same positive directions.

5a. COORDINATES OF A GRID ELEMENT

The coordinates X_A, Y_A, Z_A of a grid element ij with respect to the main system (Fig. 18) are given by the following relations:

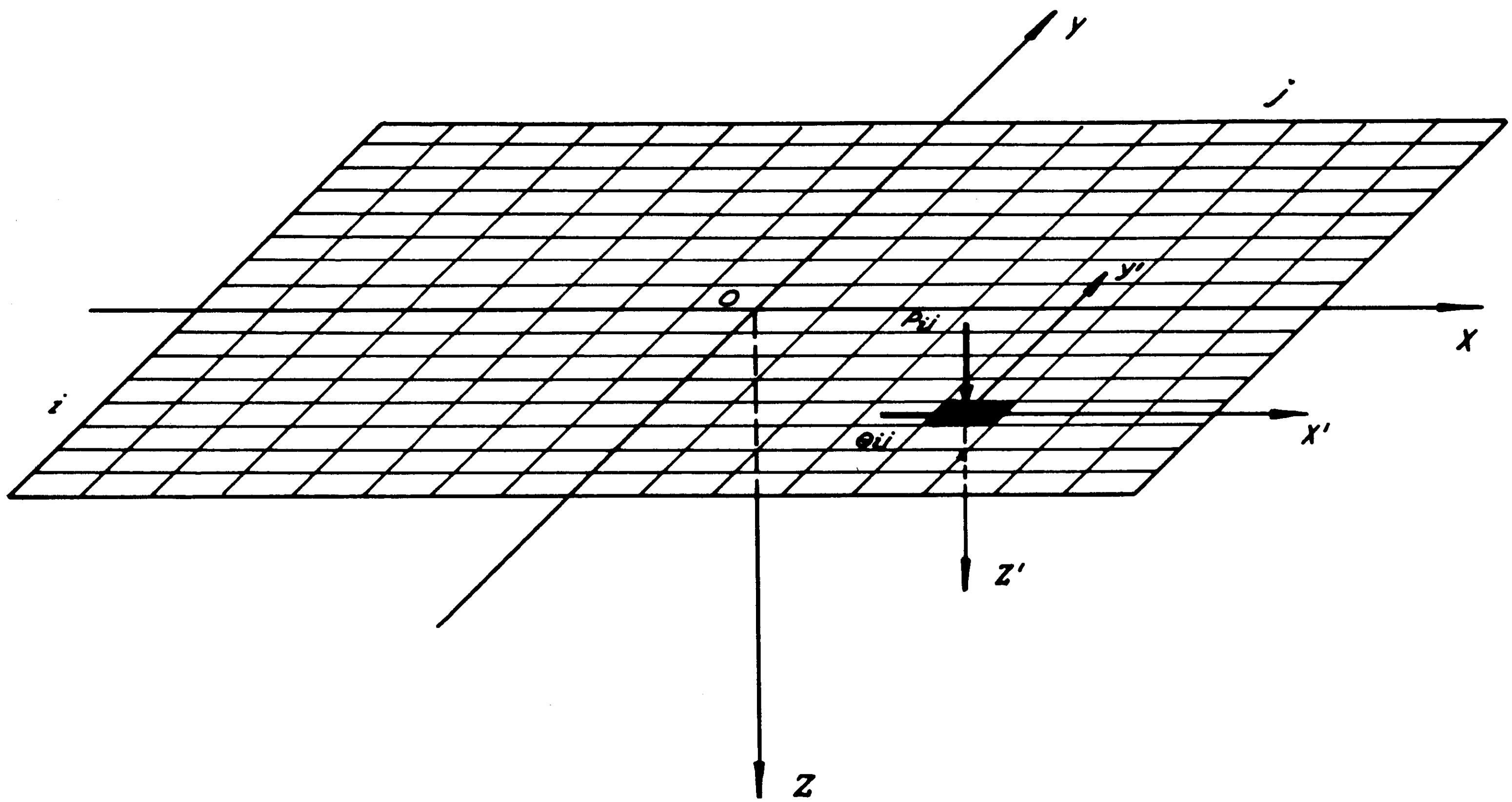


FIG. 17

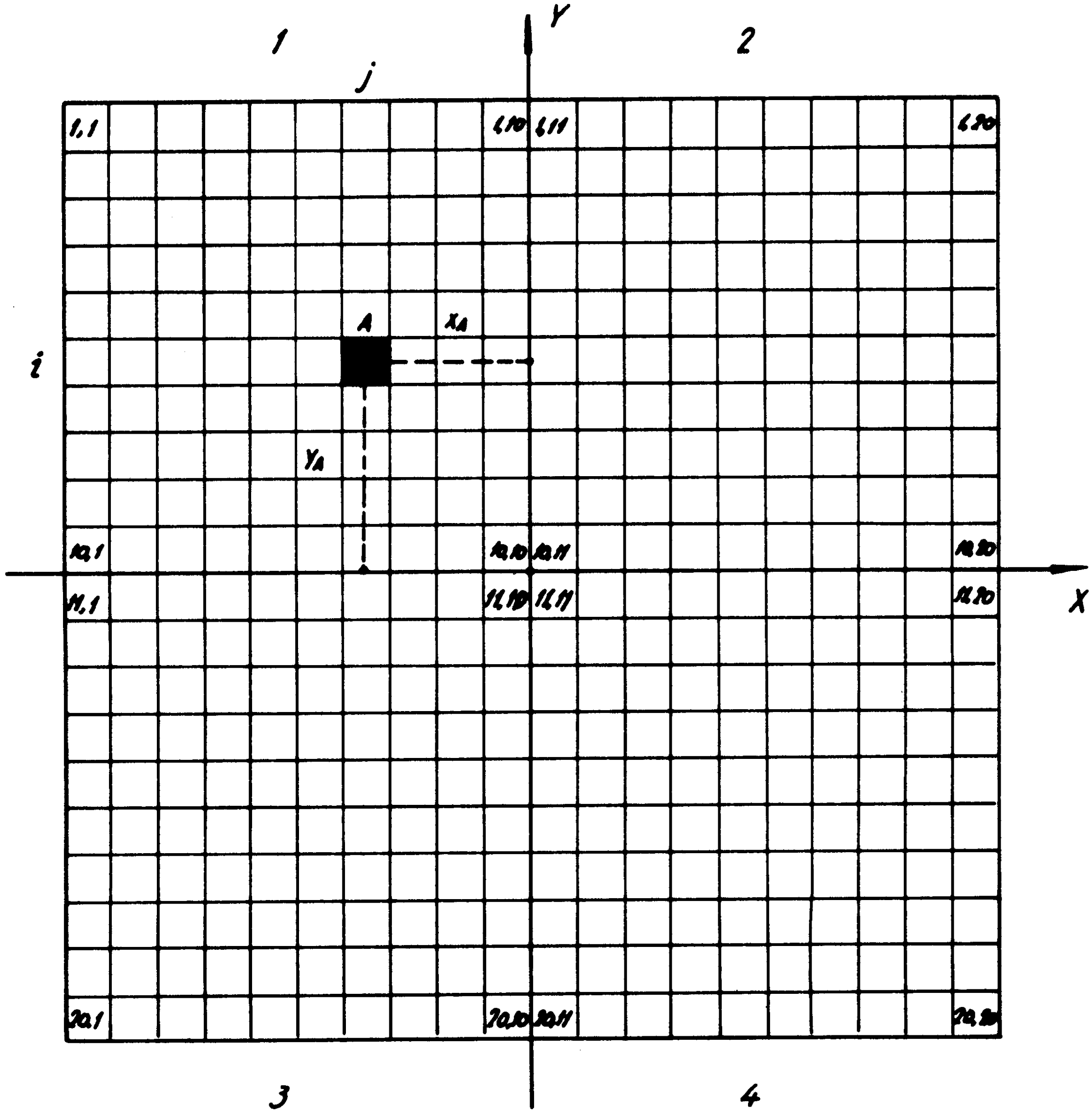

Y
1
2
j
1,1
1,10
1,11
1,20
A
x_A
i
y_A
10,1
10,10
10,11
10,20
11,1
11,10
11,11
11,20
X
20,1
20,10
20,11
20,20
3
4

FIG. 18.

a. Quadrant 1

$$1 \leq i \leq 10$$

$$1 \leq j \leq 10$$

$$XA = -(10-j).\Delta\rho - \frac{\Delta\rho}{2} = -(10-j+\frac{1}{2}).\Delta\rho = -(10.5-j).\Delta\rho$$

$$= (j-10.5).\Delta\rho$$

$$YA = (10-i).\Delta\rho + \frac{\Delta\rho}{2} = (10+\frac{1}{2}-i)\Delta\rho = (10.5-i).\Delta\rho$$

$$ZA = 0$$

b. Quadrant 2

$$1 \leq i \leq 10$$

$$11 \leq j \leq 20$$

$$XA = (j-10).\Delta\rho - \frac{\Delta\rho}{2} = (j-10-\frac{1}{2}).\Delta\rho = (j-10.5).\Delta\rho$$

$$YA = (10-i).\Delta\rho + \frac{\Delta\rho}{2} = (10+\frac{1}{2}-i)\,\Delta\rho = (10.5-i).\Delta\rho$$

$$ZA = 0$$

c. Quadrant 3

$$11 \leq i \leq 20$$

$$1 \leq j \leq 10$$

$$XA = -(10 - j).\Delta\rho - \frac{\Delta\rho}{2} = -(10 - j + \frac{1}{2})\Delta\rho = (j - 10.5).\Delta\rho$$

$$YA = -(i - 10)\Delta\rho - \frac{\Delta\rho}{2} = -(i - 10 - \frac{1}{2})\Delta\rho = (10.5 - i).\Delta\rho$$

$$ZA = 0$$

d. Quadrant 4

$$11 \leq i \leq 20$$

$$11 \leq j \leq 20$$

$$XA = (j - 10).\Delta\rho - \frac{\Delta\rho}{2} = (j - 10 - \frac{1}{2})\Delta\rho = (j - 10.5).\Delta\rho$$

$$YA = -(i - 10).\Delta\rho - \frac{\Delta\rho}{2} = -(i - 10 - \frac{1}{2}).\Delta\rho = (10.5 - i).\Delta\rho$$

$$ZA = 0$$

This shows that for the expressions for X_A, Y_A, Z_A are equal in the four quadrants:

$$XA = (j - 10.5) . \Delta\rho \quad (a)$$

$$YA = (10.5 - i) . \Delta\rho \quad (b)$$

$$ZA = 0 \quad (c)$$

5b. <u>COORDINATES OF POINT P(x,y,z) WITH RESPECT TO SYSTEM X', Y', Z'</u>.

From Fig. 17 it can be seen that the coordinates of any point P(x,y,z) of the half-space with respect to the auxiliary system X',Y',Z' are:

$$X' = X - XA$$

$$Y' = Y - YA$$

$$Z' = Z - 0 = Z$$

CHAPTER 6

PRINCIPAL STRESSES

The principal stresses are the characteristic values of the matrix :

$$\begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix}$$

The three characteristic values of these matrices can be obtained solving the following determinantal equation:

$$\begin{vmatrix} \sigma_x - \sigma & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y - \sigma & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z - \sigma \end{vmatrix} = 0$$

Developing this determinant and rearranging terms, the following characteristic equation is obtained:

$$\sigma^3 - I_1 \sigma^2 + I_2 \sigma - I_3 = 0$$

where I_1, I_2, and I_3 are the stress invariants.

$$I_1 = \sigma_x + \sigma_y + \sigma_z$$

$$I_2 = \sigma_x \sigma_y + \sigma_y \sigma_z + \sigma_z \sigma_x - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{xz}^2$$

$$I_3 = \sigma_x \sigma_y \sigma_z + 2 \tau_{xy} \tau_{yz} \tau_{xz} - \sigma_x \tau_{yz}^2 -$$

$$- \sigma_y \tau_{xz}^2 - \sigma_z \tau_{xy}^2$$

A simple direct method of solution for this cubic equation is to reduce it to the form:

$$\sigma'^3 - I_2' \sigma' - I_3' = 0$$

This is the cubic equation for the principal stresses expressed in their deviatoric form and in terms of the deviatoric stress components, where

$$I_1' = \frac{1}{3}(\sigma_x + \sigma_y + \sigma_z)$$

$$I_2' = 3 I_1^2 - I_2$$

$$I_3' = I_3 - I_2 I_1' + 2 I_1^3$$

$$\sigma_x' = \sigma_x - I_1' \qquad \tau_{xy}' = \tau_{xy}$$

$$\sigma_y' = \sigma_y - I_1' \qquad \tau_{yz}' = \tau_{yz}$$

$$\sigma_z' = \sigma_z - I_1' \qquad \tau_{xz}' = \tau_{xz}$$

The principal stresses expressed in their deviatoric form are:

$$\sigma'_I = \sigma_I - J'_1$$
$$\sigma'_{II} = \sigma_{II} - J'_1$$
$$\sigma'_{III} = \sigma_{III} - J'_1$$

These three stresses are equal to:

$$\sigma'_I = c \times \cos \alpha$$
$$\sigma'_{II} = c \times \cos \left(\alpha + \frac{2\pi}{3}\right)$$
$$\sigma'_{III} = c \times \cos \left(\alpha - \frac{2\pi}{3}\right)$$

where

$$c = 2 \sqrt{\frac{J'_2}{3}}$$

$$\cos \alpha = \frac{\sqrt{2} \,.\, J'_3}{\left[\sqrt{\frac{2}{3} J'_2} \right]^3}$$

This solution has been programmed into a subroutine together with the method for the determination of the principal direction cosines explained in section 5.

CHAPTER 7

PRINCIPAL DIRECTION COSINES

The direction cosines $l_1 m_1 n_1$, $l_2 m_2 n_2$, and $l_3 m_3 n_3$ of the principal stresses σ_I, σ_{II}, σ_{III} are the solutions of the following system of homogeneous linear equations:

$$\begin{aligned}
(\sigma_x - \sigma).l + \tau_{xy}.m + \tau_{xz}.n &= 0 \\
\tau_{yx}.l + (\sigma_y - \sigma).m + \tau_{yz}.n &= 0 \\
\tau_{zx}.l + \tau_{zy}.m + (\sigma_z - \sigma).n &= 0
\end{aligned} \qquad [1]$$

When σ_I is substituted the solution will be $l_1\ m_1\ n_1$

When σ_{II} is substituted the solution will be $l_2\ m_2\ n_2$

When σ_{III} is substituted the solution will be $l_3\ m_3\ n_3$

The direction cosines of each principal stress must satisfy the following relations:

$$\begin{aligned}
l_1^2 + m_1^2 + n_1^2 &= 1 \\
l_2^2 + m_2^2 + n_2^2 &= 1 \\
l_3^2 + m_3^2 + n_3^2 &= 1
\end{aligned} \qquad [2]$$

System [1] can be written in the following form:

$$a_1 x_i + b_1 y_i + c_1 z_i = 0 \tag{1}$$

$$a_2 x_i + b_2 y_i + c_2 z_i = 0 \tag{2}$$

$$a_3 x_i + b_3 y_i + c_3 z_i = 0 \tag{3}$$

and relations [2] can be written as:

$$x_i^2 + y_i^2 + z_i^2 = 1 \tag{4}$$

where i = 1,2,3

Assuming $x_i = A$

from (1) $$y_i = \frac{1}{b_1}(-c_1 z_i - a_1 A) \tag{5}$$

from (2) $$y_i = \frac{1}{b_2}(-c_2 z_i - a_2 A) \tag{6}$$

from (5) and (6) $$z_i = \frac{b_1 b_2 A (a_2 - a_1)}{b_2 c_1 - b_1 c_2} \tag{7}$$

substituting in (5)

$$x_i = A \tag{8}$$

$$y_i = \frac{A}{b_1}\left[\frac{-c_1 b_1 b_2 (a_2 - a_1)}{b_2 c_1 - b_1 c_2} - a_1\right] \tag{9}$$

$$z_i = \frac{b_1 b_2 A (a_2 - a_1)}{b_2 c_1 - b_1 c_2} \tag{10}$$

The constant A can be determined making use of relation (4):

$$A^2 \left\{ 1 + \frac{1}{b_1^2} \left[\frac{-c_1 b_1 b_2 (a_2 - a_1)}{b_2 c_1 - b_1 c_2} - a_1 \right]^2 + \left[\frac{b_1 b_2 (a_2 - a_1)}{b_2 c_1 - b_1 c_2} \right]^2 \right\} = 1$$

$$A = \left\{ 1 + \frac{1}{b_1^2} \left[\frac{-c_1 b_1 b_2 (a_2 - a_1)}{b_2 c_1 - b_1 c_2} - a_1 \right]^2 + \left[\frac{b_1 b_2 (a_2 - a_1)}{b_2 c_1 - b_1 c_2} \right]^2 \right\}^{-1/2}$$

Substituting A into (8), (9), and (10) we obtain X_1, Y_1, and Z_1. In case that the denominator $b_2 c_1 - b_1 c_2 = 0$ it is necessary to prepare another combination in the program, for example using equations (2) and (3).

From (2) and (3)

$$z_i = \frac{b_2 b_3 (a_3 - a_2)}{b_3 c_2 - b_2 c_3}$$

$$x_i = A$$

$$y_i = \frac{A}{b_2} \left[\frac{-c_2 b_2 b_3 (a_3 - a_2)}{b_3 c_2 - b_2 c_3} - a_2 \right]$$

The constant A again can be determined from relation (4):

$$A^2 \left\{ 1 + \frac{1}{b_2^2} \left[\frac{-c_2 b_2 b_3 (a_3 - a_2)}{b_3 c_2 - b_2 c_3} - a_2 \right]^2 + \left[\frac{b_2 b_3 (a_3 - a_2)}{b_3 c_2 - b_2 c_3} \right]^2 \right\} = 1$$

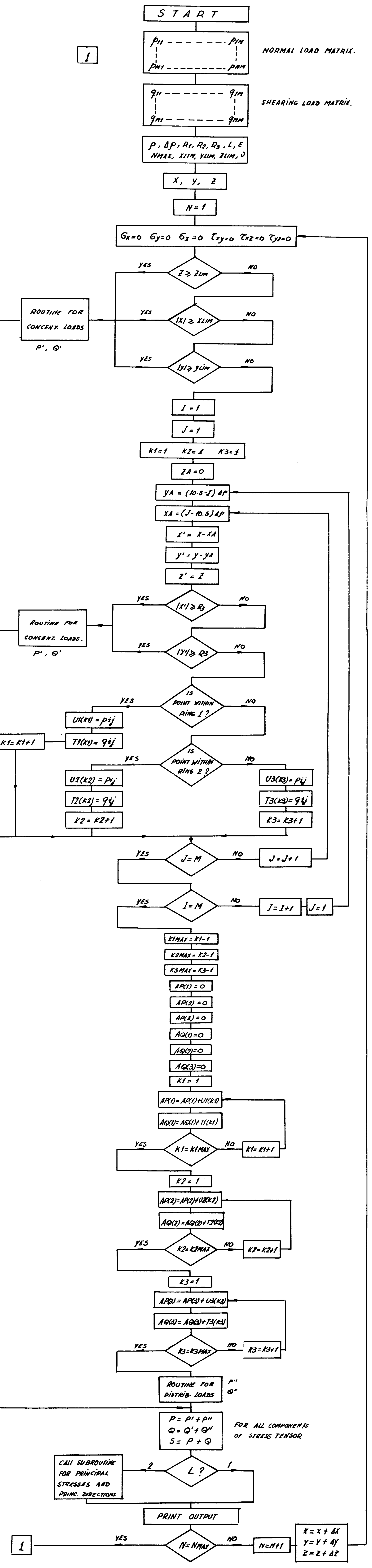
$$A = \left\{ 1 + \frac{1}{b_2^2} \left[\frac{-c_2 b_2 b_3 (a_3 - a_2)}{b_3 c_2 - b_2 c_3} - a_2 \right]^2 + \left[\frac{b_2 b_3 (a_3 - a_2)}{b_3 c_2 - b_2 c_3} \right]^2 \right\}^{-1/2}$$

CHAPTER 8

COMPUTER PROGRAM

The method described in the previous chapters was programmed in FORTRAN IV for an IBM/360 digital computer. The flow chart and the program are shown in pages 57 to 69 . Table IV (page 78) shows the output form for each point of the half-space.

Compilation time for the program is of the order of 0.96 minutes and execution time for each point is approximately 15 seconds.


START
NORMAL LOAD MATRIX.
p11 ---- p1M
pM1 ---- pMM
SHEARING LOAD MATRIX.
q11 ---- q1M
qM1 ---- qMM
ρ, Δρ, R1, R2, R3, L, E
NMAX, XLIM, YLIM, ZLIM, ν
X, Y, Z
N = 1
σx=0 σy=0 σz=0 τxy=0 τxz=0 τyz=0
Z ≥ ZLIM
|X| ≥ XLIM
|Y| ≥ YLIM
YES
NO
ROUTINE FOR CONCENT. LOADS
P', Q'
I = 1
J = 1
K1=1 K2=1 K3=1
ZA = 0
YA = (10.5-I) Δρ
XA = (J-10.5) Δρ
X' = X-XA
Y' = Y-YA
Z' = Z
|X'| ≥ R3
|Y'| ≥ R3
ROUTINE FOR CONCENT. LOADS.
P', Q'
IS POINT WITHIN RING 1?
U1(K1) = pij
T1(K1) = qij
K1= K1+1
IS POINT WITHIN RING 2?
U2(K2) = pij
T2(K2) = qij
K2 = K2+1
U3(K3) = pij
T3(K3) = qij
K3 = K3+1
J = M
J = J+1
I = M
I = I+1
J = 1
K1MAX = K1-1
K2MAX = K2-1
K3MAX = K3-1
AP(1) = 0
AP(2) = 0
AP(3) = 0
AQ(1)=0
AQ(2)=0
AQ(3)=0
K1 = 1
AP(1)= AP(1)+U1(K1)
AQ(1)= AQ(1)+T1(K1)
K1= K1MAX
K1= K1+1
K2 = 1
AP(2)=AP(2)+U2(K2)
AQ(2)=AQ(2)+T2(K2)
K2= K2MAX
K2= K2+1
K3=1
AP(3)= AP(3)+U3(K3)
AQ(3)= AQ(3)+T3(K3)
K3=K3MAX
K3 = K3+1
ROUTINE FOR DISTRIB. LOADS
P''
Q''
P = P' + P''
Q = Q' + Q''
S = P + Q
FOR ALL COMPONENTS OF STRESS TENSOR
L ?
2
1
CALL SUBROUTINE FOR PRINCIPAL STRESSES AND PRINC. DIRECTIONS
PRINT OUTPUT
N= NMAX
N=N+1
X = X + ΔX
Y = Y + ΔY
Z = Z + ΔZ
1

```
/JOB     GO                       DOMINGUEZ A B   6032   BPR
/FTC
C      STRESSES AND DISPLACEMENTS IN SEMI-INFINITE MEDIA.
C      MAIN PROGRAM.
C      CAPABILITIES OF THE SYSTEM.
C      FOR ANY POINT OF THE SEMI-INFINITE MEDIUM
C      FOR ANY SHAPE OF THE LOADED AREA
C      STRESS TENSOR DUE TO NORMAL LOADS.
C      STRESS TENSOR DUE TO SHEARING LOADS.
C      STRESS TENSOR DUE TO SUPERIMPOSED NORMAL AND SHEARING LOADS.
C      VERTICAL COMPONENT OF DISPLACEMENT DUE TO NORMAL LOADS.
C      VERTICAL COMPONENT OF DISPLACEMENT DUE TO SHEARING LOADS.
C      VERTICAL COMPONENT OF DISPLACEMENT DUE  TO SUPERIMPOSED NORMAL
C      AND SHEARING LOADS.
C      PRINCIPAL STRESSES.
C      PRINCIPAL DIRECTION COSINES.
       DIMENSION P(20,20),Q(20,20)
       DIMENSION U1(25),U2(25),U3(25),T1(25),T2(25),T3(25)
       DIMENSION AP(3),AQ(3),AT(3)
       DIMENSION S(3),B(3,3)
       COMMON S,B,SXX,SYY,SZZ,SXY,SXZ,SYZ
     1 READ(5,2) NMAX,L,POISS,ELAST,R
     2 FORMAT(2I6,F7.3,2F15.5)
       READ(5,3) ((P(I,J),J=1,10),I=1,20)
       READ(5,3) ((P(I,J),J=11,20),I=1,20)
       READ(5,3) ((Q(I,J),J=1,10),I=1,20)
       READ(5,3) ((Q(I,J),J=11,20),I=1,20)
     3 FORMAT(10F7.2)
       READ(5,30) X,Y,Z,DX,DY,DZ
    30 FORMAT(6F10.5)
C      PRINT DATA.
       WRITE(6,2) NMAX,L,POISS,ELAST,R
       WRITE(6,988)
       WRITE(6,100) ((P(I,J),J=1,20),I=1,20)
       WRITE(6,988)
       WRITE(6,100) ((Q(I,J),J=1,20),I=1,20)
       WRITE(6,988)
   988 FORMAT(1H )
   100 FORMAT(20F6.0)
       WRITE(6,789) X,Y,Z,DX,DY,DZ
   789 FORMAT(6F10.5,///////////////////////)
C      ELAST=MODULUS OF ELASTICITY.
C      POISS=POISSON'S RATIO
C      FL AND GL=LAME CONSTANTS.
       V=1.-2.*POISS
       FL=POISS*ELAST/((POISS+1.)*V)
       E628=6.28*ELAST
       GL=ELAST/(2.*(POISS+1.))
       RHO=0.2*R
       DRHO=0.5*RHO
       R1=DRHO/SQRT (3.14)
       R2=3.*R1
       R3=5.*R1
       RHO3=3.*DRHO
       DRHO2=DRHO**2
```

```
      XLIM=12.*DRHO
      YLIM=XLIM
      ZLIM=R
    4 DO 40 N=1,NMAX
C     INITIAL VALUES OF STRESS AND DISPLACEMENT COMPONENTS
C     AT POINT (X,Y,Z) OF HALF-SPACE.
      P1XX=0.
      P1YY=0.
      P1ZZ=0.
      P1XY=0.
      P1XZ=0.
      P1YZ=0.
      P2XX=0.
      P2YY=0.
      P2ZZ=0.
      P2XY=0.
      P2XZ=0.
      P2YZ=0.
      Q1XX=0.
      Q1YY=0.
      Q1ZZ=0.
      Q1XY=0.
      Q1XZ=0.
      Q1YZ=0.
      Q2XX=0.
      Q2YY=0.
      Q2ZZ=0.
      Q2XY=0.
      Q2XZ=0.
      Q2YZ=0.
      WP1=0.
      WQ1=0.
      WP2=0.
      WQ2=0.
C     IS POINT (X,Y,Z) WITHIN SENSITIVE VOLUME (2*XLIM*2YLIM*2ZLIM).
C     IF YES M=1          IF NO M=0
      IF(Z-ZLIM) 43,43,44
   43 IF(ABS(X)-XLIM) 45,44,44
   45 IF(ABS(Y)-YLIM) 46,44,44
   46 M=1
      GO TO 47
   44 M=0
   47 K1=1
      K2=1
      K3=1
      DO 9 I=1,20
      DO 9 J=1,20
C     COORDINATES OF LOAD ELEMENT (I,J) WITH RESPECT TO MAIN SYSTEM.
      ZA=0.
      HI=I
      HJ=J
      XA=(HJ-10.5)*DRHO
      YA=(10.5-HI)*DRHO
C     COORDINATES OF POINT (X,Y,Z) WITH RESPECT TO LOAD ELEMENT (I,J).
      XP=X-XA
```

```
      YP=Y-YA
      ZP=Z
      IF(M-1) 5,5,6
C     IS POINT (X,Y,Z) WITHIN SENSITIVE ZONE OF LOAD ELEMENT (I,J)
C     IN EITHER CASE GO TO CORRESPONDING ROUTINE.
    6 IF(ABS(XP)-R3) 7,5,5
    7 IF(ABS(YP)-R3) 8,5,5
C     POINT(X,Y,Z) OUTSIDE OF SENSITIVE ZONE OF LOAD ELEMENT (I,J)
C     ROUTINE FOR CONCENTRATED LOADS.
    5 D=SQRT(XP*XP+YP*YP+ZP*ZP)
      DZP2=(D+ZP)**2
      R2=XP*XP+YP*YP
      A=D**2
      AA=D**3
      C=D**5
      CCC=6.28*C
      E=XP*XP
      F=YP*YP
      G=ZP**2
      H=ZP**3
      IF(P(I,J)) 77,78,77
   77 PP=P(I,J)*DRHO2
      A1XY=0.
      TTT=3.*PP/CCC
      A1YZ=TTT*YP*G
      A1XZ=TTT*XP*G
      A1ZZ=TTT*H
      UUU=3.*ZP/C
      VVV=R2*D*(D+ZP)
      SSS=ZP/(R2*AA)
      A1YY=PP*(F*UUU-V*((F-E)/(VVV+E*SSS)))/6.28
      A1XX=PP*(E*UUU-V*((E-F)/(VVV+F*SSS)))/6.28
      DP1=PP*(1.+POISS)*(G/AA+V*2./D)/(6.28*ELAST)
      GO TO 79
   78 A1XY=0.
      A1XX=0.
      A1YY=0.
      A1ZZ=0.
      A1XZ=0.
      A1YZ=0.
      DP1=0.
   79 IF(Q(I,J)) 87,88,87
   87 QQ=Q(I,J)*DRHO2
      B1XZ=(-3.*QQ*E*ZP)/CCC
      B1YZ=(-3.*QQ*XP*YP*ZP)/CCC
      B1XY=QQ*YP*(-3.*E/A+V* (-A+E+2.*D*E/(D+ZP))/ (DZP2))
     1/(6.28*AA)
      B1ZZ=(-3.*QQ*XP*G)/CCC
      B1YY=QQ*XP*(-3.*F/A+V*(3.*A-E-2.*D*E/(D+ZP))/ (DZP2))/
     1(6.28*AA)
      DQ1=QQ*(XP*ZP/(GL*AA)+XP/((FL+GL)*D*(D+ZP)))/(4.*3.14)
      GO TO 89
   88 B1XY=0.
      B1XX=0.
      B1YY=0.
```

```
      B1ZZ=0.
      B1XZ=0.
      B1YZ=0.
      DQ1=0.
   89 P1XX=P1XX+A1XX
      P1YY=P1YY+A1YY
      P1ZZ=P1ZZ+A1ZZ
      P1XY=P1XY+A1XY
      P1XZ=P1XZ+A1XZ
      P1YZ=P1YZ+A1YZ
      Q1XX=Q1XX+B1XX
      Q1YY=Q1YY+B1YY
      Q1ZZ=Q1ZZ+B1ZZ
      Q1XY=Q1XY+B1XY
      Q1XZ=Q1XZ+B1XZ
      Q1YZ=Q1YZ+B1YZ
      WP1=WP1+DP1
      WQ1=WQ1+DQ1
      GO TO 9
C     POINT (X,Y,Z) INSIDE OF SENSITIVE ZONE OF LOAD ELEMENT (I,J)
    8 IF(ABS(XP)-R1) 101,101,102
  101 IF(ABS(YP)-R1) 103,103,102
  103 U1(K1)=P(I,J)
      T1(K1)=Q(I,J)
      GO TO 91
  102 IF(ABS(XP)-R2) 104,104,105
  104 IF(ABS(YP)-R2) 106,106,105
  106 U2(K2)=P(I,J)
      T2(K2)=Q(I,J)
      GO TO 92
  105 U3(K3)=P(I,J)
      T3(K3)=Q(I,J)
      GO TO 93
   91 K1=K1+1
      GO TO 9
   92 K2=K2+1
      GO TO 9
   93 K3=K3+1
    9 CONTINUE
C     ROUTINE FOR DISTRIBUTED LOADS.
      K1MAX=K1-1
      K2MAX=K2-1
      K3MAX=K3-1
      IF(K1MAX+K2MAX+K3MAX) 50,50,51
   50 A2XX=0.
      A2YY=0.
      A2ZZ=0.
      A2XY=0.
      A2XZ=0.
      A2YZ=0.
      B2XX=0.
      B2YY=0.
      B2ZZ=0.
      B2XY=0.
      B2XZ=0.
```

```
      B2YZ=0.
      DP2=0.
      DQ2=0.
      GO TO 52
   51 SUMP1=0.
      SUMP2=0.
      SUMP3=0.
      SUMQ1=0.
      SUMQ2=0.
      SUMQ3=0.
      DO 600 K1=1,K1MAX
      SUMP1=SUMP1+U1(K1)
  600 SUMQ1=SUMQ1+T1(K1)
      DO 601 K2=1,K2MAX
      SUMP2=SUMP2+U2(K2)
  601 SUMQ2=SUMQ2+T2(K2)
      DO 602 K3=1,K3MAX
      SUMP3=SUMP3+U3(K3)
  602 SUMQ3=SUMQ3+T3(K3)
      R1MAX=K1MAX
      R2MAX=K2MAX
      R3MAX=K3MAX
      IF(R1MAX) 900,900,901
  900 AP(1)=0.
      AQ(1)=0.
      GO TO 902
  901 AP(1)=SUMP1/R1MAX
      AQ(1)=SUMQ1/R1MAX
  902 IF(R2MAX) 903,903,904
  903 AP(2)=0.
      AQ(2)=0.
      GO TO 905
  904 AP(2)=SUMP2/R2MAX
      AQ(2)=SUMQ2/R2MAX
  905 IF(R3MAX) 906,906,907
  906 AP(3)=0.
      AQ(3)=0.
      GO TO 908
  907 AP(3)=SUMP3/R3MAX
      AQ(3)=SUMQ3/R3MAX
  908 A=ZP**2
      AA=ZP*A
      C1=R1**2
      C2=R2**2
      C3=R3**2
      D1=C1*R1
      D2=C2*R2
      D3=C3*R3
      E1=SQRT(C1+A)
      E2=SQRT(C2+A)
      E3=SQRT(C3+A)
      F1=E1**3
      F2=E2**3
      F3=E3**3
      G1=ALOG((R1+E1)/ZP)
```

```
      G2=ALOG((R2+E2)/ZP)
      G3=ALOG((R3+E3)/ZP)
      H1=ATAN(R1/ZP)
      H2=ATAN(R2/ZP)
      H3=ATAN(R3/ZP)
      DO 753 M=1,2
      IF(M-2) 300,301,301
  300 AT(1)=AP(1)
      AT(2)=AP(2)
      AT(3)=AP(3)
      D1XX=(AA/F1-2.*(1.+POISS)*ZP/E1+V+6.)/2.
      D2XX=(AA/F2-2.*(1.+POISS)*ZP/E2+V+6.)/2.
      D3XX=(AA/F3-2.*(1.+POISS)*ZP/E3+V+6.)/2.
      D1XY=0.
      D2XY=0.
      D3XY=0.
      D1XZ=2.*D1/(3.14*F1)
      D2XZ=2.*D2/(3.14*F2)
      D3XZ=2.*D3/(3.14*F3)
      D1YY=0.
      D2YY=0.
      D3YY=0.
      D1YZ=0.
      D2YZ=0.
      D3YZ=0.
      D1ZZ=1.-AA/F1
      D2ZZ=1.-AA/F2

      D3ZZ=1.-AA/F3
      D1W=(1.+POISS)*(R1/E1+2.*V*G1)/ELAST
      D2W=(1.+POISS)*(R2/E2+2.*V*G2)/ELAST
      D3W=(1.+POISS)*(R3/E3+2.*V*G3)/ELAST
      GO TO 700
  301 AT(1)=AQ(1)
      AT(2)=AQ(2)
      AT(3)=AQ(3)
      D1XX=(-4.*(1.-6.*POISS)*R1/E1+8./(3.*F1)-8.*G1+V*(8.*E1/RHO3
     1-32.*ZP/(3.*R1)-16.*H1/3.))/6.28
      D2XX=(-4.*(1.-6.*POISS)*R2/E2+8./(3.*F2)-8.*G2+V*(8.*E2/RHO3
     1-32.*ZP/(3.*R2)-16.*H2/3.))/6.28
      D3XX=(-4.*(1.-6.*POISS)*R3/E3+8./(3.*F3)-8.*G3+V*(8.*E3/RHO3
     1-32.*ZP/(3.*R3)-16.*H3/3.))/6.28
      D1XY=((4.+20.*V/3.)*R1/E1+4.*D1/(3.*F1)-4.*G1+V*(-8.*E1/RHO3+
     132.*ZP/RHO3+16.*H1/3.))/6.28
      D2XY=((4.+20.*V/3.)*R2/E2+4.*D2/(3.*F2)-4.*G2+V*(-8.*E1/RHO3+
     132.*ZP/RHO3+16.*H2/3.))/6.28
      D3XY=((4.+20.*V/3.)*R3/E3+4.*D3/(3.*F3)-4.*G3+V*(-8.*E3/RHO3+
     132.*ZP/RHO3+16.*H3/3.))/6.28
      D1XZ=0.
      D2XZ=0.
      D3XZ=0.
      D1YY=(8.*POISS*R1/E1+4.*D1/(3.*F1)-4.*G1+V*(-20.*E1/RHO3+
     144.*ZP/RHO3+4.*H1))/6.28
      D2YY=(8.*POISS*R2/E2+4.*D2/(3.*F2)-4.*G2+V*(-20.*E2/RHO3+
     144.*ZP/RHO3+4.*H2))/6.28
```

```
      D3YY=(8.*POISS*R3/E3+4.*D3/(3.*F3)-4.*G3+V*(-20.*E3/RHO3+
     144.*ZP/RHO3+4.*H3))/6.28
      D1YZ=-3.*ZP*(2./(3.*ZP)+A/(3.*F1)-1./E1)/3.14
      D2YZ=-3.*ZP*(2./(3.*ZP)+A/(3.*F2)-1./E2)/3.14
      D3YZ=-3.*ZP*(2./(3.*ZP)+A/(3.*F3)-1./E3)/3.14
      D1ZZ=-2.*D1/(3.14*F1)
      D2ZZ=-2.*D2/(3.14*F2)
      D3ZZ=-2.*D3/(3.14*F3)
      D1W=2.*(1.+POISS)*(V*G1/(V+2.*ELAST)-ZP*R1/E1+V*(2.*ZP/R1
     1-2.*E1/R1)/(V+2.*ELAST))/(3.14*ELAST)
      D2W=2.*(1.+POISS)*(V*G2/(V+2.*ELAST)-ZP*R2/E2+V*(2.*ZP/R2
     1-2.*E2/R2)/(V+2.*ELAST))/(3.14*ELAST)
      D3W=2.*(1.+POISS)*(V*G3/(V+2.*ELAST)-ZP*R3/E3+V*(2.*ZP/R3
     1-2.*E3/R3)/(V+2.*ELAST))/(3.14*ELAST)
  700 IF(AT(1)-AT(2)) 710,710,711
  710 IF(AT(1)-AT(3)) 712,712,713
  711 IF(AT(2)-AT(3)) 714,714,715
  712 IF(AT(2)-AT(3)) 716,716,717
C     MIN=AT(1)   INT=AT(2)   MAX=AT(3)
  716 ALFA=AT(1)
      BETA=AT(2)-AT(1)
      GAMMA=AT(3)-AT(2)
      C1XX=ALFA*D3XX
      C2XX=BETA*(D3XX-D1XX)
      C3XX=GAMMA*(D3XX-D2XX)
      C1XY=ALFA*D3XY
      C2XY=BETA*(D3XY-D1XY)
      C3XY=GAMMA*(D3XY-D2XY)
      C1XZ=ALFA*D3XZ
      C2XZ=BETA*(D3XZ-D1XZ)
      C3XZ=GAMMA*(D3XZ-D2XZ)
      C1YY=ALFA*D3YY
      C2YY=BETA*(D3YY-D1YY)
      C3YY=GAMMA*(D3YY-D2YY)
      C1YZ=ALFA*D3YZ
      C2YZ=BETA*(D3YZ-D1YZ)
      C3YZ=GAMMA*(D3YZ-D2YZ)
      C1ZZ=ALFA*D3ZZ
      C2ZZ=BETA*(D3ZZ-D1ZZ)
      C3ZZ=GAMMA*(D3ZZ-D2ZZ)
      C1W=ALFA*D3W
      C2W=BETA*(D3W-D1W)
      C3W=GAMMA*(D3W-D2W)
      GO TO 750
C     MIN=AT(1)   INT=AT(3)   MAX=AT(2)
  717 ALFA=AT(1)
      BETA=AT(3)-AT(1)
      GAMMA=AT(2)-AT(3)
      C1XX=ALFA*D3XX
      C2XX=BETA*(D3XX-D1XX)
      C3XX=GAMMA*(D2XX-D1XX)
      C1XY=ALFA*D3XY
      C2XY=BETA*(D3XY-D1XY)
      C3XY=GAMMA*(D2XY-D1XY)
      C1XZ=ALFA*D3XZ
```

```
      C2XZ=BETA*(D3XZ-D1XZ)
      C3XZ=GAMMA*(D2XZ-D1XZ)
      C1YY=ALFA*D3YY
      C2YY=BETA*(D3YY-D1YY)
      C3YY=GAMMA*(D2YY-D1YY)
      C1YZ=ALFA*D3YZ
      C2YZ=ALFA*(D3YZ-D1YZ)
      C3YZ=GAMMA*(D2YZ-D1YZ)
      C1ZZ=ALFA*D3ZZ
      C2ZZ=BETA*(D3ZZ-D1ZZ)
      C3ZZ=GAMMA*(D2ZZ-D1ZZ)
      C1W=ALFA*D1W
      C2W=BETA*(D3W-D1W)
      C3W=GAMMA*(D2W-D1W)
      GO TO 750
C     MIN=AT(3)    INT=AT(1)    MAX=AT(2)
  713 ALFA=AT(3)
      BETA=AT(1)
      GAMMA=AT(2)-AT(1)
      C1XX=ALFA*D3XX
      C2XX=BETA*D2XX
      C3XX=GAMMA*(D2XX-D1XX)
      C1XY=ALFA*D3XY
      C2XY=BETA*D2XY
      C3XY=GAMMA*(D2XY-D1XY)
      C1XZ=ALFA*D3XZ
      C2XZ=BETA*D2XZ
      C3XZ=GAMMA*(D2XZ-D1XZ)
      C1YY=ALFA*D3YY
      C2YY=BETA*D2YY
      C3YY=GAMMA*(D2YY-D1YY)
      C1YZ=ALFA*D3YZ
      C2YZ=BETA*D2YZ
      C3YZ=GAMMA*(D2YZ-D1YZ)
      C1ZZ=ALFA*D3ZZ
      C2ZZ=BETA*D2ZZ
      C3ZZ=GAMMA*(D2ZZ-D1ZZ)
      C1W=ALFA*D3W
      C2W=BETA*D2W
      C3W=GAMMA*(D2W-D1W)
      GO TO 750
  714 IF(AT(1)-AT(3)) 720,720,721
C     MIN=AT(2)    INT=AT(1)    MAX=AT(3)
  720 ALFA=AT(2)
      BETA=AT(2)-AT(1)
      GAMMA=AT(3)-AT(2)
      C1XX=ALFA*D3XX
      C2XX=BETA*D1XX
      C3XX=GAMMA*(D3XX-D1XX)
      C1XY=ALFA*D3XY
      C2XY=BETA*D1XY
      C3XY=GAMMA*(D3XY-D1XY)
      C1XZ=ALFA*D3XZ
      C2XZ=BETA*D1XZ
      C3XZ=GAMMA*(D3XZ-D1XZ)
```

```
      C1YY=ALFA*D3YY
      C2YY=BETA*D1YY
      C3YY=GAMMA*(D3YY-D1YY)
      C1YZ=ALFA*D3YZ
      C2YZ=BETA*D1YZ
      C3YZ=GAMMA*(D3YZ-D1YZ)
      C1ZZ=ALFA*D3ZZ
      C2ZZ=BETA*D1ZZ
      C3ZZ=GAMMA*(D3ZZ-D1ZZ)
      C1W=ALFA*D3W
      C2W=BETA*D1W
      C3W=GAMMA*(D3W-D1W)
      GO TO 750
C     MIN=AT(2)    INT=AT(3)    MAX=AT(1)
  721 ALFA=AT(2)
      BETA=AT(1)-AT(2)
      GAMMA=AT(3)-AT(2)
      C1XX=ALFA*D3XX
      C2XX=BETA*D1XX
      C3XX=GAMMA*(D3XX-D2XX)
      C1XY=ALFA*D3XY
      C2XY=BETA*D1XY
      C3XY=GAMMA*(D3XY-D2XY)
      C1XZ=ALFA*D3XZ
      C2XZ=BETA*D1XZ
      C3XZ=GAMMA*(D3XZ-D2XZ)
      C1YY=ALFA*D3YY
      C2YY=BETA*D1YY
      C3YY=GAMMA*(D3YY-D2YY)
      C1YZ=ALFA*D3YZ
      C2YZ=BETA*D1YZ
      C3YZ=GAMMA*(D3YZ-D2YZ)
      C1ZZ=ALFA*D3ZZ
      C2ZZ=BETA*D1ZZ
      C3ZZ=GAMMA*(D3ZZ-D2ZZ)
      C1W=ALFA*D3W
      C2W=BETA*D1W
      C3W=GAMMA*(D3W-D2W)
      GO TO 750
C     MIN=AT(3)    INT=AT(2)    MAX=AT(1)
  715 ALFA=AT(3)
      BETA=AT(2)-AT(3)
      GAMMA=AT(1)-AT(2)
      C1XX=ALFA*D3XX
      C2XX=BETA*D2XX
      C3XX=GAMMA*D1XX
      C1XY=ALFA*D3XY
      C2XY=BETA*D2XY
      C3XY=GAMMA*D1XY
      C1XZ=ALFA*D3XZ
      C2XZ=BETA*D2XZ
      C3XZ=GAMMA*D1XZ
      C1YY=ALFA*D3YY
      C2YY=BETA*D2YY
      C3YY=GAMMA*D1YY
```

```
      C1YZ=ALFA*D3YZ
      C2YZ=BETA*D2YZ
      C3YZ=GAMMA*D1YZ
      C1ZZ=ALFA*D3ZZ
      C2ZZ=BETA*D2ZZ
      C3ZZ=GAMMA*D1ZZ
      C1W=ALFA*D3W
      C2W=BETA*D2W
      C3W=GAMMA*D1W
  750 A2XX=C1XX+C2XX+C3XX
      A2XY=C1XY+C2XY+C3XY
      A2XZ=C1XZ+C2XZ+C3XZ
      A2YY=C1YY+C2YY+C3YY
      A2YZ=C1YZ+C2YZ+C3YZ
      A2ZZ=C1ZZ+C2ZZ+C3ZZ
      DW=C1W+C2W+C3W
      IF(M-2) 751,752,752
  751 A2YY=A2XX
      A2YZ=A2XZ
      P2XX=P2XX+A2XX
      P2XY=P2XY+A2XY
      P2XZ=P2XZ+A2XZ
      P2YY=P2YY+A2YY
      P2YZ=P2YZ+A2YZ
      P2ZZ=P2ZZ+A2ZZ
      WP2=WP2+DW
      GO TO 753
  752 A2XZ=A2YZ
      Q2XX=Q2XX+A2XX
      Q2XY=Q2XY+A2XY
      Q2XZ=Q2XZ+A2XZ
      Q2YY=Q2YY+A2YY
      Q2YZ=Q2YZ+A2YZ
      Q2ZZ=Q2ZZ+A2ZZ
      WQ2=WQ2+DW
  753 CONTINUE
C     STRESS COMPONENTS DUE TO NORMAL LOAD.
   52 PXX=P1XX+P2XX
      PYY=P1YY+P2YY
      PZZ=P1ZZ+P2ZZ
      PXY=P1XY+P2XY
      PXZ=P1XZ+P2XZ
      PYZ=P1YZ+P2YZ
C     VERTICAL DISPLACEMENT DUE TO NORMAL LOAD.
      WP=WP1+WP2
C     STRESS COMPONENTS DUE TO SHEARING LOAD.
      QXX=Q1XX+Q2XX
      QYY=Q1YY+Q2YY
      QZZ=Q1ZZ+Q2ZZ
      QXY=Q1XY+Q2XY
      QXZ=Q1XZ+Q2XZ
      QYZ=Q1YZ+Q2YZ
C     VERTICAL DISPLACEMENT DUE TO SHEARING LOAD.
      WQ=WQ1+WQ2
C     STRESS COMPONENTS DUE TO SUPERIMPOSED NORMAL AND SHEARING LOADS.
```

```
      SXX=PXX+QXX
      SYY=PYY+QYY
      SZZ=PZZ+QZZ
      SXY=PXY+QXY
      SXZ=PXZ+QXZ
      SYZ=PYZ+QYZ
C     VERTICAL DISPLACEMENT DUE TO SUPERIMPOSED NORMAL AND SHEARING LOADS.
      W=WP+WQ
C     PRINT OUTPUT.
      WRITE(6,11) X,Y,Z
   11 FORMAT(20X,2HX=F15.8,20X,2HY=F15.8,20X,2HZ=F15.8,//)
      WRITE(6,12)
   12 FORMAT(20X,2HXX,18X,2HYY,18X,2HZZ,18X,2HXY,18X,2HXZ,18X,2HYZ,//)
      WRITE(6,13) PXX,PYY,PZZ,PXY,PXZ,PYZ
   13 FORMAT(5X,1HP,6F20.8,//)
      WRITE(6,14) QXX,QYY,QZZ,QXY,QXZ,QYZ
   14 FORMAT(5X,1HQ,6F20.8,//)
      WRITE(6,15) SXX,SYY,SZZ,SXY,SXZ,SYZ
   15 FORMAT(5X,1HS,6F20.8,//)
      WRITE(6,16) WP,WQ,W
   16 FORMAT(19X,3HWP=F15.8,19X,3HWQ=F15.8,19X,3HW =F15.8,///)
      IF(L-2) 17,18,18
   18 CALL MHOR
C     PRINCIPAL STRESSES AND PRINCIPAL DIRECTIONS OF STRESS.
      WRITE(6,55)
   55 FORMAT(55X,18HPRINCIPAL STRESSES)
      WRITE(6,19) S(1),S(2),S(3)
   19 FORMAT(17X,5HS(1)=F15.8,17X,5HS(2)=F15.8,17X,5HS(3)=F15.8,//)
      WRITE(6,56)
   56 FORMAT(50X,27HPRINCIPAL DIRECTION COSINES)
      WRITE(6,20) B(1,1),B(2,1),B(3,1)
      WRITE(6,21) B(1,2),B(2,2),B(3,2)
      WRITE(6,22) B(1,3),B(2,3),B(3,3)
   20 FORMAT(15X,7HB(1,1)=F15.4,15X,7HB(2,1)=F15.4,15X,7HB(3,1)=F15.4,/)
   21 FORMAT(15X,7HB(1,2)=F15.4,15X,7HB(2,2)=F15.4,15X,7HB(3,2)=F15.4,/)
   22 FORMAT(15X,7HB(1,3)=F15.4,15X,7HB(2,3)=F15.4,15X,7HB(3,3)=F15.4,
     1////)
C     INCREASE X,Y,Z.
   17 X=X+DX
      Y=Y+DY
   40 Z=Z+DZ
      CALL EXIT
      END
```

```
/FTC
      SUBROUTINE MHOR
C     SUBROUTINE FOR PRINCIPAL STRESSES AND PRINCIPAL DIRECTIONS.
      DIMENSION S(3),B(3,3)
      COMMON S,B,SXX,SYY,SZZ,SXY,SXZ,SYZ
C     STRESS INVARIANTS.
      SNV1=(SXX+SYY+SZZ)/3.
      SNV2=(SXX*SYY+SYY*SZZ+SZZ*SXX-SXY**2 -SXZ**2 -SYZ**2 )
      SNV3=-(SXX*SYY*SZZ + 2.*SXY*SYZ*SXZ-SXX*SYZ*SYZ - SYY*SXZ*SXZ
     1-SZZ*SXY*SXY)
C     SOLUTION OF THE CUBIC EQUATION.
      X=3.*(SNV1**2 )-SNV2
      Y=SNV3-SNV2*SNV1+2.*(SNV1**3 )
      Z=SQRT(2.*X/3.)
      C3A=Y*SQRT(2.)/(Z**3)
      S3A=SQRT(1.-(C3A**2 ))
      T3A=S3A/C3A
      ALFA=(ATAN(T3A))/3.
      C=Z*SQRT(2.)
C     VOLUMETRIC PRINCIPAL STRESSES.
      SS1=C*COS(ALFA)
      SS2=C*COS(ALFA+6.28/3.)
      SS3=C*COS(ALFA+12.56/3.)
C     PRINCIPAL STRESSES.
      S(1)=SS1+SNV1
      S(2)=SS2+SNV1
      S(3)=SS3+SNV1
C     PRINCIPAL DIRECTIONS.
C     B(1,1),B(2,1),B(3,1) = DIRECTION COSINES OF PRINCIPAL STRESS S(1).
C     B(1,2),B(2,2),B(3,2) = DIRECTION COSINES OF PRINCIPAL STRESS S(2).
C     B(1,3),B(2,3),B(3,3) = DIRECTION COSINES OF PRINCIPAL STRESS S(3).
      DO 5 J=1,3
      U1=(SYY-S(J))*SXZ-SXY*SYZ
      U2=SYZ**2 -(SYY-S(J))*(SZZ-S(J))
      IF(U1) 10,11,10
   10 V1=SXZ*SXY*(SYY-S(J))*(SXY-(SXX-S(J)))
      W1=SXY*(SYY-S(J))*(SXY-(SXX-S(J)))
      A=1./SQRT(1.+((-V1/U1-(SXX-S(J)))/SXY)**2 +(W1/U1)**2 )
      GO TO 12
   11 V1=-SXY*(SYY-S(J))*SYZ*(SXZ-SXY)
      W1=(SYY-S(J))*SYZ*(SXZ-SXY)
      A=1./SQRT(1.+((V1/U2-SXY)/(SYY-S(J)))**2 +(W1/U1)**2 )
      GO TO 13
   12 B(1,J)=A
      B(3,J)=A*SXY*(SYY-S(J))*(SXY-(SXX-S(J)))/U1
      B(2,J)=(-SXZ*B(3,J)-(SXX-S(J))*A)/SXY
      GO TO 5
   13 B(1,J)=A
      B(3,J)=A*(SYY-S(J))*SYZ*(SXY-SXZ)/U2
      B(2,J)=(-SYZ*B(3,J)-SXY*A)/(SYY-S(J))
    5 CONTINUE
      RETURN
      END
```

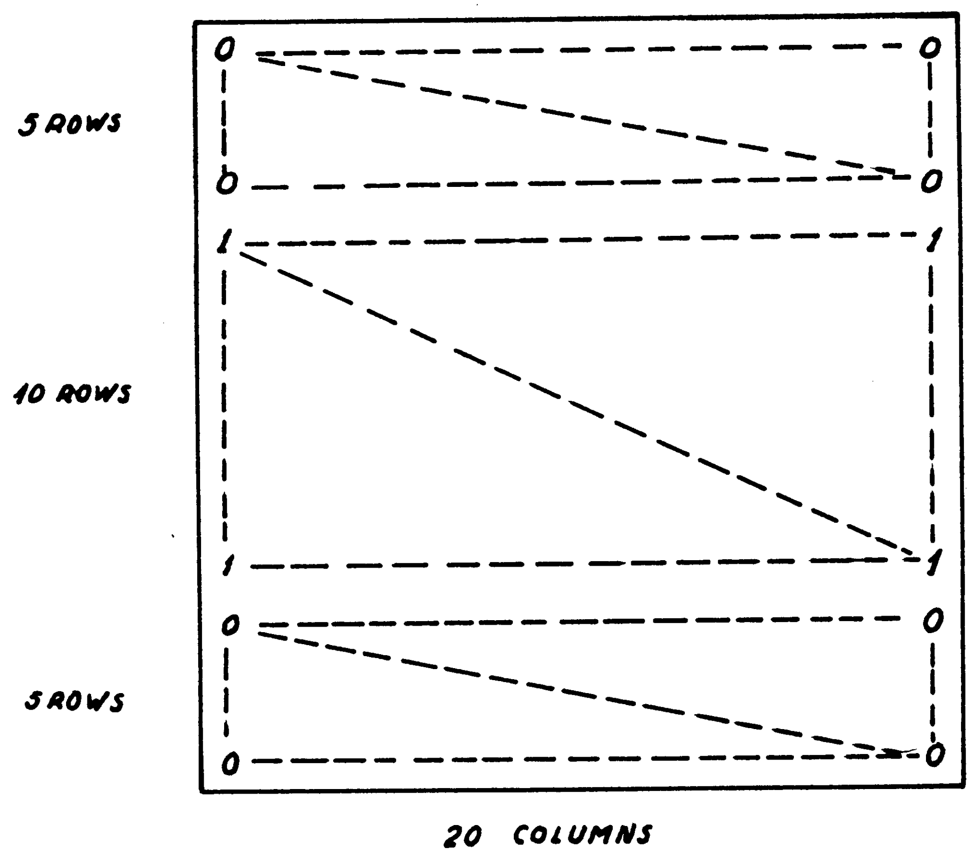
CHAPTER 9

RESULTS

The program was tested several times with different data, an example is shown in pages 72 to 76 in which the following was utilized:

Number of points to be analyzed	NMAX = 10
Use subroutine for principal stresses	L = 2
Poisson's ratio	POISS = 0
Modulus of Elasticity	ELAST = 1
Maximum diameter of loaded area	R = 1
Initial Coordinates	x = 0
	y = 0
	z = 0.2
Increments	Dx = 0
	Dy = 0
	Dz = 0

Load Matrices: The two load matrices were identical and had the following form:



X= 0.0 Y= 0.0 Z= 0.19999999

	XX	YY	ZZ	XY	XZ	YZ
P	0.51012242	0.58316684	0.97635388	0.0	-0.00000011	0.00000294
Q	0.0	-0.00000002	0.00000010	-0.00000448	-0.67576253	-0.00000001
S	0.51012242	0.58316678	0.97635394	-0.00000448	-0.67576259	0.00000294

WP= 1.33483410 WQ= -0.00000024 W = 1.33483315

PRINCIPAL STRESSES

S(1)= 1.41631317 S(2)= -0.02467448 S(3)= 0.57669755

PRINCIPAL DIRECTION COSINES

B(1,1)=	0.0000	B(2,1)=	-1.0000	B(3,1)=	0.0000
B(1,2)=	0.0000	B(2,2)=	1.0000	B(3,2)=	-0.0000
B(1,3)=	0.0000	B(2,3)=	-1.0000	B(3,3)=	0.0000

X= 0.0 Y= 0.0 Z= 0.39999998

	XX	YY	ZZ	XY	XZ	YZ
P	0.30598438	0.28801090	0.87176800	0.0	-0.00000014	0.00000401
Q	0.0	-0.00000011	0.00000015	-0.00000364	-0.42454565	-0.00000000
S	0.30598438	0.28801078	0.87176812	-0.00000364	-0.42456677	0.00000401

WP= 1.14883327 WQ= -0.00000028 W = 1.14883237

PRINCIPAL STRESSES

S(1)= 1.02869987 S(2)= -0.04928577 S(3)= 0.48535895

PRINCIPAL DIRECTION COSINES

B(1,1)=	0.0000	B(2,1)=	-1.0000	B(3,1)=	0.0000
B(1,2)=	0.0000	B(2,2)=	1.0000	B(3,2)=	-0.0000
B(1,3)=	0.0000	B(2,3)=	-1.0000	B(3,3)=	0.0000

X= 0.0	Y= 0.0	Z= 0.59999995

	XX	YY	ZZ	XY	XZ	YZ
P	0.17173719	0.15050429	0.72884732	0.0	-0.00000019	0.00000387
Q	0.0	-0.00000010	0.00000023	-0.00000033	-0.25987351	-0.00000000
S	0.17173719	0.15050417	0.72884750	-0.00000033	-0.25987369	0.00000386

WP= 0.98864913	WQ= -0.00000025	W = 0.98864888

PRINCIPAL STRESSES

S(1)= 0.79404873	S(2)= -0.03682834	S(3)= 0.29315555

PRINCIPAL DIRECTION COSINES

B(1,1)=	0.0000	B(2,1)=	-1.0000	B(3,1)=	0.0000
B(1,2)=	0.0000	B(2,2)=	1.0000	B(3,2)=	-0.0000
B(1,3)=	0.0000	B(2,3)=	-1.0000	B(3,3)=	0.0000

X= 0.0	Y= 0.0	Z= 0.79999995

	XX	YY	ZZ	XY	XZ	YZ
P	0.09370512	0.08837312	0.59389663	0.0	-0.00000022	0.00000275
Q	0.0	-0.00000004	0.00000023	-0.00000030	-0.15902418	0.00000000
S	0.09370512	0.08837306	0.59389681	-0.00000030	-0.15902436	0.00000275

WP= 0.85669965	WQ= -0.00000022	W = 0.85669941

PRINCIPAL STRESSES

S(1)= 0.62271565	S(2)= -0.02399778	S(3)= -0.17573695

PRINCIPAL DIRECTION COSINES

B(1,1)=	0.0000	B(2,1)=	-1.0000	B(3,1)=	0.0000
B(1,2)=	0.0000	B(2,2)=	1.0000	B(3,2)=	-0.0000
B(1,3)=	0.0000	B(2,3)=	-1.0000	B(3,3)=	0.0000

X=	0.0	Y=	0.0	Z=	[illegible]

	XX	YY	ZZ	XY	XZ	YZ
P	0.05042163	0.05800239	0.48160243	0.0	-0.00000020	0.00000032
Q	0.0	-0.00000001	0.00000022	-0.00000030	[illegible]	0.0
S	0.05042163	0.05800238	0.48160261	-0.00000030	[illegible]	0.00000032

WP=	0.74955177	WQ=	-0.00000023	W =	[illegible]

PRINCIPAL STRESSES

S(1)=	0.49494243	S(2)=	-0.01514882	S(3)=	[illegible]

PRINCIPAL DIRECTION COSINES

B(1,1)=	0.0000	B(2,1)=	-1.0000	B(3,1)=	[illegible]
B(1,2)=	0.0000	B(2,2)=	1.0000	B(3,2)=	[illegible]
B(1,3)=	0.0000	B(2,3)=	-1.0000	B(3,3)=	[illegible]

X=	0.0	Y=	0.0	Z=	[illegible]

	XX	YY	ZZ	XY	XZ	YZ
P	0.02663846	0.04128749	0.39230436	0.0	[illegible]	0.00000033
Q	0.0	-0.00000001	0.00000018	-0.00000026	[illegible]	-0.00000000
S	0.02663846	0.04128748	0.39230454	-0.00000026	[illegible]	0.00000033

WP=	0.66251713	WQ=	-0.00000017	W =	[illegible]

PRINCIPAL STRESSES

S(1)=	0.39895099	S(2)=	-0.00927562	S(3)=	[illegible]

PRINCIPAL DIRECTION COSINES

B(1,1)=	0.0000	B(2,1)=	-1.0000	B(3,1)=	[illegible]
B(1,2)=	0.0000	B(2,2)=	1.0000	B(3,2)=	[illegible]
B(1,3)=	0.0000	B(2,3)=	-1.0000	B(3,3)=	[illegible]

	X= 0.0		Y= 0.0		Z= 1.399[illegible]	
	XX	YY	ZZ	XY	XZ	YZ
P	0.01346328	0.03100809	0.32233775	0.0	-0.000[illegible]012	0.00000032
Q	0.0	-0.00000002	0.00000012	-0.00000020	-0.0412559[illegible]	-0.00000000
S	0.01346328	0.03100806	0.32233787	-0.00000020	-0.041255[illegible]	0.00000032
	WP= 0.59133977		WQ= -0.00000012		W = 0.591339[illegible]	

PRINCIPAL STRESSES

S(1)= 0.32601237	S(2)= -0.00525391	S(3)= 0.0458[illegible]

PRINCIPAL DIRECTION COSINES

B(1,1)= 0.0000	B(2,1)= -1.0000	B(3,1)= 0.0000
B(1,2)= 0.0000	B(2,2)= 1.0000	B(3,2)= -0.0000
B(1,3)= 0.0000	B(2,3)= -1.0000	B(3,3)= 0.0000

	X= 0.0		Y= 0.0		Z= 1.5999[illegible]943	
	XX	YY	ZZ	XY	XZ	YZ
P	0.00606450	0.02414056	0.26758820	0.0	-0.000[illegible]	0.00000033
Q	0.0	-0.00000002	0.00000000	-0.00000002	-0.02778545	0.00000000
S	0.00606450	0.02414054	0.26758820	-0.00000002	-0.02778545	0.00000033
	WP= 0.53256720		WQ= -0.00000001		W = 0.53255714	

PRINCIPAL STRESSES

S(1)= 0.26988083	S(2)= -0.00235498	S(3)= 0.033[illegible]8039

PRINCIPAL DIRECTION COSINES

B(1,1)= 0.0000	B(2,1)= -1.0000	B(3,1)= 0.0000
B(1,2)= 0.0000	B(2,2)= 1.0000	B(3,2)= -0.0000
B(1,3)= 0.0000	B(2,3)= -1.0000	B(3,3)= 0.0000

	X= 0.0		Y= 0.0		Z= 1.7999[illegible]24	
	XX	YY	ZZ	XY	XZ	YZ
P	0.00186374	0.01928773	0.22452271	0.0	-0.00000001	0.00000029
Q	0.0	-0.00000002	0.00000001	-0.00000002	-0.01921256	0.00000000
S	0.00186374	0.01928771	0.22452271	-0.00000002	-0.01921257	0.00000029
	WP= 0.48352551		WQ= -0.00000001		W = 0.48352[illegible]45	

PRINCIPAL STRESSES

S(1)= 0.22612745	S(2)= -0.00008625	S(3)= 0.01943[illegible]

PRINCIPAL DIRECTION COSINES

B(1,1)= 0.0000	B(2,1)= -1.0000	B(3,1)= 0.00[illegible]
B(1,2)= 0.0000	B(2,2)= 1.0000	B(3,2)= -0.00[illegible]
B(1,3)= 0.0000	B(2,3)= -1.0000	B(3,3)= 0.0000

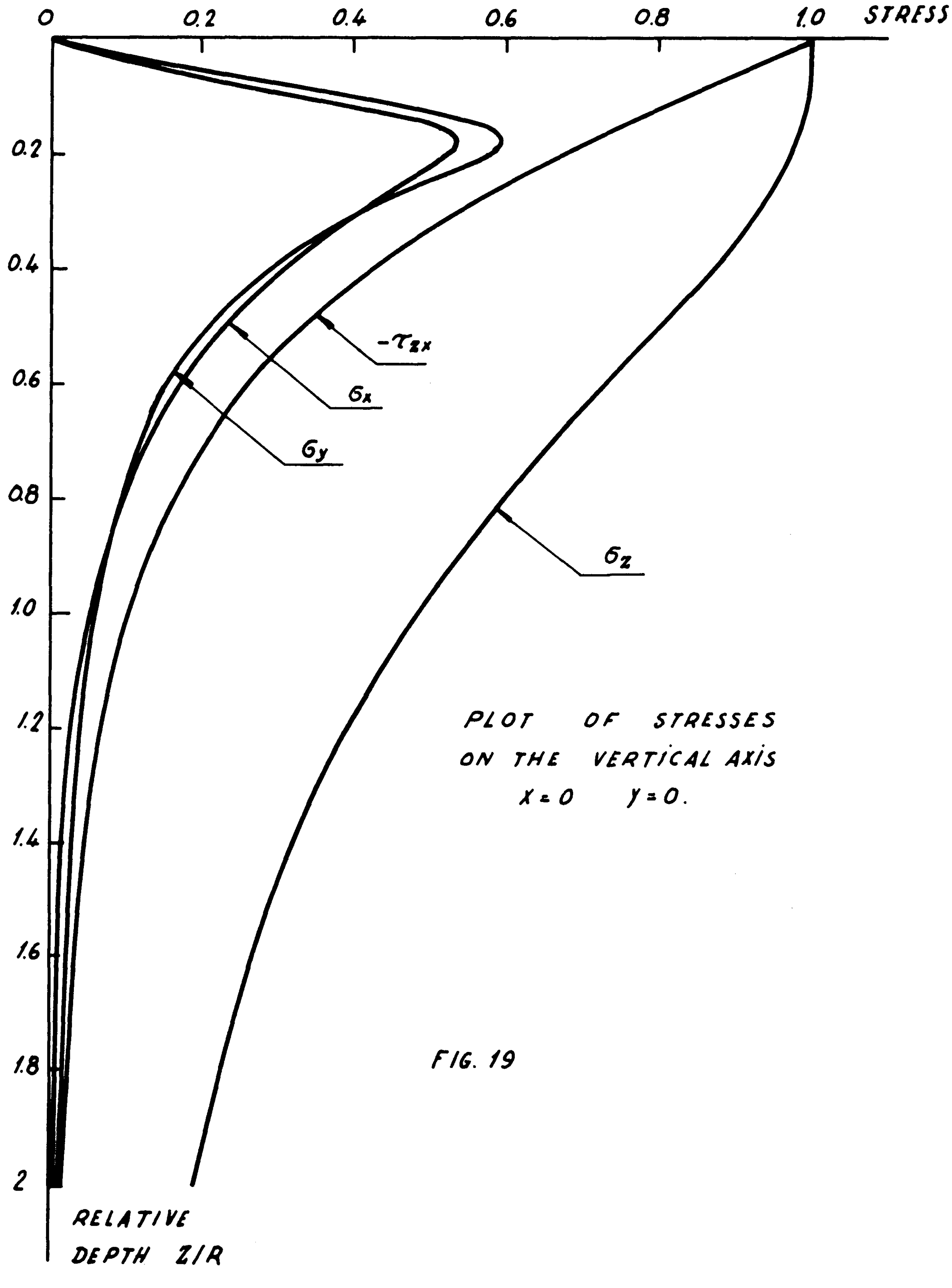
	X= 0.0		Y= 0.0		Z= 1.9999[illegible]5	
	XX	YY	ZZ	XY	XZ	YZ
P	-0.00052694	0.01572476	0.19035697	0.0	-0.00000001	0.00000030
Q	0.0	-0.00000002	0.00000001	-0.00000002	-0.01351569	0.
S	-0.00052694	0.01572474	0.19035697	-0.00000002	-0.01351570	0.00000030
	WP= 0.44216603		WQ= -0.00000001		W = 0.44216507	

PRINCIPAL STRESSES

S(1)= 0.19158775	S(2)= 0.00206238	S(3)= 0.01178230

PRINCIPAL DIRECTION COSINES

B(1,1)= 0.0000	B(2,1)= -1.0000	B(3,1)= 0.0000
B(1,2)= 0.0000	B(2,2)= -1.0000	B(3,2)= 0.0000
B(1,3)= 0.0000	B(2,3)= -1.0000	B(3,3)= 0.0000



PLOT OF STRESSES
ON THE VERTICAL AXIS
$x = 0 \quad y = 0.$

FIG. 19

TABLE IV

COORDINATES						
	X		Y		Z	
COMPONENTS OF STRESS TENSOR						
	XX	YY	ZZ	XY	XZ	YZ
Due to normal load						
Due to shearing load						
Superposition						
VERTICAL COMPONENT OF DISPLACEMENT						
	Due to normal load WP		Due to shearing load WQ		Superposition W	
PRINCIPAL STRESSES						
	σ_{I}		σ_{II}		σ_{III}	
PRINCIPAL DIRECTION COSINES						
	l_1		m_1		n_1	
	l_2		m_2		n_2	
	l_3		m_3		n_3	

COMPARISON OF RESULTS WITH NEWMARK'S METHOD

The results obtained in this example can be **verified** by Newmark's influence charts (Reference 13). The procedure, for the case of the vertical stress σ_z due to the normal load, is shown in Figs. 20 and 21 which corresponds to points $P_1(0,0,1)$ and $P_2(0,0,2)$ respectively.

<u>POINT P(0,0,1)</u>

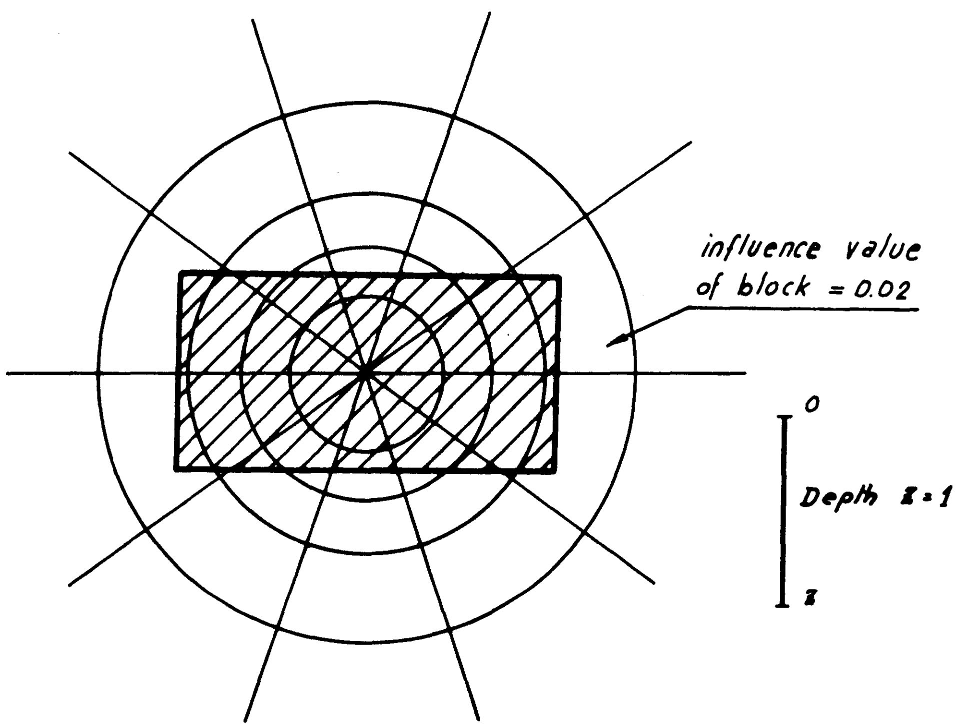


FIG. 20

$$\sigma_z = \sum_{i=1}^{\text{n: of blocks}} \text{influence value} \times \text{Surface stress intensity}$$

$$= 24 \times 0.02 \times 1 = 0.48$$

The result obtained with the computer is $\sigma_z = 0.48160243$

POINT P(0,0,2)

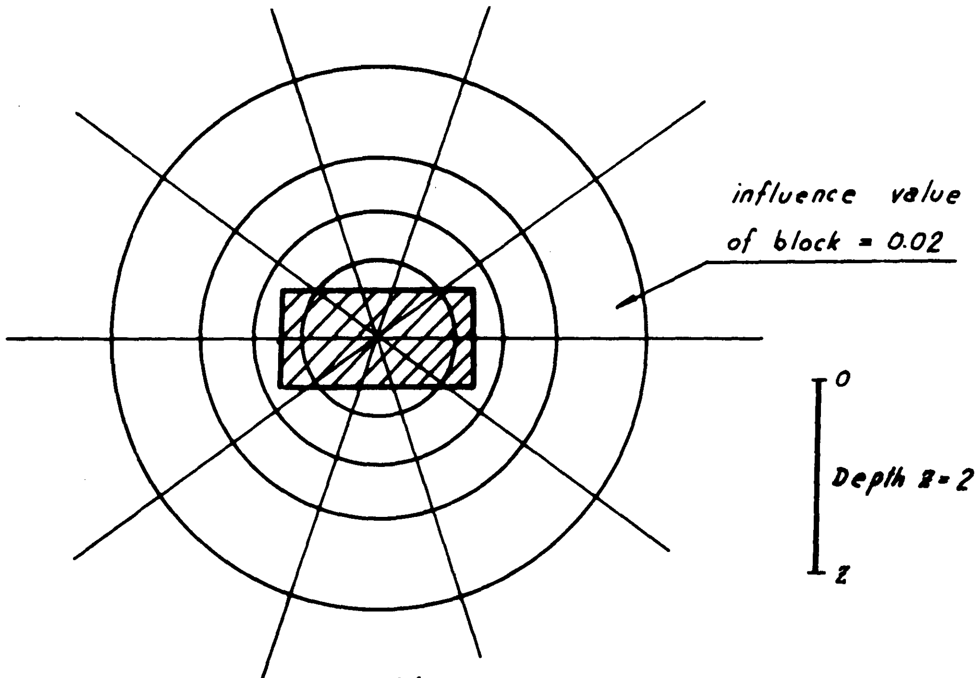


FIG. 21

$$\sigma_z = \sum_{i=1}^{\text{no of blocks}} \text{influence value} \times \text{surface stress intensity}$$

$$= 0.02 \times 1 \times 10 = 0.20$$

The result obtained with the computer is = 0.19035697

CHAPTER 10

CONCLUSIONS AND SUGGESTIONS FOR FUTURE WORK

The advantages of this method (generality, speed and accuracy) become evident after the example and verifications shown in Chapter 9. The results produced are in very good agreement with those obtained using Newmark's influence charts. To these advantages also can be added the possibility of producing the output of this program in graphical form using any type of graphical device coupled to the computer. Appendix E shows an example of computer plotting of the vertical stress σ_z along a vertical axis.

Considering the actual stage and future evolution of computer science and technology it seems important to extend the capabilities of this method towards the solution of the general problem mentioned in the introduction (layered half-space with time dependent properties, subjected to time-varying moving loads) and develop the necessary structure of commands to present it definitely in the form of a problem oriented language.

APPENDIX A

DEFINITION OF SYMBOLS

VARIABLE	SYMBOL	SYMBOL IN PROGRAM
Element of normal load	p_{ij}	P(I,J)
Element of shearing load	q_{ij}	Q(I,J)
Row number of load matrices	i	I
Column number of load matrices	j	J
Coordinates of point of half-space	x,y,z	X,Y,Z
Coord. of load elem ij with resp. to main system	x_A,y_A,z_A	X_A,Y_A,Z_A
Coord. of points of half-space with resp. to our system.	x',y',z'	XP,YP,ZP
Equiv radius of Square ring 1	R_1	R1
Equiv radius of Square ring 2	R_2	R2
Equiv radius of Square ring 3	R_3	R3
Side of error prism	ρ	RHO
Side of grid element	$\Delta\rho$	DRHO
Number of grid elements in Ring 1	K_1	K1
Number of grid elements in Ring 2	K_2	K2
Number of grid elements in Ring 3	K_3	K3
Max. No. of grid elems in Ring 1	K1MAX	K1MAX
Max. No. of grid elems in Ring 2	K2MAX	K2MAX
Max. No. of grid elems in Ring 3	K3MAX	K3MAX
Average intensity of normal load in rings 1, 2, 3	$p_{av1}, p_{av2}, p_{av3}$	AP(1), AP(2), AP(3)
Average intensity of shearing load in rings 1, 2, 3	$q_{av1}, q_{av2}, q_{av3}$	AQ(1), AQ(2), AQ(3)
Increments of Coordinates	Δx , Δy , Δz	DX, DY, DZ

VARIABLE	SYMBOL	SYMBOL IN PROGRAM
Order of grid and load matrices	M	M
Limits of total error volume	XLIM, YLIM, ZLIM	XLIM, YLIM, ZLIM
Modulus of Elasticity	E	ELAST
Poisson's Ratio	ν	POISS
1 - 2x Poisson's ratio	$1 - 2\nu$	V
Lame constants		S, T
Stress components due to normal point loads . (1)		P1XY, P1XZ, P1YZ P1XX, P1YY, P1ZZ
Stress components due to normal distribution loads . (2)		P2XX, P2YY, P2ZZ P2XY, P2XZ, P2YZ
Stress components due to shearing point loads . (3)		Q1XX, Q1YY, Q1ZZ Q1XY, Q1YZ, Q1XZ
Stress components due to shearing distribution loads. (4)		Q2XX, Q2YY, Q2ZZ Q2XY, Q2YZ, Q2XZ
Sum of 1 and 2 . (5)		PXX, PYY, PZZ PXY, PXZ, PYZ
Sum of 3 and 4 (6)		QXX, QYY, QZZ QXY, QYZ, QXZ
Sum of 5 and 6		SXX, SYY, SZZ SXY, SYZ, SXZ
Principal Stresses	$\sigma_I, \sigma_{II}, \sigma_{III}$	S(1) S(2) S(3)
Direction cosines of Princ. Stress	$x_i \quad y_i \quad z_i$	B(1,J), B(2,J), B(3,J)
Index of Principal stresses		J = 1, 2, 3
Vertical Displacement Components		
1 Due to normal point loads		WP1
2 Due to normal dist. loads		WP2
3 Sum of 1 and 2		WP

VARIABLE	SYMBOL	SYMBOL IN PROGRAM
4 Due to shearing point load		WQ1
5 Due to shearing distrib. load		WQ2
6 Sum of 4 and 5		WQ
7 Sum of 3 and 6		W
Number of points to be analyzed that program is to be repeated	nmax	NMAX

APPENDIX B

LIST OF FIGURES

APPENDIX C

LIST OF TABLES

TABLE PAGE

APPENDIX D

UNIFORM CIRCULAR LOAD

Expressions of the Cartesian components of the stress tensor and vertical components of the displacement vector for points on the vertical axis for normal and horizontal loads.

ARISTIDES BRYAN DOMINGUEZ

1966

1. NORMAL LOAD.

For a single normal point load, the cartesian components of the stress tensor are:

$$\sigma_x = \frac{P}{2\pi}\left\{\frac{3Zx^2}{D^5} - (1-2\nu)\left[\frac{x^2-y^2}{\rho^2 D(D+Z)} + \frac{Zy^2}{\rho^2 D^3}\right]\right\} \tag{1}$$

$$\sigma_y = \frac{P}{2\pi}\left\{\frac{3Zy^2}{D^5} - (1-2\nu)\left[\frac{y^2-x^2}{\rho^2 D(D+Z)} + \frac{Zx^2}{\rho^2 D^3}\right]\right\} \tag{2}$$

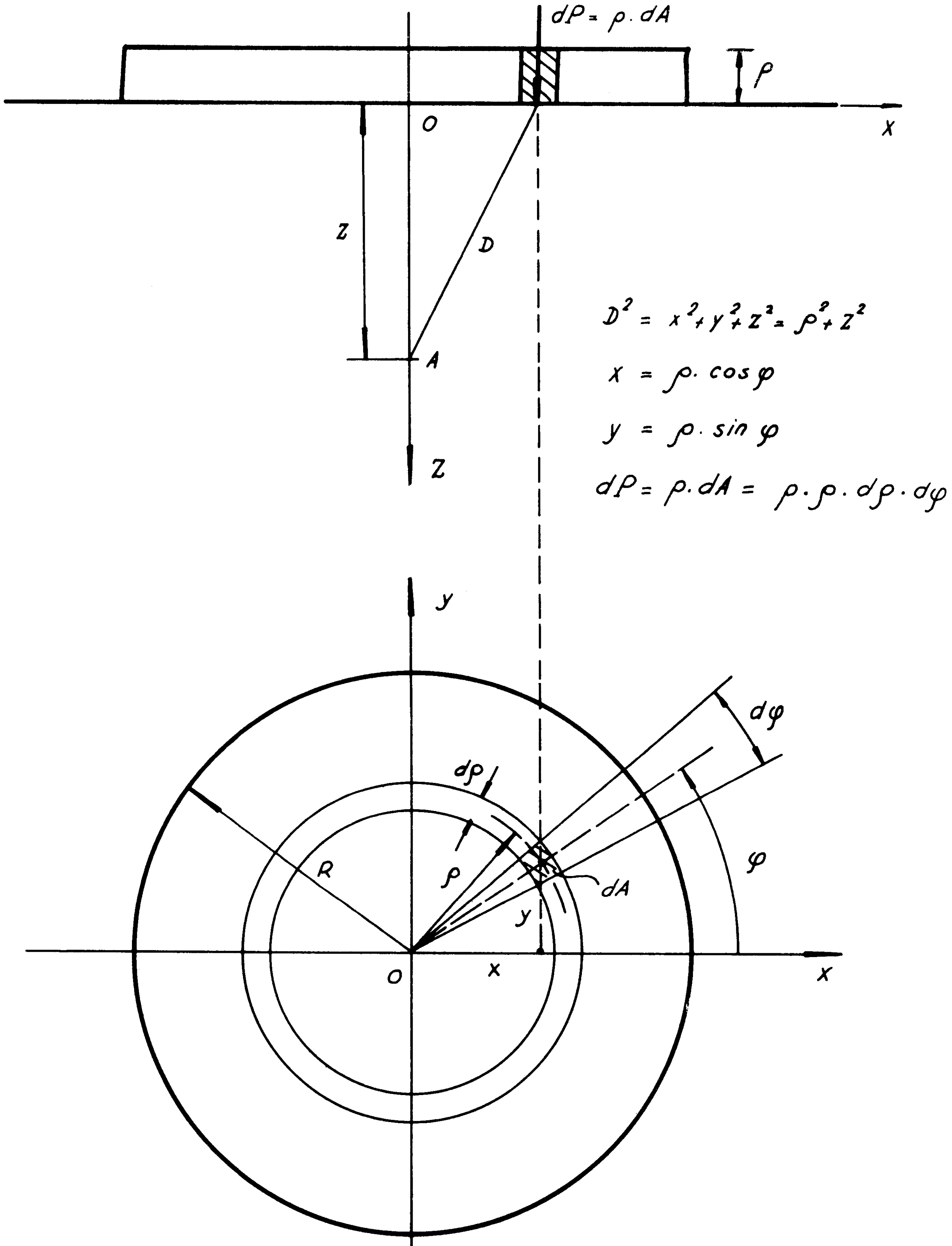
$$\sigma_z = \frac{3P}{2\pi}\cdot\frac{Z^3}{D^5} \tag{3}$$

$$\tau_{zx} = \frac{3P}{2\pi}\cdot\frac{Z^2 x}{D^5} \tag{4}$$

$$\tau_{yz} = \frac{3P}{2\pi}\cdot\frac{Z^2 y}{D^5} \tag{5}$$

$$\tau_{xy} = 0 \tag{6}$$

The corresponding expressions for the uniform circular load (for points on the z axis) are obtained by integration of these for the whole circle.



$$D^2 = x^2 + y^2 + Z^2 = \rho^2 + Z^2$$

$$x = \rho \cdot \cos\varphi$$

$$y = \rho \cdot \sin\varphi$$

$$dP = p \cdot dA = p \cdot \rho \cdot d\rho \cdot d\varphi$$

a.- σ_z. Differentiating formula (3) with respect to P and substituting the expressions on pag. 2 :

$$d\sigma_z = \frac{3P}{2\pi} \cdot z^3 \cdot \frac{\rho}{(\rho^2+z^2)^{5/2}} \cdot d\rho \cdot d\varphi$$

$$\sigma_z = \frac{3P}{2\pi} z^3 \int_{\varphi_1}^{\varphi_2} d\varphi \int_{\rho_1}^{\rho_2} \frac{\rho}{(\rho^2+z^2)^{5/2}} \cdot d\rho$$

$$= \frac{3P}{2\pi} z^3 \Big[\varphi\Big]_{\varphi_1}^{\varphi_2} \cdot \left[\frac{-1}{3(\rho^2+z^2)^{3/2}}\right]_{\rho_1}^{\rho_2}$$

Integrating over 1/4 of the circle and multiplying by 4, the limits are:

$\varphi_1 = 0 \qquad \varphi_2 = \pi/2 \qquad \rho_1 = 0 \qquad \rho_2 = R$

$$\sigma_z = 4 \cdot \frac{3P}{2\pi} z^3 \frac{\pi}{2} \cdot \frac{1}{3}\left[\frac{-1}{(R^2+z^2)^{3/2}} + \frac{1}{(z^2)^{3/2}}\right]$$

$$= p \cdot z^3 \left[\frac{-1}{(R^2+z^2)^{3/2}} + \frac{1}{z^3}\right] = p\left[1 - \frac{z^3}{(R^2+z^2)^{3/2}}\right]$$

$$\boxed{\sigma_z = p\left[1 - \frac{z^3}{(R^2+z^2)^{3/2}}\right]}$$

6.- τ_{xz}. Differentiating fórmula (4) with respect to P and substituting expressions on pag. 2.

$$d\tau_{xz} = \frac{3P}{2\pi} \, z^2 \, \frac{\rho^2 \cdot \cos\varphi}{(\rho^2 + z^2)^{5/2}} \cdot d\rho \cdot d\varphi$$

$$\tau_{xz} = \frac{3P}{2\pi} \, z^2 \int_{\varphi_1}^{\varphi_2} \cos\varphi \int_{\rho_1}^{\rho_2} \frac{\rho^2}{(\rho^2 + z^2)^{5/2}} \, d\rho \cdot d\varphi$$

$$\tau_{xz} = \frac{3P}{2\pi} \cdot z^2 \left[\sin\varphi \right]_{\varphi_1}^{\varphi_2} \cdot \frac{1}{3z^2} \left[\frac{\rho^3}{(\rho^2 + z^2)^{3/2}} \right]_{\rho_1}^{\rho_2}$$

Substituting the limits as before and multiplying by 4 :

$$\tau_{xz} = \frac{4P}{2\pi} \cdot [1] \cdot \left[\frac{R^3}{(R^2 + z^2)^{3/2}} \right]$$

$$\boxed{\tau_{xz} = \frac{2P}{\pi} \cdot \frac{R^3}{(R^2 + z^2)^{3/2}}}$$

c.- τ_{zy}. Differentiating formula (5) with respect to ρ and using expressions on pag. 2:

$$d\tau_{yz} = \frac{3p}{2\pi} Z^2 . \frac{\rho^2 \sin\varphi}{(\rho^2 + Z^2)^{5/2}} . d\rho . d\varphi$$

$$\tau_{zy} = \frac{3p}{2\pi} Z^2 \int_{\varphi_1}^{\varphi_2} \sin\varphi . \int_{\rho_1}^{\rho_2} \frac{\rho^2}{(\rho^2 + Z^2)^{5/2}} . d\rho . d\varphi$$

$$\tau_{yz} = \frac{3p}{2\pi} Z^2 \left[- \cos\varphi \right] . \frac{1}{3Z^2} \left[\frac{\rho^3}{(\rho^2 + Z^2)} \right]_{\rho_1}^{\rho_2}$$

Substituting the limits and multiplying by 4:

$$\tau_{yz} = -4 . \frac{P}{2\pi} \left[0 - 1 \right] \frac{R^3}{(R^2 + Z^2)^{3/2}}$$

$$\boxed{\tau_{yz} = \frac{2P}{\pi} . \frac{R^3}{(R^2 + Z^2)^{3/2}}}$$

d.- σ_x Differentiating fórmula (1) with respect to P and using expressions on pag. 2:

$$d\sigma_x = \frac{P}{2\pi}\left\{ 3Z\,\frac{\rho^3 \cdot \cos^2\varphi}{(\rho^2+Z^2)^{5/2}} - (1-2\nu)\left[\frac{\rho^3(\cos^2\varphi - \sin^2\varphi)}{\rho^2(\rho^2+Z^2)\left[(\rho^2+Z^2)^{1/2}+Z\right]} + \right.\right.$$

$$\left.\left. + \frac{Z\rho^3 \cdot \sin^2\varphi}{\rho^2(\rho^2+Z^2)^{3/2}}\right]\right\} d\rho \cdot d\varphi$$

$$d\sigma_x = \frac{P}{2\pi}\left\{ 3Z\,\frac{\rho^3 \cdot \cos^2\varphi}{(\rho^2+Z^2)} - (1-2\nu)\left[\frac{\rho \cdot \cos 2\varphi}{(\rho^2+Z^2)+Z(\rho^2+Z^2)^{1/2}} + \right.\right.$$

$$\left.\left. + \frac{Z \cdot \rho \cdot \sin^2\varphi}{(\rho^2+Z^2)^{3/2}}\right]\right\} d\rho \cdot d\varphi$$

1st. term.

$$3Z\int_{\varphi_1}^{\varphi_2} \cos^2\varphi \int_{\rho_1}^{\rho_2} \frac{\rho^3}{(\rho^2+Z^2)} \cdot d\rho \cdot d\varphi \quad =$$

$$= 3Z\left[\frac{\varphi}{2} + \frac{\sin 2\varphi}{4}\right]_{\varphi_1}^{\varphi_2} \cdot \left[\frac{-1}{(\rho^2+Z^2)^{1/2}} + \frac{Z^2}{3(\rho^2+Z^2)^{3/2}}\right]_{\rho_1}^{\rho_2}$$

Substituting the limits and multiplying by 4:

$$3\pi Z\left[\frac{Z^2}{3(R^2+Z^2)^{3/2}} - \frac{1}{(R^2+Z^2)^{1/2}} + \frac{2}{3Z}\right]$$

2nd. term.

$$\cos 2\varphi \; \frac{\rho(\rho^2+Z^2) - Z\rho(\rho^2+Z^2)^{1/2}}{\rho^2(\rho^2+Z^2)} \; d\rho.d\varphi =$$

$$= \cos 2\varphi \left[\frac{1}{\rho} - \frac{Z}{\rho(\rho^2+Z^2)^{1/2}} \right] d\rho.d\varphi$$

$$\int_{\varphi_1}^{\varphi_2} \cos 2\varphi . \int_{\rho_1}^{\rho_2} \left[\frac{1}{\rho} - \frac{Z}{\rho(\rho^2+Z^2)^{1/2}} \right] d\rho.d\varphi =$$

$$= \frac{1}{2} . \left[\sin 2\varphi \right]_{\varphi_1}^{\varphi_2} . \left[\ln \rho - Z\left(-\frac{1}{Z} . \ln \left| \frac{Z+(\rho^2+Z^2)^{1/2}}{\rho} \right| \right) \right]_{\rho_1}^{\rho_2}$$

Substituting the limits and multiplying by 4:

$$4 \times \frac{1}{2} . \left[0 - 0 \right] . \left[\ln \rho + \ln \left| \frac{Z+(\rho^2+Z^2)^{1/2}}{\rho} \right| \right) \Big]_{0}^{R} = 0$$

3rd. term.

$$Z \int_{\rho_1}^{\rho_2} \sin^2\varphi \int_{\rho_1}^{\rho_2} \frac{\rho}{(\rho^2+Z^2)^{3/2}} \; d\rho.d\varphi =$$

$$= Z \left[\frac{\varphi}{2} - \frac{\sin 2\varphi}{4} \right]_{\varphi_1}^{\varphi_2} . \left[\frac{-1}{(\rho^2+Z^2)^{1/2}} \right]_{\rho_1}^{\rho_2}$$

Substituting the limits and multiplying by 4:

$$4 \cdot z \left[\frac{\pi}{4}\right] \cdot \left[\frac{-1}{(R^2+Z^2)^{1/2}} + \frac{1}{Z}\right] = \pi \left[1 + \frac{Z}{(R^2+Z^2)^{1/2}}\right]$$

$$\sigma_z = \frac{P}{2\pi} \left\{ 3\pi . z \left[\frac{Z^2}{3(R^2+Z^2)^{3/2}} - \frac{1}{(R^2+Z^2)^{1/2}} + \frac{2}{3Z}\right] - \right.$$

$$\left. - (1-2\nu)\, \pi \left[1 - \frac{Z}{(R^2+Z^2)^{1/2}}\right] \right\}$$

$$\boxed{\sigma_x = \frac{P}{2}\left[\frac{Z^3}{(R^2+Z^2)^{3/2}} - \frac{2(1+\nu)Z}{(R^2+Z^2)^{1/2}} + (1-2\nu) + 6\right]}$$

e.- σ_y. Differentiating formula (2) with respect to P and substituting expressions on pag. 2:

$$d\sigma_y = \frac{P}{2\pi}\left\{ 3Z \frac{\rho^3 \cdot \sin^2\varphi}{(\rho^2+Z^2)^{5/2}} - (1-2\nu)\left[\frac{\rho^3 \cdot \cos 2\varphi}{\rho^2(\rho^2+Z^2)^{1/2} \cdot \left[(\rho^2+Z^2)^{1/2}+Z\right]} + \right.\right.$$

$$\left.\left. + \frac{Z\rho^3 \cdot \cos^2\varphi}{\rho^2(\rho^2+Z^2)^{3/2}} \right]\right\} d\rho \cdot d\varphi$$

1st. term.

$$3Z\int_{\varphi_1}^{\varphi_2} \sin^2\varphi \int_{\rho_1}^{\rho_2} \frac{\rho^3}{(\rho^2+Z^2)^{5/2}}\, d\rho \cdot d\varphi \;=$$

$$= 3Z\left[\frac{\varphi}{2} - \frac{\sin 2\varphi}{4}\right]_{\varphi_1}^{\varphi_2} \cdot \left[\frac{-1}{(\rho^2+Z^2)^{1/2}} + \frac{Z^2}{3(\rho^2+Z^2)^{3/2}}\right]_{\rho_1}^{\rho_2}$$

Substituting the limits and multiplying by 4:

$$4 \times 3Z\left[\frac{\pi}{4}\right] \cdot \left[\frac{-1}{(R^2+Z^2)^{1/2}} + \frac{1}{Z} + \frac{Z^2}{3(R^2+Z^2)^{3/2}} - \frac{Z^2}{3Z^3}\right] =$$

$$= 3\pi Z\left[\frac{Z^2}{3(R^2+Z^2)^{3/2}} - \frac{1}{(R^2+Z^2)^{1/2}} + \frac{2}{3Z}\right]$$

2nd. term.

$$\cos 2\varphi \, . \, \frac{\rho(\rho^2+Z^2) - Z\rho(\rho^2+Z^2)^{1/2}}{\rho^2(\rho^2+Z^2)} \, d\rho . d\varphi$$

$$\int_{\varphi_1}^{\varphi_2} \cos 2\varphi . \int_{\rho_1}^{\rho_2} \left[\frac{1}{\rho} - \frac{Z}{\rho(\rho^2+Z^2)} \right] d\rho . d\varphi \; =$$

$$= \frac{1}{2} \left[\sin 2\varphi \right]_{\varphi_1}^{\varphi_2} \left[\ln \rho - Z \left(-\frac{1}{Z} \ln \left| \frac{Z+(\rho^2+Z^2)^{1/2}}{\rho} \right| \right) \right]_{\rho_1}^{\rho_2}$$

Substituting the limits and multiplying by 4:

$$4 \times \frac{1}{2} \left[0 - 0 \right] . \left[\ln \rho - Z \left(-\frac{1}{Z} . \ln \left| \frac{Z+(\rho^2+Z^2)^{1/2}}{\rho} \right| \right) \right]_{\rho_1}^{\rho_2} = 0$$

3rd. term.

$$Z^2 \int_{\varphi_1}^{\varphi_2} \cos\varphi \int_{\rho_1}^{\rho_2} \frac{\rho}{(\rho^2+Z^2)^{3/2}} \, . \, d\rho . d\varphi = Z \left[\frac{\varphi}{2} + \frac{\sin 2\varphi}{4} \right]_{\varphi_1}^{\varphi_2} \bullet$$

$$\bullet \left[\frac{-1}{(\rho^2+Z^2)^{1/2}} \right]_{\rho_1}^{\rho_2}$$

Substituting the limits and multiplying by 4:

$$4 \cdot Z \left[\frac{\pi}{4}\right] \cdot \left[\frac{-1}{(R^2+Z^2)^{1/2}} + \frac{1}{Z}\right] = \pi \left[1 - \frac{Z}{(R^2+Z^2)^{1/2}}\right]$$

$$\sigma_y = \frac{P}{2\pi} \left\{ 3\pi Z \left[\frac{Z^2}{3(R^2+Z^2)^{3/2}} - \frac{1}{(R^2+Z^2)^{1/2}} + \frac{2}{3Z} \right] - \right.$$

$$\left. - (1-2\nu)\,\pi \left[1 - \frac{Z}{(R^2+Z^2)^{1/2}} \right] \right\}$$

$$\boxed{\sigma_y = \frac{P}{2} \left[\frac{Z^3}{(R^2+Z^2)^{3/2}} - \frac{2(1+\nu)\,Z}{(R^2+Z^2)^{1/2}} + (1-2\nu) + 6 \right]}$$

2.- HORIZONTAL LOAD.

For a single horizontal point load applied in the X direction, the cartesian components of the stress tensor are:

$$\sigma_x = \frac{Q.x}{2\pi D^3}\left[-\frac{3x^2}{D^2}+\frac{1-2\nu}{(D+Z)^2}\left(D^2-y^2-\frac{2Dy^2}{D+Z}\right)\right] \tag{7}$$

$$\sigma_y = \frac{Q.x}{2\pi D^3}\left[-\frac{3y^2}{D^2}+\frac{1-2\nu}{(D+Z)^2}\left(3D^2-x^2-\frac{2Dx^2}{D+Z}\right)\right] \tag{8}$$

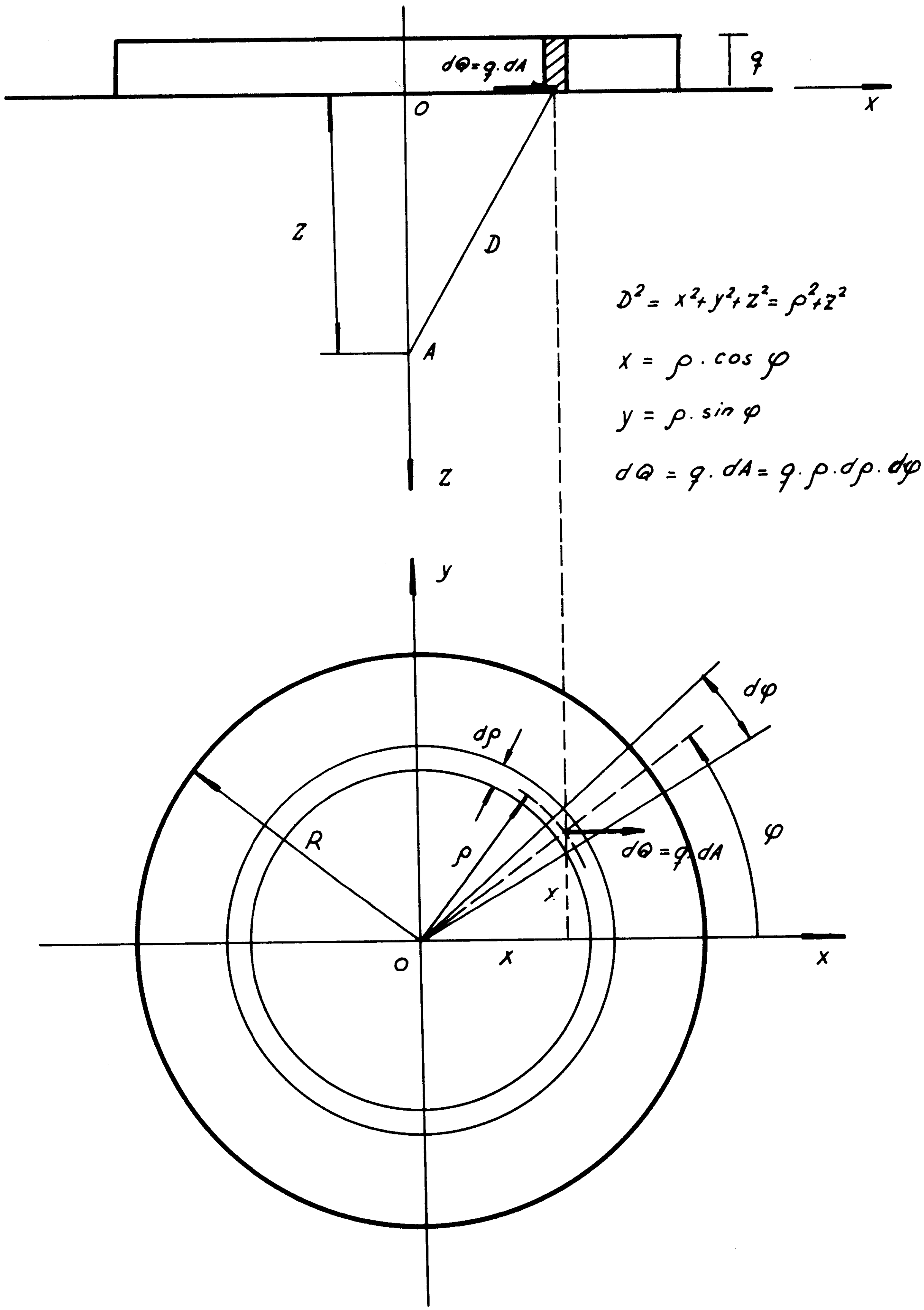
$$\sigma_z = -\frac{3QxZ^2}{2\pi D^5} \tag{9}$$

$$\tau_{xy} = \frac{Q.y}{2\pi D^3}\left[-\frac{3x^2}{D^2}+\frac{1-2\nu}{(D+Z)^2}\cdot\left(-D^2+x^2+\frac{2Dx^2}{D+Z}\right)\right] \tag{10}$$

$$\tau_{yz} = -\frac{3QxyZ}{2\pi D^5} \tag{11}$$

$$\tau_{xz} = -\frac{3Qx^2Z}{2\pi D^5} \tag{12}$$

The corresponding expressions for the uniform circular load (for points on the Z axis) are obtained by integration of these for the whole circle.



$D^2 = x^2 + y^2 + Z^2 = \rho^2 + Z^2$

$x = \rho \cdot \cos \varphi$

$y = \rho \cdot \sin \varphi$

$dQ = q \cdot dA = q \cdot \rho \cdot d\rho \cdot d\varphi$

a.- σ_z. Differentiating formula (9) with respect to Q and using expressions on pag. 13:

$$d\sigma_z = -\frac{3qZ^2}{2\pi} \cdot \frac{\rho^2 \cos\varphi}{(\rho^2+Z^2)^{5/2}} \cdot d\rho . d\varphi$$

$$\sigma_z = -\frac{3q}{2\pi} \cdot Z^2 \int_{\varphi_1}^{\varphi_2} \cos\varphi \cdot \int_{\rho_1}^{\rho_2} \frac{\rho^2}{(\rho^2+Z^2)^{5/2}} \cdot d\rho . d\varphi =$$

$$= -\frac{3q}{2\pi} \; Z^2 \left[\sin\varphi\right]_{\varphi_1}^{\varphi_2} \cdot \frac{1}{3Z^2} \left[\frac{\rho^3}{(\rho^2+Z^2)^{3/2}} \right]_{\rho_1}^{\rho_2}$$

Substuting the limits and multiplying by 4:

$$\sigma_z = -4 \times \frac{q}{2\pi} \left[1-0\right] \times \left[\frac{R^3}{(R^2+Z^2)^{3/2}} - 0 \right]$$

$$\boxed{\sigma_z = -\frac{2q}{\pi} \cdot \frac{R^3}{(R^2+Z^2)^{3/2}}}$$

6.- τ_{yz}. Differentiating formula (11) with respect to Q and substituting expressions on pag. 13:

$$d\tau_{yz} = -\frac{3qz}{2\pi} \cdot \frac{\rho^3 \cdot \sin\varphi \cdot \cos\varphi}{(\rho^2 + z^2)^{5/2}} \cdot d\rho \cdot d\varphi$$

$$\tau_{yz} = -\frac{3qz}{2\pi} \int_{\varphi_1}^{\varphi_2} \sin\varphi \cdot \cos\varphi \int_{\rho_1}^{\rho_2} \frac{\rho^3}{(\rho^2 + z^2)^{5/2}} \cdot d\rho \cdot d\varphi =$$

$$= -\frac{3q \cdot z}{2\pi} \left[\frac{\sin^2\varphi}{2} \right]_{\varphi_1}^{\varphi_2} \cdot \left[\frac{-1}{(\rho^2 + z^2)^{1/2}} + \frac{z^2}{3(\rho^2 + z^2)^{3/2}} \right]_{\rho_1}^{\rho_2}$$

Substituting the limits and multiplying by 4:

$$\tau_{yz} = -4\,\frac{3qz}{2\pi} \left[\frac{1-0}{2} \right] \cdot \left\{ \left[\frac{-1}{(R^2 + z^2)^{1/2}} + \frac{z^2}{3(R^2 + z^2)^{3/2}} \right] - \left[\frac{-1}{(z^2)^{1/2}} + \frac{z^2}{3(z^2)^{3/2}} \right] \right\}$$

$$\boxed{\tau_{yz} = -\frac{3q}{\pi} \left[\frac{2}{3} + \frac{z^3}{3(R^2 + z^2)^{3/2}} - \frac{z}{(R^2 + z^2)^{1/2}} \right]}$$

c.- τ_{zx}. Differentiating fórmula (12) with respect to Θ and substituting expressions on pag. 13:

$$d\tau_{xz} = -\frac{3q}{2\pi} z \frac{\rho^3 \cdot \cos^2\varphi}{(\rho^2+z^2)^{5/2}} \cdot d\rho . d\varphi$$

$$\tau_{xz} = -\frac{3q}{2\pi} z \int_{\varphi_1}^{\varphi_2} \cos^2\varphi \int_{\rho_1}^{\rho_2} \frac{\rho^3}{(\rho^2+z^2)^{5/2}} \cdot d\rho . d\varphi =$$

$$= -\frac{3q}{2\pi} z \left[\frac{\varphi}{2} + \frac{\sin 2\varphi}{4}\right]_{\varphi_1}^{\varphi_2} \cdot \left[\frac{-1}{(\rho^2+z^2)^{1/2}} + \frac{z^2}{3(\rho^2+z^2)^{3/2}}\right]_{\rho_1}^{\rho_2}$$

Substituting the limits and multiplying by 4:

$$\tau_{xz} = -4 . \frac{3q}{2\pi} z \left[\frac{\pi}{4} - 0\right] \cdot \left[\frac{-1}{(R^2+z^2)^{1/2}} + \frac{1}{(z^2)^{1/2}} + \frac{z^2}{3(R^2+z^2)^{3/2}} + \right.$$

$$\left. + \frac{z^2}{3(z^2)^{3/2}} \right]$$

$$\boxed{\tau_{xz} = -\frac{3q}{2}\left[\frac{2}{3} + \frac{z^3}{3(R^2+z^2)^{3/2}} - \frac{z}{(R^2+z^2)^{1/2}}\right]}$$

d.- σ_x. Differentiating fórmula (1) with respect to Q and using expressions on pag. 13:

$$d\sigma_x = \frac{q}{2\pi}\left\{-3\frac{\rho^4\cos^3\varphi}{(\rho^2+Z^2)^{5/2}} + (1-2\nu)\left[\frac{\rho^2.\cos\varphi}{(\rho^2+Z^2)^{1/2}\left[(\rho^2+Z^2)^{1/2}+Z\right]^2} - \right.\right.$$

$$\left.\left. - \frac{\rho^2\cos\varphi.\sin^2\varphi}{(\rho^2+Z^2)^{3/2}\left[(\rho^2+Z^2)^{1/2}+Z\right]^2} - \frac{2\rho^4.\cos\varphi.\sin^2\varphi}{(\rho^2+Z^2)\left[(\rho^2+Z^2)^{1/2}+Z\right]^3}\right]\right\} d\rho.d\varphi$$

First term.

$$-3\int_{\varphi_1}^{\varphi_2}\cos^3\varphi\int_{\rho_1}^{\rho_2}\frac{\rho^4}{(\rho^2+Z^2)^{5/2}}.d\rho.d\varphi =$$

$$= -3\left[\sin\varphi - \frac{\sin^3\varphi}{3}\right]_{\varphi_1}^{\varphi_2}\left[\frac{-\rho}{(\rho^2+Z^2)^{1/2}} - \frac{\rho^3}{3(\rho^2+Z^2)^{3/2}} + \right.$$

$$\left. + \ln\left[(\rho^2+Z^2)^{1/2}+\rho\right]\right]$$

Substituting the limits and multiplying by 4:

$$-8\left\{\frac{-R}{(R^2+Z^2)^{1/2}} - \frac{R^3}{3(R^2+Z^2)^{3/2}} + \ln\left[\frac{R^2+(R^2+Z^2)^{1/2}}{Z}\right]\right\}$$

2nd term.

$$\frac{\rho^2.\cos\varphi.\ d\rho.d\varphi}{(\rho^2+Z^2)^{3/2} + 2Z(\rho^2+Z^2) + Z^2(\rho^2+Z^2)^{1/2}}$$

Making : $(\rho^2+Z^2)^{1/2} = x$, $Z = a$, $\rho^2 = x^2 - a^2$

$$\frac{(x^2-a^2)\cos\varphi . d\rho . d\varphi}{x^3+2ax^2+ax} = \frac{(x+a)(x-a) . \cos\varphi . d\rho . d\varphi}{x(x+a)(x-a)} =$$

$$= \cos\varphi \left[\frac{x}{x(x+a)} - \frac{x}{x(x+a)} \right] d\rho . d\varphi = \cos\varphi \left[\frac{1}{x+a} - \right.$$

$$\left. - \frac{a}{x(x+a)} \right] d\rho . d\varphi$$

then substituting back :

$$\cos\varphi \left[\frac{1}{(\rho^2+Z^2)^{1/2}+Z} - \frac{Z}{(\rho^2+Z^2)^{1/2}\left[(\rho^2+Z^2)^{1/2}+Z\right]} \right] d\rho . d\varphi$$

Multiplying and dividing each term by the conjugate of its denominator:

$$\cos\varphi \left[\frac{(\rho^2+Z^2)^{1/2}-Z}{\rho^2} - \frac{Z(\rho^2+Z^2)-Z^2(\rho^2+Z^2)^{1/2}}{\rho^2(\rho^2+Z^2)} \right] d\rho . d\varphi =$$

$$= \cos\varphi \left[\frac{(\rho^2+Z^2)^{1/2}}{\rho^2} - \frac{Z}{\rho^2} - \frac{Z}{\rho^2} + \frac{Z}{\rho^2(\rho^2+Z^2)^{1/2}} \right] d\rho . d\varphi =$$

$$= \int_{\varphi_1}^{\varphi_2} \cos\varphi \int_{\rho_1}^{\rho_2} \left[\frac{(\rho^2+Z^2)^{1/2}}{\rho^2} - 2\frac{Z}{\rho^2} + \frac{Z}{\rho^2(\rho^2+Z^2)^{1/2}} \right] d\rho . d\varphi =$$

$$= \left[\sin\varphi \right]_{\varphi_1}^{\varphi_2} \left\{ - \frac{(\rho^2+Z^2)^{1/2}}{\rho} + \ln\left[\rho + (\rho^2+Z^2)^{1/2} \right] + \frac{4Z}{\rho} - \right.$$

$$\left. - \frac{(\rho^2+Z^2)^{1/2}}{Z^2\rho} \right\}_{\rho_1}^{\rho_2}$$

$$= \left[\sin\varphi\right]_{\varphi_1}^{\varphi_2} \left\{ -\frac{2(\rho^2+Z^2)^{1/2}}{\rho} + \frac{4Z}{\rho} + \ln\left[\rho + (\rho^2+Z^2)^{1/2}\right]\right\} d\rho . d\varphi$$

Substituting the limits, the first and second terms become undeterminate for $\rho = 0$, applying L'Hopital's rule :

$$1)\quad -2 \times \frac{1}{2}(\rho^2+Z^2)^{-1/2} \cdot 2\rho \Big|_{\rho=0} = -2\rho(\rho^2+Z^2)^{-1/2}\Big|_{\rho=0} = 0$$

$$2)\quad \frac{0}{1} = 0$$

$$\therefore 4\left[1-0\right]\left\{-\frac{2(R^2+Z^2)^{1/2}}{R} + \frac{2Z}{R} + \ln\left[R + (R^2+Z^2)^{1/2}\right] - \ln Z\right\}$$

$$= 4\left\{-\frac{2(R^2+Z^2)^{1/2}}{R} + \frac{2Z}{R} + \ln\left[\frac{R+(R^2+Z^2)^{1/2}}{Z}\right]\right\}$$

<u>3rd. term.</u>

$$\frac{\rho^4 . \cos\varphi . \sin^2\varphi . d\rho . d\varphi}{(\rho^2+Z^2)^{5/2} + 2Z(\rho^2+Z^2)^2 + Z^2(\rho^2+Z^2)^{3/2}}$$

Using the same substitution of the last case :

$$\cos\varphi . \sin^2\varphi \quad \frac{(x^2-a^2)(x^2-a^2)}{x^5 + 2ax^4 + a^2x^3} = \cos\varphi . \sin^2\varphi \, \frac{(x-a)^2}{x^3}$$

Substituting back:

$$\cos\varphi \cdot \sin^2\varphi \left\{ \frac{[(\rho^2+z^2)^{1/2} - z]^2}{(\rho^2+z^2)^{3/2}} \right\} d\rho \cdot d\varphi \quad =$$

$$= \cos\varphi \cdot \sin^2\varphi \left[\frac{(\rho^2+Z^2) - 2Z(\rho^2+Z^2)^{1/2} + Z^2}{(\rho^2+Z^2)^{3/2}} \right] d\rho \cdot d\varphi$$

$$= \int_{\varphi_1}^{\varphi_2} \cos\varphi \, \sin^2\varphi \int_{\rho_1}^{\rho_2} \left[\frac{1}{(\rho^2+Z^2)^{1/2}} - \frac{2Z}{(\rho^2+Z^2)} + \frac{Z^2}{(\rho^2+Z^2)^{3/2}} \right] d\rho \cdot d\varphi$$

$$= \left[\frac{\sin^3\varphi}{3} \right]_{\varphi_1}^{\varphi_2} : \left\{ \ln\left[\rho + (\rho^2+Z^2)^{1/2}\right] - 2\frac{Z}{Z} \cdot \tan^{-1}\frac{\rho}{Z} + \frac{+Z^2\rho}{Z^2(\rho^2+Z^2)^{1/2}} \right\}_{\rho_1}^{\rho_2}$$

$$= \left[\frac{\sin^3\varphi}{3} \right]_{\varphi_1}^{\varphi_2} : \left\{ \ln\left[\rho + (\rho^2+Z^2)^{1/2}\right] - 2\tan^{-1}\frac{\rho}{Z} + \frac{\rho}{(\rho^2+Z^2)^{1/2}} \right\}_{\rho_1}^{\rho_2}$$

Substituting the limits and multiplying by 4:

$$\frac{4}{3} \left\{ \ln\left[R + (R^2+Z^2)^{1/2}\right] - \ln Z - 2\tan^{-1}\frac{R}{Z} + \frac{R}{(R^2+Z^2)^{1/2}} \right\} =$$

$$= \frac{4}{3} \left\{ \ln \frac{R + (R^2+Z^2)^{1/2}}{Z} - 2\tan^{-1}\frac{R}{Z} + \frac{R}{(R^2+Z^2)^{1/2}} \right\}$$

4th. Term

$$2\cos\varphi . \sin^2\varphi \; \frac{\rho^4 . d\rho . d\varphi}{(\rho^2+Z^2)^{5/2} + 3Z(\rho^2+Z^2)^2 + 3Z^2(\rho^2+Z^2)^{3/2} + Z^3(\rho^2+Z^2)}$$

Using the same substitution of the last case:

$$\frac{(x^2-a^2)(x^2-a^2)}{x^5+3ax^4+3a^2x^3+a^3x^2} = \frac{(x+a)(x-a)(x+a)(x-a)}{x^2(x+a)(x+a)(x+a)} = \frac{(x-a)^2}{x^2(x+a)} =$$

$$= \frac{x^2}{x^2(x+a)} - \frac{2ax}{x^2(x+a)} + \frac{a^2}{x^2(x+a)} = \frac{1}{x+a} - \frac{2a}{x(x+a)} + \frac{a^2}{x^2(x+a)}$$

Substituting back:

$$2\cos\varphi . \sin^2\varphi \left[\frac{1}{(\rho^2+Z^2)^{1/2}+Z} - \frac{2Z}{(\rho^2+Z^2)^{1/2}\left[(\rho^2+Z^2)^{1/2}+Z\right]} + \right.$$

$$\left. + \frac{Z^2}{(\rho^2+Z^2)\left[(\rho^2+Z^2)^{1/2}+Z\right]} \right] d\rho . d\varphi$$

Multiplying and dividing each term by the conjugate of its denominator:

$$2\cos\varphi . \sin^2\varphi \left[\frac{(\rho^2+Z^2)^{1/2}-Z}{\rho^2} - \frac{2Z(\rho^2+Z^2)-2Z^2(\rho^2+Z^2)^{1/2}}{\rho^2(\rho^2+Z^2)} + \right.$$

$$\left. + \frac{Z^2(\rho^2+Z^2)\left[(\rho^2+Z^2)^{1/2}-Z\right]}{(\rho^2+Z^2)^2\left[\rho^2+Z^2-Z^2\right]} \right] d\rho . d\varphi$$

$$2.\cos\varphi.\sin^2\varphi\left[\frac{(\rho^2+Z^2)^{1/2}}{\rho^2}-\frac{Z}{\rho^2}-\frac{2Z}{\rho^2}+\frac{2Z^2}{\rho^2(\rho^2+Z^2)^{1/2}}+\frac{Z^2}{\rho^2(\rho^2+Z^2)^{1/2}}-\right.$$

$$\left.-\frac{Z^3}{\rho^2(\rho^2+Z^2)}\right]d\rho.d\varphi$$

$$2\int_{\varphi_1}^{\varphi_2}\cos\varphi.\sin^2\varphi\int_{\rho_1}^{\rho_2}\left[\frac{(\rho^2+Z^2)^{1/2}}{\rho^2}-\frac{3Z}{\rho^2}+\frac{3Z^2}{\rho^2(\rho^2+Z^2)^{1/2}}-\frac{Z^3}{\rho^2(\rho^2+Z^2)}\right]d\rho\,d\varphi$$

$$=2\left[\frac{\sin^3\varphi}{3}\right]_{\varphi_1}^{\varphi_2}\cdot\left\{-\frac{(\rho^2+Z^2)^{1/2}}{\rho}+\ln\left[\rho+(\rho^2+Z^2)^{1/2}\right]+\frac{6Z}{\rho}-\right.$$

$$\left.-\frac{3Z^2(\rho^2+Z^2)^{1/2}}{Z^2\rho}-Z^3\left[-\frac{1}{Z^2\rho}-\frac{1}{Z^3}\tan^{-1}\frac{\rho}{Z}\right]\right\}_{\rho_1}^{\rho_2}$$

Substituting the limits, the first and the second terms become undeterminate, using L'Hopital's rule:

1) $-4\times\frac{1}{2}(\rho^2+z^2)^{-1/2}\cdot 2\rho\Big|_{\rho=0}=0$

2) $\frac{0}{1}=0$

$$4\times 2\,\frac{[1-0]}{3}\left\{-\frac{4(R^2+Z^2)^{1/2}}{R}+\frac{7Z}{R}+\tan^{-1}\frac{R}{Z}+\ln\left[R+(R^2+Z^2)^{1/2}\right]-\right.$$

$$\left.-\ln Z\right\}$$

$$= \frac{8}{3}\left\{-\frac{4(R^2+Z^2)^{1/2}}{R}+\frac{7Z}{R}+\tan^{-1}\frac{R}{Z}+\ln\frac{R+(R^2+Z^2)^{1/2}}{Z}\right\}$$

$$\sigma_x = \frac{q}{2\pi}\left\{-8\left[\frac{-R}{(R^2+Z^2)^{1/2}}-\frac{R^3}{3(R^2+Z^2)^{3/2}}+\ln\frac{R+(R^2+Z^2)^{1/2}}{Z}\right]+\right.$$

$$+(1-2\nu)\left[4\left\{-\frac{2(R^2+Z^2)^{1/2}}{R}+\frac{2Z}{R}+\ln\frac{R+(R^2+Z^2)^{1/2}}{Z}\right\}-\right.$$

$$-\frac{4}{3}\left\{\ln\frac{R+(R^2+Z^2)^{1/2}}{Z}-2\tan^{-1}\frac{R}{Z}+\frac{R}{(R^2+Z^2)^{1/2}}\right\}-\frac{8}{3}\left\{-\frac{4(R^2+Z^2)^{1/2}}{R}+\right.$$

$$\left.\left.\left.+\frac{7Z}{R}+\tan^{-1}\frac{R}{Z}+\ln\frac{R+(R^2+Z^2)^{1/2}}{Z}\right\}\right]\right\}$$

$$\boxed{\begin{aligned}\sigma_x = \frac{q}{2\pi}\Bigg\{ & \frac{8}{3(R^2+Z^2)^{3/2}}-8\ln\frac{R+(R^2+Z^2)^{1/2}}{Z}+\\ & +(1-2\nu)\left[\frac{8}{3}\cdot\frac{(R^2+Z^2)^{1/2}}{R}-\frac{4}{3}\cdot\frac{R}{(R^2+Z^2)^{1/2}}-\frac{32}{3}\frac{Z}{R}-\right.\\ & \left.-\frac{16}{3}\tan^{-1}\frac{R}{Z}\right]\Bigg\}\end{aligned}}$$

e.- σ_y. Differentiating formula (8) with respect to Q and using expressions on pag. 13:

$$d\sigma_y = \frac{q}{2\pi}\left\{-\frac{3\rho^4\cos\varphi\cdot\sin^2\varphi}{(\rho^2+Z^2)} + (1-2\nu)\left[\frac{3\rho^2\cos\varphi}{(\rho^2+Z^2)^{1/2}\left[(\rho^2+Z^2)^{1/2}+Z\right]^2} -\right.\right.$$

$$\left.\left. - \frac{\rho^4\cos^3\varphi}{(\rho^2+Z^2)^{3/2}\left[(\rho^2+Z^2)^{1/2}+Z\right]^2} - \frac{2\rho^4\cos^3\varphi}{(\rho^2+Z^2)\left[(\rho^2+Z^2)^{1/2}+Z\right]^3}\right]\right\} d\rho\cdot d\varphi$$

1st. term.

$$-3\int_{\varphi_1}^{\varphi_2}\cos\varphi\cdot\sin^2\varphi\int_{\rho_1}^{\rho_2}\frac{\rho^4}{(\rho^2+Z)^{5/2}}\cdot d\rho\cdot d\varphi \quad =$$

$$= -3\left[\frac{\sin^3\varphi}{3}\right]_{\varphi_1}^{\varphi_2}\cdot\left[\frac{-\rho}{(\rho^2+Z^2)^{1/2}} - \frac{\rho^3}{3(\rho^2+Z^2)} + \ln\left[\rho+(\rho^2+Z^2)^{1/2}\right]\right]_{\rho_1}^{\rho_2}$$

Substituting the limits and multiplying by 4:

$$= -4\times 3\left[\frac{1-0}{3}\right]\cdot\left\{-\frac{R}{(R^2+Z^2)^{1/2}} - \frac{R^3}{3(R^2+Z^2)^{3/2}} + \ln\left[\frac{R+(R^2+Z^2)^{1/2}}{Z}\right]\right\} =$$

$$= -4\left\{-\frac{R}{(R^2+Z^2)^{1/2}} - \frac{R^3}{3(R^2+Z^2)^{3/2}} + \ln\frac{R+(R^2+Z^2)^{1/2}}{Z}\right\}$$

2nd term.

$$3\int_{\varphi_1}^{\varphi_2}\cos\varphi\int_{\rho_1}^{\rho_2}\frac{\rho^2 . d\rho . d\varphi}{(\rho^2+Z^2)^{3/2}+2Z(\rho^2+Z^2)+Z(\rho^2+Z^2)^{1/2}} =$$

$$= 12\left\{-\frac{2(R^2+Z^2)^{1/2}}{R}+\frac{2Z}{R}+\ln\frac{R+(R^2+Z^2)^{1/2}}{Z}\right\}$$

3rd. term.

$$\int_{\varphi_1}^{\varphi_2}\cos^3\varphi\int_{\rho_1}^{\rho_2}\frac{\rho^4 . d\rho . d\varphi}{(\rho^2+Z^2)^{5/2}+2Z(\rho^2+Z^2)^2+Z^2(\rho^2+Z^2)^{3/2}} =$$

$$\left[\sin\varphi-\frac{\sin^3\varphi}{3}\right]_{\varphi_1}^{\varphi_2}\cdot\left[\ln\left[\rho+(\rho^2+Z^2)^{1/2}\right]-2\tan^{-1}\frac{\rho}{Z}+\frac{\rho}{(\rho^2+Z^2)^{1/2}}\right]_{\rho_1}^{\rho_2}$$

Substituting the limits and multiplying by 4:

$$4\left[1-\frac{1}{3}\right]\left[\ln\frac{R+(R^2+Z^2)^{1/2}}{Z}-2\tan^{-1}\frac{R}{Z}+\frac{R}{(R^2+Z^2)^{1/2}}\right] =$$

$$\frac{8}{3}\left[\ln\frac{R+(R^2+Z^2)^{1/2}}{Z}-2\tan^{-1}\frac{R}{Z}+\frac{R}{(R^2+Z^2)^{1/2}}\right]$$

4th. term.

$$2\int_{\varphi_1}^{\varphi_2}\cos^3\varphi\int_{\rho_1}^{\rho_2}\frac{\rho^4 . d\rho . d\varphi}{(\rho^2+Z^2)^{5/2}+3Z(\rho^2+Z^2)+3Z^2(\rho^2+Z^2)^{3/2}+Z^2(\rho^2+Z^2)}$$

$$= 2\left[\sin\varphi-\frac{\sin^3\varphi}{3}\right]_{\varphi_1}^{\varphi_2} . \left\{-\frac{4(\rho^2+Z^2)^{1/2}}{\rho}+\frac{7Z}{\rho}+\tan^{-1}\frac{\rho}{Z}+\ln\left[\rho+(\rho^2+Z^2)^{1/2}\right]\right\}_{\rho_1}^{\rho_2}$$

Substituting the limits and multiplying by 4:

$$4\times 2\left[1-\frac{1}{3}\right] . \left\{-\frac{4(R^2+Z^2)^{1/2}}{R}+\frac{7Z}{R}+\tan^{-1}\frac{R}{Z}+\ln\frac{R+(R^2+Z^2)^{1/2}}{Z}\right\} =$$

$$= \frac{16}{3}\left\{-\frac{4(R^2+Z^2)^{1/2}}{R}+\frac{7Z}{R}+\tan^{-1}\frac{R}{Z}+\ln\frac{R+(R^2+Z^2)^{1/2}}{Z}\right\}$$

$$\sigma_y = \frac{q}{2\pi}\left\{-4\left[\frac{-R}{(R^2+Z^2)^{1/2}}-\frac{R^3}{3(R^2+Z^2)^{3/2}}+\ln\frac{R+(R^2+Z^2)^{1/2}}{Z}\right]+\right.$$

$$+(1-2\nu)\left\{12\left[\frac{-2(R^2+Z^2)^{1/2}}{R}+\frac{2Z}{R}+\ln\frac{R+(R^2+Z^2)^{1/2}}{Z}\right]-\right.$$

$$-\frac{8}{3}\left[\ln\frac{R+(R^2+Z^2)^{1/2}}{Z}-2\tan^{-1}\frac{R}{Z}+\frac{R}{(R^2+Z^2)^{1/2}}\right]-\frac{16}{3}\left[-\frac{4(R^2+Z^2)^{1/2}}{R}+\right.$$

$$+ \frac{7z}{R} + \tan^{-1}\frac{R}{z} + \ln \frac{R+(R^2+Z^2)^{1/2}}{Z} \Big]\Big\}$$

$$\sigma_y = \frac{q}{2\pi} \Big\{ \Big[4 - (1-2\nu)\frac{8}{3} \Big] \frac{R}{(R^2+Z^2)^{1/2}} + \frac{4R^3}{3(R^2+Z^2)^{3/2}} +$$

$$+ \Big[-4 + (1-2\nu)(12 - \frac{8}{3} - \frac{16}{3}) \Big] \ln \frac{R+(R^2+Z^2)^{1/2}}{Z} + (1-2\nu)\Big\{ (-24 + \frac{64}{3}) \cdot$$

$$\cdot \frac{(R^2+Z^2)^{1/2}}{R} + (24 - \frac{112}{3}) \frac{Z}{R} + (\frac{16}{3} - \frac{16}{3}) \tan^{-1}\frac{R}{Z} \Big\}$$

$$\sigma_y = \frac{q}{2\pi} \Big\{ \frac{4}{3}(1+2\nu) \frac{R}{(R^2+Z^2)^{1/2}} + \frac{4R^3}{3(R^2+Z^2)^{3/2}} -$$

$$- 8\nu \cdot \ln \frac{R+(R^2+Z^2)^{1/2}}{Z} + (1-2\nu) \Big[- \frac{8}{3}\frac{(R^2+Z^2)^{1/2}}{R} -$$

$$- 40 \frac{R}{Z}$$

f.- τ_{xy}. Differentiating formula (10) with respect to Q and using expressions on pag. 13:

$$d\tau_{xy} = \frac{q}{2\pi}\left\{-\frac{3\rho^4 \sin\varphi.\cos^2\varphi}{(\rho^2+Z^2)^{5/2}} + (1-2\nu)\left[\frac{-\rho^2 \sin\varphi}{(\rho^2+Z^2)^{1/2}\left[(\rho^2+Z^2)^{1/2}+Z\right]^2} + \right.\right.$$

$$\left.\left. + \frac{\rho^4 \cos^2\varphi.\sin\varphi}{(\rho^2+Z^2)\left[(\rho^2+Z^2)^{1/2}+Z\right]^2} + \frac{2\rho^4\cos^2\varphi.\sin\varphi}{(\rho^2+Z^2)\left[(\rho^2+Z^2)^{1/2}+Z\right]^3}\right]\right\} d\rho.d\varphi$$

1st. term.

$$-3\int_{\varphi_1}^{\varphi_2}\cos^2\varphi.\sin\varphi\int_{\rho_1}^{\rho_2}\frac{\rho^4}{(\rho^2+Z^2)^{3/2}}.d\rho.d\varphi =$$

$$= -3\left[-\frac{\cos^3\varphi}{3}\right]_{\varphi_1}^{\varphi_2}.\left[-\frac{\rho}{(\rho^2+Z^2)^{3/2}} - \frac{\rho^3}{3(\rho^2+Z^2)^{3/2}} + \ln\left[(\rho^2+Z^2)^{1/2}+\rho\right]\right]_{\rho_1}^{\rho_2}$$

Substituting the limits and multiplying by 4:

$$-4\times 3\left[0+\frac{1}{3}\right]\left\{\frac{-R}{(R^2+Z^2)^{1/2}} - \frac{R^3}{3(R^2+Z^2)^{3/2}} + \ln\left[\frac{R+(R^2+Z^2)^{1/2}}{Z}\right]\right\}$$

$$-4\left\{\frac{-R}{(R^2+Z^2)^{1/2}} - \frac{R^3}{3(R^2+Z^2)} + \ln\left[\frac{R+(R^2+Z^2)^{1/2}}{Z}\right]\right\}$$

2nd. term.

$$\int_{\varphi_1}^{\varphi_2}\sin\varphi\int_{\rho_1}^{\rho_2}\frac{\rho^2\,d\rho.d\varphi}{(\rho^2+Z^2)^{3/2}+2Z(\rho^2+Z^2)+Z^2(\rho^2+Z^2)^{1/2}}$$ (see pp. 17 and 18)

$$\left[-\cos\varphi\right]_{\varphi_1}^{\varphi_2}\left\{-\frac{2(\rho^2+Z^2)^{1/2}}{\rho}+\frac{2Z}{\rho}+\ln\left[\rho+(\rho^2+Z^2)^{1/2}\right]\right\}_{\rho_1}^{\rho_2}$$

$$4\left[0+1\right]\left\{\frac{-2(R^2+Z^2)^{1/2}}{R}+\frac{2Z}{R}+\ln\frac{R+(R^2+Z^2)^{1/2}}{R}\right\}=$$

$$=4\left\{-\frac{2(R^2+Z^2)^{1/2}}{R}+\frac{2Z}{R}+\ln\frac{R+(R^2+Z^2)^{1/2}}{R}\right\}$$

<u>3rd. term.</u>

$$\int_{\varphi_1}^{\varphi_2}\cos^2\varphi.\sin\varphi\int_{\rho_1}^{\rho_2}\frac{\rho^4.d\rho.d\varphi}{(\rho^2+Z^2)^{5/2}+2Z(\rho^2+Z^2)^2+Z^2(\rho^2+Z^2)^{3/2}}=$$

$$=\left[\frac{-\cos^3\varphi}{3}\right]_{\varphi_1}^{\varphi_2}.\left\{\ln\left[\rho+(\rho^2+Z^2)^{1/2}\right]-2\tan^{-1}\frac{\rho}{Z}+\frac{\rho}{(\rho^2+Z^2)^{1/2}}\right\}_{\rho_1}^{\rho_2}$$

Substituting the limits and multiplying by 4:

$$\frac{4}{3}\left\{\ln\frac{R+(R^2+Z^2)^{1/2}}{Z}-2\tan^{-1}\frac{R}{Z}+\frac{R}{(R^2+Z^2)^{1/2}}\right\}$$

4th. term.

$$2\int_{\varphi_1}^{\varphi_2} \cos^2\varphi.\sin\varphi \int_{\rho_1}^{\rho_2} \frac{\rho^4.\,d\rho.\,d\varphi}{(\rho^2+Z^2)^{5/2}+3Z(\rho^2+Z^2)^2+3Z^2(\rho^2+Z^2)^{3/2}+Z^3(\rho^2+Z^2)}$$

$$2\left[-\frac{\cos^3\varphi}{3}\right]_{\varphi_1}^{\varphi_2}\cdot\left\{\frac{-4(R^2+Z^2)^{1/2}}{R}+\frac{7Z}{R}+\tan^{-1}\frac{R}{Z}+\ln\frac{R+(R^2+Z^2)^{1/2}}{Z}\right\}$$

Substituting the limits and multiplying by 4:

$$\frac{8}{3}\left\{\frac{-4(R^2+Z^2)^{1/2}}{R}+\frac{7Z}{R}+\tan^{-1}\frac{R}{Z}+\ln\frac{R+(R^2+Z^2)^{1/2}}{Z}\right\}$$

$$\tau_{xy}=\frac{q}{2\pi}\left\{-4\left[-\frac{R}{(R^2+Z^2)^{1/2}}-\frac{R^3}{3(R^2+Z^2)^{3/2}}+\ln\frac{R+(R^2+Z^2)^{1/2}}{Z}\right]+\right.$$

$$+(1-2\nu)\left\{-4\left[-\frac{2(R^2+Z^2)^{1/2}}{R}+\frac{2Z}{R}+\ln\frac{R+(R^2+Z^2)^{1/2}}{R}\right]+\right.$$

$$+\frac{4}{3}\left[\ln\frac{R+(R^2+Z^2)^{1/2}}{Z}-2\tan^{-1}\frac{R}{Z}+\frac{R}{(R^2+Z^2)^{1/2}}\right]+\frac{8}{3}\left[\frac{-4(R^2+Z^2)^{1/2}}{R}+\right.$$

$$\left.\left.\left.+\frac{7Z}{R}+\tan^{-1}\frac{R}{Z}+\ln\frac{R+(R^2+Z^2)^{1/2}}{Z}\right]\right\}\right\}$$

$$\tau_{xy}=\frac{q}{2\pi}\left\{4(1-2\nu)\left(\frac{4}{3}+\frac{16}{3}\right)\frac{R}{(R^2+Z^2)^{1/2}}+\frac{4R^3}{3(R^2+Z^2)^{3/2}}+\right.$$

$$+\left[4+(1-2\nu)\left(-4+\frac{4}{3}+\frac{8}{3}\right)\right].\ln\frac{R+(R^2+Z^2)^{1/2}}{R}+(1-2\nu)\left[8\,\frac{(R^2+Z^2)^{1/2}}{R}+\right.$$

$$\left.+\left(-8+\frac{56}{3}\right)\frac{Z}{R}+\left(\frac{8}{3}+\frac{8}{3}\right)\tan^{-1}\frac{R}{Z}\right]\Bigg\}$$

$$\boxed{\begin{aligned}\tau_{xy}=\frac{q}{2\pi}\Bigg\{&\frac{8}{3}(4-5\nu)\frac{R}{(R^2+Z^2)^{1/2}}+\frac{4R^3}{3(R^2+Z^2)^{3/2}}-\\&-4\ln\frac{R+(R^2+Z^2)^{1/2}}{R}+(1-2\nu)\left[8\,\frac{(R^2+Z^2)^{1/2}}{R}+\frac{32}{3}\cdot\frac{Z}{R}+\right.\\&\left.+\frac{16}{3}\tan^{-1}\frac{R}{Z}\right.\end{aligned}}$$

VERTICAL COMPONENTS OF THE DISPLACEMENT

1.- NORMAL LOAD

For a single normal point load, the vertical component of the displacement vector is :

$$W = \frac{P}{2\pi E}(1+\nu)\left[\frac{Z^2}{D^3} + (1-2\nu)\frac{2}{D}\right]$$

Differentiating with respect to P and substituting expressions on pag. 2:

$$dW = \frac{1+\nu}{2\pi E}\,\rho\left[\frac{Z^2}{(\rho^2+Z^2)^{3/2}} + (1-2\nu)\frac{2}{(\rho^2+Z^2)^{1/2}}\right]d\rho . d\varphi$$

1st. term.

$$Z^2\int_{\varphi_1}^{\varphi_2} d\varphi \int_{\rho_1}^{\rho_2}\frac{d\rho}{(\rho^2+Z^2)^{3/2}} = Z^2\left[\varphi\right]_{\varphi_1}^{\varphi_2}\left[\frac{\rho}{Z^2(\rho^2+Z^2)^{1/2}}\right]_{\rho_1}^{\rho_2}$$

Substituting the limits and multiplying by 4:

$$4\,Z^2\left[\frac{\pi}{2}-0\right]\frac{R}{Z^2(R^2+Z^2)^{1/2}} = 2\pi\frac{R}{(R^2+Z^2)^{1/2}}$$

2nd. term.

$$\int_{\varphi_1}^{\varphi_2} d\varphi . \int_{\rho_1}^{\rho_2}\frac{d\rho}{(\rho^2+Z^2)^{1/2}} = \left[\varphi\right]_{\varphi_1}^{\varphi_2} . \ ln\left[\rho + (\rho^2+Z^2)^{1/2}\right]$$

Substituting the limits and multiplying by 4:

$$4\left[\frac{\pi}{2}-0\right] \ln\left[R+(R^2+Z^2)^{1/2}\right]-\ln Z =$$

$$= 2\pi \ln \frac{R+(R^2+Z^2)^{1/2}}{Z}$$

$$W = \frac{1+\nu}{2\pi E} P\left[2\pi \frac{R}{(R^2+Z^2)^{1/2}} + 2\times 2\pi(1-2\nu)\ln\frac{R+(R^2+Z^2)^{1/2}}{Z}\right]$$

$$\boxed{W = \frac{1+\nu}{E} P\left[\frac{R}{(R^2+Z^2)^{1/2}} + 2(1-2\nu).\ln\frac{R+(R^2+Z^2)^{1/2}}{Z}\right]}$$

2.- HORIZONTAL LOAD.

For a single horizontal load in the X direction, the vertical component of the displacement vector is:

$$W = \frac{Q}{4\pi}\left[\frac{1}{\mu}\;\frac{XZ}{D^3} + \frac{1}{\lambda+\mu}\cdot\frac{X}{D(D+Z)}\right]$$

where λ and μ are Lamé constants.

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} \qquad \mu = \frac{E}{2(1+\nu)}$$

Differentiating with respect to Q and substituting expressions on pag. 13:

$$dW = \frac{q}{4\pi}\left[\frac{1}{\mu}\;\frac{Z\rho^2\cos\varphi}{(\rho^2+Z^2)^{3/2}} + \frac{1}{\lambda+\mu}\cdot\frac{\rho^2\cos\varphi}{(\rho^2+Z^2)^{1/2}\left[(\rho^2+Z^2)^{1/2}+Z\right]}\right]d\rho . d\varphi$$

1st. term.

$$Z\int_{\rho_1}^{\rho_2}\cos\varphi\int_{\rho_1}^{\rho_2}\frac{\rho^2\cos\varphi}{(\rho^2+Z^2)^{3/2}} = Z\left[\sin\varphi\right]_{\varphi_1}^{\varphi_2}\left[\frac{-\rho}{(\rho^2+Z^2)^{1/2}} + \right.$$

$$\left. + \ln\left[\rho + (\rho^2+Z^2)^{1/2}\right]\right]$$

Substituting the limits and multiplying by 4:

$$4z\left[\frac{-R}{(R^2+z^2)^{1/2}}+\ln\frac{R+(R^2+z^2)^{1/2}}{z}\right]$$

<u>2nd term.</u>

$$\int_{\varphi_1}^{\varphi_2}\cos\varphi\int_{\rho_1}^{\rho_2}\frac{\rho^2}{(\rho^2+z^2)^{1/2}[(\rho^2+z^2)^{1/2}+z]}\cdot d\rho . d\varphi$$

(see pp. 18 and 19)

$$\left[\sin\varphi\right]_{\varphi_1}^{\varphi_2}\left\{-\frac{2(\rho^2+z^2)^{1/2}}{\rho}+\frac{4z}{\rho}+\ln\left[\rho+(\rho^2+z^2)^{1/2}\right]\right\}_{\rho_1}^{\rho_2}$$

substituting the limits and multiplying by 4:

$$4.\left[-\frac{2(R^2+z^2)^{1/2}}{R}+\frac{2z}{R}+\ln\frac{R+(R^2+z^2)^{1/2}}{z}\right]$$

$$W=\frac{q}{\pi}\left\{\frac{z}{\mu}\left[\frac{-R}{(R^2+z^2)^{1/2}}+\ln\frac{R+(R^2+z^2)^{1/2}}{z}\right]+\right.$$

$$\left.+\frac{1}{\lambda+\mu}\left[-\frac{2(R^2+z^2)^{1/2}}{R}+\frac{2z}{R}+\ln\frac{R+(R^2+z^2)^{1/2}}{z}\right]\right.$$

substituting Lamé's constants:

$$\boxed{W=\frac{2(1+\nu)}{E}\cdot\frac{q}{\pi}\left\{\frac{1-2\nu}{1-2\nu+2E}\cdot\ln\frac{R+(R^2+z^2)^{1/2}}{z}-\frac{zR}{(R^2+z^2)^{1/2}}+\frac{1-2\nu}{1-2\nu+2E}\left[\frac{2z}{R}-\frac{2(R^2+z^2)^{1/2}}{R}\right]\right\}}$$

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APPENDIX E

COMPUTER PLOTTING VERTICAL STRESSES

The program shown in the following pages is a simplified form of the general method described in this paper and was prepared using an IBM 1620 digital computer and a GERBER plotter. The general method was reduced accordingly to the difference in capacity and speed between this computer and the IBM/360.

```
340003200701360003200702490050001052000290010 0  SUPERMONITOR COLD START
*     ID DOMINGUEZ A B   BPR   6032
*     SUBSET
C     VERTICAL STRESSES IN SEMI INFINITE MEDIA.
C     PLOT OF VERTICAL STRESSES DUE TO NORMAL LOADS.
      DIMENSION B(1),C(2)
      CALL GPLOT(2,0,-100)
      DIMENSION P(10,10),U1(25),AP(3)
      READ 32,NMAX,R
   32 FORMAT(I6,F15.5)
      READ 3,((P(I,J),J=1,10),I=1,10)
    3 FORMAT(10F7.2)
      READ 30,CX,CY,CZ,DX,DY,DZ
   30 FORMAT(6F10.5)
C     PLOT AND NUMBER X AXIS.
      CALL GPLOT(11,0,0)
      DO 1 I=500,2500,500
      CALL GPLOT(12,I,0)
      CALL GPLOT(12,I,20)
      CALL GPLOT(11,I-15,-30)
      F=I
      H=F/500.
      CALL GNUM(I-15,-30,H,3.1,0.1,1.,0.)
    1 CALL GPLOT(11,I,0)
C     LABEL X AXIS.
      IX=2530
      IY=0
      READ 10,B(1)
   10 FORMAT(A4)
      CALL GCHAR(IX,IY,B(1),1,20,1.,0.)
C     PLOT AND NUMBER Y AXIS.
      CALL GPLOT(11,0,0)
      DO 2 I=150,1500,150
      CALL GPLOT(12,0,I)
      CALL GPLOT(12,20,I)
      CALL GPLOT(11,-50,I-5)
      F=I
      H=F/1500.
      CALL GNUM(-75,I-5,H,3.1,0.1,1.,0.)
    2 CALL GPLOT(11,0,I)
C     LABEL Y AXIS.
      IX=0
      IY=1530
      READ 11,C(1),C(2)
   11 FORMAT(A4,A4)
      CALL GCHAR(IX,IY,C(1),2,20,1.,0.)
      RHO=0.2*R
      DRHO=0.5*RHO
      R1=5.*DRHO/SQRT(3.14)
      RHO3=3.*DRHO
      DRHO2=DRHO*DRHO
      XLIM=5.4*DRHO
      YLIM=XLIM
      ZLIM=R
    4 DO 40 N=1,NMAX
```

```
C     INITIAL VALUE OF VERTICAL STRESS AT POINT P(X,Y,Z)
      P1ZZ=0.
      P2ZZ=0.
C     IS POINT P(X,Y,Z) WITHIN SENSITIVE ZONE (XLIM,YLIM,ZLIM)
C     IF YES M=1    IF NO M=0
      IF(CZ-ZLIM) 43,43,44
   43 IF(ABS(CX)-XLIM) 45,44,44
   45 IF(ABS(CY)-YLIM) 46,44,44
   46 M=1
      GO TO 47
   44 M=0
   47 K1=1
      DO 9 I=1,10
      DO 9 J=1,10
C     COORDINATES OF LOAD ELEMENT (I,J) WITH RESPECT TO MAIN SYSTEM.
      ZA=0.
      HI=I
      HJ=J
      XA=(HJ-5.5)*DRHO
      YA=(5.5-HI)*DRHO
C     COORDINATES OF POINT (X,Y,Z) WITH RESPECT TO LOAD ELEMENT (I,J).
      XP=CX-XA
      YP=CY-YA
      ZP=CZ
      IF(M-1) 5,6,6
C     IS POINT (X,Y,Z) WITHIN SENSITIVE ZONE OF LOAD ELEMENT (I,J)
C     IN EITHER CASE GO TO CORRESPONDING ROUTINE.
    6 IF(ABS(XP)-R1) 7,5,5
    7 IF(ABS(YP)-R1) 8,5,5
C     POINT (X,Y,Z) OUTSIDE OF SENSITIVE ZONE OF LOAD ELEMENT (I,J)
C     ROUTINE FOR CONCENTRATED LOADS.
    5 A=XP*XP+YP*YP+ZP*ZP
      D=SQRT(A)
      DZP2=(D+ZP)**2
      AA=A*D
      CCC=6.28*A*AA
      E=XP*XP
      F=YP*YP
      G=ZP*ZP
      H=G*ZP
      IF(P(I,J)) 77,78,77
   77 PP=P(I,J)*DRHO2
      TTT=3.*PP/CCC
      A1ZZ=TTT*H
      GO TO 89
   78 A1ZZ=0.
   89 P1ZZ=P1ZZ+A1ZZ
      GO TO 9
C     POINT (X,Y,Z) INSIDE SENSITIVE ZONE OF LOAD ELEMENT (I,J).
    8 U1(K1)=P(I,J)
   91 K1=K1+1
    9 CONTINUE
      IF(M-1) 52,60,60
C     ROUTINE FOR DISTRIBUTED LOADS.
   60 K1MAX=K1-1
```

```
      IF(K1MAX) 50,50,51
   50 A2ZZ=0.
      GO TO 52
   51 SUMP1=0.
      DO 600 K1=1,K1MAX
  600 SUMP1=SUMP1+U1(K1)
      R1MAX=K1MAX
      IF(R1MAX) 900,900,901
  900 AP(1)=0.
      A2ZZ=0.
      GO TO 750
  901 AP(1)=SUMP1/R1MAX
      IF(CZ) 150,150,908
  908 A=ZP*ZP
      AA=ZP*A
      C1=R1*R1
      D1=C1*R1
      E1=SQRT(C1+A)
      F1=E1*E1*E1
  201 A2ZZ=AP(1)*(1.-AA/F1)
  750 P2ZZ=P2ZZ+A2ZZ
   52 PZZ=P1ZZ+P2ZZ
C     PLOT OUTPUT.
      X=5.*CZ
      Y=10.*PZZ
      GO TO 151
  150 X=5.*CZ
      Y=10.*AP(1)
      CALL GPLOT(11,X,Y)
      GO TO 39
  151 CALL GPLOT(12,X,Y)
   39 CX=CX+DX
      CY=CY+DY
   40 CZ=CZ+DZ
      CALL EXIT
      END
*     DATA
    41      1.
     0.      0.      0.      0.      0.      0.      0.      0.      0.      0.
     0.      0.      0.      0.      0.      0.      0.      0.      0.      0.
     1.      1.      1.      1.      1.      1.      1.      1.      1.      1.
     1.      1.      1.      1.      1.      1.      1.      1.      1.      1.
     1.      1.      1.      1.      1.      1.      1.      1.      1.      1.
     1.      1.      1.      1.      1.      1.      1.      1.      1.      1.
     1.      1.      1.      1.      1.      1.      1.      1.      1.      1.
     1.      1.      1.      1.      1.      1.      1.      1.      1.      1.
     0.      0.      0.      0.      0.      0.      0.      0.      0.      0.
     0.      0.      0.      0.      0.      0.      0.      0.      0.      0.
   0.05000   0.05000   0.00000   0.00000   0.00000   0.10000
+Z/R
SIGMAZ/P
ZZZZ              END OF JOB 1620
```

APPENDIX F

LIST OF PROGRAMS