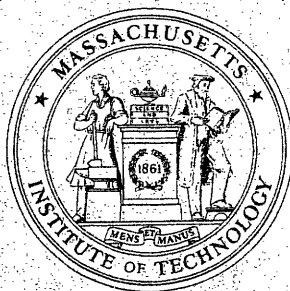# OPERATIONS RESEARCH CENTER

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Network Flows

by

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and
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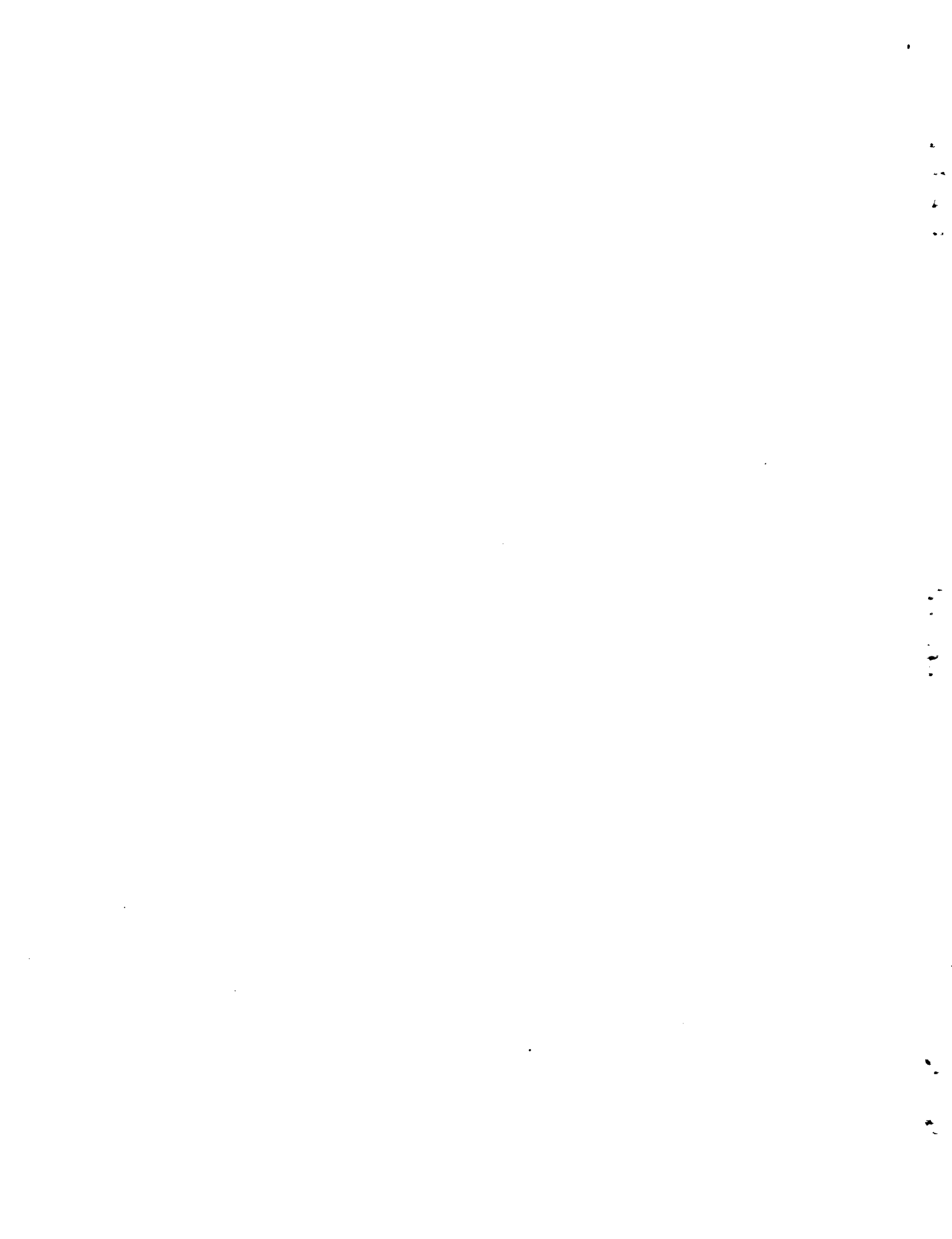
NETWORK FLOWS

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NETWORK FLOWS

OVERVIEW

Network Flows

Perhaps no subfield of mathematical programming is more alluring than network optimization. Highway, rail, electrical, communication and may other physical networks pervade in our everyday lives. As a consequence, even non-specialists recognize the practical importance and the wide ranging applicability of networks. Moreover, because the physical operating characteristics of networks (e.g., flows on arcs and mass balance at nodes) have natural mathematical representations, practitioners and non-specialists can readily understand the mathematical descriptions of network optimization problems and the basic nature of techniques used to solve these problems. This combination of widespread applicability and ease of assimilation has undoubtedly been instrumental in the evolution of network planning models as one of the most widely used modeling techniques in all of operations research and applied mathematics.

Network optimization is also alluring to methodologists. Networks provide a concrete setting for testing and devising new theories. Indeed, network optimization has inspired many of the most fundamental results in all of optimization. For example, price directive decomposition algorithms for both linear programming and combinatorial optimization had their origins in network optimization. So did cutting plane methods and branch and bound procedures of integer programming, primal-dual methods of linear and nonlinear programming, and polyhedral methods of combinatorial optimization. In addition, networks have served as the major prototype for several theoretical domains (for example, the field of matroids) and as the core model for a wide variety of min/max duality results in discrete mathematics.

Moreover, network optimization has served as a fertile meeting ground for ideas from optimization and computer science. Many results in network optimization are routinely used to design and evaluate computer systems, and ideas from computer science concerning data structures and efficient data manipulation have had a major impact on the design and implementation of many network optimization algorithms.

The aim of this paper is to summarize many of the fundamental ideas of network optimization. In particular, we concentrate on network flow problems and highlight a number of recent theoretical and algorithmic advances. We have divided the discussion into the following broad major topics:

- Applications
- Basic Properties of Network Flows
- Shortest Path Problems
- Maximum Flow Problem
- Minimum Cost Flow Problem
- Assignment Problem

Much of our discussion focuses on the design of provably good (e.g., polynomial) algorithms. Among good algorithms, we have presented those algorithms which are simple and are likely to be efficient in practice. We have attempted to structure our discussion so that it not only provides a survey of the field for the specialists, but also serves as an introduction and summary to the non-specialists who have a basic working knowledge of the rudiments of optimization, particularly linear programming.

As a prelude to our discussion in the next section, we present in this section several important preliminaries . We discuss (i) different ways to measure the performance of algorithms; (ii) graph notation and various ways to represent networks quantitively; (iii) a few basic ideas from computer science that underlie the design of many algorithms; and (iv) a couple of generic proof techniques that have been useful in designing polynomial algorithms.

1.1   Applications

Networks arise in numerous application settings and in a variety of guises. In this section, we briefly describe a few prototypical applications. Our discussion is intended to illustrate a range of applications and to be suggestive of how network flow problems arise in practice; a more extensive survey would take us far beyond the scope of our discussion. To illustrate the breadth of network applications, we consider some models requiring solution techniques that we will not describe in this chapter.

For the purposes of this discussion we will consider four different types of networks arising in practice:
- Physical networks (Streets, railbeds, pipelines, wires)
- Route networks
- Space-time networks (Scheduling networks)
- Derived networks (Through problem transformations)

These four categories are not exhaustive and overlap in coverage. Nevertheless, they provide a useful taxonomy for summarizing a variety of applications.

Network flow models are also used for several purposes:

- Descriptive modeling (answering "what is?" questions)
- Predictive modeling (answering "what will be?" questions)
- Normative modeling (answering "what should be?" questions, that is, performing optimization)

We will illustrate models in each of these categories. We first introduce the basic underlying network flow model and some useful notation.

The Network Flow Model

Let $G = (N, A)$ be a directed network with a cost c_{ij}, a lower bound l_{ij}, and a capacity u_{ij} associated with every arc $(i, j) \in A$. We associate with each node $i \in N$ an integer number $b(i)$ representing its supply or demand. If $b(i) > 0$, then node i is a *supply* node; if $b(i) < 0$, then node i is a *demand* node, and if $b(i) = 0$, then node i is a *transshipment* node. The minimum cost network flow problem can be formulated as follows:

$$\text{Minimize} \quad \sum_{(i,j) \in A} c_{ij} x_{ij} \tag{1.1a}$$

subject to

$$\sum_{\{j : (i,j) \in A\}} x_{ij} - \sum_{\{j : (j,i) \in A\}} x_{ji} = b(i), \quad \text{for all } i \in N, \tag{1.1b}$$

$$l_{ij} \leq x_{ij} \leq u_{ij}, \quad \text{for all } (i, j) \in A. \tag{1.1c}$$

We refer to x as the *flow* in the network. The constraint (1.1b) implies that the total flow out of a node minus the total flow into that node must equal the net supply/demand of the node. We henceforth refer to this constraint as the *mass balance constraint*. The flow must also satisfy the lower bound and capacity constraints (1.1c) which we refer to as the *flow bound constraints*. The flow bounds might model physical capacities, contractual obligations or simply operating ranges of interest. Frequently, the

given lower bounds l_{ij} are all zero; we show later that they can be made zero without any loss of generality.

In matrix notation, we represent the minimum cost flow problem

$$\text{minimize } \{ cx : Nx = b \text{ and } l \leq x \leq u \}, \tag{1.2}$$

in terms of a *node-arc incidence matrix* N. The matrix $N = (n_{ka})$ has one row for each node k of the network, and one column for each arc a. Since each variable x_{ij} appears in two mass balance equations, as an outflow from node i with a +1 coefficient and as an inflow to node j with a -1 coefficient, the column corresponding to an arc a has the following structure:

$$n_{ka} = \begin{cases} +1, & \text{if arc } a \text{ is directed from node } k \\ -1, & \text{if arc } a \text{ is directed into node } k \\ 0, & \text{otherwise.} \end{cases}$$

The matrix N has very special structure: only $2m$ entries out of nm total entries are nonzero, all of its nonzero entries are ± 1, and each column has exactly one +1 and one -1. Later in Sections 2.2 and 2.3, we will consider some of the consequences of this special structure. For now, we make two observations:

(i) Summing all the mass balance constraints eliminates all the flow variables and gives

$$\sum_{i \in N} b(i) = 0, \text{ or } \sum_{i \in \{N : b(i) > 0\}} b(i) = \sum_{i \in \{N : b(i) < 0\}} b(i) \, .$$

Consequently, total supply must equal total demand if the mass balance constraints are to have any feasible solution.

(ii) If the total supply does equal the total demand, then summing all the mass balance equations gives the zero equation $0x = 0$, or equivalently, any equation is equal to minus the sum of all other equations, and hence is redundant.

We shall also consider the following special cases of the minimum cost flow problem.

The shortest path problem. The shortest path problem is to determine directed paths of smallest cost from a given node 1 to all other nodes. In the minimum cost flow problem if we set $b(1) = (n - 1)$, $b(i) = -1$ for all other nodes, $l_{ij} = 0$ and $u_{ij} = n$ for all arcs, then the optimum solution sends unit flow from node 1 to every other node along a shortest path.

The maximum flow problem. The maximum flow problem is to send maximum possible flow in a network from a specified source node s to a specified sink node t. In the minimum cost flow problem, if we add an additional arc (t, s) with $c_{ts} = -1$ and $u_{ts} = \infty$, set supply/demand of all nodes and costs of all arcs to zero, then the minimum cost solution maximizes the flow on arc (t, s) which equals the maximum possible flow from the source node to the sink node.

The assignment problem. The assignment problem consists of a set N_1, say of persons, a set N_2, say of objects satisfying $|N_1| = |N_2|$, a collection of node pairs $A \subseteq N_1 \times N_2$ representing possible person-to-object assignments and a cost c_{ij} associated with each element (i, j) in A. The objective is to assign each person to exactly one object so that the cost of the assignment is minimum. The assignment problem is a minimum cost flow problem on network $G = (N_1 \cup N_2, A)$ with $b(i) = 1$ for all $i \in N_1$ and $b(i) = -1$ for all $i \in N_2$ (we set $l_{ij} = 0$ and $u_{ij} = 1$ for all (i, j) $\in$ A).

Physical Networks

The familiar city street map is perhaps the prototypical physical network, and the one that most readily comes to mind when we envision a network. Many network planning problems arise in this problem context. As one illustration, consider the problem of managing, or designing, a street network to decide upon such issues as speed limits, one way street assignments, or whether or not to construct a new road or bridge. In order to make these decisions intelligently, we need a descriptive model that tells us how to model traffic flows and measure the performance of any design as well as a predictive model for measuring the effect of any change in the system. We can then use these models to answer a variety of "what if" planning questions.

The following type of equilibrium network flow model permits us to answer these types of questions. Each line of the network has an associated delay function that specifies how long it takes to traverse this link. The time to do so depends upon traffic conditions; the more traffic that flows on the link, the longer is the travel time to traverse it. Now also suppose that each user of the system has a point of origin (e.g., their home) and a point of destination (e.g., their workplace of the central business district). Each of these users must choose a route through the network. Note, however, that these route choices affect each other; if two users traverse the same link, they add to each other;s travel time because of the added congestion on the link. Now let us make the behavioral assumption that each user wishes to travel between his or her origin and

destination as quickly as possible, that is, along a shortest travel time path. This situation leads to the following equilibrium problem with an embedded set of network optimization problems (shortest path problems); is there a flow pattern in the network with the property that no user can unilaterally change his (or her) choice of origin to destination path (that is, all other users continue to use their specified paths in the equilibrium solution) to reduce his travel time. Operations researchers have developed a set of sophisticated models for this problem setting, as well as related theory (concerning, for example, existence and uniqueness of equilibrium solutions), and algorithms for computing equilibrium solutions. Used in the mode of "what if" scenario analysis, these models permit analysts to answer the type of questions we posed previously. These models are actively used in practice. Indeed, the Urban Mass Transit Authority in the United States requires that communities perform a network equilibrium impact analysis as part of the process for obtaining federal funds for highway construction or improvement.

Similar types of models arise in many other problem contexts. For example, a network equilibrium model forms the heart of the Project Independence Energy Systems (LPIES) model developed by the U.S. Department of Energy as an analysis tool for guiding public policy on energy. The basic equilibrium model of electrical networks is another example. In this setting, Ohm's Law serves as the analog of the congestion function for the traffic equilibrium problem, and Kirkoff's Law represents the network mass balance equations.

Another type of physical network is a very large-scale integrated circuit (VLSI circuit). In this setting the nodes of the network correspond to electrical components and the links correspond to wires that connect these links. Numerous network planning problems arise in this problem context. For example, how can we lay out , or design, the smallest possible integrated circuit to make the necessary connections between its components and maintain necessary separations between the wires (to avoid electrical interference).

Route Networks

Route networks, which are one level of abstraction removed from physical networks, are familiar to most students of operations research and management science. The traditional operations research transportation problem is illustrative. A shipper with supplies at its plants must shop to geographically dispersed retail center, each with a

given customer demand. Each arc connecting a supply point to a retail center incurs costs based upon some physical network, in this case the transportation network. Rather than solving the problem directly on the physical network, we preprocess the data and construct transportation routes. Consequently, an arc connecting a supply point and retail center might correspond to a complex four leg distribution channel (1) from a plant (by truck) to a rail station, (2) from the rail station to a rail head elsewhere in the system, (3) from the rail head (by truck) to a distribution center, and (4) from the distribution center (on a local delivery truck) to the final customer (or even in some cases just to the distribution center) and assign it the composite distribution cost of all the intermediary legs, as well as a distribution capacity on this route, the problem can be modeled as a classic network transportation problem: find the flows from plants to customers that minimizes overall costs. This type of model is used in numerous applications. As but one illustration, a prize winning practice paper written several years ago reported on an application of such a network planning system by the Cahill May Roberts pharmaceutical Company (of Ireland) to reduce overall distribution costs by 20%, while improving customer service as well.

Many related problems arise in this type of problem setting. For example, determining the best design of such a system: for instance, deciding upon the location of the distribution centers? It is possible to address this type of decision problem using integer programming methodology for choosing the distribution sites and network flows to cost out (or optimize flows) for any given choice of sites; using this approach a noted study conducted several years ago permitted Hunt Wesson Foods Corporation to save over $1 million annually.

One special case of the transportation problem merits note -- the situation in which the supply and demand of each node is one (therefore, the number of supply and demand nodes are the same). This well-known assignment problem has numerous applications, particularly in problem contexts such as machine scheduling. In this application context, we would identify the supply points with jobs to be performed, the demand points with available machines, and the cost associated with arc (i, j) as the cost of completing job i on machine j. The solution to the problem specifies the minimum cost assignment of the jobs to the machines, assuming that each machine has the capacity to perform only one job.

Space Time Networks

Frequently in practice, we wish to schedule some production or service activity over time. In these instances it is often convenient to formulate a network flow problem on a "space--time network" with several nodes representing a particular facility (a machine, a warehouse, an airport) but at different points in time.

Figure 1.1, which represents a core planning model in production planning, *the economic lot size problem*, is an important example. In this problem context, we wish to meet prescribed demands d_t for a product in each of the T time periods. In each period, we can produce at level x_t and/or we can meet the demand by drawing upon inventory I_t from the previous period. The network representing this problem has T + 1 nodes: one node $t = 1, 2, \ldots, T$ represents each of the planning periods, and one node 0 represents the "source" of all production. The flow on arc (0, t) prescribes the production level x_t in period t, and the flow on arc (t, t + 1) represents the inventory level I_t to be carried from period t to period t + 1. The mass balance equation for each period t models the basic accounting equation: incoming inventory plus production in that period must equal demand plus final inventory. The mass balance equation for node 0 indicates that all demand (assuming zero beginning and final inventory over the entire planning period) must be produced in some period $t = 1, 2, \ldots, T$. Whenever the production and holding costs are linear, this problem is easily solved as a shortest path problem (for each demand period, we must find the minimum cost path of production and inventory arcs from node 0 to that demand point). If we impose capacities on production or inventory, the problem becomes a minimum cost network flow problem.
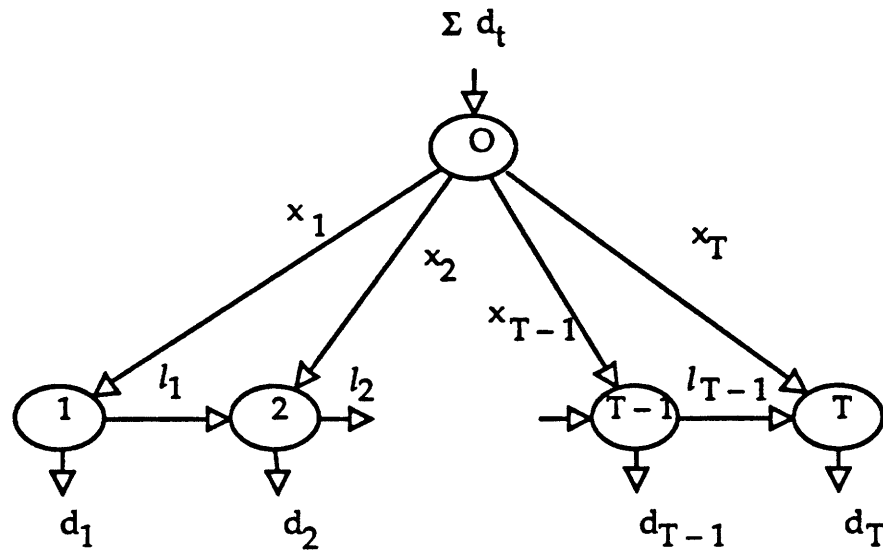
Figure 1.1. Network Flow Model of the economic lot size problem.

We next extend the economic lot sizing problem by assuming that production x_t in any period incurs a fixed cost: that is, whenever we produce in period t (i.e., $x_t > 0$), no matter how much or how little, we incur a fixed cost F_t. In addition , we may incur a per unit production cost c_t in period t and a per unit inventory cost h_t for carrying any unit of inventory from period t to period t + 1. Hence the cost on each arc for this problem is either linear (for inventory carrying arcs) or linear plus a fixed cost (for production arcs). Consequently, the objective function for the problem is concave. We prove in Section 2.2 as Theorem 2.6 that there always exists an optimum solution of the concave cost network flow problem which is a spanning tree. This problem's spanning tree solution decomposes into disjoint directed paths; the first arc on each path is a production arc (of the form (o, t)) and each other arc is an inventory carrying arc. This observation can be stated as the following *production property:* in the solution, each time we produce, we produce enough to meet the demand for an integral number of contiguous periods. Moreover, in no period do we both carry inventory from the previous period and produce.

The production property permits us to solve the problem very efficiently as a shortest path problem on the following networks G'. The network G' consists of nodes 1 to T + 1, and for every pair of nodes i and j with i < j there is an arc (i, j). The length of arc (i, j) is equal to the production and inventory cost of satisfying the demand

periods of the the periods from i t to j – 1. Observe that for every production schedule satisfying the production property there exists a directed path in G' from node 1 to node T + 1 of the same objective function value and vice-versa. Hence the optimum production schedule can be obtained by solving a shortest path problem.

Many enhancements of the model are possible, for example (i) the production facility might have limited production capacity of limited storage for inventory, or (ii) the production facility might be producing several products that are linked by common production costs or by changeover cost (for example, we may need to clean out dies when changing production from red socks to white socks), or that share common limited production facilities. In most cases, the enhanced models are quite difficult to solve (they are NP-complete), though the embedded network structure often proves to be useful in designing either heuristic or optimization methods.

Another classical network flow scheduling problem is the *airline scheduling problem* used to identify a flight schedule for an airline. In this application setting, each node represents both a geographical location (e.g., an airport) and a point in time (e.g., New York at 10 A.M.). The arcs are of two types: (i) service arcs connecting two airports, for example New York at 10 A.M. to Boston at 11 A.M.; (ii) layover in which a plane stays at New York from 10 A.M. until 11 A.M. to wait for a later flight or waits overnight at New York from 11 P.M. until 6 A.M. the next morning. If we identify revenues with each service leg, a network flow in this network (with not external supply or demand) will specify a set of flight plans (circulation of airplanes through the network). A flow that maximizes revenue will prescribe a schedule for an airline's fleet of planes. The same type of network representation arises in many other dynamic scheduling applications.

Derived Networks

This category is a "grab bag" of the specialized applications and illustrates that sometimes network flow problems arise in surprising ways from problems that on the surface might not appear to involve networks. The following examples illustrate this point.

Single Duty Crew Scheduling. Figure 1.2 illustrates a number of possible duties for the drivers of a bus company.

Time Period/Duty Number	1	2	3	4	5	6	7	8	9	10	11
9 – 10 A.M.	1	0	0	0	1	0	1	0	0	0	0
10 – 11 A.M.	0	0	0	0	1	0	1	0	0	0	0
11 – Noon	0	0	1	0	0	0	0	0	1	0	0
12 – 1 P.M.	0	0	1	0	0	0	0	0	1	0	0
1 – 2 P.M.	0	0	1	0	0	1	0	0	0	0	1
2 – 3 P.M.	0	1	0	0	0	1	0	1	0	0	1
3 – 4 P.M.	0	1	0	0	0	0	0	1	0	1	0
4 – 5 P.M.	0	0	0	1	0	0	0	0	0	1	0

Figure 1.2. Available duties for a single duty scheduling problem

For example, the first duty (the first column in the table) represents a schedule in which a driver works form 9 A.M. to 10 A.M.; the second duty specifies that a driver works from 2 P.M. to 4 P.M. Suppose each duty j has an associated cost c_j. If we wish to ensure that a driver is on duty for each hour of the planning period (9 A.M. to 5 P.M. in the example), and the cost of scheduling is minimum, then the problem is an integer program:

$$\text{minimize} \quad cx \tag{1.2a}$$

$$\text{subject to} \quad Ax = b \tag{1.2b}$$

$$x_j = 0 \quad \text{or} \quad 1 \quad \text{for all } j. \tag{1.2c}$$

In this formulation the binary variable x_j indicates whether $(x_j = 1)$ or not $(x_j = 0)$ we select the j-th duty; the matrix A represents the matrix of duties and b is a column vector whose components are all 1's. Observe that the ones in each column of A occur in consecutive rows because each driver duty contains a single work shift (no split shifts or work breaks). We show that this problem is a shortest path problem. To make this identification, we perform the following operations: In (1.2b) subtract each equation from the equation below it. This transformation does not change the solution to the system. Now add a redundant equation equal to minus the sums of all the equations in the revised system. Because of the structure of A, each column in the revised system will have a single +1 (corresponding to the first hour of the duty in the column of A) and a single -1 (corresponding to the row in A, or the added row, that lies

just below the last +1 in the column of A). Moreover, the revised right hand side vector of the problem will have a +1 in row 1 and a −1 in the last (the appended) row. Therefore, the problem is to ship one unit of flow from node 1 to node 9 at minimum cost in the network given in Figure 1.3, which is an instance of the shortest path problem.
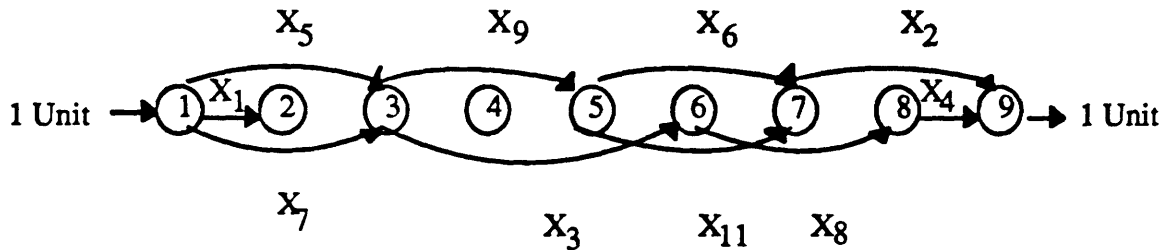

Figure 1.3. Shortest path formulation of the single duty scheduling problem.

If instead of requiring a single driver to be on duty in each period, we specified a number to be on duty in each period, the same transformation would produce a network flow problem, but in this case the right hand side coefficients (supply and demands) could be arbitrary. Therefore, the transformed problem would be a general minimum cost network flow problem, rather than a shortest path problem.

Critical Path Scheduling and Networks Derived from Precedence Conditions

In construction and many other project planning applications, workers need to complete a variety of tasks that are related by precedence conditions; for example, in constructing a house, a builder must pour the foundation before framing the house and complete the framing before beginning to install either electrical or plumbing fixtures. This type of application can be formulated mathematically as follows. Suppose we need to complete J jobs and that job j (j = 1, 2, . . . , J) requires t_j days to complete. We are to choose the start time s_j of each job j so that we honor a set of specified precedence constraints and complete the overall project as quickly as possible. If we represent the jobs by nodes, then the precedence constraints can be represented by arcs; thereby giving us a network. The precedence constraints imply that for each arc (i, j) in the network, the job j can not start until job i is completed. For the convenience of notation, we add two dummy jobs, both with zero processing time: a "start" job 0 to be completed before any other job can begin and a "completion" job J + 1 that cannot be initiated until we have completed all other jobs. Let G = (N, A) represent the network corresponding to this augmented project. Then we wish to solve the following optimization problem:

minimize $s_{J+1} - s_0$,

subject to

$s_j \geq s_i + t_i$, for each arc $(i, j) \in A$.

On the surface, this problem, which is a linear program in the variables s_j, seems to bear no resemblance to network optimization. Note, however, that if we move the variable s_i to the left hand side of the constraint, then each constraint contains exactly two variables, one with a plus one coefficient and one with a minus one coefficient. The linear programming dual of this problem has a familiar structure. If we associate the dual variable x_{ij} for each arc (i, j), then the dual of this problem is as follows.

$$\text{maximize} \quad \sum_{(i,j) \in A} t_i x_{ij} \ ,$$

subject to

$$\sum_{\{j : (i,j) \in A\}} x_{ij} \quad - \quad \sum_{\{j : (j,i) \in A\}} x_{ji} \quad = \quad \begin{cases} -1, & \text{if } i = 0 \\ 0, & \text{otherwise} \\ 1, & \text{if } i = J+1 \end{cases} \quad \text{for all } i \in N$$

$$x_{ij} \geq 0, \text{ for all } (i, j) \in A .$$

This problem is to determine the longest path in the network G from node 0 to node $J + 1$ where arc length of an arc (i, j) is t_i. This longest path has the following interpretation: it is the longest sequence of jobs needed to fulfill the specified precedence conditions. Since delaying any job in this sequence must necessarily delay the completion of the overall project, this path has become known as the "critical" path and the problem has become known as the critical path problem. This model has become a principal tool in project management, particularly for managing large-scale construction projects. The critical path itself is important because it identifies those jobs that require managerial attention in order to complete the problem as quickly as possible.
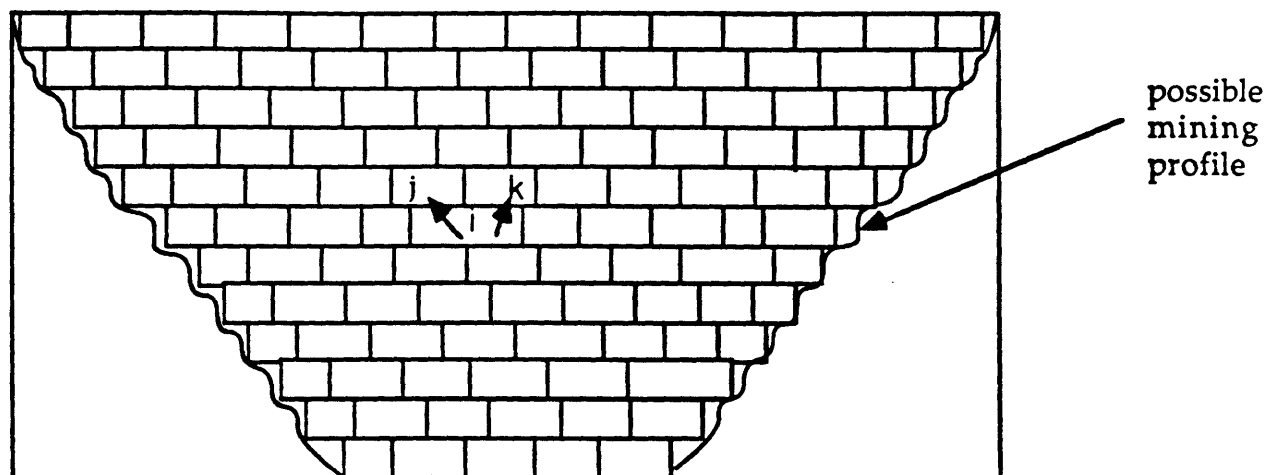
Figure 1.4. Open pit mine; we must extract blocks j and k before block i.

Researchers and practitioners have enhanced this basic model in several ways. For example, if resources are available for expediting individual jobs, we could consider the most efficient use of these resources to complete the overall project as quickly as possible. Certain versions of this problem can be formulated as minimum cost flow problems.

The *open pit mining* problem is another network flow problem that arises from precedence conditions. Consider the open pit mine shown in Figure 1.4. As shown in this figure, we have divided the region to be mined into blocks. The provisions of any given mining technology, and perhaps the geography of the mine, impose restrictions on how we can remove the blocks: for example, we can never remove a block until we have removed any block that lies immediately above it; restrictions on the "angle" of mining the blocks might impose similar precedence conditions. Suppose now that each block j has an associated revenue r_j (e.g., the value of the ore in the block minus the cost for extracting the block) and we wish to extract blocks to maximize overall revenue. If we let y_j be a zero-- one variable indicating whether or not we extract block j, the problem will contain (i) a constraint $y_j \leq y_i$ (or, $y_j - y_i \leq 0$) whenever we need to mine block j before block i and (ii) an objective function for maximizing the total revenue $r_j y_j$, summed over all blocks j. The dual linear program (obtained from the linear programming version of the problem (with the constraints $0 \leq y_j \leq 1$, rather than $y_j = 0$ or 1) will be a network flow problem with a node for each block, a variable for each precedence constraint, and the revenue r_j as the demand at node j. This network will also have a dummy "collection node" 0 with demand equal to minus the sum of the r_j's, and an arc connecting it to node j (that is, block j); this arc corresponds to the upper bound constraint $y_j \leq 1$ in the original constraint $y_j \leq 1$ in the original linear program. The dual problem is one of finding a network flow that minimizes ths sum of flows on the arcs incident to node 0.

The critical path scheduling problem and open pit mining problem illustrate one way that network flow problems arise indirectly. Whenever, two variables in a linear program are related by a precedence conditions, the variable corresponding to this precedence constraint in the dual linear program will have a network flow structure. If the only constraints in the problem are precedence constraints, the dual linear program will be a network flow problem.

Matrix Rounding of Census Information

The U.S. Census Bureau uses census information to construct thousands of different types of tables for a wide variety of purposes. By law, the Bureau has an obligation to protect the source of its information and not disclose statistics that can be attributed to any particular household. It can attempt to do so by rounding the census information contained in any table. Consider, for example, the data shown in Figure 1.5(a). Since the upper leftmost entry in this table is a 1, the tabulated information might disclose information about a particular household. We might disguise the information in this table as follows; round each entry in the table, including the row and column sums, either up or down to the nearest multiple of three, say, so that the entries in the table continue to add to the L(rounded) row and column sums. Figure 1.5(b) shows a rounded version of the data that meets this criterion. In some instances, it will be impossible to find the appropriately rounded set of data: for example, if the yy entry in the given table were a zz instead of a ww. The problem can be cast as finding a feasible flow in a network which can be solved by an application of the maximum flow algorithm. The network contains anode for each column in the table and one node for each row. It contains an arc connective node i (corresponding to column i) and node j (corresponding to row j): the flow on this arc can be the ij-th entry in the prescribed table, rounded either up or down. In addition, we add a supersource s to the network connected to each column node i: the flow on this arc must be the i-th column sum, rounded up or down. Similarly, we add a super sink t with the arc connecting each row node j to this node; the flow on this arc must be the j-th row sum, rounded up or down. If we rescale all the flows, measuring them in integral units of the rounding base (multiples of 3 in our example), then the flow on each arc must be integral at one of two consecutive integral values. Figure 1.6 illustrates the network flow problem corresponding to the census data specified in Figure 1.5. (Figures 1.5 and 1.6 will be added later.)

1.2   Complexity Analysis

There are three basic approaches for measuring the performance of an algorithm: empirical analysis, worst case analysis, and average case analysis. *Empirical analysis* typically measures the computational time of an algorithm using statistical sampling on a distribution (or several distributions) of problem instances. The major objective of empirical analysis is to estimate how algorithms behave in practice. *Worst-case analysis* aims to provide upper bounds on the number of steps that a given algorithm can take on

any problem instance. Therefore, this type of analysis provides *performance guarantees*. The objective of *average-case analysis* is to estimate the expected number of steps taken by an algorithm. Average-case analysis differs from empirical analysis because it provides rigorous mathematical proofs of average case performance, rather than statistical estimates.

Each of these three performance measures has its relative merits, and is appropriate for certain purposes. Nevertheless, this chapter will focus primarily on worst case analysis, and only secondarily on empirical behavior. Researchers designed many of the algorithms in this chapter specifically to improve worst case complexity while simultaneously maintaining good empirical behavior. Thus, for the algorithms we present, worst case analysis is the primary measure of performance.

Worst Case Analysis

For worst case analysis, we bound the running time of network algorithms in terms of several basic problem parameters: the number of nodes (n), the number of arcs (m), and upper bounds C and U on the cost coefficients and the arc capacities, respectively. Whenever C (resp. , U) appears in the complexity analysis, we assume that each cost (resp., capacity) is integer valued. As an example of a worst case result within the chapter, we will prove that the number of steps for the label correcting algorithm to solve the shortest path problem is less than pnm steps for some sufficiently large constant p.

To avoid the need to compute or mention the constant p, researchers typically use a "big O" notation, replacing the expressions: "the label correcting algorithm requires pmn steps for some constant p" with the equivalent expression "the running time of the label correcting algorithm is O(nm)." The O( ) notation avoids the need to state a specific constant; instead, this notation indicates only the *dominant* terms of the running time. By dominant, we mean the term that would dominate all other terms for *sufficiently* large values of n and m. Therefore, the time bounds are called *asymptotic* running times. For example, if the actual running time is $10nm^2 + 2^{100}n^2m$, then we would state that the running time is $O(nm^2)$, assuming that $m \geq n$. Observe that the running time indicates that the $10nm^2$ term is dominant even though for most practical values of n and m, the $2^{100}n^2m$ term would dominate. Although ignoring the constant terms may have this undesirable feature, the O( ) notation has been motivated by the following considerations:

1.   Ignoring the constants greatly simplifies the analysis. Consequently, use of O( ) notation has avoided prohibitively difficult analysis required to compute the leading constants and has led to a flourishing of research on the worst-case performance of algorithms.

2.   Estimating the constants correctly is fundamentally difficult. The least value of the constants is not determined solely by the algorithm; it is also highly sensitive to the choice of the computer language, and even to the choice of the computer.

3.   For all of the algorithms that we present, the constant terms are relatively small integers for all the terms in the complexity bound.

4.   For large practical problems, the constant factors do not contribute nearly as much to the running time as do the factors involving n, m, C or U.

Counting Steps

The running time of a network algorithm is determined by counting the number of steps it performs. The counting of steps relies on a number of assumptions, most of which are quite appropriate for most of today's computers.

A1.1   The computer carries out instructions sequentially, with at most one instruction being executed at a time.

A1.2   Each comparison and basic arithmetic operation counts as one step.

In A1.1, we assume that all computations are conducted sequentially; we will not discuss parallel implementations of network flow algorithms.

In A1.2, we implicitly assume that the only operations to be counted are comparisons and arithmetic operations. In fact, even by counting all other computer operations, on today's computers we would obtain the same asymptotic worst case results for the algorithms that we present. Our assumption that each operation, be it an addition or division, takes equal time, is justified in part by the fact that O( ) notation ignores differences in running times of at most a constant factor, which is the time difference between an addition and a multiplication on essentially all modern computers.

On the other hand, the assumption that each arithmetic operation takes one step may lead to an underestimation of the asymptotic running time of arithmetic operations involving very large numbers on real computers since, in practice, a computer must store large numbers in several words of its memory. Therefore, to perform each operation on very large numbers, a computer must access a number of words of data and thus takes more than a constant number of steps. To avoid this systematic underestimation of the running time, in comparing two running times, we will typically assume that both C and U are polynomially bounded in n, i.e., $C = O(n^k)$ and $U = O(n^k)$, for some constant k. This assumption is known as the *similarity assumption*, which is quite reasonable in practice. For example, if we were to restrict costs to be less than $100n^3$, we would allow costs to be as large as 100,000,000,000 for networks with 1000 nodes.

Polynomial Time Algorithms

An algorithm is said to be a *polynomial time* algorithm if its running time is bounded by a polynomial function of the input length. The *input length* of a problem is the number of bits needed to represent that problem. For a network problem, the input length is a low order polynomial function of n, m, log C and log U (e.g., it is $O((n + m)(\log n + \log C + \log U))$). Consequently, a network algorithm is polynomial if its running time is bounded by a polynomial function in n, m, log C and log U. For example, the running time of one of the polynomial maximum flow algorithms we consider is $O(nm + n^2 \log U)$. Other instances of polynomial bounds are $O(n^2m)$ and $O(n \log n)$. A polynomial algorithm is said to be a *strongly polynomial* algorithm if its running time is bounded by a polynomial function in n and m only, and does not involve log C or log U. The maximum flow algorithm alluded to, therefore, is not strongly polynomial. The interest in strongly polynomial algorithms is primarily theoretical. In particular, with the similarity assumption all polynomial time algorithms are strongly polynomial because $\log C = O(\log n)$ and $\log U = O(\log n)$.

An algorithm is said to be *exponential time* if its running time grows as a function that can not be polynomially bounded. Some examples of exponential time bounds are $O(nC)$, $O(2^n)$, $O(n!)$ and $O(n^{\log n})$. (Observe that nC cannot be bounded by a polynomial function of n and log C.) We say that an algorithm is *pseudopolynomial* if its running time is polynomially bounded in n, m, C and U. The class of pseudopolynomial algorithms is an important subclass of exponential time algorithms. Some instances of pseudopolynomial bounds are $O(m + nC)$ and $O(mC)$. Under the

similarity assumption, pseudopolynomial algorithms become polynomial algorithms, but the algorithms will not be attractive if C and U are high degree polynomials in n.

There are two major reasons for preferring polynomial time algorithms to exponential (i.e., nonpolynomial) time algorithms. First, any polynomial time algorithm is asymptotically superior to any exponential time algorithm. Even in extreme cases this is true. For example, n^{1000} is smaller than $n^{0.1\log n}$ if n is sufficiently large. (In this case n must be larger than $2^{100,000}$.) Figure 1.7 illustrates the asymptotic superiority of polynomial time algorithms. The second reason is more pragmatic. Much practical experience has shown that, as a rule, polynomial time algorithms perform better than exponential time algorithms. Moreover, the polynomials in practice are typically of a small degree.

APPROXIMATE VALUES							
n	log n	$n^{.5}$	n^2	n^3	$n^{\log n}$	2^n	n!
10	3·32	3·16	10^2	10^3	2·10 x 10^3	10^3	3·6 x 10^6
100	6·64	10·00	10^4	10^6	1·94 x 10^{13}	1.27 x 10^{30}	9·33 x 10^{157}
1000	9·97	31·62	10^6	10^9	7·90 x 10^{29}	1.07 x 10^{301}	4.02 x 10^{2567}
10,000	13·29	100·00	10^8	10^2	1·42 x 10^{53}	1.99 x 10^{3010}	2.85 x $10^{35,659}$

Figure 1.7. The growth of polynomial and exponential functions.

In computational complexity theory, the basic objective is to obtain polynomial algorithms, preferably ones with the lowest possible degree. For example, O(log n) is preferable to O(n^k) for any k > 0, and O(n^2) is preferable to O(n^3). However, running times involving more than one parameter, such as O(nm log n) and O(n^3), may not be comparable. If m < n^2/log n then O(nm log n) is superior; otherwise O(n^3) is superior.

Related to the O( ) notation is the Ω( ) notation or "big omega" notation. Just as O( ) notation specifies an upper bound on the computational time of an algorithm within a constant factor, similarly Ω( ) specifies a lower bound on the computational time of an algorithm within a constant factor. We say that an algorithm runs in Ω(f(n, m)) time if there exist examples of the algorithm that can indeed take q f(n, m) time for some constant q. For example, it can be shown that the label correcting algorithm for the

shortest path problem can take qnm time. Hence we write the equivalent statement, "the running time of the label correcting algorithm is $\Omega(nm)$."

1.3   Notation and Definitions

For convenience, in this subsection we collect together several basic definitions and describe some basic notation. We also state without proof some elementary properties of graphs.

We consider a *directed graph* $G = (N, A)$ consisting of a set, N, of nodes, and a set, A, of arcs whose elements are ordered pairs of distinct nodes. A *directed network* is a directed graph with numerical values attached to its nodes and arcs. Let $n = |N|$ and $m = |A|$. We associate with each arc $(i, j) \in A$, a cost c_{ij} and a capacity u_{ij}. We assume throughout that $u_{ij} \geq 0$ for each $(i, j) \in A$. Frequently, we distinguish two special nodes in a graph: the *source* s and *sink* t.

An arc (i, j) has two *end points*, i and j. The node i is called the *tail* and node j is called the *head* of arc (i, j). The arc (i, j) is said to *emanate from* node i. The arc (i, j) is an *outgoing* arc of node i and an *incoming* arc of node j. The *arc adjacency list* of node i, $A(i)$, is defined as the set of arcs emanating from node i, i.e., $A(i) = \{(i, j) \in A : j \in N\}$. The degree of a node is the number of incoming and outgoing arcs at that node.

A (undirected) *path* in $G = (N, A)$ is a sequence $i_1, i_2, i_3, \dots, i_r$ of distinct nodes of N satisfying the property that either $(i_k, i_{k+1}) \in A$ or $(i_{k+1}, i_k) \in A$ for each $k = 1, 2, \dots,$ r-1. A *directed path* is defined similarly, except that $(i_k, i_{k+1}) \in A$ for each $k = 1, 2, \dots,$ r-1. A (undirected) *cycle* is a path together with an arc (i_r, i_1) or (i_1, i_r). A *directed cycle* is a directed path together with the arc (i_r, i_1).

A graph $G = (N, A)$ is called a *bipartite graph* if its node set N can be partitioned into two subsets N_1 and N_2 such that for each arc (i, j) in A, $i \in N_1$ and $j \in N_2$.

A graph $G' = (N', A')$ is a *subgraph* of $G = (N, A)$ if $N' \subseteq N$ and $A' \subseteq A$. A graph $G' = (N', A')$ is a spanning subgraph of $G = (N, A)$ if $N' = N$ and $A' \subseteq A$.

Two nodes i and j are said to be *connected* if the graph contains at least one path from i to j. A graph is said to be *connected* if all pairs of nodes are connected; otherwise it is *disconnected*. Whenever there exists a set $Q \subseteq A$ such that the graph $G' = (N, A-Q)$ is disconnected, and no superset of Q has this property, we refer to Q as a *cutset* of G. A

cutset partitions the graph into two sets of nodes, X and N-X. Hence, the cutset can alternatively be represented as (X, N-X).

A graph is *acyclic* if it contains no cycle. A *tree* is a connected acyclic graph. A *subtree* of a tree T is a connected subgraph of T. A tree T is said to be a *spanning tree* of G if T is a spanning subgraph of G. Arcs belonging to a spanning tree T are called *tree-arcs*, and arcs not belonging to T are called *nontree-arcs*. A spanning tree of G = (N, A) has exactly n-1 tree arcs. A node in a tree with degree equal to one is called a *leaf* node. Each tree has at least two leaf nodes.

A spanning tree contains a unique path between any two nodes. The addition of any nontree-arc to a spanning tree creates exactly one cycle. Removing any arc in this cycle again creates a spanning tree. Removing any tree-arc creates two subtrees. Arcs whose end points belong to two different subtrees of a spanning tree constitute a cutset. If any arc belonging to this cutset is added to the subtrees, the resulting graph is again a spanning tree.

1.4 Network Representations

The complexity of a network algorithm depends not only on the algorithm but also upon the manner used to represent the network within a computer and the storage scheme used for maintaining and updating the intermediate results. The running time of an algorithm (either worst-case or empirical) can often be improved by representing the network more cleverly and by using improved data structures. In this section, we discuss some popular ways of representing a network.

We have already described in Section 1.1 the node-arc incidence matrix representation of a network. This scheme requires nm words to store a network, of which only 2m words have nonzero values. Clearly, this network representation is not space efficient. Another popular way to represent a network is *node-node adjacency matrix representation*. This representation stores an n × n matrix I with the property
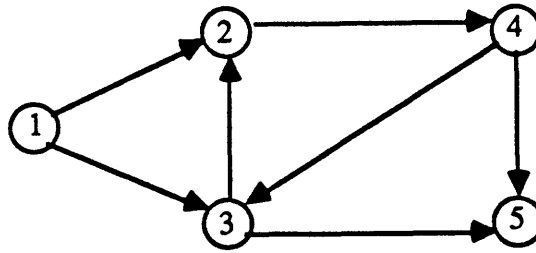
(a) The network example

arc number	point	(tail, head)	cost
1	1	(1, 2)	2
2	3	(1, 3)	3
3	4	(2, 4)	1
4	6	(3, 2)	4
5	8	(3, 5)	2
6	9	(4, 5)	4
7		(4, 3)	1
8		(5, 4)	3

(b) The forward star representation.

cost	(tail, head)	rpoint
2	(3, 2)	1
4	(1, 2)	1
3	(1, 3)	3
1	(4, 3)	5
1	(2, 4)	7
3	(5, 4)	9
4	(4, 5)	
2	(3, 5)	

(c) The reverse star representation.

arc number	point	(tail, head)	cost	trace	rpoint
1	1	(1, 2)	2	4	1
2	3	(1, 3)	3	1	1
3	4	(2, 4)	1	2	3
4	6	(3, 2)	4	7	5
5	8	(3, 5)	2	3	7
6	9	(4, 5)	4	8	9
7		(4, 3)	1	6	
8		(5, 4)	3	5	
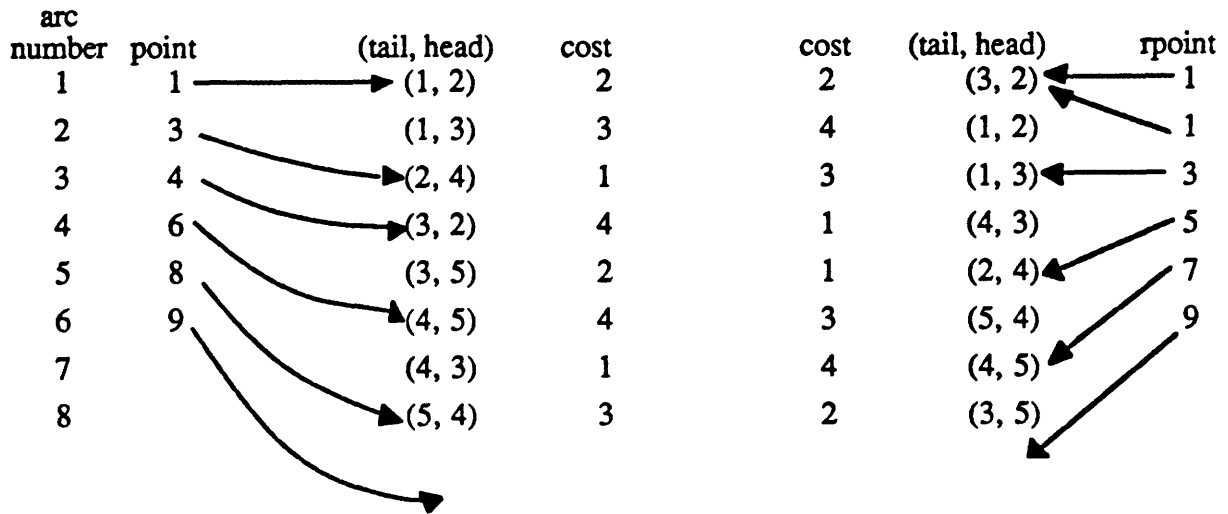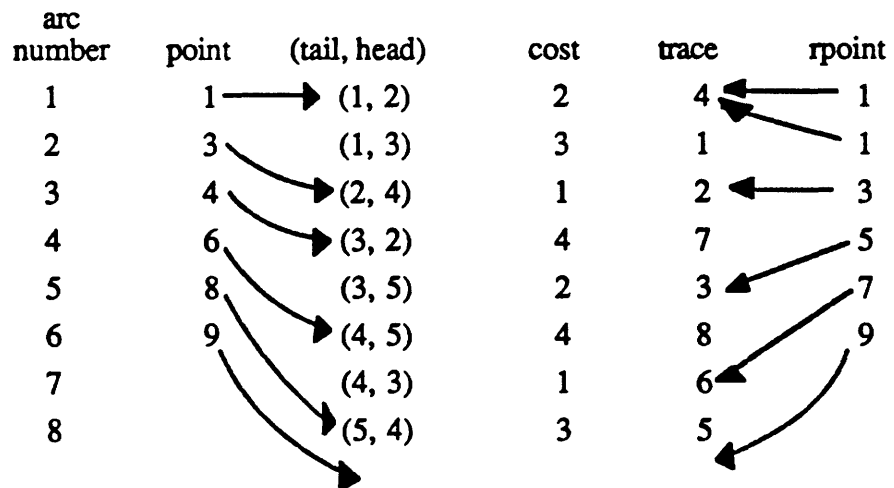
(d) The forward and reverse star representations.

Figure 1.8. Example of Forward and Reverse star Network Representations.

that the element $I_{ij} = 1$ if arc $(i, j) \in A$, and $I_{ij} = 0$ otherwise. The arc costs and capacities are also stored in $n \times n$ matrices. This representation is adequate for very dense networks but is not attractive for storing a sparse network.

The *forward star* and *reverse star representations* are probably the most popular ways to represent networks, both sparse and dense. We first describe the forward star representation. The forward star representation first numbers the arcs in a certain order. We first number the arcs emanating from node 1, then the arcs emanating from node 2, and so on. Arcs incident from the same node can be numbered arbitrarily. We also maintain a pointer with each node i, denoted by *point(i)*, that indicates the smallest number in the arc list of an arc emanating from node i. Hence the outgoing arcs of node i are stored at positions point(i) to (point(i+1) - 1) in the arc list. If point(i) > point(i+1) - 1, then node i has no outgoing arc. For consistency, we have to set point(1) = 1 and point(n+1) = m+1. Figure 1.8(b) specifies the forward star representation of our example given in Figure 1.8(a).

The forward star representation allows us to determine the set of outgoing arcs at any node efficiently. To determine, simultaneously, the set of incoming arcs at any node efficiently we need an additional data structure known as the *reverse star representation*. Starting from a forward star representation, we can create a reverse star representation as follows. We examine the nodes 1 to n in order and sequentially store the (tail, head) and the cost of incoming arcs. We also maintain a reverse pointer with each node i, denoted by *rpoint(i)*, which denotes the first position in these arrays where information about the first incoming arc at node i is stored. This gives us the representation shown in Figure 1.8(c).

For the sake of consistency, we set rpoint(1) = 1 and rpoint(n+1) = m+1. As earlier, the incoming arcs at node i are stored at positions rpoint(i) to (rpoint(i+1) - 1). Now observe that if the above representation is stored along with the forward star representation, then there is significant duplication of information. We can avoid this by storing arc numbers instead of (tail, head) and cost of the arcs. For example, arc (3, 2) has arc number 4 in the forward star representation. The arc (1, 2) has arc number 1. So instead of storing (tail, head) and cost of arcs, we can simply store the arc numbers and once we know the arc numbers, the associated information can always be retrieved from the forward star representation. We store arc numbers in an m-array *trace*. Figure 1.8(d) gives the complete trace array.

1.5   Search Algorithms

Search algorithms are fundamental graph techniques; different variants of search lie at the heart of many network algorithms. In this subsection, we discuss two of the most commonly used search techniques: *breadth-first search* and *depth-first search*.

Search algorithms attempt to find all nodes in a network that satisfy a particular property. For purposes of illustration, let us suppose that we wish to find all the nodes in a graph G = (N, A) that are reachable from a distinguished node s, called the *source*. At every point in the search procedure, all nodes in the network are in one of two states: *marked* or *unmarked*. The marked nodes are known to be reachable from the source, and the status of unmarked nodes is yet to be determined. We call an arc (i, j) *admissible* if node i is marked and node j is unmarked, and *inadmissible* otherwise. Initially, only the source node is marked. Subsequently, by examining admissible arcs the search algorithm will mark more nodes. Whenever the procedure marks a new node j by examining an admissible arc (i, j) we say that node i is a *predecessor* of node j, i.e., *pred*(j) = i. The algorithm terminates when the graph contains no admissible arcs. The following algorithm summarizes the basic iterative steps.

algorithm SEARCH;
begin
       unmark all nodes in N;
       LIST := {s};
       while LIST ≠ ∅ do
       begin
              select a node i in LIST;
              if node i is incident to an admissible arc (i, j) then
              begin
                     mark node j;
                     pred(j) := i;
                     add node j to LIST;
              end
              else delete node i from LIST;
       end;
end;

At termination, the algorithm has marked all nodes in G that are reachable from s via a directed path. The predecessor indices define a tree consisting of marked nodes.

We use the following data structure to identify admissible arcs. The same data structure is also used in the maximum flow and minimum cost flow algorithms discussed in later sections. We maintain with each node i the list A(i) of arcs emanating from it. Arcs in each list can be arranged arbitrarily. Each node has a *current arc* (i, j) which is the current candidate for being examined next. Initially, the current arc of node i is the first arc in A(i). The search algorithm examines this list sequentially and whenever the current arc is inadmissible, it makes the next arc in the arc list the current arc. When the algorithm reaches the end of the arc list, it declares that the node has no admissible arc.

It is easy to show that the search algorithm runs in O(m) time. Each iteration of the **while** loop either finds an admissible arc or does not. In the former case, the algorithm marks a new node and adds it to LIST, and in the latter case it deletes a marked node from LIST. Since the algorithm marks any node at most once, it executes the **while** loop at most 2n times. Now consider the effort spent in identifying the admissible arcs. For each node i, we scan arcs in A(i) at most once. Therefore, the search algorithm examines a total of $\sum_{i \in N} A(i) = m$ arcs, and thus terminates in O(m) time.

The algorithm, as described, does not specify the order for examining and adding nodes to LIST. Different rules give rise to different search techniques. If the set LIST is maintained as a *queue*, i.e., nodes are always selected from the front and added to the rear, then the search algorithm selects the marked nodes in the first-in, first-out order. This kind of search amounts to visiting the nodes in order of increasing distance from s; therefore this version of search is called a *breadth-first search*. It marks nodes in the nondecreasing order of the distance from s, where the distance from s to i is the minimum number of arcs in a path from s to i.

Another popular method is to maintain the set LIST as a *stack*, i.e., nodes are always selected from the front and added to the front, then the search algorithm selects the marked nodes in the last-in, first-out order. This algorithm performs a deep probe, creating a path as long as possible, and backs up one node to initiate a new probe when it can mark no new nodes from the tip of the path. Hence this version of search is called a *depth-first search*.

1.6   Developing Polynomial Time Algorithms

Researchers frequently employ two important approaches to obtain polynomial algorithms for network flow problems: the *geometric improvement* (or linear convergence) approach, and the *scaling* approach. In this subsection, we briefly outline the basic ideas underlying these two approaches. We will assume, as usual, that all data are integral and that algorithms maintain integer solutions at intermediate stages of calculation.

Geometric Improvement Approach

The geometric improvement approach shows that an algorithm runs in polynomial time if at every iteration it makes an improvement proportional to the difference between the objective function values of the current and optimum solutions. Let W be an upper bound on the difference in objective function values between any two feasible solutions. For most network problems, W is a function of n, m, C, and U. For instance, in the maximum flow problem $W = mU$, and in the minimum cost flow problem $W = mCU$.

Lemma 1.1.   Suppose z^k is the objective function value of a minimization problem of some solution at the k-th iteration of an algorithm and z^* is the minimum objective function value.   Further, suppose that the algorithm guarantees that

$$(z^k - z^{k+1}) \geq \alpha (z^k - z^*) \tag{1.3.}$$

(i.e., the improvement at the k+1 iteration is at least α times the total possible improvement) for some α with $0 < \alpha < 1$.   Then the algorithm terminates in $O((\log W)/\alpha)$ iterations.

Proof. We use the well known inequality $(1 - \alpha)^{1/\alpha} < e^{-1}$ in the proof of the lemma. The inequality (1.3) can be rewritten as $(z^{k+1} - z^*) \leq (1 - \alpha) (z^k - z^*)$. An inductive argument shows that $(z^{k+1} - z^*) \leq (1 - \alpha)^k (z^1 - z^*)$. For $k \geq (\log_e W)/\alpha = O((\log W)/\alpha)$, we have $(1 - \alpha)^k (z^1 - z^*) < 1$ because $(1 - \alpha)^{(\log_e W)/\alpha} \leq e^{-\log_e W} = 1/W$. Hence, for $k = (\log_e W)/\alpha$, $z^{k+1} - z^* < 1$ and by the integrality of z^{k+1} and z^*, $z^{k+1} = z^*$. ∎

We have stated this result for minimization versions of optimization problems. A similar result applies to maximization versions of optimization problems.

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The "geometric improvement approach" might be summarized by the statement "network algorithms that have a geometric convergence rate are polynomial time algorithms." In order to develop polynomial time algorithms using this approach, we can look for local improvement techniques that lead to large (i.e., fixed percentage) improvements in the objective function. The maximum augmenting path algorithm for the maximum flow problem and maximum improvement algorithm for the minimum cost flow problem are two examples of this approach. (See Sections 4.2 and 5.3.)

Scaling Approach

Researchers have extensively used an approach called *scaling* to derive polynomial algorithms for a wide variety of network and combinatorial optimization problems. In this discussion, we describe the simplest form of scaling which we call *bit-scaling*. Section 5.11 presents an example of bit-scaling algorithm for the assignment problem. Sections 4 and 5, using more refined versions of scaling, describe polynomial time algorithms for the maximum flow and minimum cost flow problems.

Using the bit-scaling technique, we solve a problem P parametrically as a sequence of problems $P_1, P_2, P_3, ... , P_K$: the problem P_1 approximates data to the first *bit*, the problem P_2 approximates data to the second bit, and each successive problem is a better approximation until $P_K = P$. Further, the optimum solution of problem P_{K-1} serves as the starting solution for problem P_K. The scaling technique is useful whenever reoptimization from a good starting solution is more efficient than solving the problem from scratch.

For example, consider a network flow problem whose largest arc capacity has value U. Let $K = \lceil \log U \rceil$ and suppose that we represent each arc capacity as a K bit binary number, adding leading zeros if necessary to make each capacity K bits long. Then the problem P_k would consider the capacity of each arc as the k leading bits in its binary representation. Figure 1.9 illustrates an example of this type of scaling.

The manner of defining arc capacities easily implies the following observation.

Observation. The capacity of an arc in P_k is twice of that in P_{k-1} plus 0 or 1.
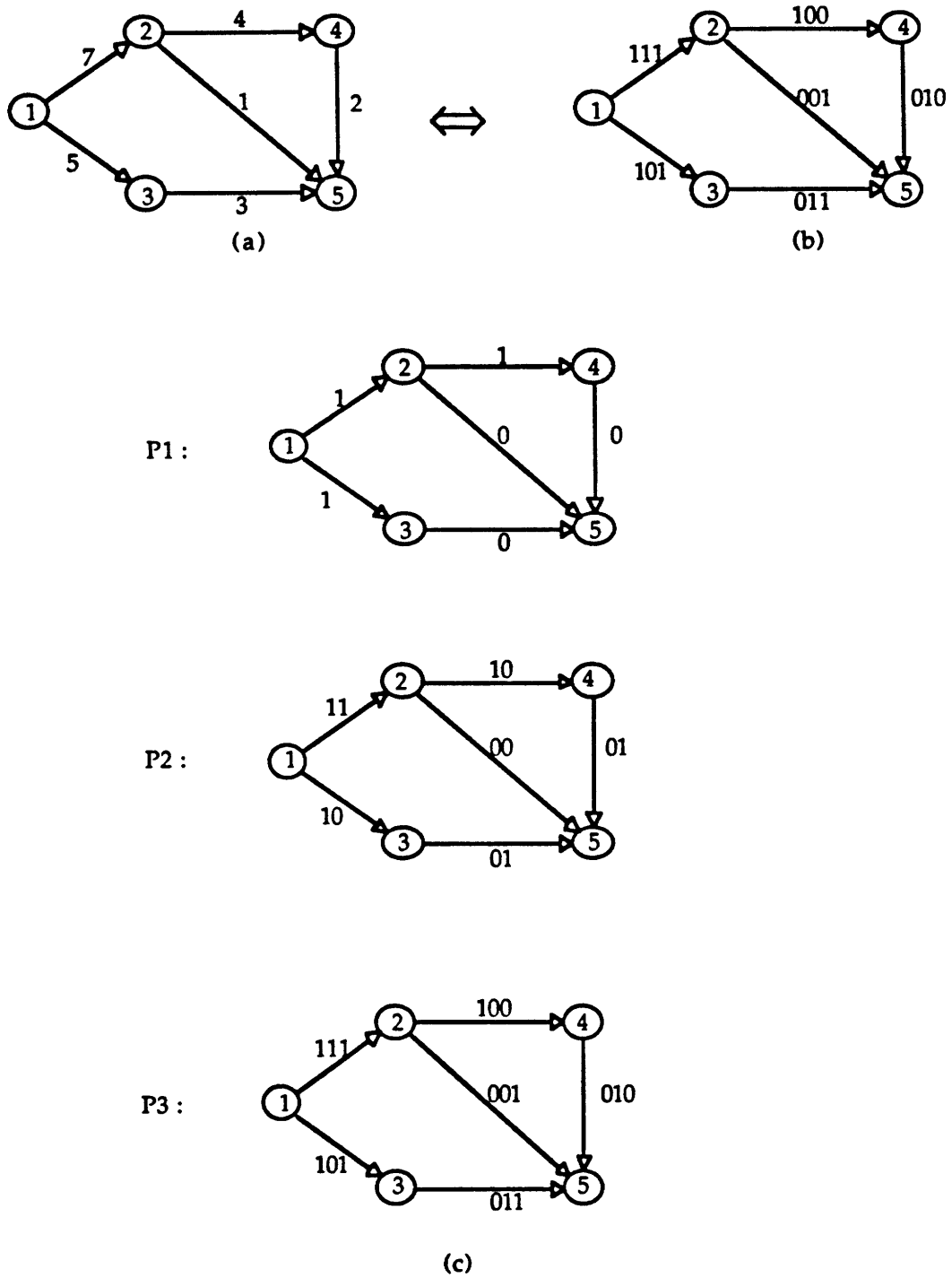
Figure 1.9. Example of a bit-scaling technique.

(a) Network with arc capacities.

(b) Network with binary expansion of arc capacities.

(c) The problems P_1, P_2, and P_3.

The following algorithm encodes a generic version of the bit-scaling procedure.

algorithm BIT-SCALING;
begin
      obtain an optimum solution of P_1;
    for $k := 2$ to K **do**
    begin
          reoptimize using the optimum solution of P_{k-1} to
             obtain an optimum solution of P_k;
    end;
end;

This approach is very robust; variants of it have led to improved algorithms for both the maximum flow and minimum cost flow problems. This approach works well for these applications, in part, because of the following reasons. (i) The optimal solution of problem P_{k-1} is an excellent starting solution for problem P_k as P_{k-1} and P_k are quite similar. Hence the optimum solution of P_{k-1} can be easily reoptimized to obtain an optimum solution of P_k. (ii) Under the similarity assumption, the number of problems solved is $O(\log n)$. Thus for this approach to work, reoptimization needs to be only a little more efficient (i.e., by a factor of $\log n$) than optimization.

Consider, for example, the maximum flow problem. Let v_k denote the maximum flow value for problem P_k and x_k an arc flow corresponding to v_k. In the problem P_k, the capacity of an arc is twice its capacity in P_{k-1} plus 0 or 1. If we multiply the optimum flow x_{k-1} for P_{k-1} by 2, we obtain a feasible flow for P_k. Moreover, $v_k - 2v_{k-1} \le m$ because multiplying the flow x_{k-1} by 2 takes care of the doubling of the capacities and the additional 1's can increase the maximum flow value by at most m units (if we add 1 to the capacity of any arc, then we increase the maximum flow from source to sink by at most 1). In general, it is easier to reoptimize such a maximum flow problem. For example, the classical labeling algorithm as discussed in Section 4.1 would perform the reoptimization in at most m augmentations, taking $O(m^2)$ time. Hence the scaling version of the labeling algorithm runs in $O(m^2 \log U)$ time, whereas the non-scaling version runs in $O(nmU)$ time. The former time bound is polynomial and the latter bound is only pseudopolynomial. Thus this simple scaling algorithm improves the running time dramatically.

2. BASIC PROPERTIES OF NETWORK FLOWS

As a prelude to the rest of this chapter, in this section we describe several basic properties of network flows. We begin by showing how network flow problems can be modeled in either of two equivalent ways: as flows on arcs as in our formulation in Section 1.2 or as flows on paths and cycles. Then we partially characterize optimal solutions to (concave cost) network flow problems and demonstrate that these problems always have certain special types of optimal solutions (so-called cycle free and spanning tree solutions). Consequently, in designing algorithms, we need only consider these special types of solutions. We next establish several important connections between network flows and linear and integer programming. Finally, we discuss a few useful transformations of network flow problems.

2.1 Flow Decomposition Properties and Optimality Conditions

It is natural to view network flow problems in either of two ways: as flows on arcs or as flows on paths and cycles. In the context of developing underlying theory, models, or algorithms, each view has its own advantages. Therefore, as the first step in our discussion we will find it worthwhile to develop several connections between these alternate formulations.

In the arc formulation (1.1), the basic decision variables are flows x_{ij} on the network's arcs (i, j). The path and cycle formulation starts with an enumeration of the paths P and cycles Q of the network. Its decision variables are h(p), the flow on path p, and f(q), the flow on cycle q, which are defined for every directed path p in P and every directed cycle q in Q.

Notice that every set of path and cycle flows uniquely determines arc flows in a natural way: the flow x_{ij} on arc (i, j) equals the sum of the flows h(p) and f(q) for all paths p and cycles q that contain this arc. We formalize this observation by defining some new notation: $\delta_{ij}(p)$ equals 1 if arc (i, j) is contained in path p and 0 otherwise; similarly, $\delta_{ij}(q)$ equals 1 if arc (i, j) is contained in cycle q and is 0 otherwise. Then

$$x_{ij} = \sum_{p \in P} \delta_{ij}(p) \, h(p) + \sum_{q \in Q} \delta_{ij}(q) \, f(q).$$

If the flow vector x is expressed in this way, we say that the flow is represented as path flows and cycle flows and that the path flow vector h and cycle flow vector f is a path and cycle flow representation of the flow.

Can we reverse this process? That is, can we decompose any arc flow into (i.e., represent it as) path and cycle flows? The following result provides an affirmative answer to this question.

Theorem 2.1: *Flow Decomposition (Directed Case). Every directed path and cycle flow has a unique representation as nonnegative arc flows. Conversely, every nonnegative arc flow x can be represented as a directed path and cycle flow (though not necessarily uniquely) with the following two properties:*

C2.1. *Every path with positive flow connects a supply node of x to a demand node of x; and*

C2.2. *At most $n+m$ paths and cycles have nonzero flow; out of these, at most m cycles have nonzero flow.*

Proof. In the light of our previous observations, we need to establish only the converse assertions. We give an algorithmic proof to show that any feasible arc flow x can be decomposed into path and cycle flows. Suppose i_0 is a supply node. Then some arc (i_0, i_1) carries a positive flow. If i is a demand node then we stop; otherwise the mass balance constraint (1.1b) of node i implies that some other arc (i_1, i_2) carries positive flow. We repeat this argument until either we encounter a demand node or we revisit a previously examined node. Note that one of these cases will occur within n steps. In the former case we obtain a path p from the supply node i_0 to some demand node i_k consisting solely of arcs with positive flow, and in the latter case we obtain a cycle q. If we obtain a path, we let $h(p) = \min [b(i_0), -b(i_k), \min \{x_{ij} : (i, j) \in p\}]$, and redefine $b(i_0) = b(i_0) - h(p)$, $b(i_k) = b(i_k) + h(p)$ and $x_{ij} = x_{ij} - h(p)$ for each arc (i, j) in p. If we get a cycle q, we let $f(q) = \min \{x_{ij} : (i, j) \in q\}$ and redefine $x_{ij} = x_{ij} - f(q)$ for each arc (i, j) in q.

We repeat this process with the redefined problem until the network contains no supply nodes (and hence no demand nodes). Then we select a transshipment node with at least one outgoing arc with positive flow as the starting node, and repeat the procedure, which in this case must find a cycle. We terminate when for the redefined problem x = 0. Clearly, the original flow is the sum of flows on the paths and cycles identified by the procedure. Now observe that each time we identify of a path we reduce the supply/demand of some node or the flow on some arc to zero; and each time we identify a cycle, we reduce the flow on some arc to zero. Consequently, the path and cycle

representation of the given flow x contains at most (n + m) total paths and cycles, of which there are at most m cycles. ■

It is possible to state the decomposition property in a somewhat more general form that permits arc flows x_{ij} to be negative. In this case, even though the underlying network is directed, the paths and cycles can be undirected, and can contain arcs with negative flows. Each undirected path p, which has an orientation from its initial to its final node, has *forward* arcs and *backward* arcs which are defined as arcs along and opposite to the path's orientation. A *path flow* will be defined on p as a flow with value h(p) on each forward arc and -h(p) on each backward arc. We define a *cycle flow* in the same way. In this more general setting, our representation using the notation $\delta_{ij}(p)$ and $\delta_{ij}(q)$ is still valid with the following provision: we now define $\delta_{ij}(p)$ and $\delta_{ij}(q)$ to be -1 if arc (i, j) is a backward arc of the path or cycle.

Theorem 2.2: **Flow Decomposition (Undirected Case).** *Every path and cycle flow has a unique representation as arc flows. Conversely, every arc flow x can be represented as an (undirected) path and cycle flow (though not necessarily uniquely) with the following three properties:*

C2.3. *Every path with positive flow connects a source node of x to a sink node of x;*

C2.4. *for every path and cycle, any arc with positive flow occurs as a forward arc and any arc with negative flow occurs as a backward arc;*

C2.5. *at most n+m paths and cycles have nonzero flow; out of these, at most m cycles have nonzero flow.*

Proof. This proof is similar to that of Theorem 2.1. The major modification is that we extend the path at some node i_{k-1} by adding an arc (i_{k-1}, i_k) with positive flow or an arc (i_k, i_{k-1}) with negative flow. The other steps can be modified accordingly. ■

The flow decomposition property has a number of important consequences. As one example, it enables us to compare any two solutions of a network flow problem in a particularly convenient way and to show how we can *build* one solution from another by a sequence of simple operations.

We need the concept of augmenting cycles with respect to a flow x. A cycle q with flow f(q) > 0 is called an *augmenting cycle* with respect to a flow x if

$$0 \leq x_{ij} + \delta_{ij}(q)\, f(q) \leq u_{ij}, \text{ for each arc (i, j)} \in q.$$

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In other words, the flow remains feasible if some positive amount of flow (namely $f(q)$) is *augmented* around the cycle q. We define the cost of an augmenting

cycle q as $c(q) = \sum_{(i,j) \in A} c_{ij} \delta_{ij}(q)$. The cost of an augmenting cycle represents the change

in cost if a unit flow is augmented along the cycle. The cost of flow of an augmenting cycle q with flow $f(q)$ is $c(q) f(q)$.

Suppose that x and y are any two solutions to a network flow problem, i.e., $Nx = b$, $0 \le x \le u$ and $Ny = b$, $0 \le y \le u$. Then the difference vector $z = y - x$ satisfies the homogeneous equations $Nz = Ny - Nx = 0$. Consequently, flow decomposition implies that z can be represented as cycle flows, i.e., we can find at most $r \le m$ cycle flows $f(q_1)$, $f(q_2), \dots, f(q_r)$ satisfying the property that for each arc (i, j) of A,

$$z_{ij} = \delta_{ij}(q_1) f(q_1) + \delta_{ij}(q_2) f(q_2) + \dots + \delta_{ij}(q_r) f(q_r).$$

Since $y = x + z$, for any arc (i, j) we have

$$0 \le y_{ij} = x_{ij} + \delta_{ij}(q_1) f(q_1) + \delta_{ij}(q_2) f(q_2) + \dots + \delta_{ij}(q_r) f(q_r) \le u_{ij}.$$

Now by condition C2.4 of the flow decomposition property, arc (i, j) is either a forward arc or a backward arc on each cycle $q_1, q_2, \dots, q_m$ that contains it. Therefore, each term between x_{ij} and the rightmost inequality in this expression has the same sign; moreover, $0 \le y_{ij} \le u_{ij}$. Consequently, for each cycle q_k, $0 \le x_{ij} + \delta_{ij}(q_k) f(q_k) \le u_{ij}$ for each arc $(i, j) \in q_k$. That is, if we add any of these cycle flows q_k to x, then the resulting solution remains feasible on each arc (i, j). Hence each cycle $q_1, q_2, \dots, q_r$ is an augmenting cycle with respect to the flow x. Further, note that

$$\sum_{(i,j) \in A} c_{ij} y_{ij} = \sum_{(i,j) \in A} c_{ij} x_{ij} + \sum_{(i,j) \in A} c_{ij} z_{ij}$$

$$= \sum_{(i,j) \in A} c_{ij} x_{ij} + \sum_{(i,j) \in A} c_{ij} \left(\sum_{k=1}^{r} \delta_{ij}(q_k) f(q_k) \right)$$

$$= \sum_{(i,j) \in A} c_{ij} x_{ij} + \sum_{k=1}^{r} c(q_k) f(q_k).$$

We have thus established the following important result.

Theorem 2.3: Augmenting Cycle Property. Let x and y be any two feasible solutions of a network flow problem. Then y equals x plus the flow on at most m augmenting cycles with respect to x. Further, the cost of y equals the cost of x plus the cost of flow on the augmenting cycles. ∎

The augmenting cycle property permits us to formulate optimality conditions characterizing the optimum solution of the minimum cost flow problem. Suppose that x is any feasible solution and x^* is an optimum solution of the minimum cost flow problem and that $x \neq x^*$. The augmenting cycle property implies that the difference vector $x^* - x$ can be decomposed into at most m augmenting cycles and the sum of the costs of these cycles equals $cx^* - cx$. If $cx^* < cx$ then one of these cycles must have a negative cost. Further, if every augmenting cycle in the decomposition of $x^* - x$ has a nonnegative cost, then $cx^* - cx \geq 0$. Since x^* is an optimum flow, $cx^* = cx$ and x is also an optimum flow. We have thus obtained the following result.

Theorem 2.4. Optimality Conditions. A feasible flow x is an optimum flow if and only if it admits no negative cost augmenting cycle. ∎

2.2   Cycle Free and Spanning Tree Solutions

We start by assuming that x is a feasible solution to the network flow problem

$$\text{minimize } \{\, cx : Nx = b \text{ and } l \leq x \leq u \,\}$$

and that $l = 0$. Much of the underlying theory of network flows stems from a simple observation concerning the example in Figure 2.1. In the example, arc flows and costs are given besides each arc.
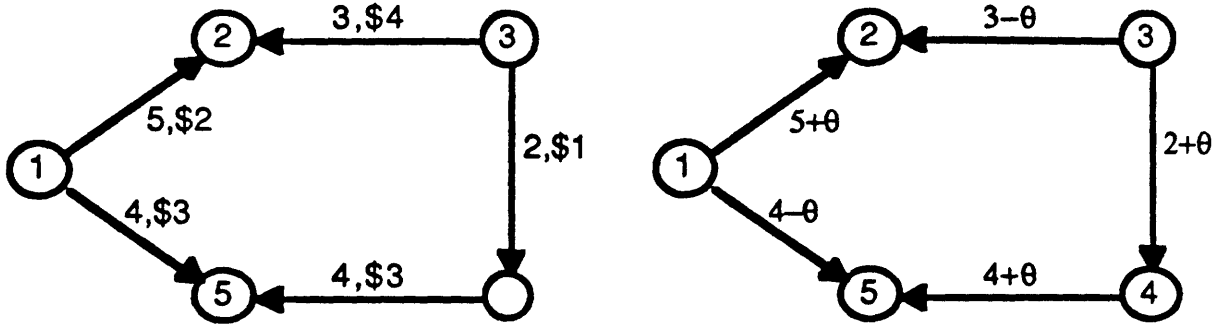
Figure 2.1. Improving Flow Around a Cycle.

Let us assume for the time being that all arcs are uncapacitated. The network in this figure contains flow around an undirected cycle. Note that adding a given amount of flow θ to all the arcs pointing in a clockwise direction and subtracting this flow from all arcs pointing in the counterclockwise direction preserves the mass balance at each node. Also, note that the per unit incremental cost for this flow change is the sum of the cost of the clockwise arcs minus the sum of the cost of counterclockwise arcs, i.e.,

$$\text{Per unit change in cost} = \Delta = \$2 + \$1 + \$3 - \$4 - \$3 = \$\,\text{-}1.$$

Let us refer to this incremental cost Δ as the *cycle cost* and say that the cycle is a *negative, positive* or *zero cost* cycle depending upon the sign of Δ. Consequently, to minimize cost in our example, we set θ as large as possible while preserving nonnegativity of all arc flows, i.e., $3 - \theta \geq 0$ and $4 - \theta \geq 0$, or $\theta \leq 3$; that is, we set $\theta = 3$. Note that in the new solution (at $\theta = 3$), we no longer have positive flow on all arcs in the cycle.

Similarly, if the cycle cost were positive (i.e., we were to change c_{12} from 2 to 4), then we would decrease θ as much as possible (i.e., $5 + \theta \geq 0$, $2 + \theta \geq 0$, and $4 + \theta \geq 0$, or $\theta \geq -2$) and again find a lower cost solution without positive flow on all arcs in the cycle. We can restate this observation in another way: to preserve nonnegativity of all flows, we must select θ in the interval $-2 \leq \theta \leq 3$. Since the objective function depends linearly on θ, we optimize it by selecting $\theta = 3$ or $\theta = -2$ at which point one arc in the cycle has a flow value of zero.

We can extend this observation in several ways:

(i) If the per unit cycle cost $\Delta = 0$, we are indifferent to all solutions in the interval $-2 \leq \theta \leq 3$ and therefore can again choose a solution as good as the original one but without positive flow on every arc in the cycle;

(ii) If we impose upper bounds on the flow, e.g., such as 6 units on all arcs, then the range of flows that preserves feasibility (i.e., mass balances, lower and upper bounds on flows) is again an interval, in this case $-2 \leq \theta \leq 1$, and we can find a solution as good as the original one by choosing $\theta = -2$ or $\theta = 1$. At these values of θ, the solution is cycle free, that is, for some arc on the cycle, either the flow is zero (the lower bound) or is at its upper bound ($x_{12} = 6$ at $\theta = 1$).

(iii) The previous observation is also valid if the cost $c(x)$ of any solution x is concave rather than linear, i.e., for any two feasible flow vectors x and y

$$c(\alpha x + (1 - \alpha) y) \leq \alpha c(x) + (1 - \alpha) c(y), \text{ for all } 0 \leq \alpha \leq 1.$$

Some additional notation will be helpful in encapsulating and summarizing our observations up to this point. Let us say that an arc (i, j) is a *free arc* with respect to a given feasible flow x if x_{ij} lies strictly between the lower and upper bounds imposed upon it. We will also say that arc (i, j) is *restricted* if its flow x_{ij} equals either its lower or upper bound. In this terminology, a solution x has the "cycle free property" if the network contains no cycle made up entirely of free arcs.

In general, our prior observations apply to *any* cycle in a network. Therefore, given any initial flow we can apply our previous argument sequentially, one cycle at a time, and establish the following fundamental result:

Theorem 2.5: Cycle Free Property. If the objective value of the network optimization problem

$$minimize \ \{ \ c(x) : Nx = b, \ l \leq x \leq u \ \}$$

is bounded from below on the feasible region and the objective function is concave, then the problem always has a cycle free solution. ∎

Note that the lower bound assumption imposed upon the objective value is necessary to rule out situations in which the flow change variable θ in our prior argument can be made arbitrarily large in a negative cost cycle, or arbitrarily small

(negative) in a positive cost cycle; for example, this condition rules out any negative cost directed cycle with no upper bounds on its arc flows.

It is useful to interpret the cycle free property in another way. Suppose that the network is connected (i.e., there is an undirected path connecting every two pairs of nodes). Then, either a given cycle free solution x contains a free arc that is incident to each node in the network, or we can add to the free arcs some restricted arcs so that the resulting set S of arcs has the following three properties:

(i)   S contains all the free arcs in the current solution,

(ii) S contains no undirected cycles, and

(iii) No superset of S satisfies properties (i) and (ii).

We will refer to any set S of arcs satisfying (i) through (iii) as a *spanning tree* of the network and any feasible solution x for the network together with a spanning tree S that contains all free arcs as a *spanning tree solution*. (At times we will also refer to a given cycle free solution x as a spanning tree solution, with the understanding that restricted arcs may be needed to form the spanning tree S.)

Figure 2.2. illustrates a spanning tree corresponding to a cycle free solution. Note that it may be possible (and often is) to complete the set of free arcs into a spanning tree in several ways (e.g., replace arc (2, 4) with arc (3, 5) in Figure 2.2(c)); therefore, a given cycle free solution can correspond to several spanning trees S. We will say that a spanning tree solution x is *nondegenerate* if the set of free arcs forms a spanning tree. In this case, the spanning tree S corresponding to x is unique. If the free arcs do not span (i.e., are not incident to) all the nodes, then any spanning tree corresponding to this solution will contain at least one arc whose flow equals the arc's lower or upper bound of the arc. In this case, we will say that the spanning tree is *degenerate*.
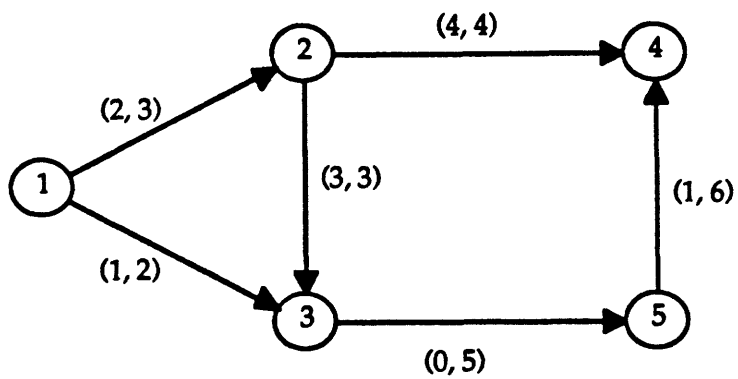
(a) The example network with arc flows and capacities represented as (x_{ij}, u_{ij}).
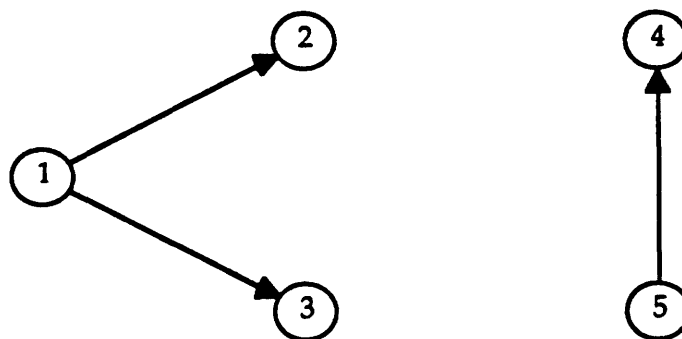
(b) The cycle free solution.
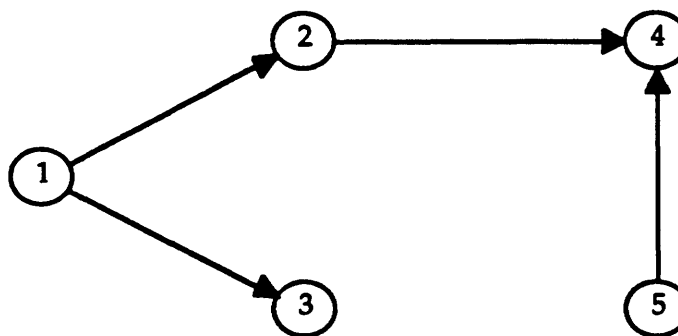
(c) A spanning tree solution.

Figure 2.2. Converting a cycle free solution to a spanning tree solution.

When restated in the terminology of spanning trees, the cycle free property becomes another fundamental result in network flow theory.

Theorem 2.6:   Spanning Tree Property.   If the objective value of the network optimization problem

$$minimize \ \{f(x): Nx = b, l \leq x \leq u\}$$

is bounded from below on the feasible region and the objective function is concave, then the problem always has a spanning tree solution. ∎

2.3   Networks, Linear and Integer Programming

The cycle free property and spanning tree property have many other important consequences. In particular, these two properties imply that the network flow theory lies at the cusp between two large and important subfields of optimization—linear and integer programming. This positioning may, to a large extent, account for the emergence of network flow theory as a cornerstone of mathematical programming.

Triangularity Property

Before establishing our first results relating network flows to linear and integer programming, we first make a few observations. Note that any spanning tree S has at least one (actually at least two) leaf nodes, that is, a node that is incident to only one arc in the spanning tree. Consequently, if we rearrange the rows and columns of the node-arc incidence matrix of S so that the leaf node is row 1 and its incident arc is column 1, then row 1 has only a single nonzero entry, a +1 or a -1, which lies on the diagonal of the node-arc incidence matrix. If we now remove this leaf node and its incident arc from S, the resulting network is a spanning tree on the remaining nodes. Consequently, by rearranging all but row and column 1 of the node-arc incidence matrix for the spanning tree, we can now assume that row 2 has +1 or -1 element on the diagonal and zeros to the right of the diagonal. Continuing in this way permits us to rearrange the node-arc incidence matrix of the spanning tree so that its first n-1 rows is lower triangular. Figure 2.3 shows the resulting lower triangular form (actually, one of several possibilities) for the spanning tree in Figure 2.2(c).

$$L = \begin{array}{c c} & \text{arcs} \\ \begin{array}{c} \text{nodes} \\ 5 \\ 4 \\ 3 \\ 2 \\ 1 \end{array} & \begin{array}{cccc} (5,4) & (2,4) & (1,3) & (1,2) \\ \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right] \end{array} \end{array}$$

Figure 2.3 The lower triangular form corresponding to a basis.

The node-arc incidence matrix of any spanning tree contains one more row than it has columns (it has n rows and n-1 columns). Therefore, after we have rearranged the matrix so that the first n-1 rows are in triangular form, the node-arc incidence matrix contains one additional row. We will, however, adopt the convention of still referring to it as lower triangular. Notice, that since each column of any node-arc incidence matrix contains exactly one +1 and one -1, the rows always sum to zero--equivalently, the last row equals -1 times the sum of the other rows and, therefore, is redundant.

Theorem 2.7: Triangularity Property. The rows and columns of the node-arc incident matrix of any spanning tree can be rearranged to be lower triangular. ∎

Integrality of Optimal Solutions

The triangularity property has several important consequences. First, let us evaluate the flows on arcs in a spanning tree solution x. By rearranging rows and columns, we partition the node-arc incident matrix N of the network as N = [L,M], where L is a lower triangular matrix corresponding to a spanning tree. Suppose that x = (x^1, x^2) is partitioned compatibly. Then

$$Nx = Lx^1 + Mx^2 = b$$
$$\text{or } Lx^1 = b - Mx^2. \tag{2.1}$$

Now further suppose that the supply/demand vector b and lower and upper bound vectors l and u have all integer components. Then since every component of x^2 equals an arc lower or upper bound and M has integer components (each equal to 0, +1, or -1), the right hand side $b - Mx^2$ is an integer vector. But this observation implies that the components of x^1 are integral as well: since the first diagonal element of U equals +1 or -1, the first equation in (2.1) implies that x_1^1 is integral; now if we move x_1^1 to the right of

the equality in (2.1), the right hand side remains integral and we can solve for x_2^1 from the second equation; continuing this forward substitution by successively solving for one variable at a time shows that x^1 is integral.

This argument shows that for problems with integral data, every spanning tree solution is integral. Since the spanning tree property ensures that network flow problems always have spanning tree solutions, we have established the following fundamental result.

Theorem 2.8. **Integrality Property.** *If the objective value of the network optimization problem*

$$\text{minimize } \{ c(x): Nx = b, l \leq x \leq u \}$$

is bounded from below on the feasible region, the objective function is concave, and the vectors b, l, and u are integer, then the problem has an integer solution. ∎

Relationship to Linear Programming

In the special case in which $c(x) = cx$, the network flow problem is a linear program which, as the last result shows, always has an integer solution. Network flow problems are distinguished as the most important large class of problems with this property.

Linear programs, or generalizations with concave cost objective functions, also satisfy another well-known property: they always have, in the parlance of convex analysis, *extreme point solutions*; that is, solutions x with the property that x cannot be expressed as a weighted combination of two other feasible solutions y and z, i.e., as $x = \alpha y + (1-\alpha)z$ for some weight $0 < \alpha < 1$. Since, as we have seen, network flow problems always have cycle free solutions, we might expect to discover extreme point solutions and cycle free solutions to be closely related, and indeed they are as shown by the next result.

Theorem 2.9. Extreme Point Property. For network flow problems, every cycle free solution is an extreme point and, conversely, every extreme point is a cycle free solution. Consequently, if the objective value of the network optimization problem

$$\text{minimize } \{ \ c(x) \colon Nx = b, \ l \leq x \leq u \ \}$$

is bounded from below on the feasible region and the objective function is concave, then the problem has an extreme point solution.

Proof. With the background developed already, this result is easy to establish. First, if x is not a cycle free solution, then it cannot be an extreme point, since by perturbing the flow by a small amount θ and by a small amount $-\theta$ around a cycle with free arcs, as in our discussion of Figure 2.1, we define two feasible solutions y and z with the property that $x = (1/2)y + (1/2)z$. Conversely, suppose that x is not an extreme point and is represented as $x = \alpha y + (1-\alpha)z$ with $0 < \alpha < 1$. Let x^1, y^1 and z^1 be the components of these vectors for which y and z differ, i.e., $l_{ij} \leq y_{ij} < x_{ij} < z_{ij} \leq u_{ij}$ or $l_{ij} \leq z_{ij} < x_{ij} < y_{ij} \leq u_{ij}$, and let N_1 denote the submatrix of N corresponding to these arcs (i, j). Then $N_1(z^1 - y^1) = 0$, which implies, by flow decomposition, that the network contains an undirected cycle with y_{ij} not equal to z_{ij} for any arc on the cycle. But by definition of the components x^1, y^1, and z^1, this cycle contains only free arcs in the solution x. Therefore, if x is not an extreme point solution, then it is not a cycle free solution. ■

In linear programming, extreme points are usually represented algebraically as *basic solutions*; for these special solutions, the columns B of the constraint matrix of a linear program corresponding to variables strictly between their lower and upper bounds are linearly independent. One can extend B to a basis of the constraint matrix by adding a maximal number of columns. Just as cycle free solutions for network flow problems correspond to extreme points, spanning tree solutions correspond to basic solutions.

Theorem 2.10: Basis Property. Every spanning tree solution to a network flow problem is a basic solution and, conversely, every basic solution is a spanning tree solution. ■

Let us now make one final connection between networks and linear and integer programming—namely, between basis and the integrality property. Consider a linear program of the form $Ax = b$ and suppose that $N = [B,M]$ for some basis B and that $x = (x^1,x^2)$ is a compatible partitioning of x. Also suppose that we eliminate the redundant row so that B is a nonsingular matrix. Then

$$Bx^1 = b - Mx^2, \text{ or } x^1 = B^{-1}(b - Mx^2).$$

Also, by Cramer's rule from linear algebra, it is possible to find each component of x^1 as sums and multiples of components of $b' = b - Mx^2$ and B, divided by det(B), the determinant of B. Therefore, if the determinant of B equals +1 or -1, then x^1 is an integer vector whenever x^2, b, and M are composed of all integers. In particular, if the partitioning of A corresponds to a basic feasible solution x and the problem data A, b, l and u are all integers, then x^2 and consequently x^1 is an integer. Let us call a matrix A *unimodular* if all of its bases have determinants either +1 or -1, and call it *totally unimodular* if all of its square submatrices have determinant equal to either 0,+1, or -1.

How are these notions related to network flows and the integrality property? Since bases of N correspond to spanning trees, the triangularity property shows that the determinant of any basis (excluding the redundant row now), equals the product of the diagonal elements in the triangular representation of the basis, and therefore equals +1 or -1. Consequently, a node-arc incident matrix is unimodular. Even more, it is totally unimodular. For let S be any square submatrix of N. If S is singular, it has determinant 0. Otherwise, it must correspond to a cycle free solution, which is a spanning tree on each of its connected components. But then, it is easy to see that the determinant of S is the product of the determinants of the spanning trees and, therefore, it must be equal to +1 or -1. (An induction argument, using an expansion of determinants by minors, provides an alternate proof of this totally unimodular property.)

Theorem 2.11: Total Unimodularity Property. The constraint matrix of a minimum cost network flow problem is totally unimodular. ∎

2.4 Network Transformations

Frequently, analysts use network transformations to simplify a network problem, to show equivalences of different network problems, or to put a network problem into a standard form required by a computer code. In this subsection, we describe some of these important transformations.

T1. (Removing Nonzero Lower Bounds). If an arc (i, j) has a positive lower bound l_{ij}, then we can replace x_{ij} by $x'_{ij} - l_{ij}$ in the problem formulation. As measured by the new variable x'_{ij}, the flow on arc (i, j) will have a lower bound of 0. This transformation has a

simple network interpretation: we begin by sending l_{ij} units of flow on the arc and then measure incremental flow above l_{ij}.
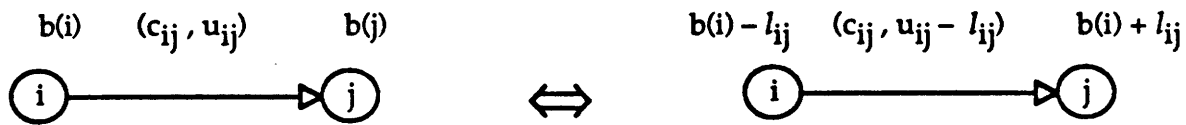
Figure 2.4. Transforming a positive lower bound to zero.

T2. (Removing Capacities). If an arc (i, j) has a positive capacity u_{ij}, then we can remove the capacity, making the arc uncapacitated, using the following ideas. The capacity constraint (i, j) can be written as $x_{ij} + s_{ij} = u_{ij}$ where the slack variable $s_{ij} \geq 0$. Multiplying both sides by -1 we get

$$-x_{ij} - s_{ij} = -u_{ij}. \tag{2.2}$$

This transformation is tantamount to turning the slack variable into an additional node where equation (2.2) is the mass balance constraint for that node. Observe that the variable x_{ij} now appears in three mass balance constraints and s_{ij} in only one. By subtracting (2.2) from the mass balance constraint of node j, we assure that each of x_{ij} and s_{ij} appear in exactly two constraints--in one with the positive sign and in the other with the negative sign. This essentially gives us the following transformation.

Figure 2.5. Removing arc capacities.

In the network context this transformation implies the following. If x_{ij} is a flow on arc (i, j) in the original network, the corresponding flow in the transformed network is $x'_{ik} = x_{ij}$ and $x'_{jk} = u_{ij} - x_{ij}$; both the flows x and x' have the same cost. Likewise, a flow x'_{ik}, x'_{jk} in the transformed network yields a flow of $x_{ij} = x'_{ik}$ of the same cost in the

original network. Further, since $x'_{ik} + x'_{jk} = u_{ij}$, and x'_{ik} and x'_{jk} are both nonnegative, x_{ij}

$= x'_{ik} \le u_{ij}$. Hence this transformation is valid.

T3. (Arc Reversal). Let u_{ij} represent the capacity of the arc (i, j) or an upper bound on the arc flow if it is uncapacitated. This transformation consists of a change in variable: replace x_{ij} by $u_{ij} - x'_{ji}$ in the problem formulation. This replaces arc (i, j) with its cost c_{ij} by the arc (j, i) with cost $-c_{ij}$. Therefore, this transformation may be used to remove arcs with negative costs. This transformation has the following network interpretation: send u_{ij} units of flow on the arc and then replace arc (i, j) by arc (j, i) with cost $-c_{ij}$. The new flow x'_{ji} measures the amount of flow we "remove" from the "full capacity" flow of u_{ij}.
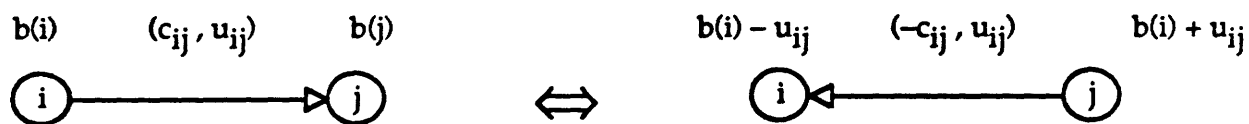
Figure 2.6. An example of arc reversal.

T4. (Node Splitting). This transformation splits each node i into two nodes i and i' and replaces each original arc (k, i) by the arc (k, i') of the same cost and capacity, and each arc (i, j) by an arc (i', j) of the same cost and capacity. We also add arcs (i, i') of cost zero for each i. Figure 2.7 illustrates the transformation.
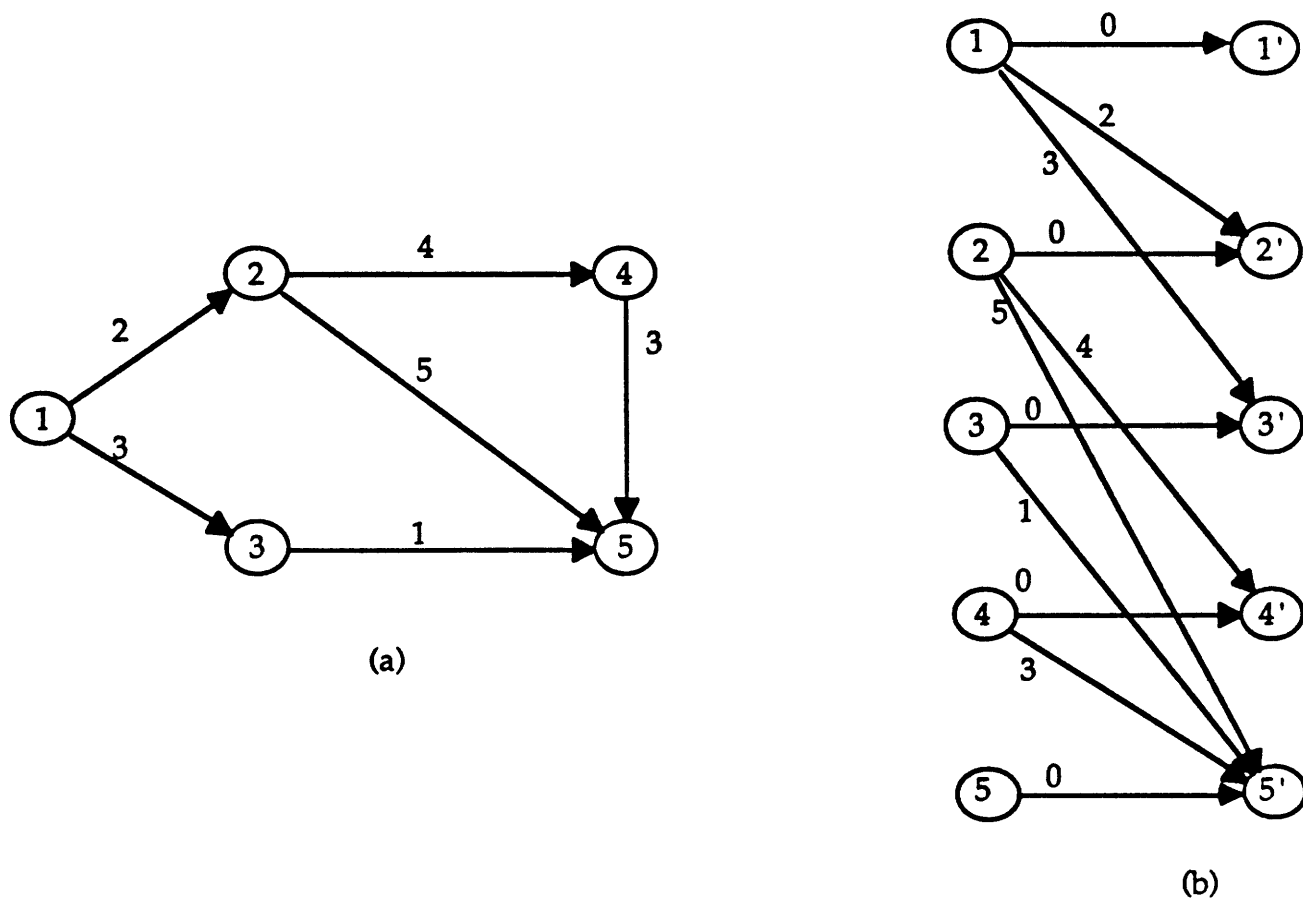
Figure 2.7. (a) The original network. (b) The transformed network.

We shall see the usefulness of this transformation in Section 5.11 in reducing a shortest path problem with arbitrary arc lengths to an assignment problem. This transformation is also used in practice for representing node activities and node data in the standard "arc flow" form of the network flow problem: we simply associate the cost or capacity for the throughput of node i with the new throughput arc (i, i').

3.   SHORTEST PATH PROBLEMS

Shortest path problems are the most fundamental and also the most commonly encountered problems in the study of transportation and communication networks. The shortest path problem arises when trying to determine the shortest, cheapest, or most reliable path between one or many pairs of nodes in a network. More importantly, algorithms for a wide variety of combinatorial optimization problems such as vehicle routing and network design often call for the solution of a large number of shortest path problems as subroutines. Consequently, designing and testing efficient algorithms for the shortest path problem has been a major area of research in network optimization.

Researchers have studied several different shortest path models. The major types of shortest path problems, in increasing order of solution difficulty, are (i) finding shortest paths from one node to all other nodes when arc lengths are nonnegative; (ii) finding shortest paths from one node to all other nodes for networks with arbitrary arc lengths; (iii) finding shortest paths from every node to every other node; and (iv) finding various types of constrained shortest paths between nodes (e.g., shortest paths with turn penalties, shortest paths visiting specified nodes, the k-th shortest path).

In this section, we discuss problem types (i), (ii) and (iii). The algorithmic approaches for solving problem types (i) and (ii) can be classified into two groups--*label setting* and *label correcting*. The label setting methods are applicable to networks with nonnegative arc lengths, whereas label correcting methods apply to networks with negative arc lengths as well. Each approach assigns tentative distance labels (shortest path distances) to nodes at each step. Label setting methods designate one or more labels as permanent (optimum) at each iteration. Label correcting methods consider all labels as temporary until the final step when they all become permanent. We will show that label setting methods have the most attractive worst case performance; nevertheless, label correcting methods have been proven to be modestly more efficient in practice.

Dijkstra's algorithm is the most popular label setting method. In this section, we first discuss a simple implementation of this algorithm that achieves a time bound of $O(n^2)$. We then describe two more sophisticated implementations that achieve improved running times in practice and in theory. Next, we consider a generic version of the label correcting method, outlining one special implementation of this general approach that runs in polynomial time and another implementation that performs very well in practice. Finally, we discuss methods to solve the all pairs shortest path problem.

3.1 Dijkstra's Algorithm

Let $G = (N,A)$ be a network and let c_{ij} represent the length of an arc $(i, j) \in A$. Let $A(i)$ represent the set of arcs emanating from node $i \in N$. Let $C = \max \{ c_{ij} : (i, j) \in A \}$. In this subsection, we assume that arc lengths are nonnegative integer numbers. Further, suppose that node s is a specially designated node. We assume without any loss of generality that there is a directed path from s to every other node in G. We can ensure this condition by adding an artificial arc (s, j), with a suitably large arc length, for each node j. We invoke this connectivity assumption throughout this section.

Dijkstra's algorithm finds shortest paths from the source node s to all other nodes. The basic idea of the algorithm is to fan out from node s and label nodes in order of their distance from s. Each node i has a label, denoted by $d(i)$: the label is *permanent* once we know that it represents the shortest distance from s to i; otherwise it is *temporary*. Initially, we give node s a permanent label of zero, and each other node j a temporary label equal to c_{sj} if $(s, j) \in A$, and ∞ otherwise. At each iteration, the label of a node i is its shortest distance from the source node along paths whose internal nodes are all permanently labeled. The algorithm selects a node i with the minimum temporary label, makes it permanent, and scans arcs in $A(i)$ to update the distance labels of adjacent nodes. The algorithm terminates when it has designated all nodes as permanently labeled. The correctness of the algorithm relies on the key observation (which we prove later) that it is always possible to designate the node with the minimum temporary label as permanent. The following algorithmic representation is a basic implementation of Dijkstra's algorithm.

algorithm DIJKSTRA;
begin
    $P := \{s\}$; $T := N - \{s\}$;
    set $d(s) := 0$ and $pred(s) := 0$;
    set $d(j) := c_{sj}$ and $pred(j) := s$ if $(s,j) \in A$, and $d(j) := \infty$ otherwise;
    while $P \neq N$ **do**
    begin
    (*node selection*) let $i \in T$ be a node for which $d(i) = \min \{d(j) : j \in T\}$;
                $P := P \cup \{i\}$; $T := T - \{i\}$;
    (*distance update*) for each $(i,j) \in A(i)$ **do**
                if $d(j) > d(i) + c_{ij}$ then $d(j) := d(i) + c_{ij}$ and $pred(j) := i$;
    end;
end;

The algorithm uses a predecessor index, denoted by *pred(i)*, for each node i ∈ N. The algorithm updates these indices to ensure that *pred(i)* is the last node prior to i on the (tentative) shortest path from node s to node i. At termination, these indices allow us to trace back the shortest path from each node to the source.

To establish the validity of Dijkstra's algorithm we use an inductive argument. At each point in the algorithm, the nodes are partitioned into two sets, P and T. Assume that the label of each node in P is the length of a shortest path from the source, whereas the label of each node j in T is the length of a shortest path subject to the restriction that each node in the path (except j) belongs to P. Then it is possible to transfer the node i in T with the smallest label d(i) to P for the following reason: any path P from the source to node i must contain a first node k that is in T. However, node k must be at least as far away from the source as node i since its label is at least that of node i; furthermore, the segment of the path P between node k and node i has a nonnegative length because arc lengths are nonnegative. This conclusion establishes that the length of path P is at least d(i) and consequently the validity of our permanently labeling node i. The subsequent use of node i to reduce the labels of adjacent nodes ensures that the label of every node in T will again be its shortest path along paths using only nodes in P as intermediary nodes.

The computational time needed by the algorithm can be split into the time required by its two basic operations—selecting nodes and updating distances. In an iteration, the algorithm requires $O(n)$ time to identify the node i with minimum temporary label and takes $O(|A(i)|)$ time to update the distance labels of adjacent nodes.

Thus, overall, the algorithm requires $O(n^2)$ time for selecting nodes and $O(\sum_{i \in N} |A(i)|) =$ $O(m)$ time for updating distances. This implementation of Dijkstra's algorithm thus runs in $O(n^2)$ time.

Dijkstra's algorithm has been a subject of much research. Researchers have attempted to reduce the node selection time without substantially increasing the time for updating distances. Consequently, they have suggested several implementations of the algorithm using clever data structures. These implementations have either dramatically reduced the running time of the algorithm in practice or improved its worst case complexity. In the following discussion, we describe Dial's algorithm, which is currently comparable to the best label setting algorithm in practice. Subsequently we describe a simple version of R-heaps, which is nearly the best implementation of Dijkstra's

algorithm from the perspective of worst-case analysis. (A more complex version of R-heaps gives the best worst-case performance for nearly all choices of the parameters n, m, and C.)

3.2   Dial's Algorithm

The bottleneck operation in Dijkstra's algorithm is node selection. To improve the algorithm's performance, we must ask the following question. Instead of scanning all temporarily labeled nodes at each iteration to find the one with the minimum distance label, can we reduce the computation time by maintaining distances in a sorted fashion? Dial's algorithm tries to accomplish this objective, and reduces the algorithm's computation time in practice, using the following fact:

FACT 3.1.   The distance labels that the algorithm designates as permanent are nondecreasing.

This fact follows from the observation that the algorithm permanently labels a node i with smallest temporary label d(i), and while scanning arcs in A(i) during the distance update step, never decreases the distance label of any node since arc lengths are nonnegative. FACT 3.1 suggests the following scheme for node selection. We maintain nC+1 *buckets* numbered 0, 1, 2, ... , nC. Bucket k stores each node whose temporary distance label is k. Recall that C represents the largest arc length in the network and, hence, nC is an upper bound on the distance labels of all the nodes. In the node selection step, we scan the buckets in increasing order until we identify the first nonempty bucket. The distance label of each node in this bucket is minimum. One by one, we delete these nodes from the bucket, making them permanent and scanning their arc lists to update distance labels of adjacent nodes. We then resume the scanning of higher numbered buckets in increasing order to select the next nonempty bucket.

By storing the content of these buckets carefully, it is possible to add, delete, and select the next element of any bucket very efficiently in O(1) time, i.e., a time bounded by some constant. One method for implemention uses a data structure known as a *doubly linked list*. In this data structure, we order the content of each bucket arbitrarily, storing two pointers for each entry: one pointer to its immediate predecessor and one to its immediate successors. Doing so permits us to select the topmost node, add a bottommost node, or delete a node easily by rearranging the pointers. Now, as we relabel nodes, decreasing any node's temporary distance label, we move it from a higher index bucket to a lower index bucket; this transfer requires O(1) time. Consequently, this

algorithm runs in $O(m + nC)$ time and uses $nC+1$ buckets. The following fact allows us to reduce the number of buckets to $C+1$.

FACT 3.2. *If $d(i)$ is the distance label that the algorithm designates as permanent at the beginning of an iteration, then at the end of that iteration $d(j) \leq d(i) + C$ for each finitely labeled node j in T.*

This fact follows by noting that $d(k) \leq d(i)$ for each $k \in P$ (by FACT 3.1), and for each finitely labeled node j in T, $d(j) = d(k) + c_{kj}$ for some $k \in P$ (by the property of distance updates). Hence, $d(j) \leq d(i) + c_{kj} \leq d(i) + C$. In other words, all finite temporary labels are bracketed from below by $d(i)$ and from above by $d(i) + C$. Consequently, $C+1$ buckets suffice to store nodes with finite temporary distance labels. We need not store the nodes with infinite temporary distance labels in any of the buckets—we can add them to a bucket when they first receive a finite distance label.

Dial's algorithm uses $C+1$ buckets numbered $0, 1, 2, \ldots, C$ which can be viewed as arranged in a circle as in Figure 3.1. This implementation stores a node j with distance label $d(j)$ in the bucket $d(j) \bmod (C+1)$. Consequently, during the entire execution of the algorithm, bucket k stores nodes with distance labels k, $k+(C+1)$, $k+2(C+1)$, and so forth; however, because of FACT 3.2, at any point in time this bucket will hold only nodes with the same distance labels. This storage scheme also implies that if bucket k contains a node with minimum distance label, then buckets $k+1, k+2, \ldots, C, 0, 1, 2, \ldots, k-1$, store nodes in increasing values of the distance labels.
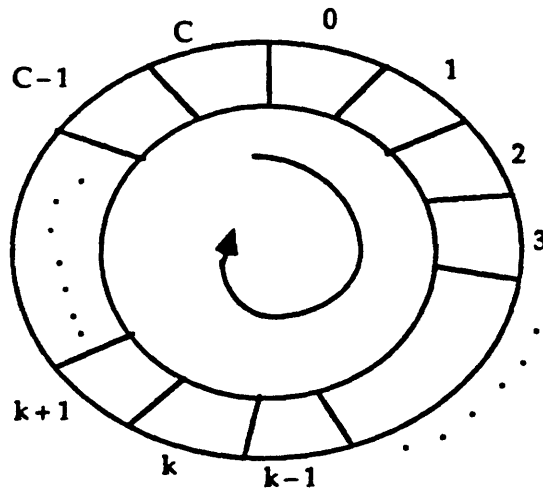

Figure 3.1. Bucket arrangement in Dial's algorithm

Dial's algorithm examines the buckets sequentially, in a wrap around fashion, to identify the first nonempty bucket. In the next iteration, it reexamines the buckets starting in the place where it left off earlier. A potential disadvantage of this scheme compared to the original algorithm is that C may be very large, necessitating large storage and increased computational time. In addition, the algorithm may wrap around as many as n-1 times, resulting in a large computation time. The algorithm, however, typically does not encounter these difficulties in practice. For most applications, C is not very large, and the number of passes through all of the buckets is much less than n. Dial's algorithm, however, is not attractive theoretically. The algorithm runs in $O(m + nC)$ time which is not even polynomial time. Rather, it is pseudopolynomial time. For example, if $C = n^4$, then the algorithm runs in $O(n^5)$ time, and if $C = 2^n$ the algorithm takes exponential time in the worst case.

The search for theoretically fastest implementations of Dijkstra's algorithm has led researchers to develop several new data structures for cases in which the network is sparse. In the next subsection, we consider an implementation using a data structure called a *redistributive heap* (R-heap) that runs in $O(m + n \log nC)$ time. The discussion of this implementation is of a more advanced nature than the previous subsections; the reader can skip it without any loss of continuity.

3.3.   R-Heap Implementation

Our first implementation of Dijkstra's algorithm and Dial's algorithm represent two extremes. In the first implementation, we consider all the temporarily labeled nodes together (in one large bucket, so to speak) and search for a node with the smallest label. In Dial's algorithm, we separate nodes by storing any two nodes with different labels in different buckets. Could we improve upon these methods by adopting an intermediate position, perhaps by storing many, but not all, labels in a bucket? For example, instead of storing only nodes with a temporary label of k in the k-th bucket, we could store temporary labels from 100k to 100k+99 in bucket k. The different temporary labels that can be stored in a bucket make up the *range* of the bucket; the cardinality of the range is called its *width*. For the preceding example, the range of bucket k is [100k .. 100k+99] and its width is 100.

Using widths of size k permits us to reduce the number of buckets needed by a factor of k. But in order to find the smallest distance label, we need to search all of the

elements in the smallest index nonempty bucket. Indeed, if k is arbitrarily large, we need only one bucket, and the resulting algorithm reduces to Dijkstra's original algorithm.

Using a width of 100, say, for each bucket reduces the number of buckets, but still requires us to search through the lowest numbered bucket to find the node with minimum temporary label. If we could devise a variable width scheme, with a width of one for the lowest numbered bucket, we could conceivably retain the advantages of both a wide bucket and narrow bucket approach. The R-heap algorithm we consider next uses variable length widths and changes the ranges dynamically. In the version of redistributive heaps that we present, the widths of the buckets are 1, 1, 2, 4, 8, 16, ... , so that the number of buckets needed is only log nC. Moreover, we dynamically modify the ranges of numbers stored in each bucket and we reallocate nodes with temporary distance labels in such a way that the minimum distance label is stored in a bucket whose width is 1. In this way, as in the previous algorithm, we avoid the need to search the entire bucket to find the minimum. In fact, the running time of this version of the R-heap algorithm is $O(m + n \log nC)$.

We now describe an R-heap in more detail. For a given shortest path problem, the R-heap consists of $1 + \lceil \log nC \rceil$ buckets. The buckets are numbered as 0, 1, 2, ... , K = $\lceil \log nC \rceil$ We represent the range of bucket k by *range(k)* which is a (possibly empty) closed interval of integers. We store a temporary node i in bucket k if $d(i) \in range(k)$. We do not store permanent nodes. The nodes in bucket k are denoted by the set CONTENTS(k). The algorithm will change the ranges of the buckets dynamically, and each time it redistributes the ranges, it redistributes the nodes in the buckets.

Initially, the buckets have the following ranges:

range(0) = [0];
range(1) = [1];
range(2) = [2 .. 3];
range(3) = [4 .. 7];
range(4) = [8 .. 15];

.

.

.

range(K) = $[2^{K-1} .. 2^{K}-1]$.

These ranges will change dynamically; however, the widths of the buckets will not increase beyond their initial widths. Suppose for example that the initial minimum distance label is determined to be in the range [8 .. 15]. We could verify this fact quickly by verifying that buckets 0 through 3 are empty and bucket 4 is nonempty. At this point, we could not identify the minimum distance label without searching all nodes in bucket 4. The following observation is helpful. Since the minimum index nonempty bucket is the bucket whose range is [8 .. 15], we know that no temporary label will ever again be less than 8, and hence buckets 0 to 3 will never be needed again. Rather than leaving these buckets idle, we can redistribute the range of bucket 4 (whose width is 8) to the previous buckets (whose combined width is 8) resulting in the ranges [8], [9], [10 .. 11], and [12 .. 15]. We then set the range of bucket 4 to 0, and we shift (or redistribute) its temporarily labeled nodes into the appropriate buckets (0, 1, 2, and 3). Thus each of the elements of bucket 4 moves to a lower indexed bucket.

Essentially, we have replaced the node selection step (i.e., finding a node with smallest temporary distance label) by a sequence of redistribute steps in which we shift nodes constantly to lower indexed buckets. Roughly speaking, the redistribution time is $O(n \log nC)$ time in total, since each node can be shifted at most $K = 1 + \log nC$ times. Eventually, the minimum temporary label is in a bucket with width one, and it is selected in an additional $O(1)$ time.

Actually, we would carry out these operations a bit differently. Since we will be scanning all of the elements of bucket 4 in the redistribute step, it makes sense to first find the minimum temporary label in the bucket. Suppose for example that the minimum label is 11. Then rather than redistributing the range [8 .. 15], we need only redistribute the subrange [11 .. 15]. In this case the resulting ranges of buckets 0 to 4 would be [11], [12], [13 .. 14], [15], ø. Moreover, at the end of this redistribution, we are guaranteed that the minimum temporary label is stored in bucket 0, whose width is 1.

To reiterate, we do not carry out the actual node selection step until the minimum nonempty bucket has width one. If the minimum nonempty bucket is bucket k, whose width is greater than 1, we redistribute the range of bucket k into buckets 0 to k-1, and then we reassign the contents of bucket k to buckets 0 to k-1. The redistribution time is $O(n \log nC)$ and the running time of the algorithm is $O(m + n \log nC)$.

We now illustrate R-heaps on the shortest path example given in Figure 3.2. In the figure, the number besides each arc indicates its length.
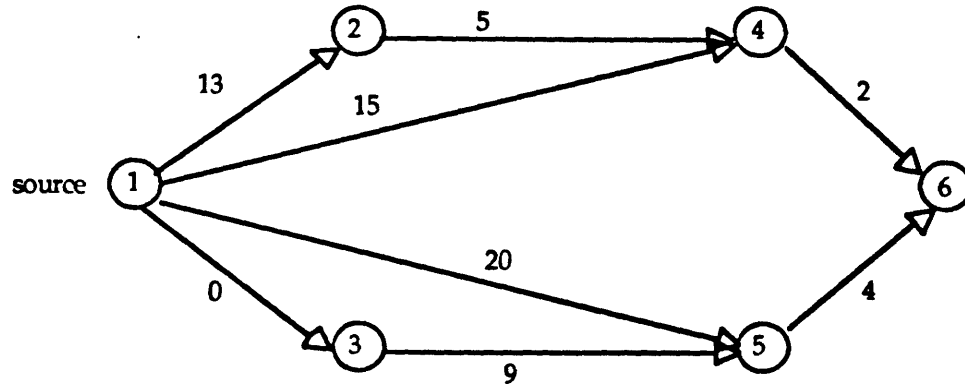
Figure 3.2 The shortest path example.

For this problem, C=20 and K = $\lceil \log 120 \rceil$ = 7. Figure 3.3 specifies the starting solution of Dijkstra's algorithm and the initial R-heap.

Node i:	1	2	3	4	5	6		
Label d(i):	0	13	0	15	20	nC = 120		
Buckets:	0	1	2	3	4	5	6	7
Ranges:	[0]	[1]	[2..3]	[4..7]	[8..15]	[16..31]	[32..63]	[64..127]
CONTENTS:	{3}	ø	ø	ø	{2, 4}	{5}	ø	{6}

Figure 3.3 The initial R-heap.

To select the node with the smallest distance label, we scan the buckets 0, 1, 2, ... , K to find the first nonempty bucket. In our example, bucket 0 is nonempty. Since the span of bucket 0 is 1, every node in this bucket has the same (minimum) distance label. So, the algorithm designates node 3 as permanent, deletes node 3 from the R-heap, and scans the arc (3,5) to change the distance label of node 5 from 20 to 9. We check whether the new distance label of node 5 is contained in the range of its present bucket, which is bucket 5. It isn't. Since its distance label has decreased, node 5 should move to a lower index bucket. So we sequentially scan the buckets from right to left, starting at bucket 5, to identify the first bucket whose range contains number 9, which is bucket 4. Node 5 moves from bucket 5 to bucket 4. Figure 3.4 shows the new R-heap.

Node i:	2	4	5	6
Label d(i):	13	15	9	120

Buckets:	0	1	2	3	4	5	6	7
Ranges:	[0]	[1]	[2..3]	[4..7]	[8..15]	[16..31]	[32..63]	[64..127]
CONTENTS:	ø	ø	ø	ø	{2, 4, 5}	ø	ø	{6}

Figure 3.4 The R-heap at the end of Iteration 1.

We again look for the node with smallest distance label. Scanning the buckets sequentially, we find that bucket 4 is the first nonempty bucket. Since the range of this bucket contains more than one integer, we cannot be guaranteed that the first node in the bucket has the minimum distance label. If $k \geq 2$ is the lowest indexed nonempty bucket, then FACT 3.1 implies that the algorithm will never use the ranges range(0), ... , range(k-1) for storing temporary distance labels. We can thus *redistribute* the range of bucket k into the buckets 0, 1, ... , k-1 and reinsert its nodes into the lower indexed buckets. In our example, the range of bucket 4 is [8 .. 15], but the smallest distance label in this bucket is 9. Again, by Fact 3.1, no temporary distance label will ever be less than 9. We therefore redistribute the range [9 .. 15] over the lower indexed buckets in the following manner:

range(0) = [9];
range(1) = [10];
range(2) = [11 .. 12];
range(3) = [13 .. 15];
range(4) = ø.

Other ranges do not change. The range of bucket 4 is now empty, and the contents of bucket 4 must be reassigned to buckets 0 through 3. This step is accomplished by successively selecting nodes in bucket 4, sequentially scanning the buckets 3, 2, ... , 0 and inserting the node in the appropriate bucket. The resulting buckets have the following contents:

CONTENTS(0) = {5},
CONTENTS(1) = ø,
CONTENTS(2) = ø,
CONTENTS(3) = {2, 4},
CONTENTS(4) = ø.

This redistribution necessarily makes bucket 4 empty, and moves the node with the smallest distance label to bucket 0.

We are now in a position to outline the general algorithm and analyze its complexity. Suppose that $j \in$ CONTENTS(k) and that d(j) decreases. If the modified d(j) $\notin$ range(k), then we sequentially scan lower numbered buckets from right to left and add the node to the appropriate bucket. Overall, this operation takes $O(m + nK)$ time. The term m reflects the number of distance updates, and the term nK arises because every time a node moves, it moves to a lower indexed bucket, and since there are K+1 buckets, a node can move at most K times. Therefore, $O(nK)$ is a bound on the total node movements.

Next we consider the node selection step. Node selection begins by scanning the buckets from left to right to identify the first nonempty bucket, say bucket k. This operation takes $O(K)$ time per iteration and $O(nK)$ time in total. If k=0 or k=1, then any node in the selected bucket has the minimum distance label. If $k \geq 2$, then we redistribute the "useful" range of bucket k into the buckets 0, 1, ... , k-1 and reinsert its contents to those buckets. If the range of bucket k is $[l .. u]$ and the smallest distance label of a node in the bucket is d_{min}, then the useful range of the bucket is $[d_{min} .. u]$.

The algorithm can redistribute the useful range in the following manner: we assign the first integer to bucket 0, the next integer to bucket 1, the next two integers to bucket 2, the next four integers to bucket 3, and so on. Since bucket k has width $\leq 2^k$ and since the widths of the first buckets can be as large as 1, 1, 2, ... , 2^{k-1} for a total potential width of 2^k, we can redistribute the useful range of bucket k over the buckets 0, 1, ... , k-1 in the manner described. This redistribution of ranges and the subsequent reinsertions of nodes empties the bucket k and moves the nodes with the smallest distance labels to bucket 0. The node selection steps take $O(nK)$ total time. Whenever we examine a node in the nonempty bucket k with the smallest index, we move it to a lower index bucket; each node can move at most K times, so all the nodes can move a total of at most nK times. Since K = $\lceil \log nC \rceil$, the algorithm runs in $O(m + n \log nC)$ time.

This algorithm requires 1 + $\lceil \log nC \rceil$ buckets. FACT 3.2 permits us to reduce the number of buckets to 1 + $\lceil \log C \rceil$. This implementation of the algorithm runs in $O(m + n \log C)$ time. Under the similarity assumption (see Section 1.3), this bound becomes $O(m + n \log n)$. Using substantially more sophisticated data structures, it is possible to reduce this bound further to $O(m+n\sqrt{\log n})$, which is a linear time algorithm for all but the sparsest classes of shortest path problems.

3.4. Label Correcting Algorithms

Label correcting algorithms, as the name implies, maintain tentative distance labels for nodes and correct the labels at every iteration. Unlike label setting methods, these methods maintain all distance labels as temporary until the end, when they all become permanent simultaneously. The label correcting algorithms are conceptually more general than the label setting algorithms and are applicable to more general situations, for example, to networks containing negative length arcs. To produce the shortest paths, these algorithms typically require that the network does not contain any negative cycle, i.e., a cycle whose arc lengths sum to a negative value. Note that the optimum solution of the shortest path problem in the presence of negative cycles is generally unbounded: if we can reach a node in such a cycle, then we can traverse the cycle as many times as desired, each time reducing the length of the path. Most label correcting algorithms have the capability to detect the presence of negative cycles.

Label correcting algorithms can be viewed as a procedure for solving the following recursive equations:

$$d(s) = 0; \tag{3.1}$$
$$d(j) = \min \{i \in N : d(i) + c_{ij}\}, \text{ for each } j \in N - \{s\}. \tag{3.2}$$

As usual, $d(j)$ denotes the length of a shortest path from the source node to node j. These equations are known as *Bellman's equations* and represent necessary conditions for optimality of the shortest path problem. These conditions are also sufficient if every cycle in the network has a positive length. We will prove an alternate version of these conditions which is more suitable from the viewpoint of label correcting algorithms.

Theorem 3.1 Let $d(i)$ for $i \in N$ be a set of labels. If $d(s) = 0$ and if in addition the labels satisfy the following conditions, then they represent the shortest path lengths from the source node:

C3.1. $d(i)$ is the length of some path from source to node i; and
C3.2. $d(j) \le d(i) + c_{ij}$ for all $(i, j) \in A$.

Proof. Since $d(i)$ is the length of some path from the source to node i, it is an upper bound on the shortest path length. We show that if the labels $d(i)$ satisfy C3.2, then they are also lower bounds on the shortest path lengths, which implies the conclusion of the theorem. Consider any directed path P from the source to node j. Let P consist of

nodes $s = i_1 - i_2 - i_3 - \dots - i_k = j$. Condition C3.2 implies that $d(i_2) \leq d(i_1) + c_{i_1 i_2} = c_{i_1 i_2}$, $d(i_3) \leq d(i_2) + c_{i_2 i_3}, \dots, d(i_k) \leq d(i_{k-1}) + c_{i_{k-1} i_k}$. Adding these inequalities yields $d(j) =$

$d(i_k) \leq \sum\limits_{(i,j) \in P} c_{ij}$. Hence $d(j)$ is a lower bound on the length of any directed path from the source to node j, including a shortest path from s to j. ∎

We note that if the network contains a negative cycle then no set of labels $d(i)$ satisfies C3.2. Suppose that there exists a negative cycle W and there exist labels $d(i)$ satisfying C3.2. Consequently, $d(i) - d(j) + c_{ij} \geq 0$ for each $(i,j) \in W$. These inequalities imply that $\sum\limits_{(i,j) \in W} (d(i) - d(j) + c_{ij}) = \sum\limits_{(i,j) \in W} c_{ij} \geq 0$, since the labels $d(i)$ cancel out in the summation. This contradicts our assumption that W is a negative cycle.

Conditions C3.1 in Theorem 3.1 correspond to primal feasibility for the linear programming formulation of the shortest path problem. Conditions C3.2 correspond to dual feasibility. The label correcting algorithms can be viewed as methods that always maintain primal feasibility and try to achieve dual feasibility. The generic label correcting algorithm that we consider first is a general procedure for successively updating distance labels $d(i)$ until they satisfy the conditions C3.2. At any point in the algorithm, the label $d(i)$ is either ∞ indicating that we are yet to discover any path from the source to node j, or it is the length of some path from the source to node j. The algorithm is based upon the simple observation that whenever $d(j) > d(i) + c_{ij}$, the current path from origin to node i, of length $d(i)$, together with the arc (i,j) is a shorter path to node j than the current path of length $d(j)$.

algorithm LABEL CORRECTING;
begin
        $d(s) := 0$; $pred(s) := 0$;
        $d(j) := \infty$ for each $j \in N - \{s\}$;
        while some arc (i,j) satisfies $d(j) > d(i) + c_{ij}$ **do**
        begin
                $d(j) := d(i) + c_{ij}$;
                $pred(j) := i$;
        end;
end;

The correctness of the label correcting algorithm follows from Theorem 3.1. At termination, the labels $d(i)$ satisfy $d(j) \leq d(i) + c_{ij}$ for all $(i, j) \in A$, and hence represent the shortest path lengths. We now note that this algorithm is finite if there are no negative cost cycles and if the data is integral. Since $d(j)$ is bounded from above by nC and below by $-nC$, $d(j)$ is updated at most $2nC$ times. Thus when all data is integral, the number of distance updates is $O(n^2C)$, and hence the algorithm is pseudopolynomial.

A nice feature of this label correcting algorithm is its flexibility: we can select the arcs that do not satisfy conditions C3.2 in any order and still assure the finite convergence of the algorithm. One drawback of the method, however, is that without a further restriction on the choice of arcs, the label correcting algorithm is not a polynomial time algorithm. Indeed, if we start with pathological instances of the problem and make a poor choice of arcs at every iteration, then the number of steps can grow exponentially with n. (Since the algorithm is pseudopolynomial, these instances do have exponentially large values of C.) To obtain a polynomial bound for the algorithm, we can organize the computations carefully in the following manner. Arrange the arcs in A in some (possibly arbitrary) order. Now make passes through A. In each pass, scan arcs in A in order and check the condition $d(j) > d(i) + c_{ij}$; if the arc satisfies the condition, then update $d(j) = d(i) + c_{ij}$. Terminate the algorithm if no distance label changes during an entire pass. We call this algorithm the *modified label correcting algorithm*.

Theorem 3.2 *When applied to a network containing no negative cycles, the modified label correcting algorithm requires $O(nm)$ time to determine shortest paths from the source to every other node.*

Proof. We show that the algorithm performs at most $n-1$ passes through the arc list. Since each pass requires $O(1)$ computations for each arc, this conclusion implies the $O(nm)$ bound. Let $d^r(j)$ denote the length of the shortest path from the source to node j consisting of r or fewer arcs. Further, let $D^r(j)$ represent the distance label of node j after r passes through the arc list. We claim, inductively, that $D^r(j) \leq d^r(j)$ for each $j \in N$, and for each $r = 1, \dots, n-1$. We perform induction on the value of r. Suppose $D^{r-1}(j) \leq d^{r-1}(j)$ for each $j \in N$. The provisions of the modified labeling algorithm imply that $D^r(j) \leq \min$

$\{ D^{r-1}(j), \min_{i \neq j} \{ D^{r-1}(i) + c_{ij} \} \}$. Next note that the shortest path to node j containing no more than r arcs (i) either has no more than $r-1$ arcs, (ii) or it contains exactly r arcs. In

case (i), $d^r(j) = d^{r-1}(j)$, and in case (ii), $d^r(j) = \min_{i \neq j} \{ d^{r-1}(i) + c_{ij} \}$. Consequently, $d^r(j) =$

$$\min \{d^{r-1}(j), \min_{i \neq j} \{d^{r-1}(i) + c_{ij}\}\} \geq \min \{D^{r-1}(j), \min_{i \neq j} \{D^{r-1}(i) + c_{ij}\}\}; \text{ the inequality follows}$$

from the induction hypothesis. Hence, $D^r(j) \leq d^r(j)$ for all $j \in N$. Finally, we note that the shortest path from the source to any node consists of at most n–1 arcs. Therefore, after at most n–1 passes, the algorithm terminates with the shortest path lengths. ■

The modified label correcting algorithm is also capable of detecting the presence of negative cycles in the network. If the algorithm does not update any distance label during an entire pass, up to the (n-1)-th pass, then it has a set of labels d(j) satisfying C3.2. In this case, the algorithm terminates with the shortest path distances and the network does not contain any negative cycle. On the other hand, when the algorithm modifies distance labels in all the (n-1)-th passes, we make one more pass. If the distance label of some node i changes in the n-th pass, then there is a directed *walk* (a path with a cycle) from node 1 to i of length greater than n–1 arcs that has smaller distance than all simple paths from the source node to i. This situation cannot occur unless the network contains a negative cost cycle.

Practical Improvements

As stated so far, the modified label correcting algorithm must consider every arc of the network during every pass through the arc list. It need not do so. Suppose we order the arcs in the arc list by their tail nodes so that all arcs with the same tail node appear consecutively on the list. Thus, while scanning the arcs, we consider one node i at a time, scanning arcs in A(i) and testing the optimality conditions. Now suppose that during one pass through the arc list, the algorithm does not change the distance label of a node i. Then, during the next pass $d(j) \leq d(i) + c_{ij}$ for every $(i, j) \in A(i)$ and the algorithm need not test these conditions. To achieve this savings, the algorithm can maintain a list of nodes whose distance labels have changed since it last examined them. It scans this list in the first-in-first-out order to assure that it performs passes through the arc list A and, consequently, terminates in O(nm) time. The following procedure is a formal description of this further modification of the modified label correcting method.

```
algorithm MODIFIED LABEL CORRECTING;
begin
        d(s) : = 0;  pred(s) : = 0;
        d(j) : = ∞  for each j ∈ N – {s};
        LIST : = {s};
        while LIST ≠ ∅ do
        begin
                select the first element  i of LIST;
                delete i from LIST;
                for each (i, j) ∈ A(i) do
                        if d(j) > d(i) + c_ij then
                        begin
                                d(j) : = d(i) + c_ij ;
                                pred(j) : = i;
                                if j ∉ LIST then add j to the end of LIST;
                        end;
        end;
end;
```

Another modification of this algorithm destroys its polynomial behavior in the worst case, but greatly improves its running time in practice. The modification alters the manner in which the algorithm adds nodes to LIST. While adding a node i to LIST, we check to see whether the algorithm has already examined it. If yes, then we add i to the *beginning* of LIST, otherwise we add it to the *end* of LIST. This heuristic rule has the following plausible justification. If the algorithm has examined node i earlier, then some nodes may have i as a predecessor. It is advantageous to update the distances for these nodes immediately, rather than update them from other nodes and then update them again when we consider node i. Empirical studies indicate that with this change alone, the algorithm is several times faster for many reasonable problem classes. Indeed, this version of the label correcting algorithm is the fastest algorithm in practice for finding the shortest path from a single source to all nodes. (For the problem of finding a shortest path from a single source node to a single sink, certain variants of the label setting algorithm are more efficient in practice.)

3.5. All Pair Shortest Path Algorithm

In certain applications of the shortest path problem, we need to determine shortest path distances between all pairs of nodes. In this subsection we describe two algorithms to solve this problem. The first algorithm is well suited for sparse graphs. It combines the modified label correcting algorithm and Dijkstra's algorithm. The second algorithm is better suited for dense graphs. It is based on dynamic programming.

If the network has nonnegative arc lengths, then the all pair shortest path problem can be solved by applying Dijkstra's algorithm n times, each node being considered as a source node once. If the network contains arcs with negative arc lengths, then the network can first be transformed to one with nonnegative arc lengths as follows. Let s be a node from which all nodes in the network are reachable. We use the modified label correcting algorithm to compute shortest path distances from s to all other nodes. The algorithm either terminates with the shortest path distances $d(j)$ or indicates the presence of a negative cycle. In the former case, we define the new length of the arc (i, j) as $\bar{c}_{ij} = c_{ij} + d(i) - d(j)$ for each $(i, j) \in A$. Condition C3.2 implies that $\bar{c}_{ij} \geq 0$ for all $(i, j) \in A$. Further, note that for any path P from node k to node l,

$$\sum_{(i, j) \in P} \bar{c}_{ij} = \sum_{(i, j) \in P} c_{ij} + d(k) - d(l)$$ since the intermediate labels $d(j)$ cancel out in

the summation. This transformation thus changes the length of all paths between a pair of nodes by a constant amount (depending on the pair) and consequently preserves the shortest paths. Since arc lengths become nonnegative after the transformation, we can apply Dijkstra's algorithm n-1 additional times to determine shortest path distances between all pairs of nodes in the transformed network. The shortest path distance between nodes k and l in the original network is obtained by adding $d(l) - d(k)$ to the corresponding shortest path distance in the transformed network. This approach requires $O(nm)$ time to solve the first shortest path problem, and if the network contains no negative cost cycle, the method takes $O(nm + n^2 \log nC)$ time to compute the remaining shortest path distances, using an R-heap implementation of Dijkstra's algorithm.

Another way to solve the all pair shortest path problem is by dynamic programming. The approach that we present is known as Floyd's algorithm. We define the variables $d^r(i, j)$ as follows:

$d^r(i, j) =$ the length of a shortest path from node i to node j subject to the condition that the path passes only through the nodes 1, 2, ... , r–1 (and i and j).

We let $d(i, j)$ denote the actual shortest path distance. To compute $d^r(i, j)$, we first observe that a shortest path from node i to node j that passes through the nodes 1, 2, ... , r either (i) does not pass through the node r, in which case $d^{r+1}(i, j) = d^r(i, j)$, or (ii) does pass through the node r, in which case $d^{r+1}(i, j) = d^r(i, r) + d^r(r, j)$. Thus we have

$$d^1(i, j) = c_{ij},$$

and

$$d^{r+1}(i, j) = \min \{d^r(i, j), d^r(i, r) + d^r(r, j)\}.$$

We assume that $c_{ij} = \infty$ for all node pairs $(i, j) \notin A$. It is possible to solve the previous equations recursively for increasing values of r, and by varying the node pairs over N x N for a fixed value of r. The following procedural representation is a formal description of Floyd's algorithm.

algorithm ALL PAIR SHORTEST PATHS;
begin
    for all node pairs $(i, j) \in N \times N$ **do** $d(i, j) := \infty$ and $pred(i, j) := 0$;
    for each $(i, j) \in A$ **do** $d(i, j) := c_{ij}$ and $pred(i, j) := i$;
    for each $r := 1$ to n **do**
        for each $(i, j) \in N \times N$ **do**
            if $d(i, j) > d(i, r) + d(r, j)$ **then**
            begin
                $d(i, j) := d(i, r) + d(r, j)$;
                if $i = j$ and $d(i, i) < 0$ **then** the network contains a negative cycle, STOP;
                $pred(i, j) := pred(r, j)$;
            end;
end;

Floyd's algorithm uses predecessor indices, *pred(i, j)*, for each node pair (i, j). The index pred(i, j) denotes the last node prior to node j in the tentative shortest path from node i to node j. The algorithm maintains the property that for each finite $d(i, j)$, the network contains a path from node i to node j of length $d(i, j)$. This path can be obtained by tracing the predecessor indices.

This algorithm performs n iterations, and in each iteration it performs O(1) computations for each node pair. Consequently, it runs in $O(n^3)$ time. The algorithm either terminates with the shortest path distances or stops when $d(i, i) < 0$ for some node i. In the latter case, for some node $r \neq i$, $d(i, r) + d(r, i) < 0$. Hence the union of the tentative shortest paths from node i to node r and from node r to node i contains a negative cycle. This cycle can be obtained by using the predecessor indices.

Floyd's algorithm is in many respects similar to the modified label correcting algorithm. This relationship becomes more transparent from the following theorem.

Theorem 3.3 *If $d(i, j)$ for $(i, j) \in N \times N$ satisfy the following conditions, then they represent the shortest path distances:*

(i)   *$d(i, i) = 0$ for all i;*
(ii)   *$d(i, j)$ is the length of some path from node i to node j;*
(iii)   *$d(i, j) \leq d(i, r) + c_{rj}$ for all i, r, and j.*

Proof. For fixed i, this theorem is a consequence of Theorem 3.1. ∎

4. MAXIMUM FLOW PROBLEM

An important characteristic of a network is its capacity to carry flow. What, given capacities on the arcs, is the maximum flow that can be sent between any two nodes? The resolution of this question determines the "best" use of arc capacities and establishes a reference point against which to compare other ways of using the network. Moreover, the solution of the maximum flow problem with capacity data chosen judiciously establishes other performance measures for a network. For example, what is the minimum number of nodes whose removal from the network destroys all paths joining a particular pair of nodes? Or what is the maximum number of node disjoint paths that join this pair of nodes? These and similar reliability measures indicate the robustness of the network to failure of its components.

In this section, we discuss several algorithms for computing the maximum flow between two nodes in a network. We begin by introducing a basic labeling algorithm for solving the maximum flow problem. The validity of these algorithms rests upon the celebrated *max-flow min-cut theorem* of network flows. This remarkable theorem has a number of surprising implications in machine and vehicle scheduling, communication systems planning and several other application domains. We then consider improved versions of the basic labeling algorithm with better theoretical performance guarantees. In particular, we describe *preflow push* algorithms that have recently emerged as the most powerful techniques for solving the maximum flow problem, both theoretically and computationally.

We consider a capacitated network $G = (N, A)$ with a *nonnegative integer* capacity u_{ij} for any arc $(i, j) \in A$. The source s and sink t are two distinguished nodes of the network. We assume that for every arc (i, j) in A, (j, i) is also in A. There is no loss of generality in making this assumption since we allow zero capacity arcs. We also assume without any loss of generality that all arc capacities are finite (since we can set the capacity of any uncapacitated arc equal to the sum of the capacities of all capacitated arcs). Let $U = \max \{u_{ij} : (i, j) \in A\}$. As earlier, the arc adjacency list defined as $A(i) = \{(i, k) : (i, k) \in A\}$ designates the arcs emanating from node i. In the maximum flow problem, we wish to find the maximum flow from the source node s to the sink node t that satisfies the arc capacities. Formally, the problem is to

Maximize v $\qquad$ (4.1a)

subject to

$$\sum_{\{j\,:\,(i,\,j)\,\in\,A\}} x_{ij} \;-\; \sum_{\{j\,:\,(j,\,i)\,\in\,A\}} x_{ji} \;=\; \begin{cases} v, & \text{if } i = s, \\ 0, & \text{if } i \neq s,\,t,\text{ for all } i \in N, \\ -v, & \text{if } i = t, \end{cases} \qquad (4.1b)$$

$$0 \le x_{ij} \le u_{ij}, \text{ for each } (i,\,j) \in A. \qquad (4.1c)$$

It is possible to relax the integrality assumption on arc capacities for some algorithms, though this assumption is necessary for others. Algorithms whose complexity bounds involve U assume integrality of data. Note, however, that rational arc capacities can always be transformed to integer arc capacities by appropriately scaling the data. Thus, the integrality assumption is not a restrictive assumption in practice.

The concept of *residual network* is crucial to the algorithms to be considered. Given a flow x, the *residual capacity*, r_{ij}, of any arc $(i,\,j) \in A$ represents the maximum additional flow that can be sent from node i to node j using the arcs $(i,\,j)$ and $(j,\,i)$. The residual capacity has two components: (i) $u_{ij} - x_{ij}$, the unused capacity of arc $(i,\,j)$, and (ii) the current flow x_{ji} on arc $(j,\,i)$ which can be cancelled to increase flow to node j. Consequently, $r_{ij} = u_{ij} - x_{ij} + x_{ji}$. The network consisting of the arcs with positive residual capacities is called the *residual network* (with respect to the flow x). We represent the residual network by $G(x)$. Figure 4.1 illustrates an example of a residual network.

4.1 Labeling Algorithm and the Max-Flow Min-Cut Theorem

One of the simplest and most intuitive algorithms for solving the maximum flow problem is the *augmenting path algorithm* due to Ford and Fulkerson. The algorithm proceeds by identifying directed paths from the source to the sink in the residual network and sending flows on these paths, until the residual network contains no such path. The following high-level (and flexible) description of the algorithm summarizes the basic iterative steps, without specifying any particular algorithmic strategy for how to determine augmenting paths.

algorithm AUGMENTING PATH;
begin
    x : = 0;
    while there is a path P from s to t in G(x) **do**
    begin
        Δ : = min {r_{ij} : (i, j) $\in$ P};
        send Δ units of flow along P and update G(x);
    end;
end;

For each (i, j) $\in$ P, sending Δ units of flow along P decreases r_{ij} by Δ and increases r_{ji} by Δ. We now discuss this algorithm in more detail. First, we need a method to identify a directed path from source to sink in the residual network or to show that the network contains no such path. Second, we need to show that the algorithm terminates finitely. Finally, we must establish that the algorithm terminates with a maximum flow. The last result follows from the proof of the *max-flow min-cut theorem*.

A directed path from source to sink in the residual network is also called an *augmenting path*. The residual capacity of an augmenting path is the minimum residual capacity of any arc on the path. The definition of the residual capacity implies that an additional flow of Δ in arc (i, j) of the residual network corresponds to (i) an increase in x_{ij} by Δ in the original network, or (ii) a decrease in x_{ji} by Δ in the original network, or (iii) a convex combination of (i) and (ii). For our purposes, it is easier to work directly with residual capacities and to compute the flows only when the algorithm terminates.

The labeling algorithm performs a search of the residual network to find a directed path from s to t. It does so by *fanning out* from the source node s to find a directed tree containing nodes that are reachable from the source along a directed path in the residual network. At any step, we refer to the nodes in the tree as *labeled* and those not in the tree as *unlabeled*. The algorithm selects a labeled node and scans its arc adjacency list (in the residual network) to label more unlabeled nodes. Eventually, the sink becomes labeled and the algorithm sends the maximum possible flow on the path from s to t. It then erases the labels and repeats this process. The algorithm terminates when it has scanned all labeled nodes and the sink remains unlabeled. The following algorithmic description specifies the steps of the labeling algorithm in detail. The
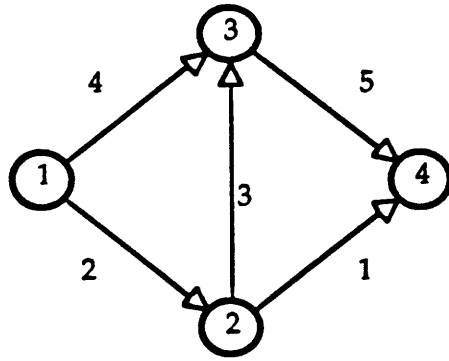
a. Network with arc capacities.
Node 1 is the source and node 4 is the sink.
(Arcs not shown have zero capacities.)
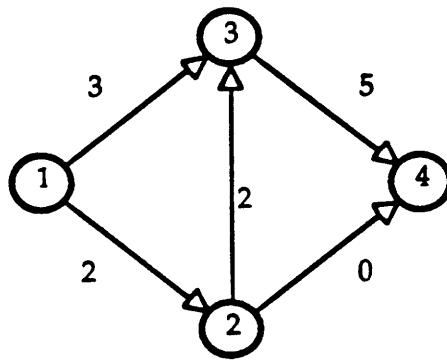
b. Network with a flow x.
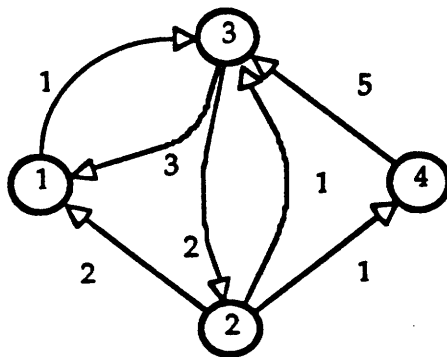
c. The residual network with residual arc capacities.

Figure 4.1 Example of a residual network.

algorithm maintains a predecessor index, *pred(i)*, for each labeled node i indicating the node that caused node i to be labeled. The predecessor indices allow us to trace back along the path from a node to a source.

algorithm LABELING;
begin
 loop
   $pred(j) := 0$ for each $j \in N$;
   $L := \{s\}$;
   while $L \neq \emptyset$ **and** t is unlabeled **do**
   begin
     select a node $i \in L$;
     for each $(i, j) \in A(i)$ **do**
       if j is unlabeled and $r_{ij} > 0$ **then**
       begin
         $pred(j) := i$;
         mark j as labeled and add this node to L;
       end
   end;

   if t is labeled **then**
   begin
     use the predecessor labels to trace back to obtain the augmenting path P
       from s to t;
     $\Delta := \min \{r_{ij} : (i, j) \in P\}$;
     send Δ units of flow along P;
     erase all labels and go to loop;
   end
   else quit the loop;
  end; {loop}
end;

The final residual capacities r can be used to obtain the arc flows as follows. Since $r_{ij} = u_{ij} - x_{ij} + x_{ji}$, $x_{ij} - x_{ji} = u_{ij} - r_{ij}$. Hence, if $u_{ij} > r_{ij}$, then we set $x_{ij} = u_{ij} - r_{ij}$ and $x_{ji} = 0$; otherwise we set $x_{ij} = 0$ and $x_{ji} = r_{ij} - u_{ij}$.

In order to show that this algorithm obtains a maximum flow, we first introduce some new definitions and notation. A *disconnecting set of arcs* between the nodes s and t is a set of arcs whose removal from the network G = (N, A) produces a new network with no *undirected* paths joining s and t. For example, the set A of all arcs is a disconnecting set of arcs. A *cut* between s and t, also referred to as an *s-t cut*, is a minimal disconnecting set of arcs. Consequently, if C is a cut, then adding an arc of C to A-C produces an undirected path joining s and t.

An s-t cut partitions the nodes of the network into two sets S and $\bar{S} = N - S$: S is the set of nodes connected to s. Conversely, any partition of the node set as S and $\bar{S}$ with $s \in S$ and $t \in \bar{S}$ defines an s-t cut. Consequently, we frequently designate a cut as $(S, \bar{S})$. An arc (i, j) with $i \in S$ and $j \in \bar{S}$ is called a *forward arc*, and an arc (i, j) with $i \in \bar{S}$ and $j \in S$ is called a *backward arc* in the cut $(S, \bar{S})$.

Any flow vector x satisfying the flow conservation and capacity constraints of (4.1) determines the net flow across a cut $(S, \bar{S})$ as

$$F_x(S, \bar{S}) = \sum_{i \in S} \sum_{j \in \bar{S}} x_{ij} - \sum_{i \in \bar{S}} \sum_{j \in S} x_{ij}. \tag{4.2}$$

The capacity $C(S, \bar{S})$ of an s-t cut $(S, \bar{S})$ is defined as

$$C(S, \bar{S}) = \sum_{i \in S} \sum_{j \in \bar{S}} u_{ij}. \tag{4.3}$$

We claim that the flow across any s-t cut equals the flow from s to t and does not exceed the cut capacity. Adding the flow conservation constraints (4.1 b) for nodes in S and noting that when nodes i and j both belong to S, x_{ij} in equation for node j cancels $- x_{ij}$ in equation for node i, we obtain

$$v = \sum_{i \in S} \sum_{j \in \bar{S}} x_{ij} - \sum_{i \in \bar{S}} \sum_{j \in S} x_{ij} = F_x(S, \bar{S}) \tag{4.4}$$

Substituting $x_{ij} \leq u_{ij}$ in the first summation and $x_{ij} \geq 0$ in the second summation shows that

$$F_x(S, \bar{S}) \leq \sum_{i \in S} \sum_{j \in \bar{S}} x_{ij} = C(S, \bar{S}). \tag{4.5}$$

This result is the weak duality property of the maximum flow problem when viewed as a linear program. Like most weak duality results, it is the "easy" half of the duality theory. The more substantive strong duality property asserts that (4.5) holds as an equality for some choice of x and some choice of an s-t cut $(S, \bar{S})$. This strong duality property is the max-flow min-cut theorem.

Theorem 4.1. *(Max-Flow Min-Cut Theorem) The maximum value of flow from s to t equals the minimum capacity of all s-t cuts.*

Proof. Consider the solution obtained at the end of the labeling algorithm. Let x denote the arc flow vector and v denote the flow value. Let S be the set of labeled nodes in the residual network and $\bar{S} = N - S$. Clearly $s \in S$ and $t \in \bar{S}$. Adding the flow conservation equations for nodes in S, we again obtain (4.4). Note that nodes in $\bar{S}$ cannot be labeled from nodes in S, hence $r_{ij} = 0$ for each forward arc (i, j) in the cut $(S, \bar{S})$. Since $r_{ij} = u_{ij} - x_{ij} + x_{ji}$, the capacity constraints imply that $x_{ij} = u_{ij}$ and $x_{ji} = 0$. Hence $x_{ij} = u_{ij}$ for each forward arc in the cut $(S, \bar{S})$ and $x_{ij} = 0$ for each backward arc in the cut. Making these substitutions in (4.4) yields

$$v = F_x(S, \bar{S}) = \sum_{i \in S} \sum_{j \in \bar{S}} u_{ij} = C(S, \bar{S}). \qquad (4.6)$$

But we have observed earlier that v is a lower bound on the capacity of any s-t cut. Consequently, v is a maximum flow and the cut $(S, \bar{S})$ is the minimum capacity s-t cut. We thus have established the theorem and, simultaneously, proved the correctness of the labeling algorithm. ∎

The proof of this theorem shows that when the labeling algorithm terminates, it has at hand both the maximum flow value (and a maximum flow vector) and the minimum capacity s-t cut. But does it terminate finitely? Each labeling iteration of the algorithm scans any node at most once, inspecting each arc in $A(i)$. Consequently, the labeling iteration scans each arc once and requires $O(m)$ computations. If all arc capacities are integral and bounded by a finite number U, then the capacity of the cut $(s, N - \{s\})$ is at most nU. The labeling algorithm increases the flow value by at least one unit in any iteration and, consequently, terminates within nU iterations. This bound on the number of iterations is not entirely satisfactory for large values of U. If $U = 2^n$ this bound is exponential in the number of nodes, and the algorithm can indeed perform that many iterations. Furthermore, if the capacities are irrational, the algorithm may not terminate, and although the successive flow values converge, they may not

converge to the maximum flow value. Thus if the method is to be effective, we must select the augmenting paths carefully. Many other algorithms, including the next algorithm, overcome this difficulty and obtain an optimum flow even if the capacities are irrational; moreover, the max-flow min-cut theorem is true even if the data are irrational.

A second drawback of the labeling algorithm is its *"forgetfulness"*. At each iteration, the algorithm generates node labels that contain information about augmenting paths from the source to other nodes. The implementation we have described erases the labels when it proceeds from one iteration to the next, even though much of this information may be valid in the updated residual network. Erasing the labels therefore destroys potentially useful information. Ideally, we should retain a label when it can be used profitably in later computations.

4.2   Decreasing the Number of Augmentations

The bound of nU on the number of augmentations in the labeling algorithm is not satisfactory from a theoretical perspective. Furthermore, without further modifications the augmenting path algorithm may take $\Omega(nU)$ augmentations, as the example given in Figure 4.2 illustrates.

Flow decomposition shows that, in principle, augmenting path algorithms should be able to find a maximum flow in no more than m augmentations. For suppose x is an optimum flow and y is any initial flow. By flow decomposition, it is possible to obtain x from y by a sequence of at most m augmentations on augmenting paths from s to t plus flow around cycles. If we define x' as the flow vector obtained from y by applying only the augmenting paths, then x' also is a maximum flow. This result shows that it is, in theory, possible to find a maximum flow with at most m augmentations. Unfortunately, to apply this flow decomposition argument we need to know a maximum flow. No algorithm developed in the literature comes close to achieving this theoretical bound. Nevertheless, it is possible to improve considerably on the bound of O(nU) augmentations of the basic labeling algorithm.

One natural specialization of the augmenting path algorithm is to augment flow along a "shortest path" from the source to the sink, defined as a path consisting of the least number of arcs. If we augment flow along a shortest path, then the length of any shortest path either stays the same or increases. Moreover, within m augmentations, the length of the shortest path is guaranteed to increase. (We will prove these results in the

(a)

(b)

(c)
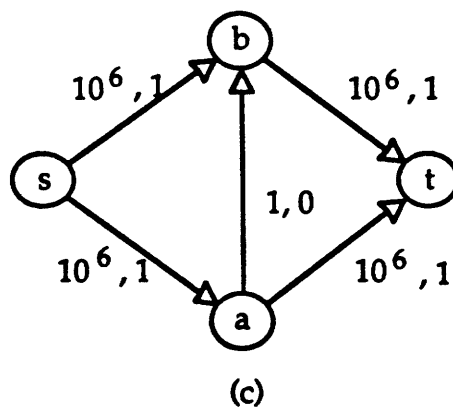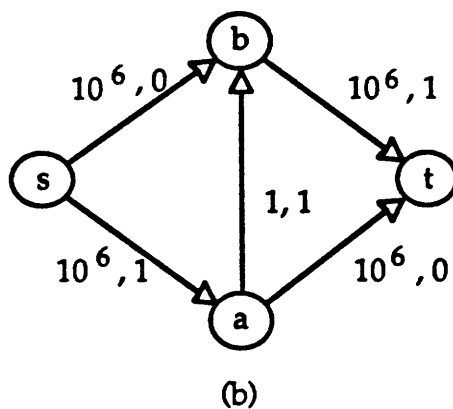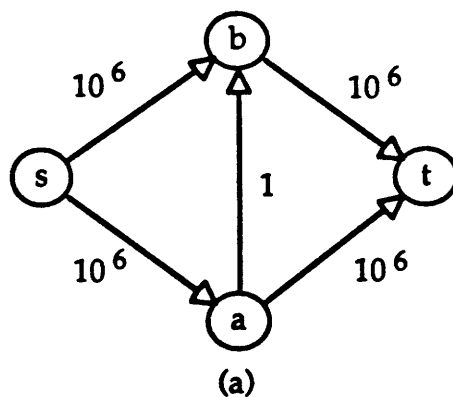
Figure 4.2 A pathological example for the labeling algorithm.

(a) The input network with arc capacities.

(b) After augmenting along the path s-a-b-t. Arc flow is indicated beside the arc capacity.

(c) After augmenting along the path s-b-a-t. After 2×10^6 augmentations, alternatively along s-a-b-t and s-b-a-t, the flow is maximum.

next section.) Since no path contains more than n-1 arcs, this rule guarantees that the number of augmentations is at most (n-1)m.

An alternative is to augment flow along a path of maximum residual capacity. This specialization also leads to improved complexity. Let v be any flow value and v^* be the maximum flow value. By flow decomposition, the network contains at most m augmenting paths whose residual capacities sum to $(v^* - v)$. Thus the maximum capacity augmenting path has residual capacity at least $(v^* - v)/m$. Now consider a sequence of 2m consecutive maximum capacity augmentations, starting with flow v. At least one of these augmentations must augment the flow by an amount $(v^* - v)/2m$ or less, for otherwise we will have a maximum flow. Thus after 2m or fewer augmentations, the maximum residual capacity implementation would reduce the capacity of a maximum capacity augmentive path by a factor of two. Since this capacity is ·initially at most U and the capacity must be at least 1 until the flow is maximum, after $O(m \log U)$ maximum capacity augmentations, the flow must be maximum. An alternative way of proving this result is to use the geometric improvement argument discussed in Section 1.7.

4.3   Shortest Augmenting Path Algorithm

A natural approach to augmenting along shortest paths would be to successively look for shortest paths by performing a breadth first search in the residual network. Each of these searches would take $O(m)$ steps both in worst case and in practice, and the resulting computation time would be $O(nm^2)$. Unfortunately, this computation time is excessive. We can improve this running time by exploiting the fact that the minimum distance from a node i to a node t is monotonically nondecreasing over all augmentations. By fully exploiting this property, we can reduce the average time per augmentation to $O(n)$.

The Algorithm

The concept of distance labels is an important construct in the maximum flow algorithms that we now discuss. A *distance function* $d : N \rightarrow Z^+$ with respect to the residual capacities r_{ij} is a function from the set of nodes to the nonnegative integers. We say that a distance function is *valid* if it satisfies the following two conditions:

C4.1 $d(t) = 0$;

C4.2. $d(i) \leq d(j) + 1$ for every arc $(i, j) \in A$ with $r_{ij} > 0$.

We refer to $d(i)$ as the *distance label* of node i and condition C4.2 as the *validity condition*. It is easy to demonstrate that $d(i)$ is a lower bound on the length of the shortest path from i to t in the residual network. Let $i = i_1 - i_2 - i_3 - \ldots -i_k - t$ be any path of length k in the residual network from node i to t. Then, from C4.2 we have $d(i) = d(i_1) \leq d(i_2) + 1$, $d(i_2) \leq d(i_3) + 1, \ldots , d(i_k) \leq d(t) + 1 = 1$. These inequalities imply that $d(i) \leq k$ for *any* path of length k in the residual network and, hence, the shortest path from node i to t contains at least $d(i)$ arcs. If for each node i, the distance label $d(i)$ equals the length of the shortest path from i to t in the residual network, then we call the distance labels *exact*. For example, in Figure 4.1(c), $d = (0, 0, 0, 0)$ is a valid distance label, though $d = (3, 1, 2, 0)$ represents the exact distance labels.

An arc (i, j) in the residual network is called *admissible* if it satisfies $d(i) = d(j) + 1$. Other arcs are called *inadmissible*. A path from s to t consisting entirely of admissible arcs is called an *admissible path*. The algorithm we describe now repeatedly augments flow along admissible paths. For any admissible path of length k, $d(s) = k$. Since $d(s)$ is a lower bound on the length of any path from the source to the sink, the algorithm augments flows along shortest paths in the residual network. Thus, we refer to the algorithm as the *shortest augmenting path algorithm*.

Whenever we augment along a path, each of the distance labels for nodes in the path is exact. However, for other nodes in the network it is not necessary to maintain exact distances; it suffices to have valid distances, which are lower bounds on the exact distances. There is no particular urgency to compute these distances exactly. By allowing the distance label of node i to be less than the distance from i to t, we maintain flexibility in the algorithm, without any significant cost.

We can compute the initial distance labels by performing a backward breadth first search of the residual network, starting at the sink node. The algorithm obtains an admissible path by successively building it up as follows. The algorithm maintains a path from the source node to some node i*, called the *current node*, consisting entirely of admissible arcs. We call this path a *partial admissible path* and store it using the *predecessor* indices, i.e., pred(j) = i for each arc (i, j) on the path. The algorithm performs one of the two steps at the current node: *advance* or *retreat*. The advance step identifies some admissible arc (i*, j*) emanating from node i*, adds it to the partial admissible path and makes j* the new current node. If no admissible arc emanates from node i*, then the algorithm performs the retreat step. This step increases the distance label of node i*

so that at least one admissible arc emanates from it (this operation is called a *relabel* operation). Increasing d(i*) makes the arc (pred(i*), i*) inadmissible if i* ≠ s. Consequently, we delete (pred(i*), i*) from the partial admissible path and node pred(i*) becomes the new current node. Whenever the partial admissible path is an admissible path (i.e., contains node t), the algorithm makes a maximum possible augmentation on this path and begins again with the source as the current node. The algorithm terminates when d(s) ≥ n, indicating that the network contains no augmenting path from the source to the sink. We next describe the algorithm formally.

algorithm SHORTEST AUGMENTING PATH;
begin
>   perform backward breadth first search of the residual network from node t to
>   obtain the distance labels d(i);
>   set i* : = s;
>   **while** d(s) < n **do**
>   **begin**
>>   **if** i* has an admissible arc **then** ADVANCE(i*)
>>   **else** RETREAT(i*);
>>   **if** i* = t **then** AUGMENT and set i* : = s;
>   **end;**
end;

procedure ADVANCE(i*);
begin
>   let (i*, j*) be an admissible arc in A(i*);
>   pred(j*) : = i* and i* : = j*;
end;

procedure RETREAT(i*);
begin
>   d(i*) : = min { d(j) + 1 : (i, j) ∈ A(i*) and r_{ij} > 0 };
>   **if** i* ≠ s **then** i* : = pred(i*);
end;

procedure AUGMENT;
begin
>   using predecessor indices identify an augmenting path P from the source to the
>   sink;

$$\Delta := \min \{r_{ij} : (i, j) \in P\};$$

augment Δ units of flow in path P;

end;

We use the following data structure to select an admissible arc emanating from a node. We maintain with each node i the list A(i) of arcs emanating from it. Arcs in each list can be arranged arbitrarily, but the order, once decided, remains unchanged throughout the algorithm. Each node i has a *current-arc* (i, j) which is the current candidate for the next advance step. Initially, the current-arc of node i is the first arc in its arc list. The algorithm examines this list sequentially and whenever the current arc is inadmissible, it makes the next arc in the arc list the current arc. When the algorithm has examined all arcs in A(i), it updates the distance label of node i and the current arc once again becomes the first arc in its arc list. In our subsequent discussion we shall always implicitly assume that the algorithms select admissible arcs using this technique.

Accuracy of the Algorithm

We first show that the shortest augmentation algorithm correctly solves the maximum flow problem.

Lemma 4.1. The shortest augmenting path algorithm maintains valid distance labels at each step. Moreover, each relabel step strictly increases the distance label of a node.

Proof. We show that the algorithm maintains valid distance labels at every step by performing induction on the number of augment and relabel steps. Initially, the algorithm constructs valid distance labels. Assume, inductively, that the distance function is valid prior to a step, i.e., satisfies the validity condition C4.2. We need to check whether these conditions remain valid (i) after an augment step (when the residual graph changes), and (ii) after a relabel step. (i) A flow augmentation on arc (i, j) might delete this arc from the residual network, but this modification to the residual network does not affect the validity of the distance function. Augmentation on arc (i, j) might, however, create an additional arc (j, i) with $r_{ji} > 0$ and, therefore, an additional condition $d(j) \leq d(i) + 1$ that needs to be satisfied. This validity condition remains satisfied since $d(i) = d(j) + 1$ by the admissibility property of the augmenting path. (ii) The algorithm performs a relabel step at node i when no arc (i, j) $\in$ A(i) satisfies $d(i) = d(j) + 1$ and $r_{ij} > 0$. Hence $d(i) < \min\{d(j) + 1 : (i, j) \in A(i)$ and $r_{ij} > 0\} = d'(i)$, thereby establishing the second part of the lemma. Finally, by the choice for changing d(i), the condition $d(i) \leq d(j) + 1$ remains valid for all (i, j) in the residual network; in addition,

since $d(i)$ increases, the conditions $d(k) \leq d(i) + 1$ remain valid for all arcs (k, i) in the residual network. ∎

Theorem 4.1. *The shortest augmenting path algorithm correctly computes a maximum flow.*

Proof. The algorithm terminates when $d(s) \geq n$. Since $d(s)$ is a lower bound on the length of the shortest augmenting from s to t, this condition implies that the network contains no augmenting path from the source to the sink. This condition is the termination criterion for the generic augmenting path algorithm.

If $d(s) \geq n$, we could obtain a minimum s-t cut as follows. For $0 \leq k \leq n$, let α_k denote the number of nodes with distance label equal to k. Note that α_{k^*} must be zero for some $k^* \leq n - 1$ since $\sum_{k=0}^{n-1} \alpha_k \leq n - 1$. Let $S = \{i \in N : d(i) > k^*\}$ and $\bar{S} = N - S$. When $d(s) \geq n$ and the algorithm terminates, $s \in S$ and $t \in \bar{S}$, and both the sets S and $\bar{S}$ are nonempty. Consider the s-t cut $(S, \bar{S})$. By construction, $d(i) > d(j) + 1$ for all $(i, j) \in (S, \bar{S})$. The validity condition C4.2 implies that $r_{ij} = 0$ for each $(i, j) \in (S, \bar{S})$. Hence, $(S, \bar{S})$ is a minimum s-t cut and the current flow is maximum. ∎

Complexity of the Algorithm

We next show that the algorithm computes a maximum flow in $O(n^2 m)$ time.

Lemma 4.2. *(a) Each distance label increases at most n times. Consequently, the total number of relabel steps is at most n^2. (b) The number of augment steps is at most $nm/2$.*

Proof. Each relabel step at node i increases $d(i)$ by at least one. After at most n relabels of node i, $d(i) \geq n$. From this point on, the algorithm never selects node i again during an advance step since for every node k in the current path, $d(k) < d(s) < n$. Thus the algorithm relabels a node at most n times and the total number of relabel steps is bounded by n^2.

Each augment step saturates at least one arc, i.e., decreases its residual capacity to zero. Suppose that the arc (i, j) becomes saturated at some iteration (at which $d(i) = d(j) + 1$). Then no more flow can be sent on (i, j) until flow is sent back from j to i (at which point $d'(j) = d'(i) + 1 \geq d(i) + 1 = d(j) + 2$). Hence, between two consecutive saturations of arc (i, j), $d(j)$ increases by at least 2 units. Consequently, any arc (i, j) can become saturated at most $n/2$ times and the total number of arc saturations is no more than $nm/2$. ∎

Theorem 4.2. The shortest augmenting path algorithm runs in $O(n^2 m)$ time.

Proof. The algorithm performs $O(nm)$ flow augmentations and each augmentation takes $O(n)$ time, resulting in $O(n^2 m)$ total effort in the augment steps. Each advance step increases the length of the partial admissible path by one, each retreat step decreases its length by one, and the length of the partial admissible path is at most n. Hence the number of advance steps is at most $O(n^2 + n^2 m)$: the first term comes from the number of retreat (relabel) steps, and the second term from the number of augmentations, which are bounded respectively by n^2 and $nm/2$ by the previous lemma.

For each node i, the algorithm performs the relabel operation $O(n)$ times, each execution requiring $O(|A(i)|)$ time. The total time spent in all relabel operations is

$$\sum_{i \in N} n|A(i)| \;=\; O(nm).$$ Finally, we consider the time spent in identifying admissible arcs. The time taken to identify the admissible arc of node i is $O(1)$ plus the time spent in scanning arcs in $A(i)$. After $|A(i)|$ such scannings, the algorithm reaches the end of the arc list and relabels node i. Thus the total time spent in all scannings is $O\left(\sum_{i \in N} n|A(i)| \right)$

$= O(nm)$. The combination of these time bounds establishes the theorem. ∎

The proof of Theorem 4.2 also suggests an alternative termination condition for the shortest augmenting path algorithm. The termination criteria of $d(s) \geq n$ is satisfactory for a worst case analysis, but may not be efficient in practice. Researchers have observed empirically that the algorithm spends too much time in relabeling, a major portion of which is done *after* it has already found the maximum flow. The algorithm can be improved by detecting the presence of a minimum cutset prior to performing these relabeling operations. We can do so by maintaining the number of nodes α_k with distance label equal to k, for $0 \leq k \leq n$. The algorithm updates this array after every relabel operation and terminates whenever it first finds a *gap* in the α array, i.e., $\alpha_k = 0$ and $k < d(s)$. As we have seen in the proof of Theorem 4.1, if $S = \{ i : d(s) > k \}$, then $(S, \bar{S})$ denotes a minimum cutset.

The idea of augmenting flows along shortest paths is intuitively appealing and easy to implement in practice. The resulting algorithms identify at most $O(nm)$ augmenting paths and this bound is tight, i.e., on particular examples these algorithms perform $\Omega(nm)$ augmentations. The only way to improve the running time of the shortest augmenting path algorithm is to perform fewer computations per

augmentation. The use of the sophisticated data structure of *dynamic trees* reduces the average time for each augmentation from O(n) to O(log n). This implementation of the maximum flow algorithm runs in O(nm log n) time; obtaining further improvements appears quite difficult except in very dense networks. These implementations with sophisticated data structures appear to be primarily of theoretical interest, however, because maintaining the data structures requires substantial overhead that tends to increase rather than reduce the computational times in practice. A detailed discussion of dynamic trees is beyond the scope of this chapter.

Potential Functions and an Alternate Proof of Lemma 4.2(b)

A powerful method for proving computational time bounds is to use *potential functions*. Potential function techniques are general purpose techniques for proving the complexity of an algorithm by analyzing the effects of different steps on an appropriately defined function. The use of potential functions enables us to obtain a relationship between the occurrences of various steps and thus a bound on steps that are difficult to bound using other methods. Rather than formally introducing potential functions, we illustrate the technique by showing that the number of augmentations in the shortest augmenting path algorithm is O(nm).

Suppose in the shortest augmenting path algorithm we kept track of the number of admissible arcs in G(x). Let F(k) denote the number of admissible arcs at the end of the k-th step; for the purpose of this argument, we count a step either as an augmentation or as a relabel operation. Let the algorithm perform K steps before it terminates. Clearly, $F(0) \leq m$ and $F(K) \geq 0$. Each augmentation decreases the residual capacity of at least one arc to zero and hence reduces F by at least one unit. Each relabeling of node i creates as many as $|A(i)|$ new admissible arcs, and increases F by the same amount. This increase in F is at most nm over all relabelings, since the algorithm relabels any node at most n times (as a consequence of Lemma 4.1) and $\sum_{i \in N} n|A(i)| = nm$. Since the initial value of F is at most m more than its terminal value, the total decrease in F due to all augmentations is m + nm. Thus the number of augmentations is at most m + nm = O(nm).

This argument is fairly representative of the potential function argument. Our objective was to bound the number of augmentations. We did so by defining a potential function that decreases whenever the algorithm performs an augmentation. The potential increases only when the algorithm relabels distances, and thus we can bound

the number of augmentations using bounds on the number of relabels. In general, we bound the number of steps of one type in terms of known bounds on the number of steps of other types.

4.3 Preflow-Push Algorithms

Augmenting path algorithms send flow by augmenting along a path. This basic step further decomposes into the more elementary operation of sending flow along an arc. Thus sending a flow of Δ units along a path of k arcs decomposes into k basic operations of sending a flow of Δ units along an arc of the path. We shall refer to each of these basic operations as a *push*.

A path augmentation has one advantage over a single push: it maintains conservation of flow at all nodes. In fact, the push-based algorithms such as those developed in this and the following section necessarily violate conservation of flow. Rather, these algorithms permit the flow into a node to exceed the flow out of this node. We will refer to any such flows as *preflows*. The two basic operations of the generic preflow-push methods are pushing the flow on an admissible arc and updating a distance label, as in the augmenting path algorithm described in the last section. (We define the distance labels and admissible arcs as in the previous subsection.)

Preflow-push algorithms have several advantages over augmentation based algorithms. First, they are more general and more flexible. Second, they can push flow closer to the sink before identifying augmenting paths. Third, they are better suited for distributed or parallel computation. Fourth, the best preflow-push algorithms currently outperform the best augmenting path algorithms in theory as well as in practice.

The Generic Algorithm

A *preflow* x is a function x: A → R that satisfies (4.1c) and the following relaxation of (4.1b):

$$\sum_{\{j\,:\,(j,\,i)\,\in\,N\}} x_{ji} \;-\; \sum_{\{j\,:\,(i,\,j)\,\in\,N\}} x_{ij} \;\geq 0 \text{ , for all } i \in N - \{s, t\}.$$

The preflow-push algorithms maintain a preflow at each intermediate stage. For a given preflow x, we define for each node $i \in N - \{s, t\}$, the *excess*

$$e(i) = \sum_{\{j\,:\,(j,\,i)\,\in\,N\}} x_{ji} \quad - \sum_{\{j\,:\,(i,\,j)\,\in\,N\}} x_{ij} \;.$$

We refer to a node with positive excess as an *active* node. We follow the convention that the source and sink nodes are never active. The preflow-push algorithms perform all operations using only local information. At each iteration of the algorithm (except its initialization and its termination), the network contains at least one active node, i.e., a node $i \in N - \{s, t\}$ with $e(i) > 0$. The goal of each iterative step is to choose some active node and to send its excess *closer* to the sink, closer being measured with respect to the current distance labels. If the method cannot send excess flow from this node to nodes with smaller distance labels, then it increases the distance label of the node. The algorithm terminates when the network contains no active nodes. The preflow-push algorithm uses the following subroutines:

procedure PREPROCESS;
begin

> perform breadth first search of the residual network, starting at node t, to determine initial distance labels d(i);
> set $x_{sj} : = u_{sj}$ for each arc $(s, j) \in A(s)$; set $d(s) : = n$;

end;

procedure PUSH/RELABEL(i);
begin

> **if** the network contains an admissible arc (i, j) **then**
> > push $\delta : = \min\{e(i),\ r_{ij}\}$ units of flow from node i to node j
> **else** replace d(i) by min $\{d(j) + 1 : (i, j) \in A(i)$ and $r_{ij} > 0\}$;

end;

We say that a push of δ units of flow on arc (i, j) is *saturating* if $\delta = r_{ij}$ and *nonsaturating* otherwise. We refer to the process of increasing the distance label of a node as a *relabel* operation. The purpose of the relabel operation is to create at least one admissible arc on which the algorithm can perform further pushes.

The following generic version of the preflow-push algorithm combines the subroutines just described.

```
algorithm PREFLOW-PUSH;
begin
        PREPROCESS;
        while the network contains an active node do
        begin
                select an active node i;
                PUSH/RELABEL(i);
        end;
end;
```

The generic preflow-push method may be visualized in terms of a network in which the arcs represent flexible water pipes, nodes represent the joints, the distance function represents distance above the ground, and water is being sent from source to sink. In addition, one can visualize flow in an admissible arc as water flowing downhill. Initially, the source node is moved upward, and water flows to its neighbors. In general, water flows downhill towards the sink; however, occasionally flow becomes trapped locally at a node in which none of its neighbors are downhill. At this point the node is moved upward, and again the water flows downhill towards the sink. Eventually, there is no additional flow than can reach the sink. As nodes continue to be moved upwards, the remaining excess flow eventually flows back towards the source. The algorithm terminates when all the water flows either into the sink or into the source.

Figure 4.3 illustrates the push/relabel steps. Figure 4.3(a) specifies the preflow determined by the preprocess step. Suppose the select step examines node 2. Since arc (2, 4) has residual capacity $r_{24} = 1$ and $d(2) = d(4) + 1$, the algorithm performs a (saturating) push of value $\delta = \min \{2, 1\}$ units. The push reduces the excess of node 2 to 1. Arc (2, 4) is deleted from the residual network and arc (4, 2) is added to the residual network. Since node 2 is still an active node, it can be selected again for further pushes. The arc (2,3) and (2, 1) have positive residual capacities, but they do not satisfy the distance condition. Hence the algorithm performs a relabel operation and gives node 2 a new distance $d'(2) = \min \{d(3) + 1, d(1) + 1\} = \min\{2, 5\} = 2$.

The pre-process step accomplishes several important tasks. First, it gives each node adjacent to node s a positive excess, so that the algorithm can begin by selecting some node with positive excess. Second, since the preprocessing step saturates arcs incident to node s, none of the arcs is admissible and setting $d(s) = n$ will satisfy the validity condition C4.2. Third, since $d(s) = n$ is a lower bound on the length of the minimum path from s to t, the residual network contains no path from s to t. Since

distances in d are nondecreasing, we are also guaranteed that in subsequent iterations the residual network will never contain a directed path from s to t, and so there never will be any need to push flow from s again.

In the push/relabel(i) step, we can identify an admissible arc in A(i) using the same data structure we used in the shortest augmenting path algorithm. We maintain with each node i a *current arc* (i, j) which is the current candidate for the push operation. We choose the current arc by sequentially scanning the arc list. We have seen earlier that scanning the arc lists takes O(nm) total time, if the algorithm relabels each node O(n) times.

d(3) = 1
$e_3 = 4$

3

4   5

d(1) = 4   1   4   d(4) = 0

3

2   1

2

d(2) = 1
$e_2 = 2$
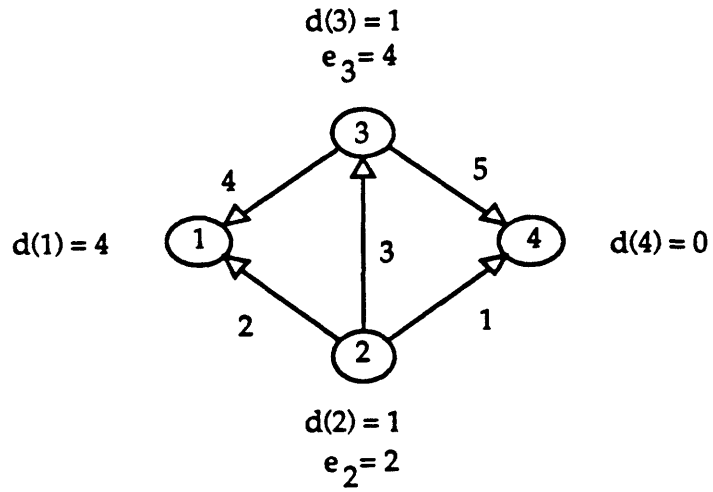
(a) The residual network after the pre-processing step.

d(3) = 1
$e_3 = 4$

3

4   5

d(1) = 4   1   4   d(4) = 0

3

2   1

2

d(2) = 1
$e_2 = 1$
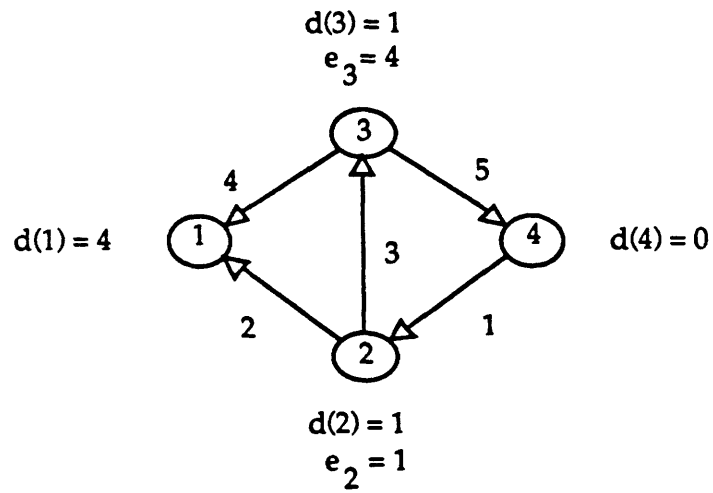
(b) After the execution of step PUSH(2).

d(3) = 1

$e_3 = 4$

```
                    d(3) = 1
                    e  = 4
                     3
                   ___
                  ( 3 )
                   ‾‾‾
           4     /  △  \    5
                /   |   \
        ___    /    |    \    ___
  d(1)=4( 1 )       | 3      ( 4 )  d(4) = 0
        ‾‾‾  \      |      /  ‾‾‾
              \     |     /
           2   \    △    /   1
                \ ___  /
                 ( 2 )
                  ‾‾‾
                d(2) = 2
                 e  = 1
                  2
```
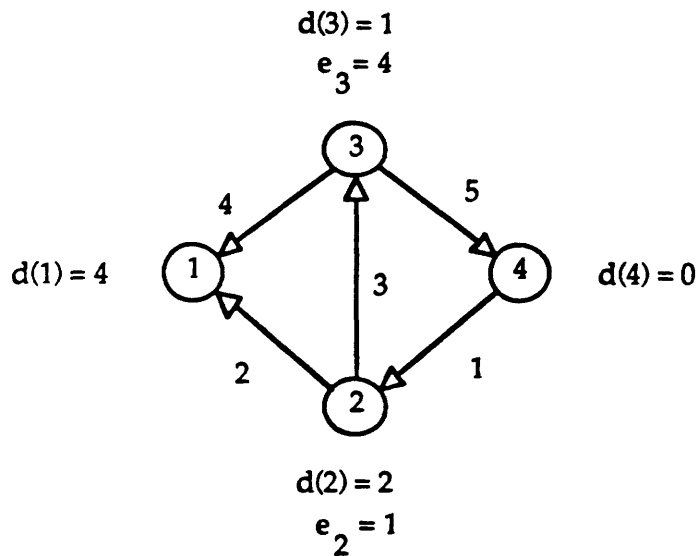
(c)   After the execution of step RELABEL(2).

Figure 4.3   Illustrations of Push and Relabel steps.

Assuming that the algorithm terminates, the correctness of the generic preflow-push algorithm is easy to show. The algorithm terminates when the excess resides either at the source or at the sink implying that the current preflow is a flow. Since $d(s) = n$, the residual network contains no path from the source to the sink. This condition is the termination criterion of the augmenting path algorithm, and thus the flow into the sink represents the maximum flow value.

Complexity of the Algorithm

We now analyze the complexity of the algorithm. One important result is to show that distance labels are always valid and do not increase too many times. The first of these conclusions follows from Lemma 4.1, because as in the shortest augmenting path algorithm, the preflow-push algorithm pushes flow only on admissible arcs and relabels a node only when no admissible arc emanates from it. The second conclusion follows from the following lemma.

Lemma 4.3.   *At any stage of the preflow-push algorithm, each node i with positive excess is connected to node s by a directed path from i to s in the residual network.*

Proof.   By the flow decomposition theory, any preflow x can be decomposed with respect to the original network G into nonnegative flows along (i) paths from the source s to t, (ii) paths from s to active nodes, and (iii) the flows around directed cycles. Let i be an

active node relative to the preflow x in G. Then there must be a path P from s to i in the flow decomposition of x, since paths from s to t and flows around cycles do not contribute to the excess at node i. Then the residual network contains the reversal of P (P with the orientation of each arc reversed), and hence a path from i to s. ∎

Lemma 4.4. *For each node* $i \in N$, $d(i) < 2n$.

Proof. The last time the algorithm relabeled node i, it had a positive excess, and hence the residual network contained a path of length at most n–1 from node i to node s. The fact that $d(s) = n$ and condition C4.2 imply that $d(i) \leq d(s) + n - 1 < 2n$. ∎

Lemma 4.5. *(a) Each distance label increases at most 2n times. Consequently, the total number of relabel step is at most* $2n^2$. *(b) The number of saturating pushes is at most* nm.

Proof. The proof is very much similar to that of Lemma 4.2. ∎

Lemma 4.6. *The number of nonsaturating pushes is* $O(n^2m)$.

Proof. We prove the theorem using an argument based on potential functions. Let I denote the set of active nodes. Consider the potential function $F = \sum_{i \in I} d(i)$. Since $|I| \leq$ n, and $d(i) \leq 2n$ for all $i \in I$, the initial value of F (after the preprocess step) is at most $2n^2$. At termination, F is zero. During the push/relabel(i) step, one of the following two cases must apply:

Case 1. The algorithm is unable to find an arc along which it can push flow. In this case, no arc (i, j) satisfies $d(i) = d(j) + 1$ and $r_{ij} > 0$ and the distance label of node i goes up by $\varepsilon \geq$ 1 units. This operation increases F by at most ε units. Since the total increase in d(i) throughout the running time of the algorithm for each node i is bounded by 2n, the total increase in F due to increases in distance labels is bounded by $2n^2$.

Case 2. The algorithm is able to identify an arc on which it can push flow, and so it performs a saturating or a nonsaturating push. A saturating push on arc (i, j) may create a new excess at node j, thereby increasing the number of active nodes by 1, and increasing F by d(j), which may be as much as 2n per saturating push, and hence $2n^2m$ over all saturating pushes. Next note that a nonsaturating push on arc (i, j) does not increase $|I|$. The nonsaturating push will decrease F by d(i) since i becomes inactive, but it simultaneously increases F by $d(j) = d(i) - 1$ if j was not active earlier. If node j was

before the push, then F decreases by an amount d(i). The net decrease in F is at least 1 unit per nonsaturating push.

We summarize these facts. The initial value of F is at most $2n^2$ and the maximum possible increase in F is $2n^2 + 2n^2m$. Each nonsaturating push decreases F by one unit and F always remains nonnegative. Hence the nonsaturating pushes can occur at most $2n^2 + 2n^2 + 2n^2m = O(n^2m)$ times, proving the theorem. ∎

Finally, we indicate how the algorithm keeps track of active nodes for push/relabel steps. The algorithm maintains a set S of active nodes. It adds to S nodes that become active following a push and are not already in S, and deletes from S nodes that become inactive following a nonsaturating push. Several data structures are available for storing S so that elements can be added, deleted, or selected from it in O(1) time. Consequently, the preflow-push algorithm can be easily implemented in $O(n^2m)$ time. We have thus established the following theorem:

Theorem 4.3. *The generic preflow-push theorem runs in $O(n^2m)$ time.* ∎

A Specialization of the Generic Algorithm

The running time of the generic preflow-push algorithm is comparable to the bound of the shortest augmenting path algorithm. However, the preflow-push algorithm has several nice features, in particular, its flexibility and its potential for further improvements. Many different algorithms can be derived from the generic version of the algorithm if we specify different rules for selecting nodes for push/relabel operations. For example, suppose that the *select* step always selects an active node with the largest distance label. Let $h^* = \max \{d(i) : e(i) > 0, i \in N\}$ at some point of the algorithm. Then nodes with distance h^* push flow to nodes with distance h^*-1, and these nodes, in turn, push flow to nodes with distance h^*-2, and so on. Thus if the algorithm relabels no node during n consecutive node examinations, then all excess reaches the sink node and the algorithm terminates. Since the algorithm requires $O(n^2)$ relabel operations, we immediately obtain a bound of $O(n^3)$ on the number of node examinations. Each node examination entails at most one nonsaturating push. Consequently, this algorithm performs $O(n^3)$ nonsaturating pushes.

We use the following data structure to efficiently select an active node with largest distance label. We maintain the lists LIST(r) = {i ∈ N : e(i) > 0 and d(i) = r}, and a variable *level* which is an upper bound on the largest index r for which LIST(r) is nonempty. We can store these lists as doubly linked lists so that adding, deleting, or

selecting an element takes $O(1)$ time. We identify the largest indexed nonempty list starting at LIST(level) and sequentially scanning the lower indexed lists. We leave it as an exercise to show that the overall effort needed to scan the lists is bounded by n plus the total increase in the distance labels which is $O(n^2)$. The following theorem is now evident.

Theorem 4.4. The preflow-push algorithm that always pushes flow from an active node with largest distance label runs in $O(n^3)$ time. ∎

We will next describe another implementation of the generic preflow-push algorithm that dramatically reduces the number of nonsaturating pushes from $O(n^2m)$ to $O(n^2 \log U)$. Recall that U represents the largest arc capacity in the network. We refer to this algorithm as the *excess-scaling algorithm* since it is based on scaling the excesses of nodes.

4.4   Excess-Scaling Algorithm

The generic preflow-push algorithm allows flows at each intermediate step to violate mass balance equations. By pushing flows from active nodes the algorithm attempts to satisfy the mass balance equations. The function $e_{max} = \max \{e(i) : i \text{ is an}$ active node} can be taken as a measure of the infeasibility of a preflow. However, during the execution of the algorithm we would observe no particular pattern in e_{max} with the exception that e_{max} eventually decreases to value 0. In this section, we develop an *excess-scaling technique* that systematically reduces e_{max} to 0.

The excess-scaling algorithm is based on the following ideas. Let Δ denote an upper bound on e_{max} and let us refer to it as the *excess-dominator*. The excess-scaling algorithm pushes flow from nodes whose excess is more than $\Delta/2 \geq e_{max}/2$. This assures that during nonsaturating pushes the algorithm sends relatively large excess closer to the sink. Pushes carrying small amounts of flow are of little benefit and can bottleneck the algorithm.

The algorithm also does not allow the maximum excess to increase beyond Δ. This property may be important for the following consideration. Suppose several nodes send flow to a single node j, creating a very large excess. It is likely that node j would not be able to send the accumulated flow closer to the sink, in which case its distance would increase and much of its excess would be returned.

The excess-scaling algorithm can be described at an abstracted level as follows:

algorithm EXCESS-SCALING;
begin

PREPROCESS;

$K := 2^{\lceil \log U \rceil}$;

for $k := K$ **down to** 0 **do**

begin (Δ-scaling phase)

$\Delta := 2^k$;

while the network contains a node i with $e(i) > \Delta/2$ **do**

perform push/relabel(i) while ensuring that no node excess exceeds Δ;

end;

end;

The algorithms performs a number of scaling phases for decreasing values of the excess-dominator Δ. We refer to a specific scaling phase with a certain value of Δ as the Δ-scaling phase. Initially, $\Delta = 2^{\lceil \log U \rceil} \geq U$. During the Δ-scaling phase, $\Delta/2 < e_{max} \leq \Delta$ and it may vary up and down during the phase. When $e_{max} \leq \Delta/2$, a new scaling phase begins. After $\lceil \log U \rceil + 1$ scaling phases, e_{max} decreases to value 0 and we obtain the maximum flow.

The excess-scaling algorithm uses the same step push/relabel(i) as in the generic preflow-push algorithm but with the slight difference that instead of pushing $\delta = \min \{e(i), r_{ij}\}$ units, it pushes $\delta = \min \{e(i), r_{ij}, \Delta - e(j)\}$ units of flow. This change will ensure that the implementation permits no excess to exceed Δ. The algorithm uses the following node selection rule to guarantee that no node excess exceeds Δ.

Selection Rule. Among all nodes with excess more than $\Delta/2$, select a node with minimum distance label (breaking the ties arbitrarily).

Lemma 4.7. *The algorithm satisfies the following two conditions:*

C4.3. *Each nonsaturating push from a node i to a node j sends at least $\Delta/2$ units of flow.*

C4.4. *No excess ever exceeds Δ.*

Proof. For every push on arc (i, j), we have $e(i) > \Delta/2$ and $e(j) \leq \Delta/2$, since node i is a node with smallest distance label among nodes whose excess is more than $\Delta/2$, and $d(j) = d(i) - 1 < d(i)$ by the design of the push operation. Hence, by sending $\min \{e(i), r_{ij}, \Delta - $

e(j)] ≥ min {Δ/2, r_{ij}} units of flow, we ensure that in a nonsaturating push the algorithm sends at least Δ/2 units of flow. Further, the push operation increases only e(j). Let e'(j) be the excess at node j after the push. Then e'(j) = e(j) + min {e(i), r_{ij}, Δ - e(j)} ≤ e(j) + Δ - e(j) ≤ Δ. All node excesses thus remain less than or equal to Δ. ∎

Lemma 4.8. *The excess-scaling algorithm performs $O(n^2)$ nonsaturating pushes per scaling phase and $O(n^2 \log U)$ pushes in total.*

Proof. Consider the potential function F = $\sum_{i \in N}$ e(i) d(i)/Δ. Using this potential function we will establish the first assertion of the lemma. The initial value of F at the beginning of Δ-scaling phase is bounded by $2n^2$ because e(i) is bounded by Δ and d(i) is bounded by 2n. During the push/relabel(i) step, one of the following two cases must apply:

Case 1. The algorithm is unable to find an arc along which it can push flow. In this case no arc (i, j) satisfies d(i) = d(j) + 1 and $r_{ij} > 0$, and thus the distance label of node i increases by ε ≥ 1 units. This relabeling operation increases F by at most ε units. Since for each i the total increase in d(i) throughout the running of the algorithm is bounded by 2n, the total increase in F due to the relabeling of nodes is bounded by $2n^2$ in the Δ-scaling phase (actually, the increase in F due to node relabelings is at most $2n^2$ over all scaling phases).

Case 2. The algorithm is able to identify an arc on which it can push flow and so it performs either a saturating or a nonsaturating push. In either case, F decreases. A nonsaturating push on arc (i, j) sends at least Δ/2 units of flow from node i to node j and since d(j) = d(i) - 1, this operation decreases F by at least 1/2 units. Since the initial value of F at the beginning of a Δ-scaling phase plus the increases in F during this scaling phase sum to at most $4n^2$ (from Case 1), the number of nonsaturating pushes is bounded by $8n^2$. ∎

This lemma implies a bound of $O(nm + n^2 \log U)$ for the excess-scaling algorithm since we have already seen that all other operations such as saturating pushes, relabel operations and finding admissible arcs require O(nm) time. Up to this point, we have ignored the method needed to identify a node whose distance label is minimum among nodes with excess more than Δ/2. This identification is easy to accomplish if we use a scheme similar to the one used in the preflow-push method in Section 4.3 to find a node with the largest distance label. We maintain the lists LIST(r) = {i ∈ N : e(i) > Δ/2 and d(i) = r}, and a variable *level* which is a lower bound on the smallest index r for which LIST(r) is nonempty. We identify the lowest indexed nonempty list starting at

LIST(level) and sequentially scan the higher indexed lists. We leave as an exercise to show that the overall effort needed to scan the lists is bounded by the number of pushes performed by the algorithm and, hence, is not a bottleneck operation. With this observation, we can summarize our discussion by the following result.

Theorem 4.5. *The preflow-push algorithm with excess-scaling runs in $O(nm + n^2 \log U)$ time.* ∎

4.5   Networks with Lower Bounds

The maximum flow problems with positive lower bounds on some or all arcs arise in several applications. We represent the lower bound of any arc (i, j) by l_{ij}, and assume that $l_{ij} \geq 0$ for all arcs (i,j) ∈ A. In this model, we replace the constraint (4.1c) in the maximum flow problem formulation by the following constraint:

$$l_{ij} \leq x_{ij} \leq u_{ij} \text{, for all (i, j) ∈ A.}$$

Although the maximum flow problem with zero lower bounds always has a feasible solution, the version of the problem with nonnegative lower bounds might not be feasible. For example, the network given by Figure 4.4 has no feasible flow.
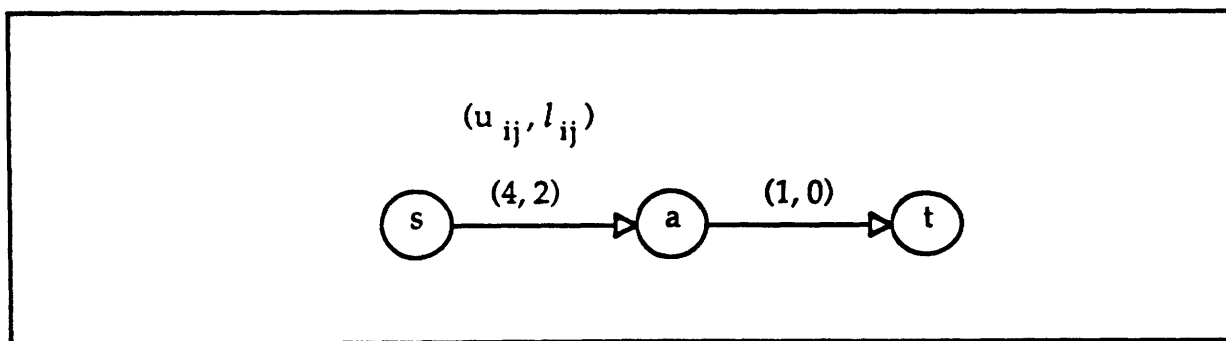
Figure 4.4. An infeasible instance of the maximum flow problem.

Indeed determining a feasible flow in the network is the crucial problem, because once we have found a feasible flow in the network, we can determine the maximum flow by a minor variation of any maximum flow algorithm. We shall show in the next section how to obtain a feasible flow, if it exists, by solving a single maximum flow problem with zero lower bounds on arc flows. Once we have found a feasible flow, we

apply any of the maximum flow algorithms with only one change: define the residual capacity of an arc (i, j) as $r_{ij} = (u_{ij} - x_{ij}) + (x_{ji} - l_{ji})$. The first and second terms in this expression denote, respectively, the residual capacity for increasing flow on arc (i, j) and for decreasing flow on arc (j, i). It is possible to establish the optimality of the solution generated by the algorithm by generalizing the max-flow min-cut theorem to accommodate situations with lower bounds.

5.   THE MINIMUM COST FLOW PROBLEM

In this section, we consider the minimum cost flow problem and summarize the important algorithmic approaches that researchers have suggested for solving this problem. We consider the following node-arc formulation of the minimum cost flow problem.

$$\text{Minimize} \quad \sum_{(i,\,j)\,\in\,A} c_{ij}\, x_{ij} \qquad\qquad (5.1a)$$

subject to

$$\sum_{\{j\,:\,(i,\,j)\,\in\,A\}} x_{ij} \;-\; \sum_{\{j\,:\,(j,\,i)\,\in\,A\}} x_{ji} = b(i), \text{ for all } i \in N, \qquad (5.1b)$$

$$0 \le x_{ij} \le u_{ij}, \text{ for each } (i,\,j) \in A. \qquad\qquad (5.1c)$$

We assume that the lower bounds l_{ij} on arc flows are all zero and that arc costs are nonnegative. Let $C = \max \{\, c_{ij} : (i,\,j) \in A \,\}$ and $U = \max [\, \max \{\, |b(i)| : i \in N \},$ $\max \{\, u_{ij} : (i,\,j) \in A \,\} \,]$. The transformations T1 and T3 in Section 2.4 imply that these assumptions cause no loss of generality. We remind the reader of our blanket assumption that all data (cost, supply/demand and capacity) are integral. We also assume that the minimum cost flow problem satisfies the following two conditions.

A5.1. Feasibility Assumption. We assume that $\displaystyle\sum_{i\in N} b(i) = 0$ and that the minimum cost flow problem has a feasible solution. The feasibility of the minimum cost flow problem can be ascertained by solving a maximum flow problem as follows. Introduce a *super source* node s, and a *super sink* node t. For each node i with $b(i) > 0$, add an arc (s, i) with capacity $b(i)$, and for each node i with $b(i) < 0$, add an arc (i, t) with capacity $-b(i)$. Now solve a maximum flow problem from s to t. If the maximum flow value equals

$$\sum_{\{i\,:\,b(i)\,>\,0\}} b(i)$$

then the minimum cost flow problem is feasible; otherwise it is infeasible.

A5.2.   Connectedness Assumption. We assume that the network G contains an uncapacitated directed path (i.e., each arc in the path has infinite capacity) between every

pair of nodes. We impose this condition, if necessary, by adding *artificial* arcs $(1, j)$ and $(j, 1)$ for each $j \in N$ and assigning a large cost and very large capacity to each of these arcs. No such arcs would appear in a minimum cost solution unless the problem contains no feasible solution without artificial arcs.

Our algorithms rely on the concept of residual networks. The residual network $G(x)$ corresponding to a flow x is defined as follows: We replace each arc $(i, j) \in A$ by two arcs (i, j) and (j, i). The arc (i, j) has cost c_{ij} and a *residual capacity* $r_{ij} = u_{ij} - x_{ij}$, and the arc (j, i) has cost $-c_{ij}$ and residual capacity $r_{ji} = x_{ij}$. The residual network consists *only* of arcs with positive residual capacity.

Observe that any directed cycle in the residual network $G(x)$ is an augmenting cycle with respect to the flow x and vice-versa (see Section 2.1 for the definition of augmenting cycle). This equivalence implies the following alternate statement of Theorem 2.4.

Theorem 5.1. *A feasible flow x is an optimum flow if and only if the residual network $G(x)$ contains no negative cost directed cycle.* ∎

The concept of residual networks, however, poses some notational difficulties. For example, if the original network contains both the arcs (i, j) and (j, i), then the residual network may contain two arcs from node i to node j and/or two arcs from node j to node i with possibly different costs. Our notation for arcs assumes that at most one arc joins one node to any other node. By using more complex notation, we can easily treat this more general case. However, we will not change our notation, but will assume that parallel arcs never arise (or, by inserting extra nodes on parallel arcs, we can produce a network without any parallel arcs).

5.1. Optimality Conditions

A flow x is *feasible* if it satisfies the mass balance constraints (5.1b), and the flow bound constraints (5.1c). A dual solution to the minimum cost flow problem is a vector π of *node potentials* and a vector $\bar{c}$ of *reduced costs* defined as $\bar{c}_{ij} = c_{ij} - \pi(i) + \pi(j)$. Since one of the mass balance constraints is redundant, we can set one node potential arbitrarily. We henceforth assume that $\pi(1) = 0$. A pair x, π of flows and node potentials is optimal if it satisfies the following linear programming optimality conditions:

C5.1   x is feasible.

C5.2   If $\bar{c}_{ij} > 0$, then $x_{ij} = 0$.

C5.3   If $\bar{c}_{ij} = 0$, then $0 \leq x_{ij} \leq u_{ij}$.

C5.4   If $\bar{c}_{ij} < 0$, then $x_{ij} = u_{ij}$.

These conditions, when stated in terms of the residual network, simplify to:

C5.5   (Primal feasibility) x is feasible.

C5.6   (Dual feasibility) $\bar{c}_{ij} \geq 0$ for each arc (i, j) in the residual network G(x).

Note that the condition C5.6 subsumes C5.2, C5.3, and C5.4. For, if $\bar{c}_{ij} > 0$ and $x_{ij} > 0$ for some (i, j) in the original network, then the residual network would contain arc (j, i) with $\bar{c}_{ji} = -\bar{c}_{ij}$. But then $\bar{c}_{ji} < 0$, contradicting C5.6. A similar contradiction arises if $\bar{c}_{ij} < 0$ and $c_{ij} < u_{ij}$ for some (i, j) in A.

It is easy to establish the equivalence between these optimality conditions and the conditions stated in Theorem 5.1. Consider any pair x, π of flows and node potentials satisfying C5.5 and C5.6. Let W be any directed cycle in the residual network. Condition

C5.6 implies that $\displaystyle\sum_{(i,\,j)\,\in\,W} \bar{c}_{ij} \geq 0$. Further , $0 \leq \displaystyle\sum_{(i,\,j)\,\in\,W} \bar{c}_{ij} = \displaystyle\sum_{(i,\,j)\,\in\,W} c_{ij} + \displaystyle\sum_{(i,\,j)\,\in\,W} (-\pi(i) + \pi(j))$

$= \displaystyle\sum_{(i,\,j)\,\in\,W} c_{ij}$. Hence the residual network contains no negative cost cycle.

To see the converse, suppose that x is feasible and G(x) does not contain a negative cycle. Hence the shortest distances from node 1, d(i) , with respect to the arc lengths c_{ij} are well defined. The shortest path optimality condition C3.2 implies that d(j) $\leq$ d(i) + c_{ij} for all (i, j) in G(x). Let $\pi = -d$. Then $0 \leq c_{ij} + d(i) - d(j) = c_{ij} - \pi(i) + \pi(j) = \bar{c}_{ij}$ for all (i, j) in G(x). Hence the pair x, π satisfies C5.5 and C5.6.

5.2.   Relationship to Shortest Path and Maximum Flow Problems

The minimum cost flow problem generalizes both the shortest path and maximum flow problems. The shortest path problem from node s to all other nodes can be formulated as a minimum cost flow problem by setting b(1) = (n − 1) , b(i) = −1 for all i $\neq$ s, and $u_{ij} = \infty$ for each (i, j) $\in$ A (in fact, setting u_{ij} equal to any integer greater

than $(n - 1)$ will suffice if we wish to maintain finite capacities). Similarly, the maximum flow problem from node s to node t can be transformed to the minimum cost flow problem by introducing an additional arc (t, s) with $c_{ts} = -1$ and $u_{ts} = \infty$ (in fact, $u_{ts} = m \cdot \max \{u_{ij} : (i, j) \in A\}$ would suffice), and setting $c_{ij} = 0$ for each arc $(i, j) \in A$. Thus, algorithms for the minimum cost flow problem solve both the shortest path and maximum flow problems as special cases.

Conversely, algorithms for the shortest path and maximum flow problems are of great use in solving the minimum cost flow problem. Indeed, many of the algorithms for the minimum cost flow problem either explicitly or implicitly use shortest path and/or maximum flow algorithms as subroutines. Consequently, improved algorithms for these two problems have led to improved algorithms for the minimum cost flow problem. This relationship will be more transparent when we discuss algorithms for the minimum cost flow problem. We have already shown in Section 5.1 how to obtain an optimum dual solution from an optimum primal solution by solving a single shortest path problem. We now show how to obtain an optimal primal solution from an optimal dual solution by solving a single maximum flow problem.

Suppose that π is an optimal dual solution and $\overline{c}$ is the vector of reduced costs. We define the *cost-residual network* $G^* = (N, A^*)$ as follows. The nodes in G^* have the same supply/demand as the nodes in G. Any arc $(i, j) \in A^*$ has an upper bound u_{ij}^* as well as a lower bound l_{ij}^*, defined as follows:

(i)   For each (i, j) in A with $\overline{c}_{ij} > 0$, A^* contains an arc (i, j) with $u_{ij}^* = l_{ij}^* = 0$.

(ii)   For each (i, j) in A with $\overline{c}_{ij} < 0$, A^* contains an arc (i, j) with $u_{ij}^* = l_{ij}^* = u_{ij}$.

(iii)   For each (i, j) in A with $\overline{c}_{ij} = 0$, A^* contains an arc (i, j) with $u_{ij}^* = u_{ij}$ and $l_{ij}^* = 0$.

The lower and upper bounds on arcs in the cost-residual network G^* are defined so that any flow in G^* satisfies the optimality conditions C5.2-C5.4. If $\overline{c}_{ij} > 0$ for some $(i, j) \in A$, then condition C5.2 dictates that $x_{ij} = 0$ in the optimum flow. Similarly, if $\overline{c}_{ij} < 0$ for some $(i, j) \in A$, then C5.4 implies the flow on arc (i, j) must be at the arc's upper bound in the optimum flow. If $\overline{c}_{ij} = 0$, then any flow value will satisfy the condition C5.3.

Now the problem is reduced to finding a flow in the cost-residual network that satisfies the lower and upper bound restrictions of arcs and, at the same time, meets the supply/demand constraints of the nodes. We first eliminate the lower bounds of arcs as

described in Section 2.4 and then transform this problem to a maximum flow problem as described in assumption A5.1. Let x^* denote the maximum flow in the transformed network. Then x^*+l^* is an optimum feasible solution of the minimum cost problem in G.

5.3. Negative Cycle Algorithm

Operations researchers, computer scientists, electrical engineers and many others have extensively studied the minimum cost flow problem and have proposed a number of different algorithms to solve this problem. Notable examples are the negative cycle, successive shortest path, primal-dual, out-of-kilter, primal simplex and scaling-based algorithms. In this and the following sections, we discuss most of these important algorithms for the minimum cost flow problem and point out the relationships between them. We first consider the negative cycle algorithm.

The negative cycle algorithm always maintains a primal feasible solution x and strives to attain dual feasibility. It does so by identifying negative cost cycles in the residual network $G(x)$ and augmenting flows in these cycles. The algorithm terminates when the residual network contains no negative cost cycles. Theorem 5.1 implies that when the algorithm terminates, it has found a minimum cost flow.

algorithm NEGATIVE CYCLE;
begin
      establish a feasible flow x in the network;
      while $G(x)$ contains a negative cycle **do**
      begin
            use some algorithm to identify a negative cycle W;
            let $\delta := \min \{r_{ij} : (i, j) \in W\}$;
            augment δ units of flow along cycle W and update $G(x)$;
      end;
end;

A feasible flow in the network can be found by solving a maximum flow problem. One algorithm for identifying a negative cost cycle is the label correcting algorithm for the shortest path problem, described in Section 3.4, which requires $O(nm)$

time to identify a negative cycle. Every iteration reduces the flow cost by at least one unit. Since mCU is an upper bound on an initial flow cost and zero is a lower bound on an optimum flow cost, the algorithm terminates after at most $O(mCU)$ iterations and requires $O(nm^2CU)$ time in total.

This algorithm can be improved in the following three ways (which we merely briefly summarize).

(i) Identifying a negative cost cycle in effort much less than $O(nm)$ time. The simplex algorithm (to be discussed later) nearly achieves this objective. It maintains a tree solution and node potentials that enable it to identify a negative cost cycle in $O(m)$ effort. However, due to degeneracy, the simplex algorithm cannot necessarily send a positive amount of flow along this cycle.

(ii) Identifying a negative cost cycle with maximum improvement in the objective function value. The improvement in the objective function due to the augmentation along a cycle W is $\left(\sum_{(i,j)\in W} c_{ij} \right) \left(\min \{r_{ij} : (i,j) \in W\} \right)$. Let x be some flow and x^* be an optimum flow. The augmenting cycle theorem (Theorem 2.3) implies that x^* equals x plus the flow on at most m augmenting cycles with respect to x. Further, improvements in cost due to flow augmentations on these augmenting cycles sum to cx^*-cx. Consequently, at least one augmenting cycle with respect to x must improve the objective function by at least $(cx^*-cx)/m$. Hence if the algorithm always augments flow along the cycle with maximum improvement, then Lemma 1.1 implies that the method would obtain the optimum flow within $O(m \log mCU)$ iterations. Finding a maximum improvement cycle is a difficult problem, but a modest variation of this approach yields a polynomial time algorithm for the minimum cost flow problem.

(iii) Identifying a negative cost cycle with minimum mean cost. We define the *mean cost* of a cycle as its cost divided by the number of arcs it contains. A *minimum mean cost* is a cycle whose mean cost is as small as possible. It is possible to identify a minimum mean cycle in $O(nm)$ or $O(\sqrt{n}\, m \log nC)$ time. Recently, researchers have shown that if the negative cycle algorithm always augments the flow along a minimum mean cycle, then from one iteration to the next the minimum mean cycle value is nondecreasing; moreover, it increases by a factor of $1-(1/n)$ within m iterations. Since the mean cost of the minimum mean (negative) cycle is bounded from below by -C and

bounded from above by $-1/n$, Lemma 1.1 implies that this algorithm will terminate in $O(nm \log nC)$ iterations.

5.4. Successive Shortest Path Algorithm

The successive shortest path algorithm maintains dual feasibility of the solution at every step and strives to attain primal feasibility. It maintains a solution x that satisfies the nonnegativity and capacity constraints, but violates the supply/demand constraints of the nodes. At each step, the algorithm selects a node i with extra supply and a node j with unfulfilled demand and sends flow from i to j along a shortest path in the residual network. The algorithm terminates when the current solution satisfies all the supply/demand constraints.

A *pseudoflow* is a function $x : A \rightarrow R$ satisfying only the capacity and nonnegativity constraints. For any pseudoflow x, we define the *imbalance* of node i as

$$e(i) = b(i) + \sum_{\{j:\ (j,\ i)\ \in\ A\}} x_{ji} - \sum_{\{j:\ (i,j)\ \in\ A\}} x_{ij}, \text{ for all } i \in N.$$

If $e(i) > 0$ (resp., $e(i) < 0$) for some node i, then $e(i)$ is called the *excess* (resp., *deficit*) of node i. A node i with $e(i) = 0$ is called *balanced*. Let S and T denote the sets of excess and deficit nodes respectively. The residual network corresponding to a pseudoflow is defined similarly to the residual network for a flow.

The successive shortest path algorithm successively augments the flow along shortest paths computed with respect to the reduced costs $\bar{c}_{ij}$. Observe that for any

directed path P from a node k to a node l, $\sum_{(i,\ j)\ \in\ P} \bar{c}_{ij} = \sum_{(i,\ j)\ \in\ P} c_{ij} - \pi(l) + \pi(k)$. Hence the

node potentials change all path lengths by a constant amount, and the shortest path with respect to c_{ij} is the same as the shortest path with respect to $\bar{c}_{ij}$. The correctness of the successive shortest path algorithm rests on the following result.

Lemma 5.1. *Suppose a pseudoflow x satisfies the dual feasibility condition C5.6 with respect to the node potentials π. Furthermore, suppose that x' is obtained from x by sending flow along a shortest path from a node k to a node l in G(x). The x' also satisfies the dual feasibility conditions with respect to some node potentials.*

Proof. Since x satisfies the dual feasibility conditions with respect to the node potentials π, we have $\overline{c}_{ij} \geq 0$ for all (i, j) in $G(x)$. Let $d(v)$ denote the shortest path distances from node k to any node v in $G(x)$ with respect to the arc lengths $\overline{c}_{ij}$. We claim that x also satisfies the dual feasibility conditions with respect to the potentials $\pi' = \pi - d$. The shortest path optimality conditions (i.e., C3.2) imply that

$$d(j) \leq d(i) + \overline{c}_{ij}, \text{ for all } (i, j) \text{ in } G(x).$$

Substituting $\overline{c}_{ij} = c_{ij} - \pi(i) + \pi(j)$ in these conditions and using $\pi'(i) = \pi(i) - d(i)$ yields

$$\overline{c}_{ij}' = c_{ij} - \pi'(i) + \pi'(j) \geq 0 \text{ for all } (i, j) \text{ in } G(x).$$

Hence x satisfies C5.6 with respect to the node potentials π'. Next note that $\overline{c}_{ij}' = 0$ for every arc (i, j) on the shortest path P from node k to node l, since $d(j) = d(i) + \overline{c}_{ij}$ for every arc $(i, j) \in P$ and $\overline{c}_{ij} = c_{ij} - \pi(i) + \pi(j)$.

We are now in a position to prove the lemma. Augmenting flow along any arc in P maintains the dual feasibility condition C5.6 for this arc. Augmenting flow on an arc (i, j) may add its reversal (j, i) to the residual network. But since $\overline{c}_{ij} = 0$ for each arc $(i, j) \in P$, $\overline{c}_{ji} = 0$, and so arc (j, i) also satisfies C5.6. ∎

The node potentials play a very important role in this algorithm. Besides using them to prove the correctness of the algorithm, we use them to ensure that the arc lengths are nonnegative, thus enabling us to solve the shortest path subproblems more efficiently. The following formal statement of the successive shortest path algorithm summarizes the steps of this method.

algorithm SUCCESSIVE SHORTEST PATH;

begin

    set x : = 0 and π : = 0;

    compute imbalances e(i) and initialize the sets S and T;

    while S $\neq \varnothing$ **do**

    begin

        select a node k $\in$ S and a node $l \in$ T;

        determine shortest path distances d(j) from node k to all

          other nodes in G(x) with respect to the residual costs $\bar{c}_{ij}$;

        let P denote a shortest path from k to l;

        update π : = π - d;

        let δ : = min [e(k), -e(l), min { r_{ij} : (i, j) $\in$ P }];

        augment δ units of flow along the path P;

        update x, S and T;

    end;

end;

To initialize the algorithm, we set x = 0, which is a feasible pseudoflow and satisfies C5.6 with respect to the node potentials π = 0 since, by assumption, all arc lengths are nonnegative. Also, if S $\neq \varnothing$ then T $\neq \varnothing$ because the sum of excesses always equals the sum of deficits. Further, the connectedness assumption implies that the residual network G(x) contains a directed path from node k to node l. Each iteration of this algorithm solves a shortest path problem with nonnegative arc lengths and reduces the supply of some node by at least one unit. Consequently, if U is an upper bound on the largest supply of any node, the algorithm terminates in at most nU iterations. Since the arc costs $\bar{c}_{ij}$ are nonnegative, the shortest path problem at each iteration can be solved using Dijkstra's algorithm. So the overall complexity of this algorithm is O(nU· S(n, m, C)), where S(n, m, C) is the time taken by Dijkstra's algorithm. Currently, the best strongly polynomial bound to implement Dijkstra's algorithm is O(m + n log n) and the best (weakly) polynomial bound is O(min {m log log C, m + n$\sqrt{\log C}$ }). The successive shortest path algorithm is pseudopolynomial since it is polynomial in n, m and the largest supply U. The algorithm is, however, polynomial for the assignment problem, a special case of the minimum cost flow problem for which U = 1. In Section 5.7, we will develop a polynomial time algorithm for the minimum cost flow problem using the successive shortest path algorithm in conjunction with scaling.

5.5. Primal-Dual and Out-of-Kilter Algorithms

The *primal-dual algorithm* is very similar to the successive shortest path problem, except that instead of sending flow on only one path during an iteration, it might send flow along many paths. To explain the primal-dual algorithm, we transform the minimum cost flow problem into a single-source and single-sink problem (possibly by adding nodes and arcs as in the assumption A5.1). At every iteration, the primal-dual algorithm solves a shortest path problem from the source to update the node potentials (i.e., as before, each $\pi(j)$ becomes $\pi(j) - d(j)$) and then solves a maximum flow problem to send the maximum possible flow from the source to the sink *using only arcs with zero reduced cost*. The algorithm guarantees that the excess of some node strictly decreases at each iteration, and also assures that the node potential of the sink strictly decreases. The latter observation follows from the fact that after we have solved the maximum flow problem, the network contains no path from the source to the sink in the residual network consisting entirely of arcs with zero reduced costs; consequently, in the next iteration $d(t) \geq 1$. These observations give a bound of min $\{nU, nC\}$ on the number of iterations since the magnitude of each node potential is bounded by nC. This bound is better than that of the successive shortest path algorithm, but, of course, the algorithm incurs the additional expense of solving a maximum flow problem at each iteration. Thus the algorithm has an overall complexity of O(min (nU S(n, m, C), nC M(n, m, U)), where S(n, m, C) and M(n, m, U) respectively denote the solution times of shortest path and maximum flow algorithms.

The successive shortest path and primal-dual algorithms maintain a solution that satisfies the dual feasibility conditions and the flow bound constraints, but that violates the mass balance constraints. These algorithms iteratively modify the flow and potentials so that the flow at each step comes closer to satisfying the mass balance constraints. However, we could just as well have violated other constraints at intermediate steps. The *out-of-kilter algorithm* satisfies only the mass balance constraints and may violate the dual feasibility conditions and the flow bound restrictions. The basic idea is to drive the flow on an arc (i, j) to u_{ij} if $\bar{c}_{ij} < 0$, drive the flow to zero if $\bar{c}_{ij} > 0$, and to permit any flow between 0 and u_{ij} if $\bar{c}_{ij} = 0$. The *kilter number*, represented by k_{ij}, of an arc (i, j) is defined as the minimum increase or decrease in the flow necessary to satisfy its flow bound constraint and dual feasibility condition. For example, for an arc (i, j) with $\bar{c}_{ij} > 0$, $k_{ij} = |x_{ij}|$ and for an arc (i, j) with $\bar{c}_{ij} < 0$, $k_{ij} = |u_{ij} - x_{ij}|$. An arc with $k_{ij} = 0$ is said to be *in-kilter*. At each iteration, the

out-of-kilter algorithm reduces the kilter number of at least one arc; it terminates when all arcs are in-kilter. Suppose the kilter number of an arc (i, j) would decrease by increasing flow on the arc. Then the algorithm would obtain a shortest path P from node j to node i in the residual network and augment at least one unit of flow in the cycle $P \cup \{(i, j)\}$. The proof of the correctness of this algorithm is similar to but more detailed than that of the successive shortest path algorithm.

5.6.   Network Simplex Algorithm

The network simplex algorithm for the minimum cost flow problem is a specialization of the bounded variable primal simplex algorithm for linear programming. The special structure of the minimum cost flow problem offers several benefits, particularly, streamlining of the simplex computations and eliminating the need to explicitly maintain the simplex tableau. The tree structure of the basis (see Section 2.3) permits the algorithm to achieve these efficiencies. The advances made in the last two decades for maintaining and updating the tree structure efficiently have substantially improved the speed of the algorithm. The performance of the simplex algorithm has also benefited from various heuristic rules for identifying entering variables determined through extensive empirical testings. Though no version of the primal network simplex algorithm is known to run in polynomial time, its best implementations are empirically comparable to or better than other minimum cost flow algorithms. (There is one variant of the dual simplex algorithm that runs in polynomial time, but its treatment is beyond the scope of this chapter.)

In this section, we describe the network simplex algorithm in detail. We first define the concept of a *basis structure* and describe a data structure to store and manipulate the basis, which is a spanning tree. We then show how to compute arc flows and node potentials for any basis structure. We next discuss how to perform various simplex operations such as the selection of entering arcs, leaving arcs and pivots using this data structure. Finally, we show how to guarantee the finiteness of the network simplex algorithm.

The simplex algorithm maintains a basic feasible solution at each stage. A basic solution of the minimum cost flow problem is defined by a triple (B, L, U); B, L and U partition the arc set A. The set B denotes the set of *basic arcs*, i.e., arcs of a spanning tree, and L and U respectively denote the sets of *nonbasic arcs* at their lower and upper bounds. We refer to the triple (B, L, U) as a *basis structure*. A basis structure (B, L, U) is called *feasible* if by setting $x_{ij} = 0$ for each $(i, j) \in L$, and setting $x_{ij} = u_{ij}$ for each $(i, j) \in U$, the problem has a feasible solution satisfying (5.1b) and (5.1c). A feasible basis structure (B, L, U) is called an *optimum* basis structure if it is possible to obtain a set of node potentials π so that the reduced costs satisfy the following optimality conditions:

$$\overline{c}_{ij} = 0 \text{ , for each } (i, j) \in B, \tag{5.2}$$

$$\overline{c}_{ij} \geq 0 \text{ , for each } (i, j) \in L, \tag{5.3}$$

$$\overline{c}_{ij} \leq 0 \text{ , for each } (i, j) \in U. \tag{5.4}$$

These optimality conditions have a nice economic interpretation. We shall see a little later that if $\pi(1) = 0$, then equations (5.2) imply that $-\pi(j)$ denotes the length of a tree path in B from node 1 to node j. Then, $\overline{c}_{ij} = c_{ij} - \pi(i) + \pi(j)$ for a nonbasic arc (i, j) in L denotes the change in the cost of flow achieved by sending one unit of flow through the tree path from node 1 to node i, through the arc (i, j), and then returning the flow along the tree path from node j to node 1. The condition (5.3) implies that this circulation of flow is not profitable for any nonbasic arc in L. The condition (5.4) has a similar interpretation.

The network simplex algorithm maintains a feasible basis structure at each iteration and successively improves the basis structure until it becomes an optimum basic structure. The following algorithmic description specifies the essential steps of the procedure.

```
algorithm NETWORK SIMPLEX;
begin
        determine an initial basic feasible flow x and the corresponding
            basis structure (B, L, U);
        compute node potentials for this basis structure;
        while some arc violates the optimality conditions do
        begin
                select an entering arc (k, l) violating the optimality conditions;
                increase flow on arc (k, l) and determine the leaving arc (p, q);
                perform a basis exchange and update node potentials;
        end;
end;
```

In the following discussion, we describe the various steps performed by the network simplex algorithm in greater detail.

Obtaining Initial Basis Structure

Our connectedness assumption A5.2 provides one way of obtaining an initial basic feasible solution. We have assumed that for every node $j \in N - \{1\}$, the network contains arcs $(1, j)$ and $(j, 1)$ with sufficiently large costs and capacities. The initial basis B includes the arc $(1, j)$ with flow $-b(j)$ if $b(j) \leq 0$ and arc $(j, 1)$ with flow $b(j)$ if $b(j) > 0$. The set L consists of the remaining arcs, and the set U is empty. The node potentials for this basis are easily computed using (5.2), as we will see later.

Maintaining the Tree Structure

The specialized network simplex algorithm is possible because of the spanning tree property of the basis. The algorithm requires the tree to be represented so that the simplex algorithm can perform operations efficiently and update the representation quickly when the basis changes. We next describe one such tree representation.

We consider the tree as "hanging" from a specially designated node, called the *root*. We assume that node 1 is the root node. See Figure 5.1 for an example of the tree. We associate three indices with each node i in the tree: a *predecessor* index *pred(i)*, a *depth* index *depth(i)*, and a *thread* index, *thread(i)*. Each node i has a unique path connecting it to the root. The predecessor index stores the first node in that path (other than node i) and the depth index stores the number of arcs in the path. For the root

node these indices are zero. The Figure 5.1 shows an example of these indices. Note that by iteratively using the predecessor indices, we can enumerate the path from any node to the root node. We say that *pred(i)* is the *predecessor* of node i and i is a *successor* of node *pred(i)*. The *descendents* of a node i consist of the node i itself, its successors, successors of its successors, and so on. For example, the arc set {5, 6, 7, 8, 9} contains the descendents of node 5 in Figure 5.1. A node with no successors is called a *leaf* node. In Figure 5.1, nodes 4, 7, 8, and 9 are leaf nodes.

The thread indices define a *traversal* of the tree, a sequence of nodes that walks or threads its way through the nodes of the tree, starting at the root and visiting nodes in a "top to bottom" and "left to right" order, and then finally returning to the root. The thread indices can be formed by performing a depth first search of the tree as described in Section 1.6 and setting the thread of a node to be the node labeled after the node itself. For our example, this sequence would read 1-2-5-6-8-9-7-3-4-1 (see the dotted lines in Figure 5.1). For each node i, thread(i) specifies the next node in the traversal visited after node i. This traversal satisfies the following two properties: (i) the predecessor of each node appears in the sequence before the node itself; and (ii) the descendents of any node are consecutive elements in the traversal. The thread provides a particularly convenient means for visiting (or finding) all descendants of a node i: We simply follow the thread from node i, recording the nodes visited until the depth of the visited node becomes at least as large as node i. For example, starting at node 5, we visit nodes 6, 8, 9, and 7 in order, which are the descendants of node 5, and then visit node 3. Since its depth equals that of node 5, we know that we have left the "descendant tree" lying below node 5. As we will see, finding the descendant tree of a node efficiently adds significantly to the efficiency of the simplex method.

The simplex method has two basic steps: (i) determining the node potentials of a given basis structure; and (ii) computing the arc flows for a given basis structure. We now describe how these steps can be performed efficiently using the tree indices.

Computing Node Potentials and Flows for a Given Basis Structure

We first consider the problem of computing node potentials π for a given basis structure (B, L, U). We assume that $\pi(1) = 0$. Note that the value of one node potential can be set arbitrarily since one constraint in (5.1b) is redundant. We compute the remaining node potentials using the conditions that $\overline{c}_{ij} = 0$ for each arc (i, j) in B. These conditions can alternatively be stated as
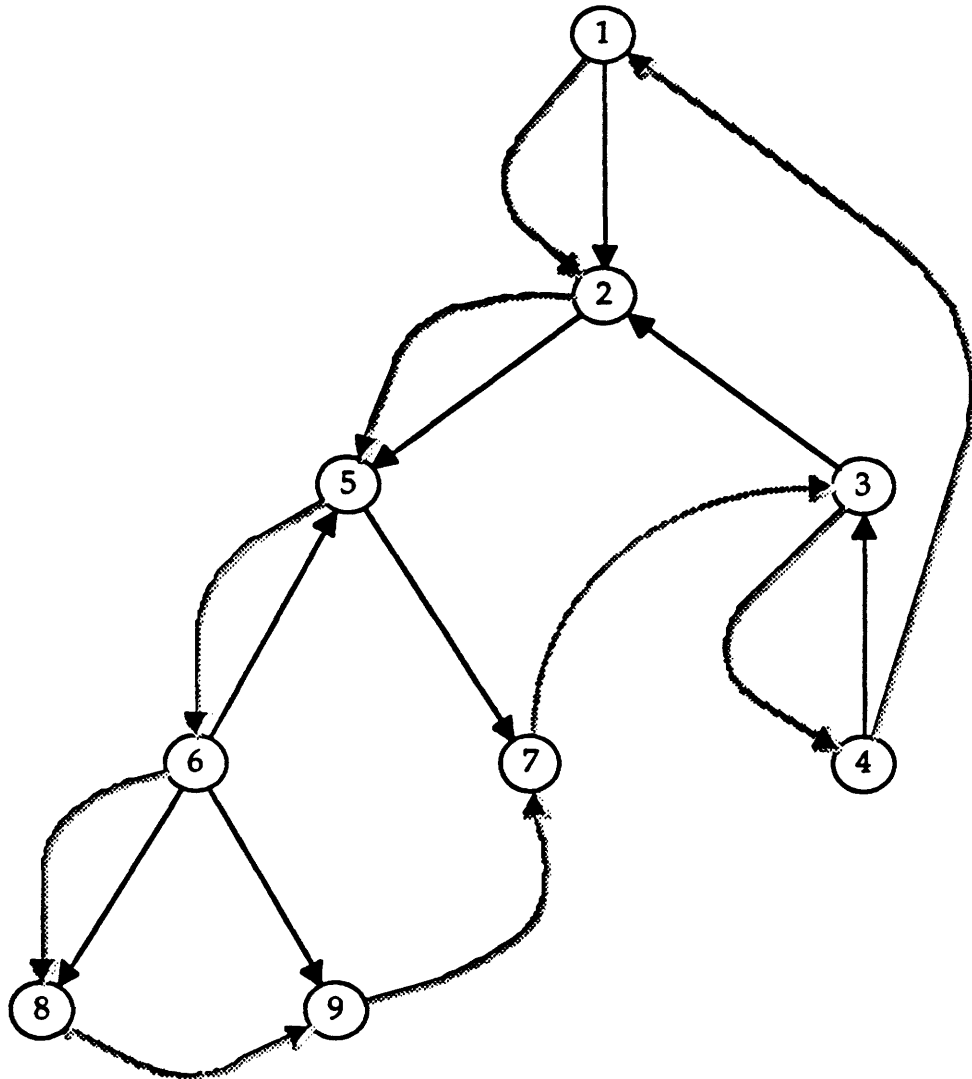
i	1	2	3	4	5	6	7	8	9
pred(i)	0	1	2	3	2	5	5	6	6
depth(i)	0	1	2	3	2	3	3	4	4
thread(i)	2	5	4	1	6	8	3	9	7

Figure 5.1. Example of a rooted tree and tree indices.

$$\pi(j) = \pi(i) - c_{ij}, \text{ for every arc } (i, j) \in B. \tag{5.5}$$

The basic idea is to start at node 1 and fan out along the tree arcs using the thread indices to compute other node potentials. This traversal assures that whenever a node j is visited, the potential of node $i = pred(j)$ has already been evaluated; hence $\pi(j)$ can be computed using (5.5). The thread indices allow us to compute all node potentials in $O(n)$ time using the following method.

procedure COMPUTE POTENTIALS;
begin
    $\pi(1) : = 0$;
    $j : = thread(1)$;
    while $j \neq 1$ **do**
    begin
        $i : = pred(j)$;
        if $(i, j) \in A$ **then** $\pi(j) : = \pi(i) - c_{ij}$;
        if $(j, i) \in A$ **then** $\pi(j) : = \pi(i) + c_{ji}$;
        $j : = thread(j)$;
    end;
end;

A similar problem is to compute flows on basic arcs for a given basis structure (B, L, U). This problem can be solved in the reverse order: start at the leaf nodes and move in toward the root using the predecessor indices, while computing flows on arcs encountered along the way. The following procedure accomplishes this task.

```
procedure COMPUTE FLOWS;
begin
        set e(i) : = b(i) for all i ∈ N;
        let T be the basis tree;
        for each (i, j) ∈ U do
            set x_ij : = u_ij, subtract u_ij from e(i) and add u_ij to e(j);
        while T ≠ {1} do
        begin
            select a leaf node j in the subtree T;
            i : = pred(j);
            if (i, j) ∈ T then x_ij : = -e(j);
            else x_ji : = e(j);
            add e(j) to e(i);
            delete node j and arc incident on it from T;
        end;
end;
```

One way of identifying leaf nodes in T is to select nodes in the reverse order of the thread indices. This task can be accomplished in $O(n)$ time by pushing all the nodes into a stack in order of their appearance on the thread, and then popping them one at a time. Note that in the thread traversal, each node appears prior to its descendants. Hence the reverse thread traversal examines each node after its descendants.

Now consider the steps of the method. The arcs in the set U must carry flow equal to their capacity. Thus, we set $x_{ij} = u_{ij}$ for these arcs. This assignment creates an additional demand of u_{ij} units at node i and makes the same amount available at node j. This effect of setting $x_{ij} = u_{ij}$ explains the initial adjustments in the supply/demand of nodes. The manner for updating e(j) implies that each e(j) represents the sum of the adjusted supply/demand of nodes in the subtree hanging from node j. Since this subtree is connected to the rest of the tree only by the arc (i, j) (or (j, i)), this arc must carry -e(j) (or e(j)) units of flow to satisfy the adjusted supply/demand of nodes in the subtree.

The procedure Compute Flows essentially solves the system of equations $\bar{B}x = b$, in which $\bar{B}$ represents the columns in the node-arc incidence matrix N corresponding to the spanning tree B. Since $\bar{B}$ is a lower triangular matrix (see Theorem 2.6 in Section 2.3), it is possible to solve these equations by forward substitution, which is precisely

what the algorithm does. Similarly, the procedure Compute Potentials solves the system of equations $\pi \bar{B} = c$ by back substitution.

Entering Arc

Any nonbasic arc at its lower bound with a negative reduced cost, or at its upper bound with a positive reduced cost, is eligible to enter the basis. These arcs violate conditions (5.3) or (5.4). The method for selecting from these eligible arcs has a major effect on the performance of the simplex algorithm. An implementation that selects the arc that violates the optimality condition the most, i.e., has the largest value of $| \bar{c}_{ij} |$ from among such arcs, might require the fewest number of iterations in practice, but must examine each arc at each iteration, which is very time-consuming. On the other hand, examining the arc list cyclically and selecting the *first* arc that violates the optimality condition would quickly find the entering arc, but might require a relatively large number of iterations due to the poor arc choice. The most successful implementations use a *candidate list* approach that strikes an effective compromise between these two strategies. This approach also offers sufficient flexibility for fine tuning to special problem classes.

The algorithm maintains a candidate list of arcs violating the optimality conditions, selecting arcs in a two-phase procedure consisting of *major* iterations and *minor* iterations. In a major iteration, we construct the candidate list. We examine arcs emanating from nodes, one node at a time, adding to the candidate list the arc emanating from node i (if any) that violates the optimality condition. We repeat this selection process for nodes i+1, i+2, ... until either we have examined all nodes or the list has reached its maximum allowable size. The next major iteration begins with the node where the previous major iteration ended. In other words, the algorithm examines nodes cyclically as it adds arcs adjacent to them to the candidate list.

Once the algorithm has formed the candidate list in a major iteration, it performs minor iterations, scanning all candidate arcs and choosing a nonbasic arc from this list that violates the optimality condition the most to enter the basis. As we scan the arcs, we update the candidate list by removing those arcs that no longer violate the optimality conditions. Once the list becomes empty or we have reached a specified limit on the number of minor iterations to be performed at each iteration, we rebuild the list with another major iteration.

Leaving Arc

Suppose we select the arc (k, l) as the entering arc. The addition of this arc to the basis B forms exactly one cycle W. We define the orientation of W as the same as that of (k, l) if $(k, l) \in L$, and opposite to the orientation of (k, l) if $(k, l) \in U$. Let $\overline{W}$ and $\underline{W}$ respectively denote the sets of arcs in W along and opposite to the cycle's orientation. Sending additional flow around W in the direction of its orientation strictly decreases the cost of the current solution. We change the flow as much as possible until one of the arcs in the cycle W reaches its lower or upper bound; this arc leaves the basis. The maximum change δ_{ij} in flow permitted on an arc $(i, j) \in W$ that satisfies the flow bound constraints is

$$\delta_{ij} = \begin{cases} u_{ij} - x_{ij}, & \text{if } (i, j) \in \overline{W}, \\ x_{ij}, & \text{if } (i, j) \in \underline{W}. \end{cases}$$

We send $\delta = \min \{\delta_{ij} : (i, j) \in W\}$ units of flow around W, and select an arc (p, q) with $\delta_{pq} = \delta$ as the leaving arc. The crucial operation in this step is to identify the cycle W. If P(i) denotes the unique path in the basis from any node i to the root node, then this cycle consists of the arcs $\{(\{(k, l)\} \cup P(k) \cup P(l)) - (P(k) \cap P(l))\}$. In other words, W consists of the arc (k, l) and the disjoint portions of P(k) and P(l). Using predecessor indices alone permits us to identify the cycle W as follows. Start at node k and using predecessor indices trace the path from this node to the root and label the nodes in this path. Repeat the same operation for node l until encountering a node already labeled, say w. The node w is the first common ancestor of nodes k and l. The cycle W contains the portions of the path P(k) and P(l) up to node w, along with the arc (k, l). This method is efficient, but it can be improved. The drawback of this method is that the portion of the path P(k) lying between node w and the root is not in the cycle, but the method still backtracks along these arcs. The simultaneous use of depth and predecessor indices, as shown below, eliminates this extra work.

```
procedure IDENTIFY CYCLE;
begin
        set i : = k and j : = l;
        while i ≠ j do
        begin
                if depth(i) > depth(j) then i : = pred(i)
                else if depth(j) > depth(i) then j : = pred(j)
                        else i : = pred(i) and j : = pred(j);
        end;
        set w : = i;
end;
```

This procedure can be easily modified so that it, besides determining the first common ancestor w of nodes k and l, also determines the flow δ that can be augmented along W. Using predecessor indices to again traverse the cycle W, the algorithm can then update flows on arcs. This operation takes $O(n)$ time in the worst-case, but typically examines only a small subset of nodes.

Basis Exchange

In the terminology of the simplex method, a basis exchange is a pivot operation. If $\delta = 0$, then the pivot is said to be *degenerate*; otherwise it is *nondegenerate*. A basis is called *degenerate* if flow on some basic arc equals its lower or upper bound, and *nondegenerate* otherwise. Observe that a degenerate pivot occurs only in a degenerate basis.

Each time the method exchanges an entering arc (k, l) for a leaving arc (p, q), it must update the basis structure. If the leaving arc is the same as the entering arc, which would happen when $\delta = u_{kl}$, the basis does not change. In this instance, the arc (k,l) merely moves from the set L to the set U, or vice versa. If the leaving arc differs from the entering arc, then more extensive changes are needed. In this instance, the arc (p, q) becomes a nonbasic arc at its lower or upper bound depending upon whether $x_{pq} = 0$ or $x_{pq} = u_{pq}$. Adding (k, l) to the basis and deleting (p, q) from it again yields a basis which is a spanning tree. The node potentials change and can be updated as follows. The deletion of the arc (p, q) from the basis partitions the set of nodes into two subtrees--one, T_1, containing the root node, and the other, T_2, not containing the root node. Note that the subtree T_2 hangs down from node p or node q. The arc (k, l) has one endpoint in T_1

and the other in T_2. As is easy to verify, the condition $c_{ij} - \pi(i) + \pi(j) = 0$ for all arcs in the new basis implies that the potentials of nodes in the subtree T_1 remain unchanged, and the potentials of nodes in the subtree T_2 change by a constant amount. If $k \in T_1$ and $l \in T_2$, then all the node potentials in T_2 change by $-\bar{c}_{kl}$; if $l \in T_1$ and $k \in T_2$, they change by the amount $\bar{c}_{kl}$. The following method, using the thread and depth indices, updates the node potentials quickly.

procedure UPDATE POTENTIALS;
begin
    if $q \in T_2$ **then** $y := q$ **else** $y := p$;
    if $k \in T_1$ **then** change $:= -\bar{c}_{kl}$ **else** change $:= c_{kl}$;
    $\pi(y) := \pi(y) +$ change;
    $z := \text{thread}(y)$;
    while depth$(z) <$ depth(y) **do**
    begin
        $\pi(z) := \pi(z) +$ change;
        $z := \text{thread}(z)$;
    end;
end;

The final step in the basis exchange is to update the various indices. This step is rather involved and we refer the reader to the reference material cited at the end of this chapter for the details. We do note, however, that it is possible to update the tree indices in $O(n)$ time.

Termination

The network simplex algorithm, as just described, moves from one basis structure to another until it obtains a basis structure that satisfies the optimality conditions (5.2)–(5.4). It is easy to show that the algorithm terminates in a finite number of steps if each pivot operation is nondegenerate. Recall that $|\bar{c}_{kl}|$ represents the net decrease in the cost per unit flow sent around the cycle W. During a nondegenerate pivot (in which $\delta > 0$), the new basis structure has a cost $\delta|\bar{c}_{kl}|$ lower than the previous basis structure. Since there are finite number of basis structures and every basis structure has a unique associated cost, the network simplex algorithm will terminate finitely assuming

nondegeneracy. Degenerate pivots, however, pose theoretical difficulties that we address next.

Strongly Feasible Basis

The network simplex algorithm does not necessarily terminate in a finite number of iterations unless we impose an additional restriction on the choice of entering and leaving arcs. Researchers have constructed very small network examples for which poor choices lead to *cycling*, i.e., an infinite repetitive sequence of degenerate pivots. Degeneracy in network problems is not only a theoretical issue, but also a practical one. Computational studies have shown that as many as 90% of the pivot operations in common networks can be degenerate. As we show next, by maintaining a special type of basis, called a *strongly feasible basis*, the simplex algorithm terminates finitely; moreover, it runs faster in practice as well.

Let (B, L, U) be a basis structure of the minimum cost flow problem with integral data. As earlier, we conceive of a basis tree as a tree hanging from the root node. The tree arcs either are *upward pointing* (towards the root) or are *downward pointing* (away from the root). We say that a basis structure (B, L, U) is *strongly feasible* if we can send a positive amount of flow from any node in the tree to the root along arcs in the tree without violating any of the flow bounds. See Figure 5.2 for an example of the strongly feasible basis. Observe that this definition implies that no upward pointing arc can be at its upper bound and no downward pointing arc can be at its lower bound.

The *perturbation technique* is a well-known method for avoiding cycling in the simplex algorithm for linear programming. This technique slightly perturbs the right-hand-side vector so that every feasible basis is nondegenerate and an optimum solution of the perturbed problem can be easily converted to an optimum solution of the original problem. We show that a particular perturbation technique for the network simplex method is equivalent to the combinatorial rule known as the *strongly feasible basis technique*.

The minimum cost flow problem can be perturbed by changing the supply/demand vector b to b+ε. We say that $\varepsilon = (\varepsilon_1, \varepsilon_2, ..., \varepsilon_n)$ is a feasible perturbation if it satisfies the following conditions:

(i) $\varepsilon_i > 0$ for all $i = 1, 2, ..., n$;

(ii) $\displaystyle\sum_{i=2}^{n} \varepsilon_i < 1;$ and

(iii) $\displaystyle\varepsilon_1 = -\sum_{i=2}^{n} \varepsilon_i.$

One possible choice for a feasible perturbation is $\varepsilon_i = 1/n$ for $i = 2, \ldots , n$ (and thus $\varepsilon_1 = -(n - 1)/n$). Another choice is $\varepsilon_i = \alpha^i$ for $i = 2, \ldots , n$, with α chosen as a very small positive number. The perturbation changes the flow on basic arcs. The procedure Compute-Flows, described earlier in this section, implies that perturbation of b by ε changes the flow on basic arcs in the following manner:

1. If (i, j) is a downward pointing arc of tree B and $D(j)$ is the set of descendants of node j, then the perturbation decreases the flow in arc (i, j) by $\displaystyle\sum_{k \in D(j)} \varepsilon_k$. Since $0 < \displaystyle\sum_{k \in D(j)} \varepsilon_k < 1$, the resulting flow is nonintegral and thus nonzero.

2. If (i, j) is an upward pointing arc of tree B and $D(i)$ is the set of descendants of node i, then the perturbation increases the flow in arc (i, j) by $\displaystyle\sum_{k \in D(i)} \varepsilon_k$. Since $0 < \displaystyle\sum_{k \in D(i)} \varepsilon_k < 1$, the resulting flow is nonintegral and thus nonzero.

Theorem 5.2. *For any basis structure (B, L, U) of the minimum cost flow problem, the following statements are equivalent:*

(i)   *(B, L, U) is strongly feasible.*

(ii)   *No upward pointing arc of the basis is at its upper bound and no downward pointing arc of the basis is at its lower bound.*

(iii)   *(B, L, U) is feasible if b is replaced by b+ε, for any feasible perturbation ε.*

(iv)   *(B, L, U) is feasible if b is replaced by b+ε, for the perturbation* $\varepsilon = (-(n-1)/n, 1/n, 1/n, \ldots , 1/n).$

Proof. (i) $\Rightarrow$ (ii). Suppose an upward pointing arc (i, j) is at its upper bound. Then node i cannot send any flow to the root, violating the definition of strongly feasible basis. For the same reason, no downward pointing arc can be at its lower bound.

(ii) ⇒ (iii). Suppose that (ii) is true. As noted earlier, perturbation increases the flow on an upward pointing arc by an amount strictly between 0 and 1. Since the flow on an upward pointing arc is integral and strictly less than the (integral) upper bound, the perturbed solution remains feasible. Similar reasoning shows that after we have perturbed the problem, downward pointing arcs also remain feasible.

(iii) ⇒ (iv). Follows directly because $\varepsilon = (-(n-1)/n, 1/n, 1/n, \ldots, 1/n)$ is a feasible perturbation.

(iv) ⇒ (i). Consider the feasible basis B of the perturbed problem. Each arc in the basis has a positive nonintegral flow. If we consider the same basis tree for the original problem, then as compared to the perturbed solution, flows on the downward pointing arcs increase, flows on the upward pointing arcs decrease and the resulting flows are integral. Consequently, $x_{ij} > 0$ for downward pointing arcs, $x_{ij} < u_{ij}$ for upward pointing arcs, and B is strongly feasible for the original problem. ∎

This theorem shows that maintaining a strongly feasible basis is equivalent to applying the ordinary simplex algorithm to the perturbed problem. This result implies that both approaches obtain exactly the same sequence of basis structures if they use the same rule to select the entering arcs. As a corollary, this equivalence shows that the simplex algorithm maintaining a strongly feasible basis performs at most nmCU pivots. To establish this conclusion, consider the perturbed problem with the perturbation $\varepsilon = (-(n-1)/n, 1/n, 1/n, \ldots, 1/n)$. With this perturbation, the flow on every arc is a multiple of $1/n$. Consequently, every pivot operation augments at least $1/n$ units of flow and therefore decreases the objective function value by at least $1/n$ units. Since mCU is an upper bound on the objective function value of the starting solution and zero is a lower bound on the minimum objective function value, the algorithm will terminate in at most nmCU iterations. Hence the simplex algorithm maintaining a strongly feasible basis is pseudopolynomial.

We can thus maintain strong feasibility by perturbing b by a suitable perturbation ε. However, there is no need to actually perform the perturbation. Instead, we can maintain strong feasibility using a "combinatorial rule" that is equivalent to applying the simplex method after we have imposed the perturbation. We now discuss this rule which will permit degenerate pivots. Figure 5.2 illustrates the discussion.

Combinatorial Version of Perturbation

The network simplex algorithm starts with a strongly feasible basis. The method described earlier to construct the initial basis always gives such a basis. The algorithm selects the leaving arc in a degenerate pivot carefully so that the next basis is also strongly feasible. Suppose that the entering arc (k, l) is at its lower bound and node w is the first common ancestor of nodes k and l. Let W be the cycle formed by adding arc (k, l) to the basis tree. We define the orientation of the cycle along the arc (k, l). After updating the flow, the algorithm identifies the blocking arcs, i.e., those arcs (i, j) in the cycle W for which $\delta_{ij} = \delta$. If the blocking arc is unique, then it leaves the basis. If the cycle contains more than one blocking arc, then the next basis will be degenerate; i.e., some basic arcs will be at their lower or upper bounds. The algorithm selects the leaving arc to be the last blocking arc, say arc (p, q), encountered in traversing W along its orientation starting at node w. We show that this rule guarantees that the next basis is strongly feasible.

To show that the next basis is strongly feasible, we show that in this basis every node in the cycle W can send positive flow to the root node. Notice that since the previous basis was strongly feasible, this conditions ensures that every node can send positive flow to the root node. Let W_1 be the segment of the cycle W between the node w and arc (p, q) when traversing the cycle along its orientation. Further, let $W_2 = W - W_1 - \{(p, q)\}$. Define the orientation of segments W_1 and W_2 along that of W. See Figure 5.2 for the segments W_1 and W_2 for our example. Since arc (p, q) is the last blocking arc in W, every node contained in the segment W_2 can augment positive flow to the root along the orientation of W_2 and via node w. Now consider nodes contained in the segment W_1. If the current pivot was a *nondegenerate* pivot, then the pivot augmented a positive amount of flow along the arcs in W_1; hence every node in the segment W_1 can augment flow back to the root opposite to the orientation of W_1 and via node w. If the current pivot was a *degenerate* pivot, then W_1 is contained in the segment of W between node w and node k, because by the property of strong feasibility no arc in the segment from node l to node w can be a blocking arc in a degenerate pivot. Now observe that before the pivot, every node in W_1 could send positive flow to the root and this must be true after the pivot too, since the pivot does not change flow values. This conclusion completes the proof that the next basis is strongly feasible.

We now study the effect of the basis change on node potentials during a degenerate pivot. Since arc (k, l) enters the basis at its lower bound, $\bar{c}_{kl} < 0$. The leaving arc belongs to the path from node k to node w. Hence node k lies in the subtree T_2 and

the potentials of all nodes in T_2 change by the amount $\bar{c}_{kl} < 0$. Consequently, this degenerate pivot strictly decreases the sum of all node potentials (which by our prior assumptions is integral). Since the sum of all node potentials is bounded from below, the number of successive degenerate pivots are finite.

So far we have assumed that the entering arc is at its lower bound. If the entering arc (k, l) is at its upper bound, then we define the orientation of the cycle W opposite to the orientation of arc (k, l). The criteria to select the leaving arc remains unchanged—the leaving arc is the last blocking arc encountered in traversing W along its orientation starting at node W.

Complexity Results

The strongly feasible basis technique implies some nice theoretical results about the network simplex algorithm implemented using Dantzig's pivot rule, i.e., pivoting in the arc with maximum violation of the optimality condition. This technique also yields polynomial time simplex algorithms for the shortest path and assignment problems.
We have already shown that any version of network simplex algorithm that maintains a strongly feasible basis performs $O(nmCU)$ pivots. Using Dantzig's pivot rule and geometric improvement arguments, we can reduce the number of pivots to $O(nmC \log W)$, in this expression , $W = mCU$. As earlier, we consider the perturbed problem with perturbation $\varepsilon = (-(n-1)/n, 1/n, 1/n, \ldots , 1/n)$. Let z^k denote the objective function of the perturbed minimum cost flow problem at the k-th iteration of the simplex algorithm, x denote the current flows and (B, L, U) denote the basis structure. Let $\Delta > 0$ denote the maximum violation of the optimality condition of any nonbasic arc. If the algorithm next pivots on a nonbasic arc corresponding to the maximum violation, then the objective function value decreases by at least Δ/n units. Hence

$$z^k - z^{k+1} \geq \Delta/n \tag{5.6}$$

We now need an upper bound on the total possible improvement in the objective function after the k-th iteration. It is easy to show that

$$\sum_{(i,\,j)\,\in\,A} \bar{c}_{ij} x_{ij} = \sum_{(i,\,j)\,\in\,A} c_{ij} x_{ij} - \sum_{i\,\in\,N} \pi(i)\, b(i).$$
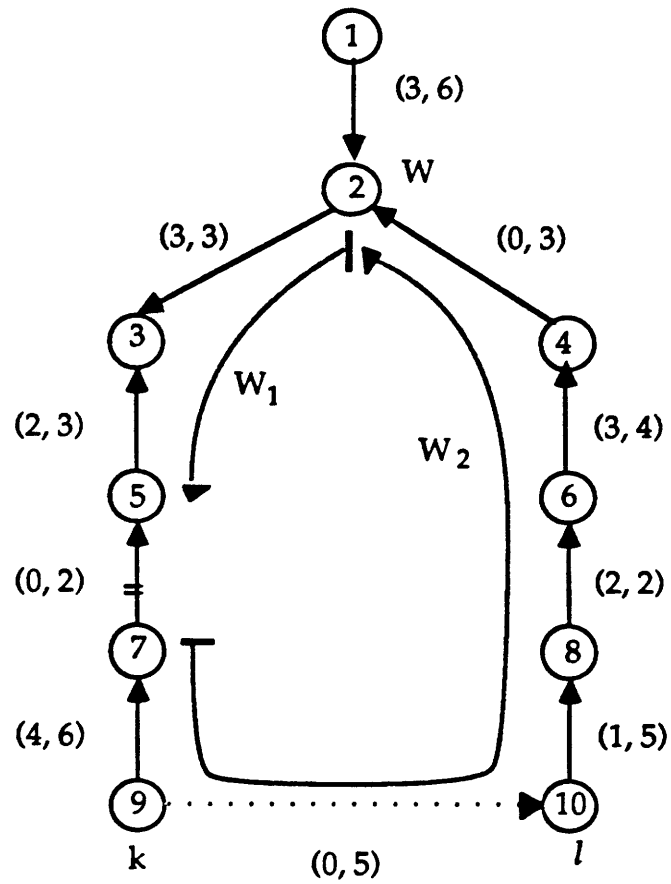
Figure 5.2. A strongly feasible basis. The figure shows the flows and capacities represented as (x_{ij}, u_{ij}). The entering arc is $(9, 10)$; the blocking arcs are $(2, 3)$ and $(7, 5)$; and the leaving arc is $(7, 5)$. This pivot is a degenerate pivot. The segments W_1 and W_2 are as shown.

Consequently, the total improvement with respect to the objective function

$\sum\limits_{(i,\,j)\,\in\,A} c_{ij} x_{ij}$ is equal to the total improvement with respect to the objective

function $\sum\limits_{(i,\,j)\,\in\,A} \bar{c}_{ij} x_{ij}$. Further, the total improvement in the objective

function $\sum\limits_{(i,\,j)\,\in\,A} \bar{c}_{ij} x_{ij}$ is bounded by the total possible improvement in the following

relaxed problem:

$$\text{minimize} \quad \sum\limits_{(i,\,j)\,\in\,A} \bar{c}_{ij} x_{ij}, \tag{5.7a}$$

subject to

$$0 \le x_{ij} \le u_{ij}, \text{ for all } (i,\,j) \in A. \tag{5.7b}$$

We obtain an optimum solution of (5.7) in the basis structure (B, L, U) by setting $x_{ij} = u_{ij}$ for all arcs $(i,\,j) \in L$ with $\bar{c}_{ij} < 0$, by setting $x_{ij} = 0$ for all arcs $(i,\,j) \in U$ with $\bar{c}_{ij} > 0$, and by leaving the flow on basis arcs unchanged. This readjustment of flows decreases the objective function by at most $m\Delta U$. We have thus shown that

$$z^k - z^* \le m\Delta U. \tag{5.8}$$

Combining (5.6) and (5.8) we obtain

$$(z^k - z^{k+1}) \ge \frac{1}{nmU} (z^k - z^*).$$

By Lemma 1.1, if $W = mCU$, the network simplex algorithm terminates in $O(nmU \log W)$ iterations. We summarize our discussion as follows.

Theorem 5.3. *The network simplex algorithm that maintains a strongly feasible basis and uses Dantzig's pivot rule performs $O(nmU \log W)$ pivots.* ∎

This result gives polynomial time bounds for the shortest path and assignment problems since both can be formulated as minimum cost flow problems with $U = n$ and

U = 1 respectively. In fact, the previous arguments and the algorithm can be modified to show that the simplex algorithm solves these problems in $O(n^2 \log C)$ pivots and the algorithm can be implemented to run in $O(nm \log C)$ total time. These results can be found in the references cited at the end of this chapter.

5.7 Right-Hand-Side Scaling Algorithm

Scaling techniques are one of the most effective algorithmic strategies for designing polynomial time algorithms for the minimum cost flow problem. In this subsection, we describe an algorithm based on a right-hand-side scaling (RHS-scaling) technique. The next two subsections will present polynomial algorithms based upon cost scaling, and simultaneous right-hand-side and cost scaling.

The RHS-scaling algorithm is an improved version of the successive shortest path algorithm. The inherent drawback in the successive shortest path problem is that augmentations may carry relatively small amounts of flow, resulting in a fairly large number of augmentations in the worst-case. The RHS-scaling algorithm guarantees that each augmentation carries sufficiently large flow and thereby reduces the number of augmentations substantially. We shall illustrate RHS-scaling on the uncapacitated minimum cost flow problem, i.e., a problem with $u_{ij} = \infty$ for each $(i, j) \in A$. This algorithm can be applied to the capacitated minimum cost flow problem after it has been converted into an uncapacitated problem (as described in Section 2.4).

The algorithm uses the pseudoflow x and the imbalances e(i) as defined in Section 5.4. It performs a number of scaling phases. Much like what we did in the excess scaling algorithm for the maximum flow problem, we let Δ be the least power of 2 satisfying either (i) e(i) < 2Δ for all i, or (ii) e(i) > -2Δ for all i, but not necessarily both. Initially, $\Delta = 2^{\lceil \log U \rceil}$. This definition implies that the sum of excesses (whose magnitude is equal to the sum of deficits) is bounded by $2n\Delta$. Let S(Δ) = { i : e(i) $\geq \Delta$ } and let T(Δ) = { j : e(j) $\leq -\Delta$ }. Then at the beginning of the Δ-scaling phase, either S(2Δ) = $\emptyset$ or T(2Δ) = $\emptyset$. In the given Δ-scaling phase, we perform a number of augmentations, each from a node i $\in$ S(Δ) to a node j $\in$ T(Δ), and each of these augmentations carries Δ units of flow. It follows by the definition of Δ that within n augmentations Δ will decrease by a factor of at least 2. At this point, we begin a new scaling iteration. Hence within O(log U) scaling phases, Δ < 1. By the integrality of data all imbalances are now zero and the algorithm ends with an optimum flow.

The driving force behind this scaling technique is an invariant property (which we will prove later) that each arc flow in the Δ-scaling phase is a multiple of Δ. This flow invariant property and the connectedness assumption (A5.2) ensure that we can always send Δ units of flow from a node in $S(\Delta)$ to a node in $T(\Delta)$. The following algorithmic description is a formal statement of the RHS-scaling algorithm.

algorithm RHS-SCALING;
begin
       let $x := 0$, $e := b$, and π be the shortest path distances in $G(0)$;
       let $\Delta := 2^{\lceil \log U \rceil}$;
       while the network contains a node with flow imbalance **do**
       begin
              $S(\Delta) := \{ i \in N : e(i) \geq \Delta \}$;
              $T(\Delta) := \{ i \in N : e(i) \leq -\Delta \}$;
              while $S(\Delta) \neq \emptyset$ and $T(\Delta) \neq \emptyset$ **do**
              begin
                     select a node $k \in S(\Delta)$ and a node $l \in T(\Delta)$;
                     determine shortest path distances d from node k to all other nodes
                         in the residual network $G(x)$ with respect to the reduced costs
                         $\overline{c}_{ij}$;
                     let P denote the shortest path from node k to node l;
                     update $\pi := \pi - d$;
                     augment Δ units of flow along the path P;
                     update x, $S(\Delta)$ and $T(\Delta)$;
              end;
              $\Delta := \Delta / 2$;
       end;
end;

The RHS-scaling algorithm correctly solves the problem because during the Δ-scaling phase, it is able to send Δ units of flow on the shortest path from a node $k \in S(\Delta)$ to a node $l \in T(\Delta)$. This fact follows from the following result.

Lemma 5.2. The residual capacities of arcs in the residual network are always integer multiples of Δ.

Proof. We use induction on the number of augmentations and scaling phases. The initial residual capacities are a multiple of Δ because they are either 0 or ∞. Each augmentation changes the residual capacities by 0 or Δ units and preserves the inductive hypothesis. A decrease in the scale factor by a factor of 2 also preserves the inductive hypothesis. This result implies the conclusion of the lemma. ∎

Theorem 5.4. The RHS-scaling algorithm correctly computes a minimum cost flow and performs $O(n \log U)$ augmentations.

Proof. The RHS-scaling algorithm is a special case of the successive shortest path algorithm and thus terminates with a minimum cost flow. We show that the algorithm performs at most n augmentations per scaling phase. Since the algorithm requires $1 + \lceil \log U \rceil$ scaling phases, this fact would imply the conclusion of the theorem. At the beginning of the Δ-scaling phase, either $S(2\Delta) = \emptyset$ or $T(2\Delta) = \emptyset$. We consider the case when $S(2\Delta) = \emptyset$. A similar proof applies when $T(2\Delta) = \emptyset$. Observe that $\Delta \leq e(i) < 2\Delta$ for each node $i \in S(\Delta)$. Since each augmentation starts at a node in $S(\Delta)$, ends at a node with a deficit and carries Δ units of flow, it decreases $|S(\Delta)|$ by one. Consequently, each scaling phase can perform at most n augmentations. ∎

This RHS-scaling approach yields an $O(n \log U \cdot S(n, m, C))$ algorithm for the uncapacitated minimum cost flow problem, where $S(n, m, C)$ is the time to solve a shortest path problem with nonnegative arc lengths.

Applying the scaling algorithm directly to the capacitated minimum cost flow problem introduces some subtlety, because Lemma 5.2 does not apply for this situation. The inductive hypothesis fails to be true initially since the residual capacities are 0 or u_{ij}. One method of solving the capacitated minimum cost flow problem is to first transform the capacitated problem to an uncapacitated one using the technique described in Section 2.4. We then apply the RHS-scaling algorithm on the transformed problem. The transformed network contains n+m nodes, and each scaling phase performs at most n+m augmentations. The shortest path problem on the transformed problem can be solved (using some clever techniques) in $S(n, m, C)$ time. Consequently, the RHS-scaling algorithm solves the capacitated minimum cost flow problem in $O(m \log U \cdot S(n, m, C))$ time. A recently developed modest variation of the RHS-scaling algorithm solves the capacitated minimum cost flow problem in $O(m \log n$

(m + n log n)) time. This method is currently the best *strongly* polynomial algorithm for solving the minimum cost flow problem.

5.8. Cost Scaling Algorithm

We now describe a cost scaling algorithm for the minimum cost flow problem. This algorithm can be viewed as a generalization of the preflow-push algorithm for the maximum flow problem.

This algorithm relies on the concept of *approximate optimality*. A flow x is said to be ε-*optimal* for some $\varepsilon > 0$ if some node potentials π satisfy the following conditions.

C5.7 (Primal feasibility) x is feasible.
C5.8. (ε-Dual feasibility) $\bar{c}_{ij} \geq -\varepsilon$ for each arc (i, j) in the residual network G(x).

We refer to these conditions as the ε-*optimality* conditions. These conditions are a relaxation of the original optimality conditions and reduce to C5.5 and C5.6 when ε is 0. The ε-optimality conditions permit $-\varepsilon \leq \bar{c}_{ij} < 0$ for an arc (i, j) at its lower bound and $\varepsilon \geq \bar{c}_{ij} > 0$ for an arc (i, j) at its upper bound, which is a relaxation of the usual optimality conditions. The following facts are useful for the cost scaling algorithm.

Lemma 5.3. *Any feasible flow is ε-optimal for $\varepsilon \geq C$. Any ε-optimal feasible flow for $\varepsilon < 1/n$ is an optimum flow.*

Proof. Clearly, any feasible flow with zero node potentials satisfies C5.8 for $\varepsilon \geq C$. Now consider an ε-optimal flow with $\varepsilon < 1/n$. The ε-dual feasibility conditions imply that for

$$\text{any directed cycle } W \text{ in the residual network, } \sum_{(i,j) \in W} c_{ij} = \sum_{(i,j) \in W} \bar{c}_{ij} \geq -n\varepsilon > -1. \text{ Since all}$$

$$\text{arc costs are integral, it follows that } \sum_{(i,j) \in W} c_{ij} \geq 0. \text{ Hence the residual network contains}$$

no negative cost cycle and from Theorem 5.1 the flow is optimum. ∎

The cost scaling algorithm treats ε as a parameter and iteratively obtains ε-optimal flows for successively smaller values of ε. Initially $\varepsilon = C$ and finally $\varepsilon < 1/n$. The algorithm performs cost scaling phases by repeatedly applying an *Improve-Approximation* procedure that transforms an ε-optimal flow into an $\varepsilon/2$-optimal flow.

After $1+\lceil \log nC \rceil$ cost scaling phases, $\varepsilon < 1/n$ and the algorithm terminates with an optimum flow. More formally, we can state the algorithm as follows.

algorithm COST SCALING;
begin
  set $\pi := 0$ and $\varepsilon := C$;
  let x be any feasible flow;
  while $\varepsilon \geq 1/n$ **do**
  begin
    IMPROVE-APPROXIMATION-I(ε, x, π);
    $\varepsilon := \varepsilon/2$;
  end;
  x is an optimum flow for the minimum cost flow problem;
end;

The Improve-Approximation procedure transforms an ε-optimal flow into an $\varepsilon/2$-optimal flow. It does so by (i) first converting an ε-optimal flow into an 0-optimal pseudoflow (a pseudoflow x is called ε-optimal if it satisfies the ε-dual feasibility conditions C5.8) and then (ii) gradually converting the pseudoflow into a flow while always maintaining the $\varepsilon/2$-dual feasibility conditions. We call a node i with $e(i) > 0$ *active* and call an arc (i, j) in the residual network *admissible* if $-\varepsilon/2 \leq \bar{c}_{ij} < 0$. The basic operations are selecting active nodes and pushing flows on admissible arcs. We shall see later that pushing flows on admissible arcs preserves the $\varepsilon/2$-dual feasibility conditions. The Improve-Approximation procedure uses the following subroutine.

procedure PUSH/RELABEL(i);
begin
  if G(x) contains an admissible arc (i, j) **then**
    push $\delta := \min \{ e(i), r_{ij} \}$ units of flow from node i to node j;
  else $\pi(i) := \pi(i) + \varepsilon/2 + \min \{ \bar{c}_{ij} : (i, j) \in A(i) \text{ and } r_{ij} > 0 \}$;
end;

Recall that r_{ij} denotes the residual capacity of an arc (i, j) in G(x). As in our earlier discussion of preflow-push algorithms for the maximum flow problem, if $\delta = r_{ij}$, then we refer to the push as *saturating*; otherwise it is *nonsaturating*. We also refer to the updating of the potential of a node as a *relabel* operation. The purpose of a relabel operation is to create new admissible arcs. Moreover, we use the same data structure as

used in the maximum flow algorithms to identify admissible arcs. For each node i, we maintain a *current arc* (i, j) which is the current candidate for pushing flow out of node i. The current arc is found by sequentially scanning the arc list A(i).

The following generic version of the Improve-Approximation procedure summarizes its essential operations.

procedure IMPROVE-APPROXIMATION-I(ε, x, π);
begin

  if $\bar{c}_{ij} > 0$ **then** $x_{ij} := 0$

  else if $\bar{c}_{ij} < 0$ **then** $x_{ij} := u_{ij}$;

  compute node imbalances;

  while the network contains an active node **do**

  begin

    select an active node i;

    PUSH/RELABEL(i);

  end;

end;

The correctness of this procedure rests on the following result.

Lemma 5.4. The Improve-Approximation procedure always maintains $\varepsilon/2$-optimality of the pseudoflow, and at termination yields an $\varepsilon/2$-optimal flow.

Proof. This proof is similar to that of Lemma 4.1. At the beginning of the procedure, the algorithm adjusts the flows on arcs to obtain an $\varepsilon/2$-pseudoflow (in fact, it is a 0-optimal pseudoflow). We use induction on the number of push/relabel steps to show that the algorithm preserves $\varepsilon/2$-optimality of the pseudoflow. Pushing flow on arc (i, j) might add its reversal (j, i) to the residual network. But since $\bar{c}_{ij} < 0$ (by the criteria of admissibility), $\bar{c}_{ji} > 0$ and the condition C5.8 is satisfied. The algorithm relabels node i when $\bar{c}_{ij} \geq 0$ for every arc (i, j) in the residual network. By our choice of price increase, increasing $\pi(i)$ by $\varepsilon/2 + \min \{ \bar{c}_{ij} : (i, j) \in A(i) \text{ and } r_{ij} > 0 \}$ units still satisfies $\bar{c}_{ij} \geq -\varepsilon/2$ for every such arc (i, j) with $r_{ij} > 0$. Therefore, the procedure preserves $\varepsilon/2$-optimality of the pseudoflow throughout and, at termination, yields an $\varepsilon/2$-optimal flow. ∎

We next analyze the complexity of the Improve-Approximation procedure. We will show that the complexity of the generic version is $O(n^2m)$ and then describe a

specialized version running in time $O(n^3)$. These time bounds are comparable to those of the preflow-push algorithms for the maximum flow problem.

Lemma 5.5. *No node potential increases more than 3n times during an execution of the Improve-Approximation procedure.*

Proof. Let x be the current $\varepsilon/2$-optimal pseudoflow and x' be the ε-optimal flow at the end of the previous cost scaling phase. Let π (resp., π') be the node potentials corresponding to the pseudoflow x (resp., flow x'). It is possible to show using a variation of the flow decomposition properties discussed in Section 2.1 that for every node v with positive imbalance in x there exists a node w with negative imbalance in x and a path P with the property that P is an augmenting path with respect to x and its reversal $\overline{P}$ is an augmenting path with respect to x'. This fact in terms of the residual networks implies that there exists a sequence of nodes $v = v_0, v_1, ... , v_l = w$ with the property that $P = v_0 - v_1 - ... - v_l$ is a path in $G(x)$ and its reversal $\overline{P} = v_l - v_{l-1} - ... - v_1$ is a path in $G(x')$. Applying $\varepsilon/2$- optimality conditions to arcs on the path P in $G(x)$, we obtain

$\sum_{(i,j) \in P} \overline{c}_{ij} \geq -l(\varepsilon/2)$. Alternatively,

$$\pi(v) \leq \pi(w) + l(\varepsilon/2) + \sum_{(i,j) \in P} c_{ij}. \qquad (5.9)$$

Applying the ε- optimality conditions to arcs on the path $\overline{P}$ in $G(x')$, we obtain

$$\pi'(w) \leq \pi'(v) + l\varepsilon + \sum_{(i,j) \in \overline{P}} c_{ji} = \pi'(v) + l\varepsilon - \sum_{(i,j) \in P} c_{ij}. \qquad (5.10)$$

Combining (5.9) and (5.10) gives

$$\pi(v) \leq \pi'(v) + (\pi(w) - \pi'(w)) + (3/2)l\varepsilon. \qquad (5.11)$$

Now we use the facts that (i) $\pi(w) = \pi'(w)$ (the potentials of a node with a negative imbalance does not change because the algorithm never selects it for push/relabel), (ii) $l \leq n$, and (iii) each increase in potential increases $\pi(v)$ by at least $\varepsilon/2$ units. The lemma is now immediate. ∎

Lemma 5.6. The Improve-Approximation procedure performs O(nm) saturating pushes.

Proof. This proof is similar to that of Lemma 4.2(b) and essentially amounts to showing that between two consecutive saturations of an arc (i, j), the potentials of both the nodes i and j increase at least once. Since any node potential increases $O(n)$ times, the algorithm also saturates any arc $O(n)$ times resulting in $O(nm)$ total saturating pushes. ■

To bound the number of nonsaturating pushes, we need one more result. We define the *admissible network* as the network consisting solely of admissible arcs. The following result is crucial to the cost scaling algorithms.

Lemma 5.7. The admissible network is acyclic.

Proof. We establish this result by an induction argument applied to the number of pushes and relabels. The result is true at the beginning of the cost scaling phase since the flow is 0-optimal and the network contains no admissible arc. We push flow on arcs (i, j) with $\bar{c}_{ij} < 0$; hence, if their reversals are added to the residual network, then $\bar{c}_{ij} > 0$. This conclusion preserves the inductive hypothesis. A relabel operation at node i may create new admissible arcs (i, j), but also deletes all admissible arcs (k, i). The latter result follows from the fact that any arc (k, i) with $\bar{c}_{ki} \geq -\epsilon/2$ before a relabel operation has $\bar{c}_{ki} \geq 0$ after the relabel operation. Hence the algorithm can create no directed cycles. ■

Lemma 5.8. The Improve-Approximation procedure performs $O(n^2 m)$ nonsaturating pushes.

Proof Sketch. Let $g(i)$ be the number of nodes reachable from node i in the admissible network and let the potential function $F = \sum_{i \text{ active}} g(i)$. The proof amounts to showing that a relabel operation or a saturating push can increase F by at most n units and each nonsaturating push decreases F by at least 1 unit. This yields a bound of $O(n^2 m)$ on the number of nonsaturating pushes. ■

As in the maximum flow algorithm, the bottleneck operation in the Improve-Approximation procedure is the nonsaturating pushes, which take $O(n^2 m)$ time. The algorithm takes $O(nm)$ time to perform saturating pushes, and the same time to scan arcs while identifying admissible arcs. Since the cost scaling algorithm calls Improve-Approximation $1 + \lceil \log nC \rceil$ times, we obtain the following result.

Theorem 5.5. *The generic cost scaling algorithm runs in $O(n^2 m \log nC)$ time.* ∎

The cost scaling algorithm illustrates an important connection between the maximum flow and the minimum cost flow problems. Solving an Improve-Approximation problem is very much similar to solving a maximum flow problem. Just as in the generic preflow-push algorithm for the maximum flow problem, the bottleneck operation is the number of nonsaturating pushes. Researchers have suggested improvements based on examining nodes in some specific order, or using clever data structures. We describe one such improvement which is called the *wave algorithm*.

The wave algorithm is the same as the Improve-Approximation procedure, but it selects active nodes for the push/relabel step in a specific order. The algorithm uses the acyclicity of the admissible network. As is well known, nodes of an acyclic network can be ordered so that for each arc (i, j) in the network, i < j. It is possible to determine this ordering, called the *topological ordering* of nodes, in $O(m)$ time. Observe that pushes do not change the admissible network since they do not create new admissible arcs. The relabel operations, however, may create new admissible arcs and consequently may affect the topological ordering of nodes.

The wave algorithm examines each node in the topological order and if the node is active then it performs a push/relabel step. When examined in this order, active nodes push flow to higher numbered nodes, which in turn push flow to even higher numbered nodes, and so on. Thus if within n consecutive node examinations, the algorithm performs no relabel operation then all active nodes discharge their excesses and the algorithm obtains a flow. Since the algorithm requires $O(n^2)$ relabel operations, we immediately obtain a bound of $O(n^3)$ on the number of node examinations. Each node examination entails at most one nonsaturating push. Consequently, the wave algorithm performs $O(n^3)$ nonsaturating pushes per Improve-Approximation.

We now describe a procedure for obtaining the topological order of nodes after each relabel operation. The initial topological ordering is determined using an $O(m)$ algorithm. Suppose that while examining node i, it becomes relabeled. Note that after the relabel operation at node i, the network contains no incoming admissible arc at node i (see proof of Lemma 5.7). We then move node i from its present position in the topological order to the first position. Notice that this changed ordering is a topological ordering of the new admissible network. This result follows from the facts (i) node i has

no incoming admissible arc; (ii) for each outgoing admissible arc (i, j), node i precedes node j in the order; and (iii) the rest of the admissible network does not change and the previous order is still valid. Thus the algorithm maintains an ordered set of nodes (possibly as a doubly linked list) and examines nodes in this order. Whenever it relabels a node i, the algorithm moves it to the first place in this order and again examines nodes in this order starting at node i. We have thereby established the following result.

Theorem 5.6. The cost scaling approach using wave algorithm as a subroutine solves the minimum cost flow problem in $O(n^3 \log nC)$ time. ∎

5.9. Double Scaling Algorithm

The double scaling approach combines ideas from both the RHS-scaling and cost scaling algorithms and obtains an improvement not obtained by either algorithm alone. For the sake of simplicity, we shall describe the double scaling algorithm on the uncapacitated transportation network $G = (N_1 \cup N_2, A)$, with N_1 and N_2 as the sets of supply and demand nodes respectively. A capacitated minimum cost flow problem can be solved by first transforming the problem into an uncapacitated transportation problem (as described in Section 2.4) and then applying the double scaling algorithm.

The double scaling algorithm is the same as the cost scaling algorithm discussed in the previous section except that it uses a more efficient version of the Improve-Approximation procedure. The Improve-Approximation procedure in the previous section relied on a "pseudoflow-push" method. A natural alternative would be to try an augmenting path based method. This approach would send flow from a node with excess to a node with deficit over an *admissible path*, i.e., a path in which each arc is admissible. A natural implementation of this approach would result in $O(nm)$ augmentations since each augmentation would saturate at least one arc and by Lemma 5.6 the algorithm requires $O(nm)$ arc saturations. Thus this approach does not seem to improve the $O(n^2m)$ bound of the generic Improve-Approximation procedure.

We can, however, use ideas from the RHS-scaling algorithm to reduce the number of augmentations to $O(n \log U)$ for an uncapacitated problem by ensuring that each augmentation carries *sufficiently large* flow. This approach gives us an algorithm that does cost scaling in the outer loop and within each cost scaling phase performs a number of RHS-scaling phases; hence this algorithm is called the *double scaling algorithm*. The driving force underlying the double scaling algorithm is the fact that we

identify an augmenting path in $O(n)$ time on average over a sequence of n augmentations, in contrast with solving a shortest path problem in the RHS-scaling algorithm. In fact, the double scaling algorithm appears to be very much like the shortest augmenting path algorithm for the maximum flow problem; this algorithm also requires $O(n)$ time on average to find each augmenting path. The double scaling algorithm uses the following Improve-Approximation procedure.

procedure IMPROVE-APPROXIMATION-II(ϵ, x, π);
begin

      set x := 0 and compute node imbalances;

      set $\pi(j) := \pi(j) + \epsilon$, for all $j \in N_2$;

      set $\Delta := 2^{\lceil \log U \rceil}$;

      while the network contains an active node **do**

      begin

            $S(\Delta) := \{ i \in N_1 \cup N_2 : e(i) \geq \Delta \}$;

            while $S(\Delta) \neq \emptyset$ **do**

            begin (RHS-scaling phase)

                select a node k in $S(\Delta)$ and delete it from $S(\Delta)$;

                determine an admissible path P from node k to some node l

                    with $e(l) < 0$;

                augment Δ units of flow on P and update x;

            end;

            $\Delta := \Delta / 2$;

      end;

end;

We shall describe a method to determine admissible paths after first commenting on the correctness of this procedure. First observe that $\bar{c}_{ij} \geq -\epsilon$ for all $(i, j) \in A$ at the beginning of the procedure and, by adding ϵ to $\pi(j)$ for each $j \in N_2$, we obtain an $\epsilon/2$-optimal (in fact, 0-optimal) pseudoflow. The procedure always augments flow on admissible arcs and, from Lemma 5.4, this choice preserves the $\epsilon/2$-optimality of the pseudoflow. Thus at the termination of the procedure, we obtain an $\epsilon/2$-optimal flow. Further, as in the RHS-scaling algorithm, the procedure maintains the invariant property that all residual capacities are integer multiples of Δ and thus each augmentation can carry Δ units of flow.

The algorithm identifies an admissible path by gradually building the path. We maintain a *partial* admissible path P using *predecessor indices*, i.e., if (u, v) ∈ P then *pred*(v) = u. At any general step, we perform one of the following two steps, whichever is applicable, at the last node of P, say node i, terminating when the last node has a deficit.

advance(i). If the residual network contains an admissible arc (i, j), then add (i, j) to P. If e(j) < 0, then stop.

retreat(i). If the residual network does not contain an admissible arc (i, j), then update $\pi(i)$ to $\pi(i) + \varepsilon/2 + \min \{ \bar{c}_{ij} : (i, j) \in A(i)$ and $r_{ij} > 0\}$. If P has at least one arc, then delete (pred(i), i) from P.

The retreat step relabels (increases the potential of) the node i for the purpose of creating new admissible arcs at node i; in the process, the arc (pred(i), i) becomes inadmissible. Hence we delete this arc from P. The proof of Lemma 5.4 implies that increasing the node potential maintains ε/2-optimality of the pseudoflow.

We next consider the complexity of the procedure. An execution of the procedure performs $1+\lceil \log U \rceil$ RHS-scaling phases. At the beginning of the Δ-scaling phase, S(2Δ) = ø, i.e., Δ ≤ e(i) < 2Δ for each node i ∈ S(Δ). During the scaling phase, the algorithm augments Δ units of flow from a node k in S(Δ) to a node *l* with e(*l*) < 0. This operation decreases the excess at node k below Δ and keeps the excess at node *l* below Δ. Consequently, each augmentation deletes a node from S(Δ) and after at most n augmentations, the method begins a new scaling phase. The algorithm thus performs a total of O(n log U) augmentations.

We next count the number of advance steps. Each advance step adds an arc to the partial admissible path, and a retreat step deletes an arc from the partial admissible path. Thus there are two types of advance steps: (i) those that add arcs to the admissible paths on which an augmentation is later performed; and (ii) those that are later cancelled by retreat steps. Since the set of admissible arcs is acyclic (by Lemma 5.7), after at most n advance steps of the first type the algorithm will discover an admissible path and will perform an augmentation. Since the algorithm requires a total of O(n log U) augmentations, the number of the first type of advance steps is at most $O(n^2 \log U)$. The second type of advance steps are at most $O(n^2)$ as each retreat step increases a node

potential and by Lemma 5.5, node potentials increase $O(n^2)$ times. The total number of advance steps, therefore, is $O(n^2 \log U)$.

The amount of time needed to identify admissible arcs is $O(\sum_{i=1}^{n} |A(i)| n) = $

$O(nm)$ since between a potential increase of a node i, $|A(i)|$ arcs will be examined for testing admissibility. We have therefore shown the following result.

Theorem 5.7. *The double scaling algorithm solves the uncapacitated transportation problem in $O((nm + n^2 \log U) \log nC)$ time.* ∎

To solve the capacitated minimum cost flow problem we first transform it to the uncapacitated transportation problem and then apply the double scaling algorithm. We leave it as an exercise for the reader to show that how to solve the capacitated minimum cost flow problem can be solved by the double scaled algorithm in $O(nm \log nU \log nC)$ time. The references describe further modest improvements of the algorithm. Under the similarity assumption, a variant of this algorithm using more sophisticated data structures is currently the fastest polynomial algorithm for most classes of the minimum cost flow problem.

5.10 Sensitivity Analysis

The purpose of the sensitivity analysis is to determine the change in the optimum solution of a minimum cost flow problem when the data (supply/demand vector, capacity or cost of any arc) changes. For simplicity, we limit our discussion to unit changes of only a particular type. In a sense, however, this discussion is quite general: it is possible to reduce more complex changes to a sequence of the simple changes we consider. We show that the sensitivity analysis of the minimum cost flow problem essentially reduces to solving shortest path or maximum flow problems.

Let x^* denote an optimum solution of a minimum cost flow problem. Let π^* be the corresponding node potentials and $\overline{c}_{ij} = c_{ij} - \pi(i) + \pi(j)$ denote the reduced costs. Further, let $d(k, l)$ denote the shortest distance between a pair of nodes k and l in the residual network with respect to the original arc lengths c_{ij}. Recall from Section 2.4 that this shortest distance equals the shortest distance with respect to the arc lengths $\overline{c}_{ij}$ plus $(\pi^*(l) - \pi^*(k))$. At optimality, the reduced costs $\overline{c}_{ij}$ of all arcs in the residual

network are nonnegative. Hence, we can compute d(k, *l*) for all pairs of nodes k and *l* by solving n single-source shortest path problems with nonnegative arc lengths.

Supply/Demand Sensitivity Analysis

We first study the change in the supply/demand vector. Suppose that the supply/demand of a node k becomes b(k) + 1 and the supply/demand of another node *l* becomes b(*l*) − 1. (Recall from Section 1.2 that feasibility of the minimum cost flow problem dictates that $\sum_{i \in N}$ b(i) = 0; hence we must change the supply/demand values of two nodes by equal magnitudes but with opposite signs. Then x* is a pseudoflow for the modified problem; moreover, this vector satisfies the dual feasibility conditions C5.6. Augmenting one unit of flow from node k to node *l* along the shortest path in the residual network G(k*) converts this pseudoflow into a flow. This augmentation would change the objective function value by d(k, *l*) units. Lemma 5.1 implies that this flow is optimum for the modified minimum cost flow problem.

Arc Capacity Sensitivity Analysis

We next consider a change in an arc capacity. Suppose that the capacityof an arc (p, q) increases by one unit . The flow x* is feasible for the modified problem; in addition, if $\overline{c}_{pq} \geq 0$, it satisfies the optimality conditions C5.2 − C5.4; hence it is an optimum flow for the modified problem. If $\overline{c}_{pq} < 0$, then condition C5.4 dictates that flow on the arc must equal its capacity. We satisfy this requirement by increasing flow on the arc (p, q) by one unit, which produces a pseudoflow with an excess of one unit at node q and a deficit of one unit at node p. We convert the pseudoflow into a flow by augmenting one unit of flow from node q to node p along the shortest path in the residual network which changes the objective function value by an amount c_{pq} + d(q, p). Optimality of this flow follows from that in the supply/demand sensitivity analysis.

When the capacity of the arc (p, q) decreases by one unit and flow on the arc is at its capacity, we decrease the flow by one unit and augment one unit of flow from node p to node q along the shortest path in the residual network. This augmentation changes the objective function value by an amount $-c_{pq}$ + d(p, q).

The preceding discussion shows how to determine changes in the optimum solution value due to unit changes of any two supply/demand values or a unit change in any arc capacity by solving n single source shortest path problems. We can, however, obtain useful upper bounds on these changes by solving only two shortest path problems. This observation uses the fact that $d(k, l) \leq d(k, 1) + d(1, l)$ for all pairs of nodes k and l. Consequently, we need to determine shortest path distances from node 1 to all other nodes, and from all other nodes to node 1 to compute upper bounds on $d(k, l)$. Recent empirical studies have suggested that these upper bounds are very close to the actual values; often these upper bounds are equal to the actual values and usually they are within 5% of the actual values.

Cost Sensitivity Analysis

Finally, we discuss changes in arc costs, which we assume are integral. Suppose that the cost of an arc (p, q) increases by one unit. This change increases the reduced cost of arc (p, q) by one unit as well. If $\bar{c}_{pq} < 0$ before the change, then after the change $\bar{c}'_{pq}$ ≤ 0. Similarly, if $\bar{c}_{pq} > 0$, before the change, then $\bar{c}'_{pq} \geq 0$ after the change. In both the cases we preserve the optimality conditions. However, if $\bar{c}_{pq} = 0$ before the change and $x_{pq} > 0$, then after the change $\bar{c}_{pq} > 0$ and the solution violates the condition C5.2. To satisfy the optimality condition of the arc, we must either reduce the flow on arc (p, q) to zero, or change the potentials so that the reduced cost of arc (p, q) becomes zero.

We first try to reroute the flow x_{pq}^* from node p to node q without violating any of the optimality conditions. We do so by solving a maximum flow problem , defined as follows: (i) the flow on the arc (p, q) to zero, thus creating an excess of x_{pq}^* at node p and a deficit of x_{pq}^* at node q , (ii) identify node p as the source node, node q as the sink node, and (iii) send a maximum of x_{pq}^* units from the source to the sink. We permit the maximum flow algorithm, however, to change flows only on arcs with zero reduced costs, since otherwise it would generate a solution that violates C5.2 and C5.4. Let v° denote the flow sent from node p to node q and x° denote the resulting arc flow. If $v^\circ = x_{pq}^*$, then x° denotes a minimum cost flow of the modified problem. In this case, the optimal objective function values of the original and modified problems are the same.

On the other hand, if $v^\circ < x_{pq}^*$ then the maximum flow algorithm yields an s-t cut $(X, N-X)$ with the properties that $p \in X$, $q \in N - X$, and every forward arc in the

cutset with zero reduced cost is capacitated. We then decrease the node potential of every node in $N - X$ by one unit. It is easy to verify by case analysis that this change in node potentials maintains the optimality conditions and, furthermore, decreases the reduced cost of arc (p, q) to zero. Consequently, we can set the flow on arc (p, q) equal to $x_{pq}^* - v^\circ$ and obtain a feasible minimum cost flow. In this case, the objective function value of the modified problem is $x_{pq}^* - v^\circ$ units more than that of the original problem.

5.11 Assignment Problem

The *assignment problem* is one of the best-known and most intensively studied special case of the minimum cost network flow problem. This problem consists of a set N_1, say of persons, a set N_2, say of objects ($|N_1| = |N_2| = n$), a collection of node pairs $A \subseteq N_1 \times N_2$ representing possible person-to-object assignments and a cost c_{ij} (possibly negative) associated with each element (i, j) in A. The objective is to assign each person to one object, choosing the assignment with minimum possible cost. The problem can be formulated as the following linear program.

$$\text{Minimize} \quad \sum_{(i, j) \in A} c_{ij} x_{ij} \qquad (5.11a)$$

subject to

$$\sum_{\{j : (i, j) \in A\}} x_{ij} = 1, \text{ for all } i \in N_1, \qquad (5.11b)$$

$$\sum_{\{i : (i, j) \in A\}} x_{ij} = 1, \text{ for all } j \in N_2, \qquad (5.11c)$$

$$x_{ij} \geq 0, \text{ for all } (i, j) \in A. \qquad (5.11d)$$

The assignment problem is a minimum cost flow problem defined on a network G with node set $N = N_1 \cup N_2$, arc set A, arc costs c_{ij}, and supply /demand specified as $b(i)$ = 1 if $i \in N_1$ and $b(i) = -1$ if $i \in N_2$. Let $m = |A|$. The network G has $2n$ nodes and m arcs. The assignment problem is also known as the *bipartite matching problem*.

We use the following notation. A 0-1 solution x of (5.11) is an *assignment*. If $x_{ij} = 1$, then i is *assigned to* j and j is *assigned to* i. A node not assigned to any other node

is *unassigned*. A 0-1 solution x satisfying $\sum_{\{j \,:\, (i,\,j) \,\in\, A\}} x_{ij} \leq 1$ for all i $\in N_1$ and $\sum_{\{i \,:\, (i,\,j) \,\in\, A\}} x_{ij} \leq$

1 for all j $\in N_2$ is called a *partial assignment*. Associated with any partial assignment x is an index set X defined as X = {(i, j) $\in$ A : x_{ij} = 1}.

Researchers have suggested numerous algorithms for solving the assignment problem. Several of these algorithms apply, either explicitly or implicitly, the successive shortest path algorithm for the minimum cost flow problem. These algorithms typically select the initial node potentials with special values: $\pi(i) = 0$ for all i $\in N_1$ and $\pi(j) = \min \{c_{ij} : (i, j) \in A\}$ for all j $\in N_2$. The successive shortest path algorithm solves the assignment problem as a sequence of n shortest path problems with nonnegative arc lengths.

One well known solution procedure for the assignment problem, the *Hungarian method*, is essentially the primal-dual variant of the succesive shortest path algorithm. The network simplex algorithm, with provisions for maintaining a strongly feasible basis ,is another solution procedure for the assignment problem. This approach is fairly efficient in practice; moreover some implementation of it provide polynomial time bounds. Under the similarity assumption, however, a cost scaling algorithm provides the best-known time bound for the assignment problem. Since these algorithms are special cases of other algorithms we described earlier, we will not specify their details. Rather, in this section, we will discuss a different type of algorithm based upon the notion of an *auction*. Before doing so, we show another intimate connection between the assignment problem and the shortest path problem.

Assignments and Shortest Paths

As we have seen that by solving a sequence of shortest path problems, we can solve any assignment problem. Interestingly, we can also use any algorithm for the assignment problem to solve the shortest path problem with arbitrary arc lengths. To do so, we apply the assignment algorithm twice. The first application determines if the network contains a negative cycle; if it doesn't, the second application identifies shortest paths. Both the applications use the node splitting transformations described in Section 2.4.

The node splitting transformation replaces each node i by two nodes i and i', replaces each arc (i, j) by the arc (i, j') and adds (artificial) zero cost arcs (i, i'). We first note that the transformed network always has a feasible solution with cost zero : namely, the assignment containing all artificial arcs (i, i'). We next show that the optimal value of the assignment problem is negative if and only if the original network has a negative cost cycle.

First, suppose the original network contains a negative cost cycle, say 1-2-3- ... - k-1. Then the assigment { (1, 2'), (2, 3'), ... (k, 1'), ((k + 1), (k + 1)') , ... , (n, n') } has a negative cost. Therefore, the cost of the optimal assignment must be negative. Conversely, suppose the cost of an optimal assignment is negative. This solution must contain at least one arc of the form (j, j_2') with $j_1 \neq j_2$. Consequently, the assignment must contain a set of arcs of the form $PA = \{ (j_1 , j_2') , (j_2 , j_3') , \ldots , (j_k, j_1') \}$. The cost of this "partial" assignment is nonpositive, because it can be no more expensive that the partial assignment $\{ (j_1 , j_1') , (j_2 , j_2') , \ldots , (j_k, j_k') \}$. Since the optimal assignment cost is negative, some partial assignment PA must be negative. But then by construction of the transformed network, the cycle $j_1 - j_2 - \ldots - j_k - j_1$ is a negative cycle in the original network.

If the original network contains no negative cycle, then we can obtain a shortest path between a specific pair of nodes, say from node 1 to node n, we can obtain as follows. We consider the transformed network as described earlier and delete the nodes 1 and n' and the arcs incident to these nodes. See Figure 5.3 for an example of this transformation. Now observe that each path from node 1 to node n in the original network has a corresponding assignment of the same cost in the transformed network; the converse is also true. For example, the path 1-2-5 in Figure 5.3(a) has the corresponding assignment {(1, 2'), (2, 5'), (3, 3'), (4, 4')} in Figure 5.3(b), and an assignment {(1, 2'), (2, 4'), (4, 5'), (3, 3')} in Figure 5.3(b) has the corresponding path 1-2-4-5 in Figure 5.3(a). Consequently, an optimum assignment in the transformed network gives a shortest path in the original network.

The Auction Algorithm

We now describe an algorithm for the assignment problem known as the *auction algorithm*. We first describe a pseudopolynomial version of the algorithm and then incorporate scaling to make the algorithm polynomial. This scaling algorithm is an instance of the *bit* scaling algorithm described in Section 1.6. To describe the auction
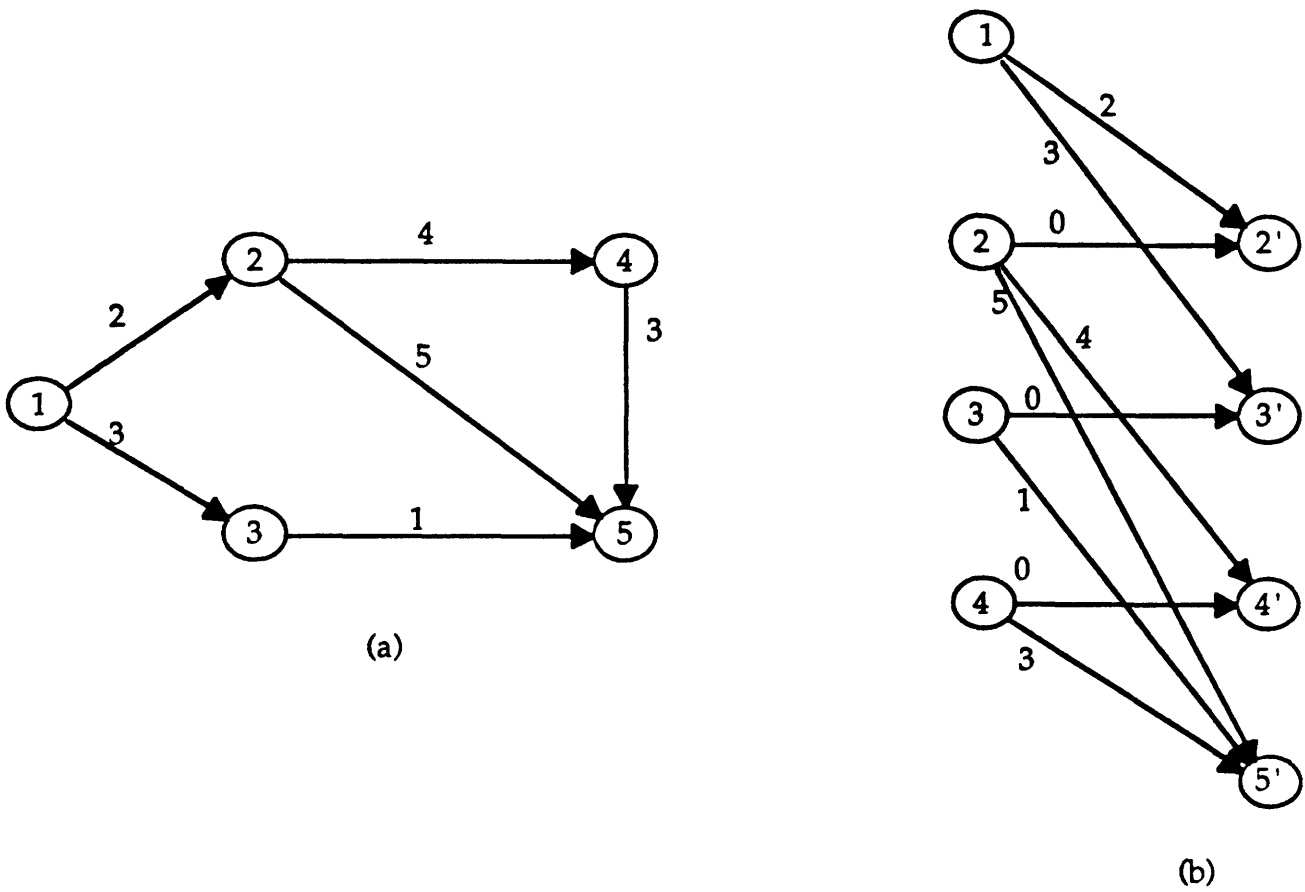
(a)

(b)

Figure 5.3. (a) The original network. (b) The transformed network.

algorithm, we consider the maximization version of the assignment problem, since this version appears more natural for interpreting the algorithm.

Suppose n persons want to buy n cars that are to be sold by auction. Each person i is interested in a subset $A(i)$ of cars ; each person i has a nonnegative utility u_{ij} for car j for each $(i, j) \in A(i)$. The objective is to find an assignment with maximum utility. We can set $c_{ij} = -u_{ij}$ to reduce this problem to (5.11). Let $C = \max \{|u_{ij}| : (i, j) \in A\}$. At each stage of the algorithm, there is an asking price for car j, represented by *price(j)*. For a given set of asking prices, the *marginal utility* of person i for buying car j is u_{ij} - price(j). At each iteration, an unassigned person bids on a car that has the highest marginal utility. We assume that all utilities and prices are measured in dollars.

We associate with each person i a number *value(i)*, which is an upper bound on that person's highest marginal utility, i.e., value(i) $\geq$ max $\{u_{ij}$ - price(j) : $(i, j) \in A(i)\}$. We call a bid (i, j) *admissible* if value(i) = u_{ij} - price(j) and *inadmissible* otherwise. The algorithm requires every bid in the auction to be admissible. If person i is next in turn to bid and has no admissible bid, then value(i) is too high and we decrease this value to max $\{u_{ij}$ - price(j) : $(i, j) \in A(i)\}$.

So the algorithm proceeds by persons bidding on cars. If a person i makes a bid on car j, then the price of car j goes up by $1, therefore, subsequent bids are of higher value. Also, person i is assigned to car j. The person k who was the previous bidder for car j, if there was one, becomes unassigned. Subsequently, person k must bid on another car. As the auction proceeds, the prices of cars increase and hence the marginal values to the persons decrease. The auction stops when each person is assigned a car. We now describe this bidding procedure algorithmically. The procedure starts with some valid choices for value(i) and price(j). For example, we can set price(j) = 0 for each car j and value(i) = max $\{u_{ij} : (i, j) \in A(i)\}$ for each person i. Although this initialization is sufficient for the pseudopolynomial version, the polynomial version requires a more clever initialization . At termination, the procedure yields an *almost* optimum assignment x°.

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```
procedure BIDDING(u, x°, value, price);
begin
        let the initial assignment be a null assignment;
        while some person is unassigned do
        begin
                select an unassigned person i;
                if some bid (i, j) is admissible then
                begin
                        assign person i to car j;
                        price(j) : = price(j) + 1;
                        if person k was already assigned to car j, then
                                person k becomes unassigned;
                end
                else update value(i) : = max {u_ij - price(j) : (i, j) ∈ A(i)};
        end;
        let x° be the current assignment;
end;
```

We now show that this procedure gives an assignment whose utility is within $\$n$ of the optimum utility. Let $x°$ denote a partial assignment at some point during the execution of the auction algorithm and x^* denote the optimum assignment. Recall that value(i) is always an upper bound on the highest marginal utility of person i, i.e., value(i) $\geq u_{ij}$ - price(j) for all (i, j) ∈ x^*. Consequently,

$$\sum_{(i, j) \in x^*} u_{ij} \leq \sum_{i \in N_1} value(i) + \sum_{j \in N_2} price(j) \qquad (5.12)$$

The partial assignment $x°$ also satisfies the condition

$$value(i) = u_{ij} - price(j) + 1, \text{ for all } (i, j) \in x°, \qquad (5.13)$$

because at the time of bidding value(i) = u_{ij} - price(j) and immediately after the bid, price(j) goes up by $\$1$. Let $UB(x°)$ be defined as follows.

$$UB(x°) = \sum_{(i, j) \in x°} u_{ij} + \sum_{i \in N_1^o} value(i). \qquad (5.14)$$

In this expression N_1^o denotes the unassigned persons in N_1. Using (5.13) in (5.14) and observing that unassigned cars in N_2 have zero prices, we obtain

$$UB(x^o) \geq \sum_{i \in N_1} value(i) + \sum_{j \in N_2} price(j) - n. \tag{5.15}$$

Combining (5.12) and (5.15) yields

$$UB(x^o) \geq -n + \sum_{(i,j) \in x^*} u_{ij}. \tag{5.16}$$

As we show in our discussion to follow, the algorithm can change the node values, and prices at most a finite number of times. Since the algorithm will either modify a node value or node price whenever x^o is not an assignment, within a finite number of steps the method must terminate with a complete assignment x^o. Then $UB(x^o)$ represents the utility of this assignment (since N_1^o is empty). Hence the utility of the assignment x^o is at most \$n less than the maximum utility.

It is easy to modify the method, however, to obtain an optimum assignment. Suppose we multiply all utilities u_{ij} by $n+1$ before applying the Bidding procedure. Since all utilities are now multiples of $n+1$, two assignments with distinct total utility will differ by at least $n+1$ units. The procedure yields an assignment that is within n units of the optimum value and, hence, must be optimal.

We next discuss the complexity of the Bidding procedure as applied to the assignment problem with all utilities multiplied by $n+1$. Hence the new largest utility is $C' = (n+1)C$. We first show that the value of any person decreases $O(nC')$ times. Since all utilities are nonnegative, (5.16) implies $UB(x^o) \geq -n$. Substituting this inequality in (5.14) yields

$$\sum_{i \in N_1^o} value(i) \geq -n(C' + 1).$$

Since value(i) decreases by at least one unit each time it changes, this inequality shows that the value of any person decreases $O(nC')$ times. Since decreasing the value of a

person i once takes $O(|A(i)|)$ time, the total time needed to update values of all persons

is $O\left(\sum_{i \in N_1} n \ |A(i)| \ C' \right) = O(nmC')$.

We next examine the number of iterations performed by the procedure. Each iteration of the procedure either decreases the value of a person i or assigns the person to some car j. By our previous arguments, the values change $O(n^2C')$ times in total. Further, since value(i) $> u_{ij} - $ price(j) after person i has been assigned to car j and the price of car j increases by one unit, a person i can be assigned at most $|A(i)|$ times between two consecutive decreases in value(i). This observation gives us a bound of $O(nmC')$ on the total number of times all bidders become assigned. As can be shown, using the "current arc" data structure permits us to locate admissible bids in $O(nmC')$ time. Since $C' = nC$, we have established the following result.

Theorem 5.8. *The auction algorithm solves the assignment problem in $O(n^2mC)$ time.*
■

The auction algorithm is potentially very slow because if increases prices (and thus decreases values) in small increments of \$1 and the final prices can be as large as n^2C (the values as small as $-n^2C$). Using scaling technique in the auction algorithm will permit us to that the prices and values do not change *too many* times. As in the *bit scaling* technique described in Section 1.7, we decompose the original problem into a sequence of $O(\log nC)$ assignment problems and solve each problem by the auction algorithm. We use the optimum prices and values of a problem as a starting solution of the subsequent problem and show that the prices and values change only $O(n)$ times per scaling phase. Thus, we solve each problem is solved in $O(nm)$ time and solve the original problem in $O(nm \log nC)$ time.

The scaling version of the auction algorithm first multiplies all utilities by n+1 and then solves a sequence of $K = \lceil \log (n+1)C \rceil$ assignment problems $P_1, P_2, \dots, P_K$. The problem P_k is an assignment problem in which arc utilities are the k leading bits in the binary representation of u_{ij}. In other words, the problem P_k has the utilities $u_{ij}^k = \lfloor u_{ij} / 2^{K-k} \rfloor$. Note that in the problem P_1, all utilities are 0 or 1, and subsequently $u_{ij}^{k+1} = 2u_{ij}^k + \{0 \text{ or } 1\}$, depending upon whether the newly added bit is 0 or 1. The scaling algorithm works as follows:

```
algorithm  ASSIGNMENT;
begin
        set K : = ⌈log (n+1)C⌉;
        set price(j) : = 0 for each car j;
        set value(i) : = 0 for each person i;
        for k : = 1 to K do
        begin
                       k
                let u  : = ⌊(n+1) u   / 2^{K-k}⌋ for each (i, j) ∈ A;
                     ij             ij

                price(j) : = 2 price(j) for each car j;
                value(i) : = 2 value (i) + 1 for each person i;
                BIDDING(u^k, x°, value, price);

        end;
end;
```

The assignment algorithm performs a number of cost scaling phases. In the k-th scaling phase, it obtains a near-optimum solution of the problem with the utilities u_{ij}^k. It is easy to verify that before the algorithm invokes the Bidding procedure, prices and values satisfy value(i) ≥ max $\{u_{ij}$ - price(j) : (i, j) ∈ A(i)$\}$, for each person i. The Bidding procedure maintains these conditions throughout the execution. In the last scaling phase, the algorithm solves the assignment problem with original utilities and obtains an optimum solution of the original problem. Observe that in each scaling phase, the algorithm starts with a null assignment; the purpose of each scaling phase is to obtain good prices and values for the subsequent scaling phase.

We next discuss the complexity of this assignment algorithm. The crucial result is that the prices and values change only O(n) times during each execution of the Bidding procedure. We define the reduced utility of an arc (i, j) in the k-th scaling phase as

$$\overline{u}_{ij} = u_{ij}^k - price(j) - value(i).$$

In this expression, price(j) and value(i) have the values computed just before calling the Bidding procedure. For any assignment X, we have

$$\sum_{(i,\,j)\,\in\,X} \overline{u}_{ij} = \sum_{(i,\,j)\,\in\,X} u_{ij}^k - \sum_{j\,\in\,N_2} price(j) - \sum_{i\,\in\,N_1} value(i).$$

Hence the reduced utility of an assignment differs from the utility of that assignment by a constant amount. Consequently, an assignment that maximizes the reduced utility also maximizes the utility. Since value(i) $\geq u_{ij}^k$ - price(j) for each (i, j) $\in$ A, we have

$$\bar{u}_{ij} \leq 0, \text{ for all } (i, j) \in A. \tag{5.17}$$

Now consider the reduced utilities of arcs in the assignment x^{k-1} (the final assignment at the end of the (k-1)-st scaling phase). The equality (5.13) implies that

$$u_{ij}^{k-1} - \text{price}'(j) - \text{value}'(i) = -1, \text{ for all } (i, j) \in x^{k-1}, \tag{5.18}$$

where price'(j) and value'(i) are the corresponding values at the end of the (k-1)-st scaling phase. Before calling the Bidding procedure, we set price(j) = 2 price'(j), value(i) = 2 value'(i) + 1, and $u_{ij}^k = 2 u_{ij}^{k-1}$ + {0 or 1}. Substituting these relationships in (5.18), we find that the reduced utilities $\bar{u}_{ij}$ of arcs in x^{k-1} are either -2 or -3. Hence the optimum reduced utility is at least -3n. If x^o is some partial assignment in the k-th scaling phase, then it (5.16) implies that UB(x^o) $\geq$ -4n. Using this result and (5.17) in (5.14) yields

$$\sum_{i \in N_1^o} \text{value}(i) \geq -4n. \tag{5.19}$$

Hence any value(i) decreases O(n) times. Using this result in the proof of Theorem 5.7, we observe that the Bidding procedure would terminate in O(nm) time. The assignment algorithm applies the Bidding procedure O(log nC) times and, consequently, runs in O(nm log nC) time. We summarize our discussion.

Theorem 5.9. The scaling version of the auction algorithm solves the assignment problem in O(nm log nC) time. ∎

The scaling version of the auction algorithm can be further improved to run in O($\sqrt{n}$ m log nC) time. This improvement is based on the following implication of (5.19). If we prohibit persons from bidding if value(i) $\geq 4\sqrt{n}$, then by (5.19) the number of unassignment persons is at most $\sqrt{n}$. Hence the algorithm takes O($\sqrt{n}$ m) time to assign n- $\lceil \sqrt{n} \rceil$ persons and O((n - $\lceil \sqrt{n} \rceil$)m) time to assign the remaining $\lceil \sqrt{n} \rceil$ persons. For example, if n = 10,000, then the auction algorithm would assign the first 99% of the

persons in 1% of the overall running time and would assign the remaining 1% of the persons in the remaining 99% of the time. We therefore terminate the execution of the auction algorithm when it has assigned all but $\lceil\sqrt{n}\rceil$ persons and use successive shortest path algorithms to assign these persons. It so happens that the shortest paths have length $O(n)$ and thus Dial's algorithm, as described in Section 3.2, will find these shortest paths in $O(m)$ time. This version of the auction algorithm solves a scaling phase in $O(\sqrt{n}\,m)$ time; its overall running time is $O(\sqrt{n}\,m\,\log\,nC)$. If we invoke the similarity assumption, then this version of the algorithm currently has the best known time bound for solving the assignment problem .

6.   Reference Notes

In this section, we present reference notes on topics covered in the previous sections. We review important theoretical contributions on each topic, point out inter-relationships among different algorithms and comment on the empirical aspects of the algorithms.

6.1   Introduction

The study of network flow models predates the development of linear programming techniques. The transportation problem was initially studied by Kantorovich [1939], Hitchcock [1941], and Koopmans [1947]. These studies provided some insight into the problem structure and yielded incomplete algorithms. Interest in the network problems grew with the advent of the simplex algorithm by Dantzig in 1947. Dantzig [1951] specialized the simplex algorithm for the tranportation problem. He noted the traingularity of the basis and integrality of the optimum solution. Orden [1956] generalized this work by specializing the simplex algorithm for the uncapacitated minimum cost flow problem. The network simplex algorithm for the capacitated minimum cost flow problem followed from the development of the bounded variable simplex method for linear programming by Dantzig [1955]. The book by Dantzig [1962] contains a thorough description of these contributions along with historical perspectives.

At this time, researchers begain to exhibit increasing interest in the minimum cost flow problem as well as its special cases--the shortest path problem, the maximum flow problem and the assignment problem -- mainly because of their important applications. Soon researchers developed special purpose algorithms to solve these problems. Dantzig, Ford and Fulkerson pioneered those efforts. Whereas Dantzig focused on the primal simplex based algorithms, Ford and Fulkerson developed primal-dual type combinatorial algorithms to solve these problems. Their book, Ford and Fulkerson [1962], presents a thorough discussion of the early research conducted by them and by others. The development of flow decomposition theory, which is credited to Ford and Fulkerson, is also covered in their book.

Since these pioneering works, network flow problems and their generalizations emerged as major research topics in operations research which has

been documented in thousands of papers and many text and reference books. We shall be surveying the important research papers in the following sections. Several important books summarize developments in the field and serve as a guide to the literature: Ford and Fulkerson [1962] (Flows in Networks), Berge and Ghouila-Houri [1962] (Programming , Games and Transportation Networks), Iri [1969] (Network Flows, Transportation and Scheduling), Hu [1969] (Integer Programming and Network Flows), Frank and Frisch [1971] (Communication, Transmission and Transportation Networks), Potts and Oliver [1972] (Flows in Transportation Networks), Christophides [1975] (Graph Theory: An Algorithmic Approach), Murty [1976] (Linear and Combinatorial Programming), Lawler [1976] (Combinatorial Optimization: Networks and Matroids), Bazaraa and Jarvis [1978] (Linear Programming and Network Flows), Minieka [1978] (Optimization Algorithms for Networks and Graphs), Kennington and Helgason [1980] (Algorithms for Network Programming), Jensen and Barnes [1980] (Network Flow Programming), Phillips and Garcia-Diaz [1981] (Fundamentals of Network Analysis), Swamy and Thulsiraman [1981] (Graphs, Networks and Algorithms), Papadimitriou and Steiglitz [1982] (Combinatorial Optimization: Algorithms and Complexity), Smith [1982] (Network Optimization Practice), Syslo, Deo, and Kowalik [1983] (Discrete Optimization Algorithms), Tarjan [1983] (Data Structures and Network Algorithms), Gondran and Minoux [1984] (Graphs and Algorithms), Rockafellar [1984] (Network Flows and Monotropic Optimization), and Derigs [1988] (Programming in Netorks and Graphs). As an additional source of references, the reader might consult the bibliography on network optimization prepared by Golden and Magnanti [1977] and the extensive set of references on integer programming compiled by researchers at the University of Bonn (Kastning [1976], Hausman [1978], and Von Randow [1982, 1985]).

Since the applications of network flow modelsa are so pervasive, no single source provides a comprehensive account of network flow models and their impact on practice. Several researchers have prepared general surveys of selected application areas. Notable among this is the paper by Glover and Klingman [1976] on the applications of minimum cost flow and generalised minimum cost flow problems. A number of books written in special problem domains also contain valuable insight about the range of applications of network flow models. Examples in this category are the paper by Bodin, Golden, Assad and Ball [1983] on vehicle routing and scheduling problems, books on communication networks by Bertsekas

and Gallager [1987] and on transportation planning by Sheffi [1985], as well as a collection of survey articles on facility location by Francis and Mirchandani [1988].

6.2   Shortest Path Problem

The shortest path problem and its generalizations have a voluminous research literature. As a guide to these results, we refer the reader to the extensive bibliographies compiled by Gallo, Pallattino, Ruggen and Starchi [1982] and Deo and Pang [1984]. This section, which summarizes some of this literature, focuses especially on issues of computational complexity.

Label Setting Algorithms

The first label setting algorithm was suggested by Dijkstra [1959], and independently by Dantzig [1960] and Whiting and Hillier [1960]. The original implementation of Dijkstra's algorithm runs in $O(n^2)$ time which is the optimal running time for fully dense networks (e.g., $m = \Omega(n^2)$), since any algorithm must examine every arc. However, improved running times are possible for sparse networks. The following table summarizes various implementations of Dijkstra's algorithm designed to improve the running time in the worst-case or in practice. In the table, $d = [2 + m/n]$ represents the average degree of a node in the network plus 2.

#	Discoverers	Running Time
1	Dijkstra [1959]	$O(n^2)$
2	Williams [1964]	$O(m \log n)$
3	Dial [1969]	$O(m + n C)$
4	Johnson [1977a]	$O(m \log_d n)$
5	Johnson [1977b]	$O((m + n \log C)\log \log C)$
6	Boas, Kaas and Zijlstra [1977]	$O(nC + m \log \log nC)$
7	Denardo and Fox [1979]	$O(m \log\log C + n\log C)$
8	Johnson [1982]	$O(m \log \log C)$
9	Fredman and Tarjan [1984]	$O(m + n \log n)$
10	Gabow [1985]	$O(m \log_d C)$
11	Ahuja, Mehlhorn, Orlin and Tarjan [1988]	(a) $O(m + n \log C)$ (b) $O\left(m + \dfrac{n \log C}{\log \log C}\right)$ (c) $O(m + n \sqrt{\log C})$

Figure 6.1. Running times of the label setting algorthms.

Computer scientists have tried to improve the worst-case complexity of Dijkstra's algorithm by using improved data structures. When implemented with a *binary heap* data structure, the algorithm takes $O(\log n)$ time for each node selection (and the subsequent deletion) step and each distance update; consequently, this implementation of Dijkstra's algorithm runs in $O(m \log n)$ time. For sparse networks, this time is better than $O(n^2)$, but it is worse for dense networks. The *d-heap* data structure suggested by Johnson [1977a] takes $O(d \log_d n)$ time for each node selection (and the subsequent deletion) step and $O(\log_d n)$ for each distance update. For $d = 2 + m/n$, this approach leads to a time bound of $O(m \log_d n)$. For very sparse networks (e.g., $m = O(n)$), this time reduces to $O(n \log n)$ and for very dense networks (e.g., $m = \Omega(n^2)$), it becomes $O(n^2)$. For all other ranges of densities as well, the running time of this algorithm is better than that of the original implementation as well as binary heap implementation of Dijkstra's algorithm.

Under the similarity assumption, Gabow's [1985] scaling algorithm achieves the same time bound. Gabow decomposed the original problem into $\lceil \log_d C \rceil$ scaled problems and solved each scaled problem in $O(m)$ time by Dial's algorithm; thus yielding an $O(m \log_d C)$ algorithm for the shortest path problem.

Boas, Kaas and Zijlstra [1977] suggested a data structure whose analysis depends upon the largest key D stored in a heap. The initialization of this algorithm takes O(D) time and each heap operation takes O(log log D). When Dijkstra's algorithm is implemented using this data structure, it runs in $O(nC + m \log \log nC)$ time. Johnson [1982] suggested an improvement of this data structure and used it to implement Dijkstra's algorithm in $O(m \log \log C)$ time.

The best strongly polynomial algorithm to date is due to Fredman and Tarjan [1984] who use a *Fibonacci heap* data structure. The Fibonacci heap is an ingenious, but somewhat complex data structure that takes an average of $O(\log n)$ time for each node selection (and the subsequent deletion) step and an average of $O(1)$ time for each distance update. Consequently, this data structure implements Dijkstra's algorithm in $O(m + n \log n)$ time.

Dial [1969] suggested his implementation of Dijkstra's algorithm because of its encouraging empirical performance. Dial, Glover, Karney and Klingman [1979] have proposed an improved version of Dial's algorithm, which runs better in practice. Though Dial's algorithm is only pseudopolynomial, its successors have had improved worst-case behavior. Denardo and Fox [1979] suggest several such improvements. Observe that if $w = \max [1, \min\{c_{ij}: (i,j) \in A\}]$, then we can use buckets of width w in Dial's algorithm, hence reducing the number of buckets from $1+ C$ to $1+(C/w)$. The correctness of this observation follows from the fact that if d^* is the current minimum temporary distance labels, then the algorithm will modify no other temporary distance label in the range $[d^*, d^* + w - 1]$ since each arc has length at least $w - 1$. Then, using a multiple level bucket scheme, Denardo and Fox implemented the shortest path algorithm in $O(\max\{k \ C^{1/k}, m \log (k+1), nk(1+C^{1/k}/w)\}$ time for any choice of k. Choosing $k = \log C$ yields a time bound of $O(m \log \log C + n \log C)$. Depending on n,m and C, other choices might lead to a modestly better time bound.

Johnson [1977b] proposed a related bucket scheme with exponentially growing widths and obtained the running time of $O((m+n \log C)\log \log C)$. This data structure is the same as the R-heap data structure described in Section 3.3, except that it performs binary search over $O(\log C)$ buckets to insert nodes into buckets during the redistribution of ranges and the distance updates. The R-heap implementation replaces the binary search by a sequential search and improves the running time by a

factor of O(log log C). Ahuja, Mehlhorn, Orlin and Tarjan [1988] suggested the R-heap implementation and its further improvements, as described below.

The R-heap implementation described in section 3.3 uses a single level bucket system. A two-level bucket system improves further on the R-heap implementation of Dijkstra's algorithm. The two-level data structure consists of K (big) buckets, each bucket being further subdivided into L (small) *subbuckets*. During redistribution, the two-level bucket system redistributes the range of a subbucket over all of its previous buckets. This approach permits the selection of much larger width of buckets, thus reducing the number of buckets. By using $K = L = 2 \log C / \log \log C$, this two-level bucket system version of Dijkstra's algorithm runs in $O(m+n \log C / \log \log C)$ time. Incorporating a generalization of Fibonacci heaps in the two-level bucket system with appropriate choices of K and L further reduces the time bound to $O(m + n\sqrt{\log C})$. If we invoke the similarity assumption, this approach currently gives the fastest worst-case implementation of Dijkstra's algorithm for all classes of graphs except very sparse ones, for which the algorithm of Johnson [1982] appears more attractive. The Fibonacci heap version of two-level R-heap is very complex, however, and so it is unlikely that this algorithm would perform well in practice.

Label Correcting Algorithm

Ford [1956] suggested, in skeleton form, the first label correcting algorithm for the shortest path problem. Subsequently, several other researchers - Ford and Fulkerson [1962] and Moore [1957] - studied the theoretical properties of the algorithm. Bellman's [1958] algorithm can also be regarded as a label correcting algorithm. Though specific implementations of label correcting algorithms run in O(nm) time, the most general form is nonpolynomial, as shown by Edmonds [1970].

Researchers have exploited the flexibility inherent in the generic label correcting algorithm to obtain algorithms that are very efficient in practice. The modification that adds a node to the LIST (see the description of the Modified Label Correcting Algorithm given in Section 3.4.) at the front if the algorithm has previously examined the node earlier and at the end otherwise, is probably the most popular. This modification was conveyed to Pollack and Wiebenson [1960] by D'Esopo, and later refined and tested by Pape [1974]. We shall subsequently refer to this algorithm as D'Esopo and Pape's algorithm. A FORTRAN listing of this algorithm can be found in Pape [1980]. Though this modified label correcting

algorithm has excellent computational behavior in the worst-case it runs in exponential time, as shown by Kershenbaum [1981].

Glover, Klingman and Phillips [1985] proposed a new polynomially bounded label correcting algorithm, called the *partitioning shortest path (PSP) algorithm*. For general networks, the PSP algorithm runs in $O(nm)$ time, while for networks with nonnegative arc lengths it runs in $O(n^2)$ time and has excellent computational behavior. Other variants of the label correcting algorithms and their computational attributes can be found in Glover, Klingman, Phillips and Schneider [1985].

Researchers have been interested in developing polynomial time simplex algorithms for the shortest path problem. Dial, Glover, Karney and Klingman [1979] and Zadeh [1979] showed that Dantzig's pivot rule (i.e., pivoting in the arc with largest violation of optimality condition) for the shortest path problem starting from an artificial basis leads to Dijkstra's algorithm. Thus the number of pivots is $O(n)$ if all arc cots are nonnegative. Akgul [1985a] developed a simplex algorithm for the shortest path problem that performs $O(n^2)$ pivots. Using simple data structures, Akgul's algorithm runs in $O(n^3)$ time which can be reduced to $O(nm + n^2 \log n)$ using Fibonacci heaps. Goldfarb, Hao and Kai [1986] describe another simplex algorithm for the shortest path problem whose number of pivots and running times are comparable to that of Akgul's algorithm. Orlin [1985] showed that the simplex algorithm with Dantzig's pivot rule solves the shortest path problem in $O(n^2 \log nC)$ pivots. Ahuja and Orlin [1988] recently discovered a scaling variation of this approach that performs $O(n^2 \log C)$ pivots and runs in $O(nm \log C)$ time. This algorithm uses simple data structures, permits partial pricing and uses very natural pricing strategies.

All Pair Shortest Path Algorithms

Most algorithms that solve the all pair shortest path problem involve matrix manipulation. The first such algorithm, which performs matrix multiplication, appears to be a part of the folklore. Lawler [1976] describes this algorithm in his textbook. The complexity of this algorithm is $O(n^3 \log n)$, which can be improved slightly by using more sophisticated matrix multiplication procedures. The algorithm we have presented is due to Floyd [1962] and is based on a theorem by Warshall [1962]. This algorithm runs in $O(n^3)$ time and is also capable of detecting the presence of negative cycles. Dantzig [1967] devised another procedure requiring

exactly the same order of calculations. The bibliography by Deo and Pang [1984] contains references for several other all pair shortest path algorithms.

From a worst-case complexity point of view, however, it might be desirable to solve the all pair shortest path problem as a sequence of single source shortest path problems. As pointed out in the text, this approach takes $O(nm)$ time to construct an equivalent problem with nonnegative arc lengths and $O(nm + n^2 \log nC)$ time to solve the n shortest path problems using an R-heap implementation of Dijkstra's algorithm.

Computational Results

Researchers have extensively tested shortest path algorithms on a variety of network classes. The studies due to Gilsinn and Witzgall [1973], Pape [1974], Kelton and Law [1978], Van Vliet [1978], Dial, Glover, Karney and Klingman [1979], Denardo and Fox [1979], Glover, Klingman, Phillips and Schneider [1985] , and Gallo and Pallottino [1988] are representatives of these contributions.

Unlike the worst-case results, the computational performance of an algorithm depends upon a number of factors such as the manner in which the program is written; the language, compiler and the computer used; and the distribution of networks on which the algorithm is tested. Hence, the results of computational studies are only suggestive, rather than conclusive. The results of these studies also depend greatly upon the density of the network. These studies generally suggest that Dial's algorithm is the best label setting algorithm for the shortest path problem. It is faster than the original $O(n^2)$ implementation, the binary heap, d-heap as well as the Fibonacci heap implementation of Dijkstra's algorithm for all network classes tested by these researchers. Denardo and Fox [1979] also find that Dial's algorithm is faster than their two-level bucket implementation for all of their test problems; however, extrapolating the results, they observe that their implementation would be faster for very large shortest path problems. Researchers have not yet tested the R-heap implementation and so at this moment no comparison with Dial's algorithm is available.

Among the label correcting algorithms, the algorithms by D'Esopo and Pape and by Glover, Klingman, Phillips and Schneider [1985] are the two fastest algorithms. The study by Glover et al. finds their algorithm superior to D'Esopo and

Pape's algorithm. Other researchers have also compared label setting algorithms with label correcting algorithms. Studies generally suggest that for very dense networks label setting algorithms are superior and for sparse networks label correcting algorithms perform better.

Kelton and Law [1978] have conducted a computational study of several all pair shortest path algorithms. This study indicates that Dantzig's [1967] algorithm with a modification due to Tabourier [1973] is faster (up to two times) than the Floyd-Warshall algorithm described in Section 3.5. This study also finds that matrix manipulation algorithms are faster than a successive application of a single-source shortest path algorithm for very dense networks, but slower for sparse networks.

6.3   Maximum Flow Problem

The maximum flow problem is distinguished by the long line of successive contributions researchers have made in improving the worst-case complexity of algorithms; some, but not all, of these improvements have produced improvements in practice.

Several researchers - Dantzig and Fulkerson [1956], Ford and Fulkerson [1956] and Elias, Feinstein and Shannon [1956] - independently established the max-flow min-cut theorem. Fulkerson and Dantzig [1955] solved the maximum flow problem by specializing the primal simplex algorithm, whereas Ford and Fulkerson [1956] and Elias et al. [1956] solved it by augmenting path algorithms. Since then, researchers have developed a number of algorithms for this problem; Figure 6.2 summarizes the running times of some of these algorithms. In the figure, n is the number of nodes, m is the number of arcs, and U is an upper bound on the integral arc capacities. The algorithms whose time bounds involve U assume integral capacities; the bounds specified for the other algorithms apply to problems with arbitrary rational or real capacities.

#	Discoverers	Running Time
1	Edmonds and Karp [1972]	$O(nm^2)$
2	Dinic [1970]	$O(n^2m)$
3	Karzanov [1974]	$O(n^3)$
4	Cherkasky [1977]	$O(n^2 \sqrt{m})$
5	Malhotra, Kumar and Maheshwari [1978]	$O(n^3)$
6	Galil [1980]	$O(n^{5/3}m^{2/3})$
7	Galil and Naamad [1980]; Shiloach [1978]	$O(nm \log^2 n)$
8	Shiloach and Vishkin [1982]	$O(n^3)$
9	Sleator and Tarjan [1983]	$O(nm \log n)$
10	Tarjan [1984]	$O(n^3)$
11	Gabow [1985]	$O(nm \log U)$
12	Goldberg [1985]	$O(n^3)$
13	Goldberg and Tarjan [1986]	$O(nm \log (n^2/m))$
14	Bertsekas [1986]	$O(n^3)$
15	Cheriyan and Maheshwari [1987]	$O(n^2 \sqrt{m})$
16	Ahuja and Orlin [1987]	$O(nm + n^2 \log U)$
17	Ahuja, Orlin and Tarjan [1988]	(a) $O\left(nm + \dfrac{n^2 \log U}{\log \log U}\right)$ (b) $O\left(nm + n^2 \sqrt{\log U}\right)$ (c) $O\left(nm \log\left(\dfrac{n \sqrt{\log U}}{m} + 2\right)\right)$

Figure 6.2. Running times of the maximum flow algorithms.

Ford and Fulkerson [1956] observed that the labeling algorithm can perform as many as $O(n U)$ augmentations for networks with integer arc capacities. For arbitrary irrational arc capacities, the labeling algorithm can perform an infinite sequence of augmentations and might converge to a value different from the maximum flow value. Edmonds and Karp [1972] suggested two specializations of the labeling algorithm, both with improved computational complexity. They showed that if the algorithm augments flow along a shortest path (i.e., one containing the smallest possible number of arcs) in the residual network, then the algorithm performs $O(nm)$ augmentations. A breadth first search of the network will determine a shortest augmenting path; consequently, this version of the labeling algorithm runs in

in $O(nm^2)$ time. Edmonds and Karp's second idea was to augment flow along a path with maximum residual capacity. They proved that this algorithm performs $O(m \log U)$ augmentations. It is shown in Tarjan [1986] how to determine a path with maximum residual capacity in $O(m)$ time on average; hence, this version of the labeling algorithm runs in $O(m^2 \log U)$ time.

Dinic [1970] independently introduced the concept of shortest path networks, called *layered networks* , for the maximum flow problem. A layered network is a subgraph of the residual network that contains only those nodes and arcs that lie on at least one shortest path from the source to the sink. The nodes in a layered network can be partitioned into layers of nodes $N_1, N_2, \ldots$, so that for every arc (i, j) in the layered network, $i \in N_k$ and $j \in N_{k+1}$ for some k. A blocking flow in a layered network $G' = (N', A')$ is a flow for which G' contains no directed path with positive residual capacity from the source node to the sink node. Dinic showed how to construct a blocking flow in a layered network by performing at most m augmentations requiring a total of $O(nm)$ time. Dinic's algorithm proceeds by constructing layered networks and establishing blocking flows in these networks. Dinic showed that at each iteration, the length of the layered network increases and after at most n iterations, the source is disconnected from the sink. Consequently, his algorithm runs in $O(n^2m)$ times.

The shortest augmenting path algorithm presented in Section 4.3 achieves the same time bound as Dinic's algorithm, but instead of constructing layered networks it maintains distance labels. Goldberg [1985] introduced distance labels in the context of his preflow push algorithm. Distance labels offer several advantages: They are simpler to understand than layered networks, are easier to manipulate, and have led to more efficient algorithms. Orlin and Ahuja [1987] developed the distance label based augmenting path algorithm given in Section 4.3. They also showed that this algorithm is equivalent both to Edmonds and Karp's algorithm and to Dinic's algorithm in the sense that all three algorithms enumerate the same augmenting paths in the same sequence. The algorithms differ only in the manner in which they obtain these augmenting paths.

Researchers have made several subsequent improvements in maximum flow algorithms by developing more efficient algorithms to establish blocking flows in layered networks. Karzanov [1974] introduced the concept of preflows in a layered network. (See the technical report of Even [1976] for a comprehensive description of

this algorithm and the paper by Tarjan [1984] for a simplified version.) Karzanov showed that an implementation that maintains preflows and pushes flows from nodes with excesses, constructs a blocking flow in $O(n^2)$ time. Malhotra, Kumar and Maheshwari [1978] present a conceptually simple maximum flow algorithm that runs in $O(n^3)$ time. Cherkasky [1977] and Galil [1980] presented further improvements of Karzanov's algorithm.

The search for more efficient maximum flow algorithms has stimulated researchers to develop new data structure for implementing Dinic's algorithm. The first such data structures were suggested independently by Shiloach [1978] and Galil and Naamad [1980]. Dinic's algorithm (or the shortest augmenting path algorithm described in Section 4.3) takes $O(n)$ time on average to identify an augmenting path and during the augmentation it saturates some arcs in this path. If we delete the saturated arcs from this path, we obtain a set of *path fragments*. The basic idea is to store these path fragments using some data structure, for example, 2-3 trees (see Aho, Hopcroft and Ullman [1974] for a discussion of 2-3 trees) and use them later to identify augmenting paths quickly. Shiloach [1978] and Galil and Naamad [1980] showed how to augment flows through path fragments in a way that finds a blocking flow in $O(m(\log n)^2)$ time. Hence, their implementation of Dinic's algorithm runs in $O(nm (\log n)^2)$ time. Sleator and Tarjan [1983] improved this approach by using a data structure called *dynamic trees* to store and update path fragments. Sleator and Tarjan's algorithm establishes a blocking flow in $O(m \log n)$ time and thereby yields an $O(nm \log n)$ time bound for Dinic's algorithm.

Gabow [1985] obtained a similar time bound by applying a bit scaling approach to the maximum flow problem. As outlined in Section 1.7, this approach solves a maximum flow problem at each scaling phase with one more bit of every arc's capacity. During a scaling phase, the initial flow value differs from the maximum flow value by at most m units and so the shortest augmenting path algorithm (and also Dinic's algorithm) performs at most m augmentations. Consequently, each scaling phase takes $O(nm)$ time and the algorithm runs in $O(nm \log C)$ time. If we invoke the similarity assumption, this time bound is comparable to that of Sleator and Tarjan's algorithm, but the scaling algorithm is much simpler to implement. Orlin and Ahuja [1987] have presented a variation of Gabow's algorithm achieving the same time bound.

Goldberg and Tarjan [1986] developed the generic preflow push algorithm and the highest-label preflow push algorithm. Previously, Goldberg [1985] had shown that the FIFO version of the algorithm that pushes flow from active nodes in the first-in-first-out order runs in $O(n^3)$ time. (This algorithm maintains a *queue* of active nodes; at each iteration, it selects a node from the front of the queue, performs a push/relabel step at this node, and adds the newly active nodes to the rear of the queue.) Using a dynamic tree data structure, Goldberg and Tarjan [1986] improved the running time of the FIFO preflow push algorithm to $O(nm \log (n^2/m))$. This algorithm currently gives the best strongly polynomial bound for solving the maximum flow problem.

Bertsekas [1986] obtained another maximum flow algorithm by specializing his minimum cost flow algorithm; this algorithm closely resembles the Goldberg and Tarjan's FIFO preflow push algorithm. Recently, Cheriyan and Maheshwari [1987] showed that Goldberg and Tarjan's highest-label preflow push algorithm actually performs $O(n^2\sqrt{m})$ nonsaturating pushes and hence runs in $O(n^2\sqrt{m})$ time.

Ahuja and Orlin [1987] improved the Goldberg and Tarjan's algorithm using the excess scaling technique to obtain an $O(nm + n^2 \log U)$ time bound. If we invoke the similarity assumption, this algorithm improves Goldberg and Tarjan's $O(nm \log (n^2/m))$ algorithm by a factor of $\log n$ for networks that are both non-sparse and nondense. Further, this algorithm does not use any complex data structure. Scaling excesses by a factor of $\log U/\log \log U$ and pushing flow from a large excess node with the highest distance label, Ahuja, Orlin and Tarjan [1988] reduced the number of nonsaturating pushes to $O(n^2 \log U/ \log \log U)$. Ahuja, Orlin and Tarjan obtained another variation of original excess scaling algorithm which further reduces the number of nonsaturating pushes to $O(n^2 \sqrt{\log U})$.

The use of the dynamic tree data structure will improve the running times of the excess scaling algorithm and its variations, though the improvements are not as dramatic as they have been for Dinic's and the FIFO preflow push algorithms. For example, the $O(nm + n^2 \sqrt{\log U})$ algorithm improves to $O\left(nm \log\left(\frac{n\sqrt{\log U}}{m}+2\right)\right)$ by using dynamic trees, as shown in Ahuja, Orlin and Tarjan [1988]. Tarjan [1987] conjectures that any preflow push algorithm that performs p nonsaturating pushes can be implemented in $O(nm \log (2+p/nm)$ time using dynamic trees. Although this

conjecture is true for all known preflow push algorithms, it is still open for the general case.

Developing a polynomial time primal simplex algorithm for the maximum flow problem has been an outstanding open problem for quite some time. Recently, Goldfarb and Hao [1988] developed such an algorithm. This algorithm is essentially based on selecting pivot arcs so that flow is augmented along a shortest path from the source to the sink. As one would expect, this algorithm performs $O(nm)$ pivots and can be implemented in $O(n^2m)$ time.

Researchers have also investigated the following special cases of the maximum flow problems: the maximum flow problem (i) on unit capacity networks (i.e., $U=1$); (ii) unit capacity simple networks (i.e., $U=1$, and, for every node in the network except source and sink, there is one incoming arc or one outgoing arc) ; (iii) bipartite networks; and (iv) planar networks. Observe that the unit capacity networks have maximum flow value less than n and so the shortest augmenting path algorithm will solve these problems in $O(nm)$ time; therefore, these problems are easier than problems with large capacities. Even and Tarjan [1975] showed that Dinic's algorithm solves the maximum flow problem on unit capacity networks in $O(n^{2/3}m)$ time and on unit capacity simple networks in $O(n^{1/2} m)$ time. Orlin and Ahuja [1987] have achieved the same time bounds using a modification of the shortest augmenting path algorithm. Fernandez-Baca and Martel [1987] have generalized these ideas for networks with small integer capacities.

Versions of the maximum flow algorithms run considerably faster on a bipartite networks $G = (N_1 \cup N_2, A)$ if $|N_1| << |N_2|$ (or $|N_2| << |N_1|$). Let $n_1 = |N_1|, n_2 = |N_2|$ and $n = n_1 + n_2$. Suppose that $n_1 < n_2$. Gusfield, Martel and Fernandez-Baca [1985] obtained the first such results by showing how the running times of Karzanov's and Malhotra et al.'s algorithms reduce from $O(n^3)$ to $O(n_1^2 n_2)$ and $O(n_1^3 + nm)$ respectively. Ahuja, Orlin, Stein and Tarjan [1988] improved their ideas by showing how n can be substituted by n_1 in the time bounds for all preflow push algorithms to obtain the respective time bounds for bipartite networks. This result implies that the FIFO preflow push algorithm and the original excess scaling algorithm, respectively, solve the maximum flow problem on bipartite graphs in $O(n_1 m + n_1^3)$ and $O(n_1 m + n_1^2 \log U)$ time.

It is possible to solve the maximum flow problem on planar networks much more efficiently than on general networks. (A network is called *planar* if it can be drawn in a two-dimensional plane so that arcs intersect one another only at the nodes.) A planar network has at most 6n arcs; hence the running times of the maximum flow algorithms on planar networks appear more attractive. Specialized solution techniques, that have even better running times, are quite different than those for the general networks. Some important references for planar maximum flow algorithms are Itai and Shiloach [1979], Johnson and Venkatesan [1982] and Hassin and Johnson [1985].

Researchers have also investigated whether the worst-case bounds of the maximum flow algorithms are *tight*, i.e., whether there are families of networks for which the algorithms do take time equal to their worst-case bounds. Zadeh [1972] showed that the bound of Edmonds and Karp algorithm is tight when $m = n^2$. Even and Tarjan [1975] noted that the same examples imply that the bound of Dinic's algorithm is tight when $m = n^2$. Baratz [1977] showed that the bound on Karzanov's algorithm is tight. Galil [1981] constructed an interesting class of examples and showed that the algorithms of Edmonds and Karp, Dinic, Karzanov, Cherkasky, Galil and Malhotra et al. achieve their worst-case bounds on those examples.

Other researchers have made some progress in constructing worst-case examples for preflow push algorithms. Martel [1987] showed that the FIFO preflow push algorithm can take $\Omega(nm)$ time to solve a class of unit capacity networks. Cheriyan and Maheshwari [1987] have recently shown that the bound of $O(n^2\sqrt{m})$ for the highest-label preflow push algorithm is tight. Cheriyan [1988] has also constructed a family of examples to show that the bound $O(n^3)$ for FIFO preflow push algorithm and the bound $O(n^2m)$ for the generic preflow push algorithm is tight. Similar results are unknown for other preflow push algorithms, especially for the excess scaling algorithms. It is worth mentioning, however, that these known worst-case examples are quite artificial and are not likely to arise in practice.

Several computational studies have assessed the empirical behavior of the maximum flow algorithms. The studies performed by Hamacher [1979], Cheung [1980], Glover, Klingman, Mote and Whitman [1979, 1984], Imai [1983] and Goldfarb and Grigoriadis [1986] are noteworthy. These studies were conducted prior to the development of algorithms that use distance labels. These studies rank Edmonds and Karp, Dinic's and Karzanov's algorithms in increasing order of performance for

most classes of networks. Dinic's algorithm is competitive with Karzanov's algorithm for sparse networks, but slower for dense networks. Imai [1983] noted that Galil and Naamad's [1980] implementation of Dinic's algorithm, using sophisticated data structures, is slower than the original Dinic's algorithm. Sleator and Tarjan [1983] reported a similar finding; they observed that their implementation of Dinic's algorithm using dynamic tree data structure is slower than the original Dinic's algorithm by a constant factor. Hence, the sophisticated data structures improve only the worst-case performance of algorithms, but are not useful empirically. Researchers have also tested the Malhotra et al. algorithm and the primal simplex algorithm due to Fulkerson and Dantzig [1955] and found these algorithms to be slower than Dinic's algorithm for most classes of networks.

A number of researchers are currently evaluating the computational performance of preflow push algorithms. Derigs and Meier [1988], Grigoriadis [1988], and Ahuja, Kodialam and Orlin [1988] have found that the preflow push algorithms are substantially (often 2 to 10 times) faster than Dinic's and Karzanov's algorithms for most classes of networks. Among all nonscaling preflow push algorithms the highest-label preflow push algorithm runs the fastest. The excess scaling algorithm and its variations have not been tested thoroughly. We do not anticipate that dynamic tree implementations of preflow push algorithms would be useful in practice, in this case, as in others, their contribution is to improve the worst-case performances of algorithms.

Finally, we discuss some important generalizations of the maximum flow problem: (i) the multi-terminal flow problem; (ii) the maximum dynamic flow problem; and (iii) the generalized maximum flow problem.

The multi-terminal flow problem is to determine the maximum flow value between every pair of nodes. Gomory and Hu [1961] showed how to solve the multi-terminal flow problem by solving (n-1) maximum flow problems. Recently, Gusfield [1987] has suggested a simpler multi-terminal flow algorithm.

In the simplest version of maximum dynamic flow problem, we associate with each arc (i, j) in the network a number t_{ij} denoting the time needed to travel that arc. The objective is to send the maximum possible flow from the source node to the sink node within a given time period T. Ford and Fulkerson [1958] first showed that the maximum dynamic flow problem can be solved by solving a

minimum cost flow problem. (Ford and Fulkerson [1962] give a nice treatment of this problem). Orlin [1983] has considered infinite horizon dynamic flow problems in which the objective is to minimize the average cost per period.

The generalized maximum flow problem considers networks with gains on arcs. Each arc (i, j) in the network has a gain factor r_{ij}. If x units of flow enter the arc at node i, then $r_{ij}x$ units of flow arrive at node j. The objective is to send the maximum possible flow into the sink. For the ordinary maximum flow problem we considered, every arc has a gain factor equal to one. The generalized maximum flow problem is considerably harder than the ordinary maximum flow problem. Jewell [1962] first formulated and studied this problem. Some important references on this subject are Onaga [1967], Grinold [1975], Truemper [1977], Elam, Glover and Klingman [1979], and Bertsekas and Tseng [1988]. The recent paper by Goldberg, Plotkin and Tardos [1988] describes the first polynomial time combinatorial algorithms for the generalized maximum flow problem.

6.4 Minimum Cost Flow Problem

The minimum cost flow problem has a rich history. The classical transportation problem, a special case of the minimum cost flow problem, was posed and solved (though incompletely) by Kantorovich [1939], Hitchcock [1941], and Koopmans [1947]. Dantzig [1951] developed the first complete solution procedure for the transportation problem by specializing his simplex algorithm for linear programming. He observed the spanning tree property of the basis and the integrality property of the optimum solution. Later he developed the upper bounded technique for linear programming which allowed an efficient specialization of the simplex algorithm for the minimum cost flow problem. Dantzig's book [1962] discusses these topics.

Ford and Fulkerson [1956, 1957] suggested the first combinatorial algorithms for the uncapacitated and capacitated transportation problem; these algorithms are known as the primal-dual algorithms. Ford and Fulkerson [1962] describe the primal-dual algorithm for the minimum cost flow problem. The successive shortest path problem can be attributed independently to Jewell [1958], Iri [1960] and Busaker and Gowen [1961]. These researchers showed how to solve the minimum cost flow problem as a sequence of shortest path problems with arbitrary arc lengths. Tomizava

[1971] and Edmonds and Karp [1972] independently pointed out that if node potentials are used in the computations, then these algorithms can be implemented so that the shortest path problems have nonnegative arc lengths.

The out-of-kilter algorithm was independently discovered by Minty [1960] and Fulkerson [1961]. The negative cycle algorithm is credited to Klein [1967]. The specialization of the linear programming dual simplex algorithm for the minimum cost flow problem (not discussed in this chapter) can be found in Helgason and Kennington [1977] and Armstrong, Klingman and Whitman [1980]. Each of these algorithms perform iterations that can (apparently) not be polynomially bounded. Zadeh [1973a] describes one such example on which each of the following algorithms -- the primal simplex algorithm with Dantzig's pivot rule, the dual simplex algorithm, the negative cycle algorithm (which augments flow along a most negative cycle), the successive shortest path algorithm, the primal-dual algorithm, and the out-of-kilter algorithm -- performs an exponential number of iterations. In Zadeh [1973b], he describes more pathological examples for network algorithms.

The fact that one example is bad for many network algorithms suggests inter-relationship among the algorithms. The insightful paper by Zadeh [1979] shows this relationship by pointing out that each of the algorithms just mentioned are indeed equivalent in the sense that they perform the same sequence of augmentations (provided ties are broken arbitrarily). All these algorithms essentially consist of identifying shortest paths between appropriately defined nodes and augmenting flow along these paths. Further, the algorithms obtain shortest paths using a method that can be regarded as an application of Dijkstra's algorithm.

The network simplex algorithm and its practical implementations have been most popular with operations researchers. The first data structure suggested for implementing the simplex algorithm can be found in Johnson [1966]. The first implementations using these ideas are due to Srinivasan and Thompson [1973] and Glover, Karney, Klingman and Napier [1974]. These tree manipulating data structures significantly enhanced the running time of the simplex algorithm. Improved data structures were subsequently discovered by Glover, Klingman and Stutz [1974], Bradley, Brown and Graves [1977], and Barr, Glover, and Klingman [1979]. The book of Kennington and Helgason [1980] is an excellent source of reference on these data structures.

Researchers have conducted extensive studies to determine the most effective pricing strategy, i.e., selection of the entering variable. These studies show that the choice of the pivot strategy has a significant effect on both solution time and the number of pivots required to solve minimum cost flow problems. The candidate list strategy we described is due to Mulvey [1978a]. Goldfarb and Reid [1977], Bradley, Brown and Graves [1978], Grigoriadis and Hsu [1979], Gibby, Glover, Klingman and Mead [1983] and Grigoriadis [1986] have described other strategies that have been found to be effective in practice. It appears that the best pivot strategy depends both upon the network structure and the network size.

Experience with solving large scale minimum cost flow problems has established that more than 90% of the pivoting steps in the simplex method can be degenerate (see Bradley, Brown and Graves [1978], Gavish, Schweitzer and Shlifer [1977] and Grigoriadis [1986]). Thus degeneracy is both a computational and a theoretical issue. The strongly feasible basis technique by Cunningham [1976], and independently by Barr, Glover and Klingman [1977, 1978] showed that maintaining strongly feasible basis substantially reduces the number of degenerate pivots. On the theoretical front, it led to a finitely converging primal simplex algorithm. Orlin [1985] showed, using perturbation technique, that for integer data the primal simplex algorithm maintaining strongly feasible basis performs $O(nmCU)$ pivots for any arbitrary pricing strategy and $O(nm\,C\log(mCU))$ for Dantzig's pricing strategy.

The strongly feasible basis technique prevents cycling during a sequence of consecutive degenerate pivots, but the number of consecutive degenerate pivots may be exponential. This phenomenon is known as *stalling*. Cunningham [1979] described an example of stalling and suggested several rules for selecting the entering variable to avoid stalling. One such rule is the LRC (Least Recently Considered) rule which orders the arcs in an arbitrary but fixed manner. The algorithm then examines the arcs in the wrap-around fashion, each iteration starting at a place where it left off earlier, and introduces the first eligible arc into the basis. Cunningham shows that this rule admits at most nm consecutive degenerate pivots. Goldfarb, Hao and Kai [1987] describe more anti-stalling pivot rules for the minimum cost flow problem.

Researchers have also been interested in developing polynomial time simplex algorithms for the minimum cost flow problem or its special cases. The only polynomial time simplex algorithm for the minimum cost flow problem is a dual

simplex algorithm due to Orlin [1984] which performs $O(n^3 \log n)$ pivots for the uncapacitated minimum cost flow problem. Developing a polynomial time primal simplex algorithm for the minimum cost flow problem is still open. However, researchers have developed such algorithms for the shortest path problem, the maximum flow problem and the assignment problem. Polynomial time simplex algorithms for the shortest path problem have been discovered by Dial et al. [1979], Zadeh [1979], Orlin [1985], Akgul [1985a], Goldfarb, Hao and Kai [1986] and Ahuja and Orlin [1988]; for the maximum flow problem by Goldfarb and Hao [1988]; and for the assignment problem by Roohy-Laleh [1980], Hung [1983], Orlin [1985], Akgul [1985b] and Ahuja and Orin [1988].

The recent *relaxation* algorithm by Bertsekas and Tseng [1988] is another interesting algorithm for solving the minimum cost flow problem. This algorithm maintains a pseudoflow satisfying the optimality conditions. The algorithm proceeds by either augmenting flow from an excess node to a deficit node along a path consisting of arcs with zero reduced cost, or changing the potentials of a subset of nodes. In the latter case it resets flows on some arcs to their lower or upper bounds so as to satisfy the optimality conditions; however, this flow assignment might change the excesses and deficits at nodes. The algorithm operates so that each change in the node potentials increases the dual objective function value and when it finally has determined the optimum dual objective function value, it has also obtained an optimum primal solution. The relaxation algorithm has exhibited nice empirical behaviour and has been generalized for the minimum cost flow with gains and the convex cost flow problems.

A number of empirical studies have extensively tested minimum cost flow algorithms for wide variety of network structures, data distributions and problem sizes. The most common problem generator is NETGEN due to Klingman, Napier and Stutz [1974], which is capable of generating assignment, and capacitated or uncapacitated transportation and minimum cost flow problems. Glover, Karney and Klingman [1974] and Aashtiani and Magnanti [1976] have tested the primal-dual and out-of-kilter algorithms. Helgason and Kennington [1977] and Armstrong, Klingman and Whitman [1980] report extensive studies of the dual simplex algorithm. The primal simplex algorithm has been a subject of more rigorous investigation; studies due to Glover, Karney, Klingman and Napier [1974] , Glover, Karney and Klingman [1974], Bradley, Brown and Graves [1977], Mulvey [1978b], Grigoriadis and Hsu [1979]

and Grigoriadis [1986] are noteworthy. Bertsekas and Tseng [1988] present the computational results of the relaxation algorithm.

In view of Zadeh's [1979] result, we would expect that the successive shortest path algorithm, the primal-dual algorithm, the out-of-kilter algorithm, the dual simplex algorithm, and the primal simplex algorithm with Dantzig's pivot rule should have comparable running times. By using more effective pricing strategies that determine a good entering arc without examining all arcs, we would expect that the primal simplex algorithm should outperform other algorithms. All the computational studies have verified this expectation and until very recently the primal simplex algorithm has been a clear winner for almost all classes of network problems. Bertsekas and Tseng [1988] report their relaxation algorithm to be substantially faster than the primal simplex algorithm. However, Grigoriadis [1986] finds his new version of primal simplex algorithm faster than the relaxation algorithm. At this time, it appears that the relaxation algorithm of Bertsekas and Tseng, and the primal simplex algorithm due to Grigoriadis are the two fastest algorithms for solving the minimum cost flow problem in practice.

Computer codes for some minimum cost flow problem are available in the public domain. These include the primal simplex codes RNET and NETFLOW by Grigoradis and Hsu [1979] and Kennington and Helgason [1980] respectively, and the relaxation code RELAX by Bertsekas and Tseng [1987].

Polynomial Algorithms

In the recent past, researchers have actively pursued the design of fast (weakly) polynomial and strongly polynomial algorithms for the minimum cost flow problem. Recall that an algorithm is strongly polynomial if its running time is polynomial in the number of nodes and arcs, and does not evolve terms containing logarithms of C or U. The table given in Figure 6.3 summarizes these theoretical developments in solving the minimum cost flow problem. The table reports running times for networks with n nodes and m arcs of which m' arcs are capacitated. It assumes that the integral cost coefficients are bounded in absolute value by C, and the integral capacities, supplies and demands are bounded in absolute value by U. The term S(·) is the running time for the shortest path problem and the

term M($\cdot$) represents the corresponding running time to solve a maximum flow problem.

Polynomial Combinatorial Algorithms

#	Discoverers	Running Time
1	Edmonds and Karp [1972]	$O\big((n + m') \log U \; S(n, m, C)\big)$
2	Rock [1980]	$O\big((n + m') \log U \, S(n, m, C)\big)$
3	Rock [1980]	$O\big(n \log C \, M(n, m, U)\big)$
4	Bland and Jensen [1985]	$O\big(n \log C \, M(n, m, U)\big)$
5	Goldberg and Tarjan [1985]	$O\big(nm \log (n^2/m) \log nC\big)$
6	Bertekas and Eckstein [1988]	$O\big(n^3 \log nC\big)$
7	Gabow and Tarjan [1987]	$O\big(nm \log n \; \log U \log nC\big)$
8	Goldberg and Tarjan [1988]	$O\big(nm \log n \log nC\big)$
9	Ahuja, Goldberg, Orlin and Tarjan [1988]	$O\big(nm \, (\log U/\log \log U) \log nC\big)$ and $O\big(nm \log \log U \log nC\big)$

Strongly Polynomial Combinatorial Algorithms

#	Discoverers	Running Time
1	Tardos [1985]	$O(m^4)$
2	Orlin [1984]	$O\big((n + m')^2 \, S(n, m)\big)$
3	Fujishige [1986]	$O\big((n + m')^2 \, S(n, m)\big)$
4	Galil and Tardos [1986]	$O(n^2 \log n \, S(n,m))$
5	Goldberg and Tarjan [1987]	$O\big(nm^2 \log n \; \log(n^2/m)\big)$
6	Goldberg and Tarjan [1988]	$O\big(nm^2 \log^2 n\big)$
7	Orlin [1988]	$O\big((n + m') \log n \; S(n, m)\big)$

Figure 6.3   Polynomial algorithms for the minimum cost flow problem.

For the sake of comparing the polynomial and strongly polynomial algorithms, we assume that $C = O(n^{O(1)})$ and $U = O(n^{O(1)})$ which is known as the *similarity assumption* (see Gabow [1985]). For problems that satisfy the similarity assumption, the best bounds for the shortest path and maximum flow problems are:

Polynomial Bounds	Discoverers
$S(n,m, C) = \min \{m \log\log C, m + n\sqrt{\log C}\}$	Johnson [1982], and
	Ahuja, Mehlhorn, Orlin and Tarjan [1988]
$M(n, m, C) = nm \log\left(\frac{n\sqrt{\log U}}{m} + 2\right)$	Ahuja, Orlin and Tarjan [1987]

Strongly Polynomial Bounds	Discoverers
$S(n, m) = m + n \log n$	Fredman and Tarjan [1984]
$M(n, m) = nm \log (n^2/m)$	Goldberg and Tarjan [1986]

Edmonds and Karp [1972] developed the first (weakly) polynomial algorithm for the minimum cost flow problem based on capacity and right-hand-side scaling. The RHS-scaling algorithm presented in Section 5.7 is a variant of the Edmonds-Karp algorithm suggested by Orlin [1988]. This technique did not initially capture the interest of many researchers, since they regarded it as having little practical utility. However, researchers gradually recognized scaling technique to be of great theoretical value and also potential practical significance. Rock [1980] developed two different bit scaling algorithms for the minimum cost flow problem, one using capacity scaling and the other using cost scaling. This cost scaling algorithm reduces the minimum cost flow problem to a sequence of O(n log C) maximum flow problems. Bland and Jensen [1985] independently discovered a similar cost scaling algorithm.

The cost scaling algorithm due to Goldberg and Tarjan [1987], described in Section 5.8, differs from the previous cost algorithms in the sense that it relies on the concept of *approximate optimality* introduced by Bertsekas [1979]. This cost scaling approach showed that solving the minimum cost flow problem is *almost* O(log nC) times harder than solving the maximum flow problem. Goldberg and Tarjan obtained an $O(n^2m \log nC)$ time bound for the generic algorithm and $O(n^3 \log nC)$ bound for the wave algorithm. Using both *finger tree* (see Mehlhorn [1984]) and

dynamic tree data structures, Goldberg and Tarjan obtained an $O(nm \log (n^2/m) \log nC)$ bound for the minimum cost flow problem.

Goldberg and Tarjan's bound for the minimum cost flow problem is very attractive, but the algorithm requires sophisticated data structures with a very high computational overhead. Researchers investigated the possibility of obtaining a comparable time bound without using any complex data structure. The first success in this direction was due to Gabow and Tarjan [1987], who developed a triple scaling algorithm running in time $O(nm \log n \ \log U \log nC)$. The second success in this direction was due to Ahuja, Goldberg, Orlin and Tarjan [1988], who developed the double scaling algorithm. The double scaling algorithm, as described in Section 5.9, runs in $O(nm \log U \log nC)$ time. Scaling costs by an appropriate larger factor improves the algorithm to $O(nm(\log U/\log \log U) \log nC)$ and a dynamic tree implementation improves the bound further by $O(nm \log\log U \log nC)$. For problems satisfying the similarity assumption, the double scaling algorithm is faster than all other algorithms for all network topologies except for very dense networks; in these instances, algorithms by Goldberg-Tarjan appear more attractive.

Goldberg and Tarjan [1988] and Barahona and Tardos [1987] have developed other polynomial algorithms. Both the algorithms are based on the negative cycle algorithm due to Klein [1967]. Goldberg and Tarjan [1988] show that if flow is always augmented along a minimum mean cycle (a cycle W for which $\displaystyle\sum_{(i,j) \in W} c_{ij} / |W|$ is minimum), then the negative cycle algorithm is strongly polynomial. Goldberg and Tarjan describe an implementation of this approach running in time $O(nm(\log n) \min\{\log nC, m \log n\})$. Barahona and Tardos [1987], analysing an algorithm suggested by Weintraub [1974], show that if flow is augmented along a cycle with maximum improvement in the objective function, then the negative cycle algorithm performs $O(m \log mCU)$ iterations. Since identifying a cycle with maximum improvement is difficult (i.e., NP-hard), they describe a method (based upon solving an auxiliary assignment problem) to determine a disjoint set of augmenting cycles with the property that augmenting flows along these cycles results in an improvement at least as large as augmenting along any single cycle. Their algorithm runs in $O(m^2 \log (mCU) S(n, m, C))$ time.

Edmonds and Karp [1972] proposed the first polynomial time algorithm for the minimum cost flow problem, and also highlighted the desire to develop a strongly polynomial algorithm. This desire was motivated primarily by theoretical considerations. (Indeed, in practice, the terms log C and log U typically range from 1 to 20, and are sublinear in n.) Strongly polynomial algorithms are theoretically attractive for at least two reasons: (i) they might provide, in principle, network flow algorithms that can run on real valued data as well as integer valued data, and (ii) they might, at a more fundamental level, identify the source of the underlying complexity in solving a problem; i.e., are problems more difficult or equally difficult to solve as the values of the underlying data becomes increasingly larger.

The first strongly polynomial minimum cost flow algorithm is due to Tardos [1985]. Several researchers Orlin [1984], Fujishige [1986], Galil and Tardos [1986], and Orlin [1988] provided subsequent improvements in the running time. Goldberg and Tarjan [1988] proposed a simple, but inefficient algorithm along with an efficient, but more complex, implementation of their simple algorithm. Currently, the fastest strongly polynomial algorithm is due to Orlin [1988]. This algorithm solves the minimum cost flow problem as a sequence of $O(\min(m\log U, m\log n))$ shortest path problems. Whenever the network is very sparse, the worst case running time of this algorithm is nearly as low as the best weakly polynomial algorithm, even under the similarity assumption.

Interior point linear programming algorithms are another source of polynomial algorithms for the minimum cost flow problem. Kapoor and Vaidya [1986] have shown that Karmarkar's [1984] algorithm, when applied to the minimum cost flow problem performs $O(n^{2.5} mK)$ operations, where $K = \log n + \log C + \log U$. Vaidya [1986] suggests another algorithm for linear programming that solves the minimum cost flow problem in $O(n^{2.5} \sqrt{m} K)$ time. Asymptotically, these time bounds are worse than that of the double scaling algorithm.

At this time, sufficient evidence is not available to fully assess the computational worth of scaling and interior point linear programming algorithms for the minimum cost flow problem. According to the folklore, even though they might provide the best worst case bounds on running times, the scaling algorithms are not as efficient as the non-scaling algorithms. Boyd and Orlin [1986] have obtained contradictory results. Testing the right-hand-side scaling algorithm for the minimum cost flow problem, they found the scaling algorithm to be competitive

with the relaxation algorithm for some classes of problems. Bland and Jensen [1985] also report encouraging results with their cost scaling algorithm. We believe that the use of appropriate speed-up techniques have the potential to make the scaling algorithms competitive with the best other algorithms.

6.5   Assignment Problem

The assignment problem has been a popular research topic. The primary emphasis in the literature has been on the development of empirically efficient algorithms rather than the development of algorithms with improved worst-case complexity. Although the research community has developed several different algorithms for the assignment problem, many of these algorithms share common features. The successive shortest path algorithm, described in Section 5.4 for the minimum cost flow problem, appears to be at the heart of many assignment algorithms. This algorithm is implicit in the first assignment algorithm due to Kuhn [1955], known as the *Hungarian method*, and is explicit in the papers by Tomizava [1971] and Edmonds and Karp [1972].

When applied to an assignment problem on the network $G = (N_1 \cup N_2, A)$, the successive shortest path algorithm operates as follows. We first transform the assignment problem into a minimum cost flow problem by adding a source node s, a sink node t and introducing arcs (s,i) for all $i \in N_1$, and (j,t) for all $j \in N_2$ of zero cost and unit capacity. The algorithm successively obtains a shortest path from s to t with respect to the linear programming reduced costs, updates the node potentials and augments one unit of flow along the shortest path. This algorithm solves the assignment problem by n applications of the shortest path algorithm (with nonnegative arc lengths) and runs in $O(nS(n,m,C))$ time, where $S(n,m,C)$ is the time needed to solve a shortest path problem. For a naive implementation of Dijkstra's algorithm, $S(n,m,C)$ is $O(n^2)$ and for Fibonacci heap implementation it is $O(m+n\log n)$. For problems satisfying the similarity assumption, $S(n,m,C)$ is $\min\{m\log\log C, m+n\sqrt{\log C}\}$.

The fact that the assignment problem can be solved as a sequence of n shortest path problems with arbitrary arc lengths follows from the works of Jewell [1958], Iri [1960] and Busaker and Gowen [1961] on the minimum cost flow problem. However, Tomizava [1971] and Edmonds and Karp [1972] independently pointed out that

working with reduced costs leads to shortest path problems with nonnegative arc lengths. Weintraub and Barahona [1979] worked out the details of Edmonds-Karp algorithm for the assignment problem. The more recent *threshold assignment algorithm* by Glover, Glover and Klingman [1986] is also a successive shortest path algorithm which integrates their threshold shortest path algorithm (see Glover, Glover and Klingman [1984]) with the flow augmentation process. Carraresi and Sodini [1986] also suggested a similar threshold assignment algorithm.

Kuhn's [1955] Hungarian method is the primal-dual version of the successive shortest path algorithm. After solving a shortest path problem and updating the node potentials, the Hungarian method solves a (particularly simple) maximum flow problem to send maximum possible flow from the source node s to the sink node t using arcs with zero reduced cost. Whereas the successive shortest path problem augments flow along one path in an iteration, the Hungarian method augments flow along all the shortest paths from the source node to the sink node. If we use the labeling algorithm to solve the resulting maximum flow problems, then these applications take a total of $O(nm)$ time overall, since there are n augmentations and each augmentation takes $O(m)$ time. Consequently, Hungarian method, too, runs in $O(nm + nS(n,mC)) = O(nS(n,m,C))$ time. (For some time after the development of the Hungarian method as described by Kuhn, the research community considered it to be $O(n^4)$ method. Lawler [1976] described an $O(n^3)$ implementation of the method. Subsequently, many researchers realized that the Hungarian method in fact runs in $O(nS(n,m,C))$ time.) Jonker and Volgenant [1986] suggest some practical improvements of the Hungarian method.

The relaxation approaches due to Dinic and Kronrod [1969], Hung and Rom [1980] and Engquist [1982] are also closely related to the successive shortest path approach. The relaxation approach relaxes the constraint (5.11c) and allows an object to be assigned to more than one person. This relaxed problem can be easily solved by assigning each person i to an object j with smallest c_{ij} value. As a result, there may be unassigned as well as overassigned objects. The algorithm gradually makes this infeasible assignment feasible by identifying shortest paths from overassigned objects to unassigned objects and augmenting flows on these paths. This approach always maintains the optimality conditions and hence it can solve the shortest path problems by implementations of Dijkstra's algorithm. The algorithms of Dinic and Kronrod [1969] and Engquist [1982] are essentially the same as the one we just

described, but the shortest path computations are somewhat disguised in the Dinic and Kronrod paper. The algorithm of Hung and Rom [1980] maintains a strongly feasible basis rooted at an overassigned node and after each augmentation reoptimizes over the previous basis to obtain another strongly feasible basis. All of these algorithms run in $O(nS(n,m,C))$ time.

Another algorithm worth mentioning is due to Balinski and Gomory [1964]. This algorithm is a primal algorithm that maintains a feasible assignment and gradually converts it into an optimum assignment by augmenting flows along negative cycles or by modifying node potentials. Derigs [1985] notes that the shortest path computations underlie this method and it runs in $O(nS(n,m,C))$ time.

Researchers have also studied primal simplex algorithms for the assignment problem. The basis of the assignment problem is highly degenerate; of its $2n-1$ variables, only n are nonzero. Probably due to this excessive degeneracy, the mathematical programming community did not conduct much research on the network simplex method for the assignment problem until the advent of the strongly feasible basis technique by Barr, Glover and Klingman [1977]. They developed the details of the network simplex algorithm when implemented to maintain a strongly feasible basis for the assignment problem and reported encouraging computational performance. Subsequent research was focused on developing polynomial time simplex algorithms. Roohy-Laleh [1980] developed a simplex pivot rule requiring $O(n^3)$ pivots. Hung [1983] describes a pivot rule that performs at most $O(n^2)$ consecutive degenerate pivots and at most $O(nlog\ nC)$ nondegenerate pivots. Hence his algorithm performs $O(n^3lognC)$ pivots. Akgul [1985b] suggests another primal simplex algorithm performing $O(n^2)$ pivots. This algorithm essentially amounts to solving n shortest path problems and runs in $O(nS(n,m,C))$ time.

Orlin [1985] studied the theoretical properties of Dantzig's pivot rule for the network simplex algorithm and showed that this rule results in $O(n^2lognC)$ pivots for the assignment problem. A naive implementation of the algorithm runs in $O(n^2mlognC)$. Ahuja and Orlin [1988] describe a scaling version of Dantzig's pivot rule that performs $O(n^2logC)$ pivots; this algorithm can be implemented to run in $O(nmlogC)$ time using simple data structures. The algorithm essentially consists of pivoting in any arc with *sufficiently large* reduced cost. The author's define the term

"sufficiently large" iteratively; initially this threshold value equals C and within $O(n^2)$ pivots its value is halved.

Balinski [1985] developed the *signature method*, which is a dual simplex algorithm for the assignment problem. (Although his basic algorithm maintains a dual feasible basis, but is not really a dual simplex algorithm in the traditional sense because it does not necessarily increase the dual objective at every iteration; some variants of this algorithm do have this property.) Balinski's algorithm performs $O(n^2)$ pivots and runs in $O(n^3)$ time. Goldfarb [1985] also describes some implementations of Balinski's algorithm that run in $O(n^3)$ time using simple data structures and in $O(nm + n^2\log n)$ time using Fibonacci heaps.

The auction algorithm is due to Bertsekas and uses basic ideas originally suggested by Bertsekas [1979]. Bertsekas [1981] presented a hybrid of the auction algorithm with the Hungarian method. A more recent version of the auction algorithm can be found in Bertsekas and Eckstein [1988].

Currently, the best strongly polynomial bound to solve the shortest path problem is $O(nm + n^2 \log n)$ which is achieved by many assignment algorithms. Scaling algorithms can do better for problems that satisfy the similarity assumption. Gabow [1985] developed the first scaling algorithm for the assignment problem based on bit scaling of costs. Gabow's algorithm performs $O(\log C)$ scaling phases, solves each phase in $O(n^{3/4}m)$ time, thereby achieving an $O(n^{3/4}m \log C)$ time bound. Using the concept of ε-optimality, Gabow and Tarjan [1987] developed another scaling algorithm running in time $O(n^{1/2}m \log nC)$. Bertsekas and Eckstein [1988] showed that the scaling version of the auction algorithm runs in $O(nm \log nC)$. Section 5.11 has presented a modified version of this algorithm in Orlin and Ahuja [1988] improved the time bound of the auction algorithm to $O(n^{1/2}m \log nC)$. This time bound is comparable to that of Gabow and Tarjan's algorithm, but the two algorithms would probably have different computational attributes. For problem's satisfying the similarity assumption, these two algorithms achieve the best time bound to solve the assignment problem without using any sophisticated data structure.

As mentioned previously, most of the research effort devoted to assignment algorithms has stressed the development of empirically faster algorithms. Over the

years many computational studies have appeared comparing one algorithm with a few other algorithms. The computational studies by Barr, Glover and Klingman [1977] on the network simplex method, by McGinnis [1983] on the primal-dual method, by Engquist [1982] on the relaxation methods, and by Glover et al. [1986] and Jonker and Volgenant [1987] on the successive shortest path methods are some representatives. Since no paper has compared all of these algorithms , it is difficult to assess their computational merits. Nevertheless, results to date seem to justify the following observations about the algorithm's relative performance. The primal simplex algorithm is slower than the primal-dual, relaxation and successive shortest path algorithms. Among the latter three approaches, the successive shortest path algorithms due to Glover et al. [1986] and Jonker and Volgenant [1987] appear to be the fastest. Bertsekas and Eckstein [1988] found that the scaling version of the auction algorithm is substantially superior to Jonker and Volgenant's algorithm for sparse networks but worse for dense networks.

6.6   Other Topics

Our discussion in this paper has featured on single commodity network flow problems with linear costs. Several other generic topics in the broader problem domain of network optimization are of considerable theoretical and practical interests. In particular, three other topics deserve mention:

i)   **Multicommodity flow problems.** For this class of problems, several commodities use the same underlying network, sharing common arc capacities . That is, the problem formulation contains "bundle constraints" that specify that the total flow on certain arcs cannot exceed the arc's capacity. The text by Kennington and Helgason [1980] describes the basic approaches to this problem, as do surveys by Assad [1978], Kennington [1978], and Ali et al. [1984].

(ii)   **Convex Cost Network Flow Problems.** One of the most natural extensions of the network flow models we have considered would be to replace the linear objective functions by more general convex cost functions. In some instances, for example when the cost function separates by arcs, that is $f(x) = \sum_{(i,j) \in A} f_{ij}(x_{ij})$, approximating each arc by a series of parallel arcs with linear costs would

permit us to use the techniques we have considered to solve these problems approximately to within any desired degree of accuracy. More elaborate algorithms are also possible. The general convex cost case requires solution techniques from nonlinear programming that are quire different that those we have described. As an overview to this literature, the reader might refer to the Kennington and Helgason [1980], Florian [1986] and the monograph by Rockafellar [1984].

iii) **Network Design.** We have focused on solution methods for finding optimal routings in a network; that is, on analysis rather than design. The design problem itself is of considerable importance in practice and has generated an extensive literature of its own. Many design problems can be stated as fixed cost network flow problems: (some) arcs have an associated fixed cost which is incurred whenever the arc carries *any* flow. This class of problem requires solution techniques from integer programming and other type of solution methods from combinatorial optimization. Magnanti and Wong [1984] describe the broad range of applicability of network design models and summarize solution methods for those problems and many references from the network design literature.

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References

Aashanti, H.A., and T. L. Magnanti. 1976. Implementing Primal-Dual Network Flow Algorithms. Technical Report OR 055-76, Operations Research Center, M.I.T., Cambridge, MA.

Aho, A.V. , J.E. Hopcroft, and J.D. Ullman. 1974. *The Design and Analysis of Computer Algorithms*. Addison-Wesley, Reading, MA.

Ahuja, R.K., A.V. Goldberg, J.B. Orlin, and R.E. Tarjan. 1988. Finding Minimum-Cost Flows by Double Scaling. To appear.

Ahuja, R.K., M. Kodialam, and J.B. Orlin. 1988. Personal Communication.

Ahuja, R.K., K. Mehlhorn, J.B. Orlin, and R.E. Tarjan. 1988. Faster Algorithms for the Shortest Path Problem. Technical Report No. 193, Operations Research Center, M.I.T., Cambridge, MA.

Ahuja, R.K., and J.B. Orlin. A Fast and Simple Algorithm for the Maximum Flow Problem. Working Paper 1905-87, Sloan School of Management, M.I.T., Cambridge, MA. 1987. To appear in *Oper. Res.*

Ahuja, R.K., and J.B. Orlin. 1988. Improved Primal Simplex Algorithms for the Shortest Path, Assignment and Minimum Cost Flow Problems. To appear.

Ahuja, R.K., J.B. Orlin, C. Stein, and R.E. Tarjan. Improved Algorithms for Bipartite Network Flow Problems. To appear.

Ahuja, R.K., J.B. Orlin, and R.E. Tarjan. 1988. Improved Time Bounds for the Maximum Flow Problem. Working Paper 1966-87, Sloan School of Management, M.I.T., Cambridge, MA.

Akgul, M. 1985a. Shortest Path and Simplex Method. Research Report, Department of Computer Science and Operations Research, North Carolina State University, Raleigh, N.C.

Ali,I., D. Barnett, K. Farhangian, J. Kennington, B. Patty, B. Shetty, B. McCarl and P. Wong. 1984. Multicommodity Network Problems: Applications and Computations. *I.I.E. Trans.* 16, 127-134.

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Akgul, M. 1985b. A Genuinely Polynomial Primal Simplex Algorithm for the Assignment Problem. Research Report, Department of Computer Science and Operations Research, North Carolina State University, Raleigh, N.C.

Armstrong, R.D., D. Klingman, and D. Whitman. 1980. Implementation and Analysis of a Variant of the Dual Method for the Capacitated Transshipment Problem. *Euro. J. Oper. Res.* 4, 403-420.

Assad, A. 1978. Multicommodity Network Flows – A Survey. *Networks* 8, 37-91.

Balinski, M.L. 1985. Signature Methods for the Assignment Problem. *Oper. Res.* 33, 527-536.

Balinski, M.L., and R.E. Gomory. 1964. A Primal Method for the Assignment and Transportation Problems. *Man. Sci.* 10, 578-593.

Barahona, F., and E. Tardos. 1987. Note on Weintraub's Minimum Cost Flow Algorithm. Research Report, Dept. of Mathematics, M.I.T., Cambridge, MA.

Baratz, A.E. 1977. Construction and Analysis of Network Flow Problem Which Forces Karzanov Algorithm to $O(n^3)$ Running Time. Technical Report TM-83, Laboratory for Computer Science, MIT, Cambridge, MA.

Barr, R., F. Glover, and D. Klingman. 1977. The Alternating Path Basis Algorithm for the Assignment Problem. *Math. Prog.* 12, 1-13.

Barr, R., F. Glover, and D. Klingman. 1978. Generalized Alternating Path Algorithm for Transportation Problems. *Euro. J. Oper. Res.* 2, 137-144.

Barr, R., F. Glover, and D. Klingman. 1979. Enhancement of Spanning Tree Labeling Procedures for Network Optimization. *INFOR* 17, 16-34.

Bazaraa, M., and J.J. Jarvis. 1978. *Linear Programming and Network Flows.* John Wiley & Sons.

Bellman, R. 1958. On a Routing Problem. *Quart. Appl. Math.* 16, 87-90.

Berge, C., and A. Ghouila-Houri. 1962. *Programming, Games and Transportation Networks.* John Wiley & Sons.

Bertsekas, D.P. 1979. A Distributed Algorithm for the Assignment Problem. Working Paper, Laboratory for Information Decision Systems, M.I.T., Cambridge, MA.

Bertsekas, D.P. 1981. A New Algorithm for the Assignment Problem. *Math. Prog.* 21, 152-171.

Bertsekas, D.P. 1986. Distributed Relaxation Methods for Linear Network Flow Problems. *Proc. of 25th IEEE Conference on Decision and Control*, Athens, Greece.

Bertsekas, D.P., and J. Eckstein. 1988. Dual Coordinate Step Methods for Linear Network Flow Problems. To appear in *Math. Prog., Series B.*

Bertsekas, D., and R. Gallagher. 1987. *Data Networks.* Prentice-Hall.

Bertsekas, D.P., and P. Tseng. 1987. Relax : A Code for Linear Network Flow Problems. In B. Simeone, (ed.), *FORTRAN Codes for Network Optimization.* To be published by the *Annals of Operations Research.*

Bertsekas, D.P., and P. Tseng. 1988. Relaxation Methods for Minimum Cost Ordinary and Generalized Network Flow Problems. *Oper. Res.* 36, 93-114.

Bland, R.G., and D.L. Jensen. 1985. On the Computational Behavior of a Polynomial-Time Network Flow Algorithm. Techical Report 661, School of Operations Research and Industrial Engineering, Cornell University, Ithaca, N.Y.

Boas, P. Van Emde, R. Kaas, and E. Zijlstra. 1977. Design and Implementation of an Efficient Priority Queue. *Math. Sys. Theory* 10, 99-127.

Bodin, L. D., B. L. Golden, A. A. Assad, and M. O. Ball. 1983. Routing and Scheduling of Vehicles and Crews. *Comp. and Oper. Res.* 10, 65-211.

Boyd, A., and J.B. Orlin. 1986. Personal Communication.

Bradley, G., G. Brown, and G. Graves. 1977. Design and Implementation of Large Scale Primal Transshipment Algorithms. *Man. Sci.* 21, 1-38.

Busaker, R.G., and P.J. Gowen. 1961. A Procedure for Determining a Family of Minimal-Cost Network Flow Patterns. O.R.O. Technical Report No. 15, Operational Research Office, John Hopkins University, Baltimore, MD.

184

Carraresi, P., and C. Sodini. 1986. An Efficient Algorithm for the Bipartite Matching Problem. *Eur. J. Oper. Res.* 23, 86-93.

Cheriyan, J. 1988. Parametrized Worst Case Networks for Preflow Push Algorithms. Technical Report, Computer Science Group, Tata Institute of Fundamental Research, Bombay, India.

Cheriyan, J., and S.N. Maheshwari. 1987. Analysis of Preflow Push Algorithms for Maximum Network Flow. Technical Report, Dept. of Computer Science and Engineering, Indian Institute of Technology, New Delhi, India.

Cherkasky, R.V. 1977. Algorithm for Construction of Maximum Flow in Networks with Complexity of $O(V^2 \sqrt{E}$) Operation, *Mathematical Methods of Solution of Economical Problems* 7, 112-125 (in Russian).

Cheung, T. 1980. Computational Comparison of Eight Methods for the Maximum Network Flow Problem. *ACM Trans. on Math. Software*, 6, 1-16.

Christophides, N. 1975. *Graph Theory : An Algorithmic Approach*. Academic Press.

Cunningham, W.H. 1976. A Network Simplex Method. *Math. Prog.* 11, 105-116.

Cunningham, W.H. 1979. Theoretical Properties of the Network Simplex Method. *Math. of Oper. Res.* 4, 196-208.

Dantzig, G.B. 1951. Application of the Simplex Method to a Transportation Problem. In T.C. Koopmans (ed.), *Activity Analysis of Production and Allocation*, John Wiley & Sons, Inc., 359-373.

Dantzig, G.B. 1955. Upper Bounds, Secondary Constraints, and Block Triangularity in Linear Programming. *Econometrica* 23, 174-183.

Dantzig, G.B. 1960. On the Shortest Route through a Network. *Man. Sci.* 6, 187-190.

Dantzig, G.B. 1962. *Linear Programming and Extensions*. Princeton University Press, Princeton, NJ.

Dantzig, G.B. 1967. All Shortest Routes in a Graph. In P. Rosenthiel (ed.), *Theory of Graphs*, Gordon and Breach, NY, 91-92.

Dantzig, G.B., and D.R. Fulkerson. 1956. On the Max-Flow Min-Cut Theorem of Networks. In H.W. Kuhn and A.W. Tucker (ed.), *Linear Inequalities and Related Systems*, Annals of Mathematics Study 38, Princeton University Press, 215-221.

Deo, N., and C. Pang. 1984. Shortest Path Algorithms: Taxonomy and Annotation. *Networks* 14, 275-323.

Denardo, E.V., and B.L. Fox. 1979. Shortest-Route Methods: 1. Reaching, Pruning and Buckets. *Oper. Res.* 27, 161-186.

Derigs, U. 1985. The Shortest Augmenting Path Method for Solving Assignment Problem : Motivation and Computational Experience. *Annals of Operations Research* 4, 57-102.

Derigs, U. 1988. *Programming in Networks and Graphs*. Lecture Notes in Economics and Mathematical Systems, Vol. 300, Springer-Verlag.

Derigs, U., and W. Meier. 1988. Implementing Goldberg's Max-Flow algorithm: A Computational Investigation. Technical Report, Institute of Operations Research, University of Bonn, West Germany.

Dial, R. 1969. Algorithm 360: Shortest Path Forest with Topological Ordering. *Comm. ACM* 12, 632-633.

Dial, R., F. Glover, D. Karney, and D. Klingman. 1979. A Computational Analysis of Alternative Algorithms and Labeling Techniques for Finding Shortest Path Trees. *Networks* 9, 215-248.

Dijkstra, E. 1959. A Note on Two Problems in Connexion with Graphs. *Numeriche Mathematics* 1, 269-271.

Dinic, E.A. 1970. Algorithm for Solution of a Problem of Maximum Flow in Networks with Power Estimation, *Soviet Math. Dokl.* 11, 1277-1280.

Dinic, E.A., and M.A. Kronrod. 1969. An Algorithm for Solution of the Assignment Problem. *Soviet Maths. Doklady* 10, 1324-1326.

Edmonds, J. 1970. Exponential Growth of the Simplex Method for the Shortest Path Problem. Unpublished paper, University of Waterloo, Ontario, Canada.

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Edmonds, J., and R.M. Karp. 1972. Theoretical Improvements in Algorithmic Efficiency for Network Flow Problems. *J. ACM* 19, 248-264.

Elam, J., F. Glover, and D. Klingman. 1979. A Strongly Convergent Primal Simplex Algorithm for Generalized Networks. *Math. of Oper. Res.* 4, 39-59.

Elias, P., A. Feinstein, and C.E. Shannon. 1956. Note on Maximum Flow Through a Network. *IRE Trans. on Infor. Theory* IT-2, 117-119.

Engquist, M. 1982. A Successive Shortest Path Algorithm for the Assignment Problem. *INFOR* 20, 370-384.

Even, S. 1976. The Max-Flow Algorithm of Dinic and Karzanov: An Exposition. Technical Report TM-80, Laboratory for Computer Science, M.I.T., Cambridge, MA.

Even, S. 1979. *Graph Algorithms*. Computer Science Press, Maryland.

Even, S., and R.E. Tarjan. 1975. Network Flow and Testing Graph Connectivity. *SIAM J. Comput.* 4, 507-518.

Fernandez-Baca, D., and C.U. Martel. 1987. On the Efficiency of Maximum Flow Algorithms on Networks with Small Integer Capacities. Research Report, Department of Computer Science, Iowa State University, Ames, IA. To appear in *Algorithmica*.

Florian, M. 1986. Nonlinear Cost Network Models in Transportation Analysis. *Math. Prog. Study* 26, 167-196.

Floyd, R.W. 1962. Algorithm 97: Shortest Path. *Comm. ACM* 5, 345.

Ford, L.R., Jr. 1956. Network Flow Theory. Report P-923, Rand Corp., Santa Monica, CA.

Ford, L.R., Jr., and D.R. Fulkerson. 1956. Maximal Flow through a Network. *Canad. J. Math.* 8, 399-404.

Ford, L.R., Jr., and D.R. Fulkerson. 1956. Solving the Transportation Problem. *Man. Sci.* 3, 24-32.

Ford, L.R., Jr., and D.R. Fulkerson. 1957. A Primal-Dual Algorithm for the Capacitated Hitchcock Problem. *Naval Res. Logist. Quart.* 4, 47-54.

Ford, L.R., Jr., and D.R. Fulkerson. 1958. Constructing Maximal Dynamic Flows from Static Flows. *Oper. Res.* 6, 419-433.

Ford, L.R., Jr., and D.R. Fulkerson. 1962. *Flows in Networks* . Princeton University Press, Princeton, NJ.

Francis, R., and P. Mirchandani (eds.). 1988. *Discrete Location Theory*. John Wiley & Sons. To appear.

Frank, H., and I.T. Frisch. 1971. *Communication, Transmission, and Transportation Networks*. Addison-Wesley.

Fredman, M.L., and R.E. Tarjan. 1984. Fibonacci Heaps and Their Uses in Improved Network Optimization Algorithms. *25th Annual IEEE Symp. on Found. of Comp. Sci* , 338-346, also in *J. of ACM* 34(1987), 596-615.

Fujishige, S. 1986. An $O(m^3 \log n)$ Capacity-Rounding Algorithm for the Minimum Cost Circulation Problem: A Dual Framework of Tardos' Algorithm. *Math. Prog.* 35, 298-309.

Fulkerson, D.R. 1961. An Out-of-Kilter Method for Minimal Cost Flow Problems. *SIAM J. Appl. Math.* 9, 18-27.

Fulkerson, D.R., and G.B. Dantzig. 1955. Computation of Maximum Flow in Networks. *Naval Res. Log. Quart.* 2, 277-283.

Gabow, H.N. 1985. Scaling Algorithms for Network Problems. *J. of Comput. Sys. Sci.* 31, 148-168.

Gabow, H.N., and R.E. Tarjan. 1987. Faster Scaling Algorithms for Network Problems. *SIAM J. Comput.* (submitted).

Galil, Z. 1980. $O(V^{5/3} E^{2/3})$ Algorithm for the Maximum Flow Problem, *Acta Informatica* 14, 221-242.

Galil, Z. 1981. On the Theoretical Efficiency of Various Network Flow Algorithms. *Theoretical Comp. Sci.* 14, 103-111.

Galil, Z., and A. Naamad. 1980. An $O(VE \log^2 V)$ Algorithm for the Maximum Flow Problem. *J. of Comput. Sys. Sci.* 21, 203-217.

Galil, Z., and E. Tardos. 1986. An $O(n^2(m + n \log n) \log n)$ Min-Cost Flow Algorithm. *Proc. 27th Annual Symp. on the Found. of Comp. Sci.* , 136-146.

Gallo, G., and S. Pallottino. 1988. Shortest Path Algorithms. To appear in *Annals of Operations Research.*

Gallo, G., S. Pallottino, C. Ruggen, and G. Starchi. 1982. Shortest Paths: A Bibliography. Sofmat Document 81-P1-4-SOFMAT-27, Rome, Italy.

Gavish, B., P. Schweitzer, and E. Shlifer. 1977. The Zero Pivot Phenomenon in Transportation Problems and Its Computational Implications. *Math. Prog.* 12, 226-240.

Gibby, D., F. Glover, D. Klingman, and M. Mead. 1983. A Comparison of Pivot Selection Rules for Primal Simplex Based Network Codes. *Oper. Res. Letters* 2, 199-202.

Gilsinn, J., and C. Witzgall. 1973. A Performance Comparison of Labeling Algorithms for Calculating Shortest Path Trees. Technical Note 772, National Bureau of Standards, Washington, D.C.

Glover, F., R. Glover, and D. Klingman. 1984. The Threshold Shortest Path Algorithm. *Networks* 14, No. 1.

Glover, F., R. Glover, and D. Klingman. 1986. Threshold Assignment Algorithm. *Math. Prog. Study* 26, 12-37.

Glover, F., D. Karney, and D. Klingman. 1974. Implementation and Computational Comparisons of Primal, Dual and Primal-Dual Computer Codes for Minimum Cost Network Flow Problem. *Networks* 4, 191-212.

Glover, F., D. Karney, D. Klingman, and A. Napier. 1974. A Computational Study on Start Procedures, Basis Change Criteria, and Solution Algorithms for Tranportation Problem. *Man. Sci.* 20, 793-813.

Glover, F., and D. Klingman. 1976. Network Applications in Industry and Government. Technical Report, Center for Cybernetic Research, University of Texas, Austin, TX.

Glover, F., D. Klingman, J. Mote, and D. Whitman. 1979. Comprehensive Computer Evaluation and Enhancement of Maximum Flow Algorithms. Research Report CCS356, Center for Cybernetic Studies, University of Texas, Austin, 1979.

Glover, F., D. Klingman, J. Mote, and D. Whitman. 1984. A Primal Simplex Variant for the Maximum Flow Problem. *Naval Res. Logis. Quart.* 31, 41-61.

Glover, F., D. Klingman, and N. Phillips. 1985. A New Polynomially Bounded Shortest Path Algorithm. *Oper. Res.* 33, 65-73.

Glover, F., D. Klingman, N. Phillips, and R.F. Schneider. 1985. New Polynomial Shortest Path Algorithms and Their Computational Attributes. *Man. Sci.* 31, 1106-1128.

Glover, F., D. Klingman, and J. Stutz. 1974. Augmented Threaded Index Method for Network Optimization. *INFOR* 12, 293-298.

Goldberg, A.V. 1985. A New Max-Flow Algorithm. Technical Report MIT/LCS/TM-291, Laboratory for Computer Science, M.I.T., Cambridge, MA.

Goldberg, A.V., S.A. Plotkin, and E. Tardos. 1988. Combinatorial Algorithms for the Generalized Circulation Problem. Research Report. Laboratory for Computer Science, M.I.T., Cambridge, MA.

Goldberg, A.V., and R.E. Tarjan. 1986. A New Approach to the Maximum Flow Problem. *Proc. 18th ACM Symp. on the Theory of Comput.*, 136-146. To appear in *J. ACM*.

Goldberg, A.V., and R.E. Tarjan. 1987. Solving Minimum Cost Flow Problem by Successive Approximation. *Proc. 19th ACM Symp. on the Theory of Comp.* 136-146.

Goldberg, A.V., and R.E. Tarjan. 1988. Finding Minimum-Cost Circulations by Canceling Negative Cycles. *Proc. 20th ACM Symp. on the Theory of Comp.*, 388-397.

Golden, B., and T. L. Magnanti. 1977. Deterministic Network Optimization: A Bibliography. *Networks* 7, 149-183.

Goldfarb, D. 1985. Efficient Dual Simplex Algorithms for the Assignment Problem. *Math. Prog.* 33, 187-203.

Goldfarb, D., and M.D. Grigoriadis. 1986. A Computational Comparison of the Dinic and Network Simplex Methods for Maximum Flow. Technical Report No. LCSR-TR-94, Dept. of Computer Science, Rutgers University, New Brunswick, NJ.

Goldfarb, D., J. Hao, and S. Kai. 1986. Efficient Shortest Path Simplex Algorithms. Research Report, Dept. of OR and IE, Columbia University, New York, NY.

Goldfarb, D., J. Hao, and S. Kai. 1987. Anti-Stalling Pivot Rules for the Network Simplex Algorithm. Research Report, Dept. of OR and IE, Columbia University, New York, NY.

Goldfarb, D., and J. Hao. 1988. A Primal Simplex Algorithm that Solves the Maximum Flow Problem in At Most nm Pivots and $O(n^2m)$ Time. Technical Report, Dept. of IE and OR, Columbia University, New York.

Goldfarb, D., and J.K. Reid. 1977. A Practicable Steepest Edge Simplex Algorithm. *Math. Prog.* 12, 361-371.

Gomory, R. E., and T. C. Hu. 1961. Multi-Terminal Network Flows. *J. of SIAM* 9, 551-570.

Gondran, M., and M. Minoux. 1984. *Graphs and Algorithms.* Wiley-Interscience.

Grigoriadis, M. D. 1986. An Efficient Implementation of the Network Simplex Method. *Math. Prog. Study* 26, 83-111.

Grigoriadis, M. D. 1988. Personal Communication.

Grigoriadis, M. D., and T. Hsu. 1979. The Rutgers Minimum Cost Network Flow Subroutines. *SIGMAP Bulletin of the ACM* 26, 17-18.

Grinold, R. C. 1973. Calculating Maximal Flows in a Network with Positive Gains. *Oper. Res.* 21, 528-541.

Gusfield, D. 1987. Very Simple Algorithms and Programs for All Pairs Network Flow Analysis. Research Report No. CSE-87-1. Dept. of Computer Science and Engg., University of California, Davis, CA.

Gusfield, D., C. Martel, and D. Fernandez-Baca. 1985. Fast Algorithms for Bipartite Network Flow. Technical Report No. YALEN/DCS/TR-356, Yale University, New Haven, CT.

Hamachar, H. 1979. Numerical Investigations on the Maximal Flow Algorithm of Karzanov. *Computing* 22, 17-29.

Hassin, R., and D. B. Johnson. 1985. An $O(n\log^2 n)$ Algorithm for Maximum Flow in Undirected Planar Networks. *SIAM J. Comput.* 14, 612-624.

Hausman, D. 1978. *Integer Programming and Related Areas: A Classified Bibliography.* Lecture Notes in Economics and Mathematical Systems, Vol. 160. Springer-Verlag.

Helgason, R. V., and J. L. Kennington. 1977. An Effiicient Procedure for Implementing a Dual-Simplex Network Flow Algorithm. *AIIE Trans.* 9, 63-68.

Hitchcock, F. L. 1941. The Distribution of a Product from Several Sources to Numerous Facilities. *J. Math. Phys .* 20, 224-230.

Hu, T.C. 1969. *Integer Programming and Network Flows.* Addison-Wesley.

Hung, M. S. 1983. A Polynomial Simplex Method for the Assignment Problem. *Oper. Res.* 31, 595-600.

Hung, M. S., and W. O. Rom. 1980. Solving the Assignment Problem by Relaxation. *Oper. Res .* 28, 969-892.

Imai, H. 1983. On the Practical Efficiency of Various Maximum Flow Algorithms, *J. Oper. Res. Soc. Japan* 26, 61-82.

Iri, M. 1960. A New Method of Solving Transportation-Network Problems. *J. Oper. Res. Soc. Japan* 3, 27-87.

Iri, M. 1969. *Network Flows, Transportation and Scheduling.* Academic Press.

Itai, A., and Y. Shiloach. 1979. Maximum Flow in Planar Networks. *SIAM J. Comput.* 8, 135-150.

Jensen, P.A., and W. Barnes. 1980. *Network Flow Programming.* John Wiley & Sons.

Jewell, W. S. 1958. Optimal Flow Through Networks. Interim Technical Report No. 8, Operation Research Center, M.I.T., Cambridge, MA.

Jewell, W. S. 1962. Optimal Flow Through Networks with Gains. *Oper. Res.* 10, 476-499.

Johnson, D. B. 1977a. Efficient Algorithms for Shortest Paths in Sparse Networks. *J. ACM* 24, 1-13.

Johnson, D. B. 1977b. Efficient Special Purpose Priority Queues. *Proc. 15th Annual Allerton Conference on Comm., Control and Computing,* 1-7.

Johnson, D. B. 1982. A Priority Queue in Which Initialization and Queue Operations Take O(log log D) Time. *Math. Sys. Theory* 15, 295-309.

Johnson, D. B., and S. Venkatesan. 1982. Using Divide and Conquer to Find Flows in Directed Planar Networks in $O(n^{3/2}\log n)$ time. In *Proceedings of the 20th Annual Allerton Conference on Comm. Control, and Computing.* Univ. of Illinois, Urbana-Champaign, IL.

Johnson, E. L. 1966. Networks and Basic Solutions. *Oper. Res.* 14, 619-624.

Jonker, R., and T. Volgenant. 1986. Improving the Hungarian Assignment Algorithm. *Oper. Res. Letters* 5, 171-175.

Jonker, R., and A. Volgenant. 1987. A Shortest Augmenting Path Algorithm for Dense and Sparse Linear Assignment Problems. *Computing* 38, 325-340.

Kantorovich, L. V. 1939. Mathematical Methods in the Organization and Planning of Production. Publication House of the Leningrad University, 68 pp. Translated in *Man. Sci.* 6(1960), 366-422.

Kapoor, S., and P. Vaidya. 1986. Fast Algorithms for Convex Quadratic Programming and Multicommodity Flows, *Proc. of the 18th ACM Symp. on the Theory of Comp.* , 147-159.

Karmarkar, N. 1984. A New Polynomial-Time Algorithm for Linear Programming. *Combinatorica* 4, 373-395.

Karzanov, A.V. 1974. Determining the Maximal Flow in a Network by the Method of Preflows. *Soviet Math. Doklady* 15, 434-437.

Kastning, C. 1976. *Integer Programming and Related Areas: A Classified Bibliography.* Lecture Notes in Economics and Mathematical Systems. Vol. 128. Springer-Verlag.

Kelton, W. D., and A. M. Law. 1978. A Mean-time Comparison of Algorithms for the All-Pairs Shortest-Path Problem with Arbitrary Arc Lengths. *Networks* 8, 97-106.

Kennington, J.L. 1978. Survey of Linear Cost Multicommodity Network Flows. *Oper. Res.* 26, 209-236.

Kennington, J. L., and R. V. Helgason. 1980. *Algorithms for Network Programming,* Wiley-Interscience, NY.

Kershenbaum, A. 1981. A Note on Finding Shortest Path Trees. *Networks 11,* 399-400.

Klein, M. 1967. A Primal Method for Minimal Cost Flows. *Man. Sci.* 14, 205-220.

Klingman, D., A. Napier, and J. Stutz. 1974. NETGEN: A Program for Generating Large Scale Capacitated Assignment, Tranportation, and Minimum Cost Flow Network Problems. *Man. Sci.* 20, 814-821.

Koopmans, T. C. 1947. Optimum Utilization of the Transportation System. *Proceedings of the International Statistical Conference,* Washington, DC. Also reprinted as supplement to *Econometrica* 17 (1949).

Kuhn, H. W. 1955. The Hungarian Method for the Assignment Problem. *Naval Res. Log. Quart.* 2, 83-97.

Lawler, E.L. 1976. *Combinatorial Optimization: Networks and Matroids.* Holt, Rinehart and Winston.

Magnanti, T. L. 1981. Combinatorial Optimization and Vehicle Fleet Planning: Perspectives and Prospects. *Networks* 11, 179-214.

Magnanti, T.L., and R. T. Wong. 1984. Network Design and Tranportation Planning: Models and Algorithm. *Trans. Sci.* 18, 1-56.

Malhotra, V. M., M. P. Kumar, and S. N. Maheshwari. 1978. An $O(|V|^3)$ Algorithm for Finding Maximum Flows in Networks. *Inform. Process. Lett.* 7 , 277-278.

Martel, C. V. 1987. A Comparison of Phase and Non-Phase Network Flow Algorithms. Research Report, Dept. of Electrical and Computer Engineering, University of California, Davis, CA.

McGinnis, L.F. 1983. Implementation and Testing of a Primal-Dual Algorithm for the Assignment Problem. *Oper. Res.* 31, 277-291.

Mehlhorn, K. 1984. *Data Structures and Algorithms.* Springer Verlag.

Minieka, E. 1978. *Optimization Algorithms for Networks and Graphs.* Marcel Dekker, New York.

Minty, G. J. 1960. Monotone Networks. *Proc. Roy. Soc. London* , 257 Series A, 194-212.

Moore, E. F. 1957. The Shortest Path through a Maze. In *Proceedings of the International Symposium on the Theory of Switching Part II; The Annals of the Computation Laboratory of Harvard University* 30, Harvard University Press, 285-292.

Mulvey, J. 1978a. Pivot Strategies for Primal-Simplex Network Codes. *J. ACM* 25, 266-270.

Mulvey, J. 1978b. Testing a Large-Scale Network Optimization Program. *Math. Prog.* 15, 291-314.

Murty, K.G. 1976. *Linear and Combinatorial Programming.* John Wiley & Sons.

Onaga, K. 1967. Optimal Flows in General Communication Networks. *J. Franklin Ins.* 283, 308-327.

Orden, A. 1956. The Transshipment Problem. *Man. Sci.* 2, 276-285.

Orlin, J.B. 1983. Maximum-Throughput Dynamic Network Flows. *Math. Prog.* 27, 214-231.

Orlin, J. B. 1984. Genuinely Polynomial Simplex and Non-Simplex Algorithms for the Minimum Cost Flow Problem. Technical Report No. 1615-84, Sloan School of Management, M.I.T., Cambridge, MA.

Orlin, J. B. 1985. On the Simplex Algorithm for Networks and Generalized Networks. *Math. Prog. Study* 24, 166-178.

Orlin, J. B. 1988. A Faster Strongly Polynomial Minimum Cost Flow Algorithm. *Proc. 20th ACM Symp. on the Theory of Comp.*, 377-387.

Orlin, J. B., and R. K. Ahuja. 1987. New Distance-Directed Algorithms for Maximum Flow and Parametric Maximum Flow Problems. Working Paper 1908-87, Sloan School of Management, Massachusetts Institute of Technology, Cambridge, MA.

Orlin, J. B., and R. K. Ahuja. 1988. New Scaling Algorithms for the Assignment and Minimum Cycle Mean Problems. Working Paper No. OR 178-88, Operations Research Center, M.I.T., Cambridge, MA.

Papadimitriou, C.H., and K. Steiglitz. 1982. *Combinatorial Optimization: Algorithms and Complexity*. Prentice-Hall.

Pape, U. 1974. Implementation and Efficiency of Moore-Algorithms for the Shortest Route Problem. *Math. Prog.* 7, 212-222.

Pape, U. 1980. Algorithm 562: Shortest Path Lenghts. *ACM Trans. Math. Software* 6, 450-455.

Phillips, D.T., and A. Garcia-Diaz. 1981. *Fundamentals of Network Analysis*. Prentice-Hall.

Pollack, M., and W. Wiebenson. 1960. Solutions of the Shortest-Route Problem--A Review. *Oper. Res.* 8, 224-230.

Potts, R.B., and R.M. Oliver. 1972. *Flows in Transportation Networks*. Academic Press.

Rock, H. 1980. Scaling Techniques for Minimal Cost Network Flows. In V. Page (ed.), *Discrete Structures and Algorithms* . Carl Hansen, Munich, 101-191.

Rockafellar, R.T. 1984. *Network Flows and Monotropic Optimization*. Wiley-Interscience.

Roohy-Laleh, E. 1980. *Improvements to the Theoretical Efficiency of the Network Simplex Method.* Unpublished Ph.D. Dissertation, Carleton University, Ottawa, Canada.

Sheffi, Y. 1985. *Urban Transportation Networks: Equilibrium Analysis with Mathematical Programming Methods.* Prentice-Hall.

Shiloach, Y., 1978. An $O(nI \log^2(I))$ Maximum Flow Algorithm. Technical Report STAN-CS-78-702, Computer Science Dept., Stanford University, CA.

Shiloach, Y., and U. Vishkin. 1982. An $O(n^2 \log n)$ Parallel Max-Flow Algorithm. *J. Algorithms 3* , 128-146.

Sleator, D. D., and R. E. Tarjan. 1983. A Data Structure for Dynamic Trees, *J. Comput. Sys. Sci.* 24, 362-391.

Smith, D. K. 1982. *Network Optimisation Practice: A Computational Guide.* John Wiley & Sons.

Srinivasan, V., and G. L. Thompson. 1973. Benefit-Cost Analysis of Coding Techniques for Primal Transportation Algorithm, *J. ACM* 20, 194-213.

Swamy, M.N.S., and K. Thulsiraman. 1981. *Graphs, Networks, and Algorithms.* John Wiley & Sons.

Syslo, M.M., N. Deo, and J.S. Kowalik. 1983. *Discrete Optimization Algorithms.* Prentice-Hall, New Jersey.

Tabourier, Y. 1973. All Shortest Distances in a Graph: An Improvement to Dantzig's Inductive Algorithm. *Disc. Math.* 4, 83-87.

Tardos, E. 1985. A Strongly Polynomial Minimum Cost Circulation Algorithm. *Combinatorica* 5, 247-255,

Tarjan, R.E. 1983. *Data Structures and Network Algorithms.* SIAM, Philadelphia, PA.

Tarjan, R. E. 1984. A Simple Version of Karzanov's Blocking Flow Algorithm, *Oper. Res. Letters 2* , 265-268.

Tarjan, R. E. 1986. Algorithms for Maximum Network Flow. *Math. Prog. Study* 26, 1-11.

Tarjan, R. E. 1987. Personal Communication.

Tomizava, N. 1972. On Some Techniques Useful for Solution of Transportation Network Problems. *Networks* 1, 173-194.

Truemper, K. 1977. On Max Flow with Gains and Pure Min-Cost Flows. *SIAM J. Appl. Math.* 32, 450-456.

Vaidya, P. 1987. An Algorithm for Linear Programming which requires $O(((m+n)n^2 + (m+n)^{1.5}n)L)$ arithmetic operations, *Proc. of the 19th ACM Symp. on the Theory of Comp.*, 29-38.

Van Vliet, D. 1978. Improved Shortest Path Algorithms for Transport Networks. *Transp. Res.* 12, 7-20.

Von Randow, R. 1982. *Integer Programming and Related Areas: A Classified Bibliography 1978-1981.* Lecture Notes in Economics and Mathematical Systems, Vol. 197. Springer-Verlag.

Von Randow, R. 1985. *Integer Programming and Related Areas: A Classified Bibliography 1981-1984.* Lecture Notes in Economics and Mathematical Systems, Vol. 243. Springer-Verlag.

Warshall, S. 1962. A Theorem on Boolean Matrices. *J. ACM* 9, 11-12.

Weintraub, A. 1974. A Primal Algorithm to Solve Network Flow Problems with Convex Costs. *Man. Sci.* 21, 87-97.

Weintraub, A., and F., Barahona. 1979. A Dual Algorithm for the Assignment Problem. Departmente de Industrias Report No. 2, Universidad de Chile-Sede Occidente, Chile.

Whitting, P. D. , and J. A. Hillier. 1960. A Method for Finding the Shortest Route Through a Road Network. *Oper. Res. Quart.* 11, 37-40.

Williams, J. W. J. 1964. Algorithm 232: Heapsort. *Comm. ACM 7* , 347-348.

Zadeh, N. 1972. Theoretical Efficiency of the Edmonds-Karp Algorithm for Computing Maximal Flows. *J. ACM* 19, 184-192.

Zadeh, N. 1973a. A Bad Network Problem for the Simplex Method and other Minimum Cost Flow Algorithms. *Math. Prog.* 5, 255-266.

Zadeh, N. 1973b. More Pathological Examples for Network Flow Problems. *Math. Prog.* 5, 217-224.

Zadeh, N. 1979. Near Equivalence of Network Flow Algorithms. Technical Report No. 26, Dept. of Operations Research, Stanford University, CA.