HYDROMAGNETIC STABILITY OF PARALLEL FLOW OF AN IDEAL HETEROGENEOUS FLUID

By

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ABSTRACT

A theoretical treatment of the dynamical stability of plane parallel flow of an ideal, Boussinesq, magnetic fluid is presented. The magnetohydrodynamic approximation is used and the basic magnetic field is taken to be parallel to the flow. All variations of magnetic field, velocity and density occur in the direction of a uniform gravity vector which is perpendicular to the flow. It is shown by an extension of Squire's theorem that the two-dimensional version of the problem exhibits the greatest instability for normal mode disturbances whenever the stratification is stable. A scaling analysis is applied to a gaseous atmosphere and conditions are derived for showing when the gas behaves like the Boussinesq fluid. Reference is made to the case of the solar atmosphere.

Two simple problems are solved. A three layer model for a jet exhibits several of the characteristics of continuous jets. Instability may be manifest through a varicose wave and a sinuous mode. It is found that the magnetic field acts solely as a stabilizing influence. A surprisingly accurate heuristic formula describing the behavior of the sinuous mode for small wave numbers is derived. The double shear layer model represents the first problem with a complete solution in which the magnetic field can destabilize the motion. Instability can occur through three modes. One of the modes degenerates as the magnetic field in the central layer approaches zero. It is this third mode which is mainly responsible for the destabilizing effect which the magnetic field produces. It is shown in some cases that this destabilizing effect is connected with the fact that a magnetic field in the central region produces a coherency in the waving motions throughout the fluid.

Since the normal modes do not constitute the complete solution, the problem is treated as one of initial values. Considering the case of a fluid in which the square of the derivative of the velocity is always greater than the square of the derivative of the magnetic field, the initial perturbation amplitude is found to decrease algebraically in time for any gravitationally stable stratification. At zero stratification, the solution is neutral.

The Nyquist stability criterion is applied to Couette flow with various magnetic field configurations. A piecewise linear magnetic field can produce instability whenever it has a maximum which is less than half of the maximum of the velocity. The Nyquist technique strictly applies for a homogeneous fluid at zero wave number only, but can be used for arbitrary velocity and magnetic field configurations.

A theory for long wave unbounded flow is presented when certain restrictions are placed on the behavior of the velocity density, and magnetic field configurations near plus and minus infinity. A very simple, convergent eigenvalue relation in powers of wave number and overall Richardson number is obtained by two approaches. This relation gives results which agree with the approximations for small wave numbers of the solutions to the two three layer models considered in the dissertation. A formula for determining approximately the critical Richardson number is derived for a shearing fluid with antisymmetric velocity and density but symmetric magnetic field profiles. For the two continuous velocity profiles considered a magnetic field which increases the critical stratification is found. Finally, a critique of the dissertation is presented and suggestions for future research are mentioned.

Thesis Advisor: Victor P. Starr
Title: Professor of Meteorology

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Finally, although this is theoretically an apersonal document, it is a fact that when I am happier I work better so I therefore dedicate this dissertation to Bernice.

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Table of Symbols

x, y, z	cartesian coordinates
g	gravity vector in z direction
u, v, w	cartesian perturbation velocity components
$\vec{v}$	vector velocity
$U(z)$	basic velocity in direction
D	derivative operator with respect to
$c = c_r + ic_i$	complex wave speed
i	$\sqrt{-1}$
k	wave number in x direction
Ri	Richardson number
$F = w/(U-c)$	perturbation amplitude
$\vec{B}(z)$	magnetic field
b_x, b_y, b_z	cartesian perturbation magnetic field components
$M(z)$	scaled magnetic field Alfven number
h_x, h_y, h_z	scaled perturbation field components
X	$(U-c)^2 - M^2$
$\vec{E}$	electric field
$\vec{J}$	electric current
ε	charge density
c	speed of light
P	pressure

Table of Symbols (continued)

ρ	density
T	temperature
Const	constant
sgn	sign of a quantity
$P\int$	principal value of an integral
$\prod_{i=1}^{n}$	product of terms

LIST OF FIGURES

I. Introduction

I take it as a fundamental principle in nature that physical processes act in a manner which serves to relieve tensions. Whatever the external forces acting on fluid may be, the latter will react in such a way as to most suitably accommodate itself to these outside influences. It may be, as in the case of our atmosphere, that heating differences will force the fluid into a relatively well organized motion pattern such as the jet stream.

It often happens that the velocity and density patterns produced by various outside forces cannot be maintained and the patterns change very suddenly. The attempts to understand such breakdowns constitute the study of hydrodynamic stability. A fluid state of precarious balance needs to be only slightly disturbed to undergo complete alteration. Such disturbances may well be provided by the very forces which produced the fluid situation in the first place but from the point of breakdown the changes which occur indicate little dependence on external forcing and seem to depend mainly on the fluid characteristics.

The general approach, guided by mathematical convenience, has been to consider only the initial reaction of a basically steady fluid state to arbitrary but small wavelike disturbances. If these perturbations grow with time, we have an unstable situation; if they maintain their amplitude the basic state is neutral and if they gradually damp out, the fluid will return to its basic state and is therefore stable.

There are two reasons that small disturbances are chosen. If a large disturbance is superimposed on the basic state the question of the nature of the disturbance becomes relevant because almost any state, when bombarded by a sufficiently large disturbance will be completely altered. So long as the question of importance concerns the liklihood of maintaining a given situation when no gross attempt is made to change it, it is clearly desirable to use a disturbance which nature itself might provide by chance and which often enough is some small vibration. Secondly, the mathematical problem of considering small disturbances makes possible the approximation that the products of the disturbance quantities may be neglected. Mathematically, this introduces the simplification that is inherent in linear equations; physically, it limits the investigation to an analysis of the effects of the interactions between the perturbation and the basic state and excludes all consideration of the mutual interactions of the perturbations. The limitations of the linear technique prohibit the investigation to proceed in time when the basic state is unstable because of the fact that the mutual interactions of the disturbances soon become important and may behave in a manner which restricts further development of the instability. Such a situation must then be considered by nonlinear techniques. In this paper the linear technique is used exclusively.

A complete analysis of the linearized problem must take into account the various diffusive effects which are at work. In the body of this thesis the fluid is taken to be ideal so that instability will be a function solely of the dynamical processes. The complete diffusive problem for a heterogeneous magnetic fluid yields an equation of the twelfth order which is rather difficult to work with. In the ideal problem we are left with an equation of second order which is singular whenever there is no unstable solution. Further justification for ignoring the diffusive effects lies partially in the fact that so little work has been done on the subject. Also, the larger the scale of motions, the smaller the effect of diffusive forces.

The fluid model consists of the infinite plane parallel flow of an ideal incompressible heterogeneous magnetic fluid. The fluid is taken to obey the standard approximations of magnetohydrodynamics. Only small percentual density changes are permitted so that density variations assume importance only as buoyancy effects and are taken to have no inertial effects. This is known as the Boussinesq approximation. The basic magnetic field is taken to be parallel to the flow. All variations of velocity, magnetic field and density are considered to be parallel to a constant gravity vector perpendicular to the basic flow. A detailed discussion of these restrictions is given in chapter 2.

The instability problem considered in this paper is one in which three distinct physical processes operate. In briefest terms, these

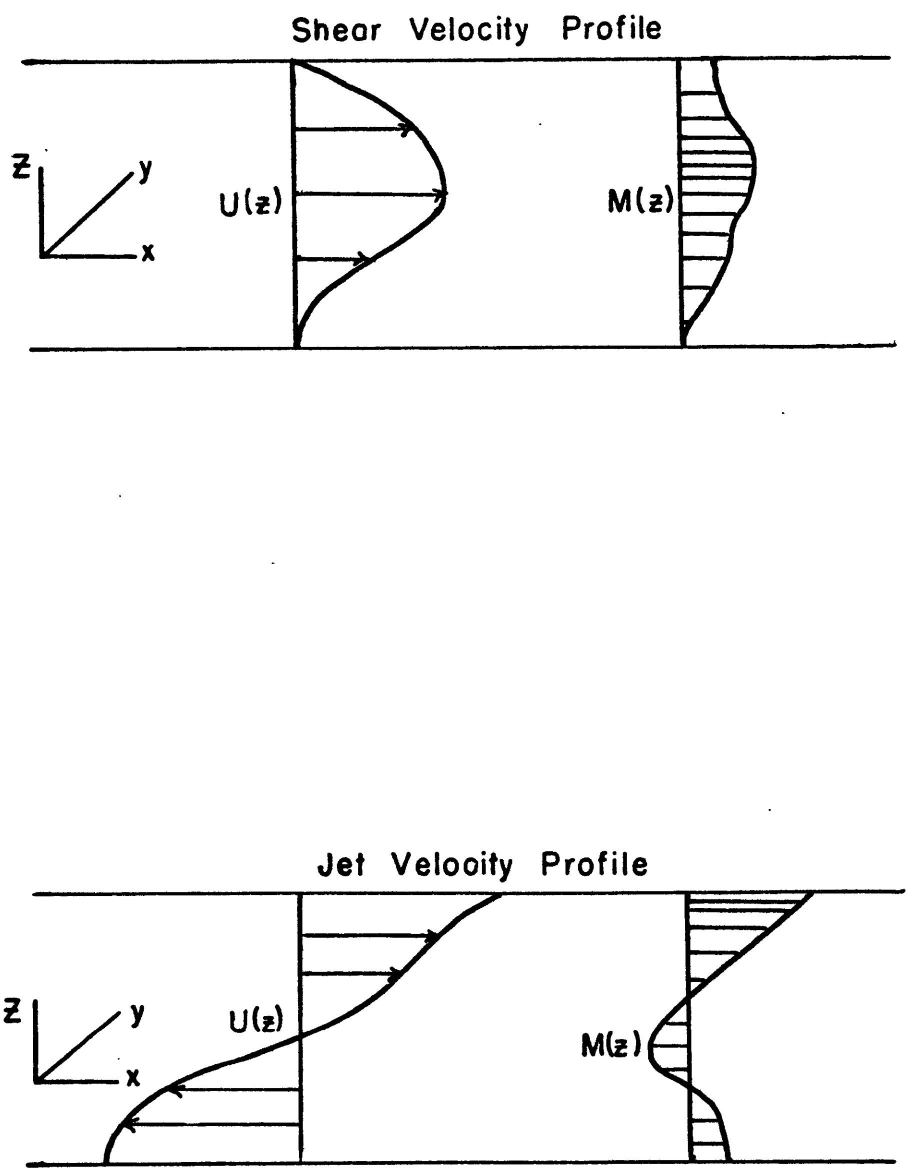

Figure 1. The basic fluid model.

three processes and their effects are: 1, the presence of a density gradient in the fluid coupled with gravity so that a redistribution of the fluid would cause a change in the potential energy of the system, 2, the presence of shear in the fluid so that redistribution would cause a change in the kinetic energy of the basic state, and 3, the presence of a magnetic field so that a redistribution of the fluid would cause a change in the magnetic energy of the system. Because the preponderance of the literature has dealt with the nonmagnetic version of the problem, emphasis will be placed on the role of the magnetic field throughout this thesis. It is appropriate at this point to present a review of the literature as one means of introducing the topic and setting the tone for the work in the body of the thesis.

A. Review of the Literature

Rayleigh (1916) showed that as soon as the stratification in an ideal fluid was such that density increased with height, gravitational instability set in. When viscosity and thermal conductivity were included in the analysis, he could no longer solve for the time behavior as a function of the stratification and other parameters. He then assumed that there was a critical value for the stratification beyond which instability would set in. He arrived at the following equation for perturbation velocity

$$(D^2 - K^2)^3 W = -R_a K^2 W \qquad (1.1)$$

R_a is the Rayleigh number, and it becomes the eigenvalue of this problem.

In problems of hydrodynamic stability it is the aim to solve for the wave speed. Lacking the ability to do so, as is almost always the case, we proceed in one of several manners. One is to proceed as Rayleigh did, namely, by finding some neutral solution and then make assumptions about stability characteristics for parameters with neighboring values. Several perturbation schemes, with varying degrees of validity, are also employed to find the solution in the neighborhood of the neutral point. Actually, it is not often possible to find neutral solutions. Another technique is to find some sufficient condition for stability or instability. These conditions, however, are generally restricted to rather simple cases; there are a large class of flows for which the stability characteristics cannot be determined by these "general" conditions.

In the case of the Benard problem a simple neutral solution could be found and we are guaranteed by the principle of Exchange of Stabilities that instability will result for any increase in the stratification. The dissipative forces, moreover, act purely as stabilizing influences and thus, the problem is rather straightforward. Generally, for parallel flows of an ideal fluid, the Exchange of Stabilities is not valid and diffusive effects may exhibit seemingly anomalous behavior as in the case of plane Poisseuille flow, where viscosity is the cause of the instability.

Squire's theorem has enabled us to consider two-dimensional disturbances of plane parallel flow as being the most unstable. We assume that the solution is composed of normal modes and thus write

$$\mathcal{F}(x,t,z) = F(z)\, e^{ik(x-ct)} \tag{1.2}$$

The governing equation for an ideal homogeneous fluid is given by

$$(U-c)\left[D^2-k^2\right]w - (D^2U)\,w = 0 \tag{1.3}$$

We use the boundary conditions that $w=0$ on both of the horizontal boundaries. The main difficulty in solution lies in the fact that there are singularities present in the equation when $c_i=0$ and $U=c_r$. The inherent difficulty may become clearer by considering what would appear to be a simple problem. Plane Couette flow, ($U \propto z$), yields an equation

$$(U-c)\left[D^2-k^2\right]w = 0 \tag{1.4}$$

which would seem to admit exponential solutions with no regard to c. Actually, we find it impossible to satisfy the boundary conditions. If, for example, $z = \pm 1/2$, then the general solution

$$w = Ae^{-kz} + Be^{kz} \tag{1.5}$$

becomes, at $z = \pm 1/2$ respectively

$$0 = Ae^{-\frac{1}{2}k} + Be^{\frac{1}{2}k}$$

$$0 = Ae^{\frac{1}{2}k} + Be^{-\frac{1}{2}k}$$

whose only solution is $A=B=0$. We have incurred our first major problem. The normal mode solution yields, at most, an incomplete picture of the stability problem. This incompleteness is reflected by the fact that in casually dropping the $(U-c)$ term from consideration one solution is lost. The problem is rendered complete when we solve it by the method of Laplace transforms as an initial value problem. The continuum which the initial value approach yields generally exhibits algebraic time behavior so that if we can find unstable distinct modes we need look no further.

The Kelvin-Helmholtz problem is one of the very few which has been solved completely. Without including density effects we find that

$$c = \frac{U_1 + U_2}{2} \pm \sqrt{-\frac{(U_1 - U_2)^2}{4}} \tag{1.6}$$

The equation is seen to have exponential solutions which can be satisfied by the boundary conditions at the interface of the two distinct fluid layers. All other problems which have been completely solved possess this layered nature. Lin has shown that there does exist value in these problems in the long wavelength limit in connection with problems possessing continuous velocity distributions. Since short waves tend to be stable for continuous velocity distributions, the behavior is seen to be somewhat different for the two cases.

We now modify our aims and seek weaker but more general results. Since only k^2 appears in the equation and k appears in a symmetric way in the boundary conditions, we suffer no limitations by considering the case $k \geq 0$ only. Furthermore, if we have a wave speed, c, corresponding to a solution, w, then there will also be a wave speed, c^*, corresponding to a solution, w^*. This means that if we ever find a $c_i \neq 0$, then the flow is unstable.

Integral theorems provide several results. Multiplying (1.3) by $w^*/(U-c)$, integrating between the boundaries and making use of the boundary conditions, we get the classical result first derived by Rayleigh (1880) for the imaginary part of the equation.

$$c_i \int_{z_1}^{z_2} \frac{D^2U \, |w|^2}{|U-c|^2} \, dz = 0 \tag{1.7}$$

If c_i is to be nonzero, then D^2U must change sign somewhere in the interval. Fjortoft has extended Rayleigh's result by considering the real part. He obtained

$$\int_{z_1}^{z_2} \frac{D^2U(U-U_s)\, |w|^2}{|U-c|^2} \, dz = -\int_{z_1}^{z_2} \left[|Dw|^2 + k^2 |w|^2 \right] dz \tag{1.8}$$

$$< 0 \qquad ; \quad U_s = U \text{ @ } D^2U = 0$$

In order to have instability the fluid must have regions where

$D^2U(U-U_s) < 0$ and if the velocity distribution is monotonic and has one inflection point (point where $D^2U = 0$) then $D^2U(U-U_s) \leq 0$ everywhere. When the conditions of these theorems are met, we are still not guaranteed that the fluid will be unstable because sinusoidal flow becomes stable once the boundaries are sufficiently close. If, however, the conditions are not met, then the fluid will be stable to normal mode disturbances. Graphically, various profiles are shown in figure 2 and the possibility of instability is ruled out in all cases but (d); (c) satisfies Rayleigh's theorem, but not Fjortoft's and it is interesting to note that Kent (1968) has shown that a small constant magnetic field may destabilize some flows in this category.

These arguments can be phrased in terms of vorticity considerations. A fluid parcel interacts with the basic flow in such a way as to seek out its own vorticity level. In a fluid with a monotonic vorticity profile, a fluid parcel will oscillate around its point of origin; only when the vorticity has an extremum can a parcel, when forced across it, be forced even further from its initial position.

The fact that ideal plane Poisseuille flow has $D^2U = \text{const} \neq 0$ indicates its stability to normal mode disturbances. Since in actuality it is unstable we can only conclude that if the continuum solution isn't unstable, then viscosity was the cause of the instability. Much

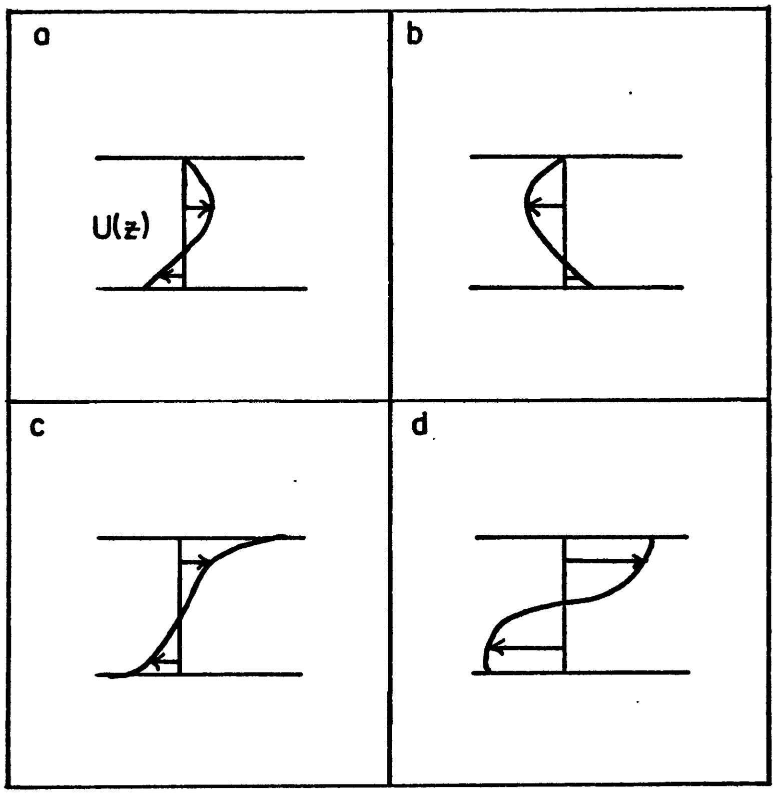

Figure 2. Four shearing profiles. c. and d. have an inflection point but only d. satisfies Fjortoft's theorem and only d. can be unstable (borrowed essentially from Drazin and Howard 1966).

difficulty has been encountered in taking the inviscid limit of the viscous Orr-Sommerfeld equation. Part of this difficulty arises from the fact that on considering only the ideal problem the order of the governing equation is reduced so that certain solutions are lost. Nevertheless, we shall continue to ignore all diffusive effects in the body of the thesis and consider only those solutions of the complete equations whose limits are expressed in the ideal equations.

Tollmien and later Lin (1945) have found sufficient conditions for instability. For symmetric flows in which Fjortoft's theorem is satisfied a neutral solution is given by $c = U_s$. Finding the wave number corresponding to this neutral solution and then investigating the behavior of c with wave members in the vicinity of the neutral point gives instability. Lin developed the general formula

$$\left.\frac{dk^2}{dc}\right)_{k_s} = \left\{ -P\int_{z_1}^{z_2} \frac{(D^2U)\, W_s^2}{(U-U_s)^2}\, dz - \right.$$
$$\left. - i\pi\left[\operatorname{sgn}(DU)_{z_s}\right]\left[\frac{(D^3U)\, W_s^2}{(DU)^2}\right]_{z_s}\right\} \Big/ \int_{z_1}^{z_2} W_s^2\, dz \qquad (1.9)$$

For symmetric flows of the Tollmien variety we are guaranteed that for k just smaller than k_s we have instability.

Rosenbluth and Simon (1964) have extended this result by use of an interesting technique. The Nyquist stability criterion can be applied to problems for which the general form of the solution is known.

For wave number zero the general solution is known for arbitrary velocity profile. For a fluid in which the velocity profile is monotonic and which satisfies Fjortoft's condition throughout, instability is guaranteed if

$$\left. \frac{1}{DU\,(U_s - U)} \right|_{z_1}^{z_2} - \int_{z_1}^{z_2} \frac{D^2 U}{(DU)^2\,(U - U_s)}\,dz > 0$$

From this it is possible to see that if the boundaries are sufficiently close the first term dominates and the expression is negative. Stability is thus guaranteed for boundaries sufficiently close.

The form of the equation they used is given in terms of the displacement $F = w/(U-c)$ and appears as

$$D\left[(U-c)^2 DF\right] - k^2 (U-c)^2 F = 0 \qquad (1.10)$$

From this equation we can derive one further integral theorem. Multiplying it by F^*, integrating and making use of the boundary conditions, we arrive at Howard's semicircle theorem. This states that we can place limitations on the ranges of both c_r and c_i for unstable solutions and it is given by

$$\left\{ c_r - \frac{1}{2}\left(U_{min} + U_{max}\right) \right\}^2 + c_i^2 \le \left\{ \frac{1}{2}\left(U_{max} - U_{min}\right) \right\}^2 \qquad (1.11)$$

Equation (1.10) proves useful for expansions in small wave number. Drazin and Howard (1962) have derived a rather simple formula for c for a certain class of unbounded flows given by

$$0 = -k\left[(U_{\infty}-c)^2 + (U_{-\infty}-c)^2\right] - k^2\int_{-\infty}^{\infty}\left[(U-c)^2 - (U_{\infty}-c)^2\right]\left[(U-c)^2-(U_{-\infty}-c)^2\right]dz + \cdots \tag{1.12}$$

where the subscript indicates the value of z at which the velocity is evaluated. For the Kelvin-Helmholtz problem this gives the exact value for c .

When we include the effects of stratification we add to the complexity of the problem. If the density gradient is anywhere gravitationally unstable we will have instability, for the velocity profile has no effect on disturbances normal to it. When the stratification is stable, Squire's theorem is once again valid and the two-dimensional equation governing the flow may take the following forms corresponding respectively to (1.3) and (1.10)

$$(U-c)^2\left[D^2-k^2\right]w - (U-c)D^2U\cdot w - \overline{R}_i\, w = 0 \tag{1.13}$$

and

$$D\left[(U-c)^2 DF\right] - k^2\left[(U-c)^2\right]F - \overline{R}_i F = 0 \tag{1.14}$$

$\overline{R}_i/(DU)^2 \equiv R_i$ is the Richardson number, a nondimensional parameter relating the effects of stratification to those of shear. It seems reasonable to assume that if the homogeneous problem exhibits an instability, then we may be able to suppress that instability if we superimpose a strong enough stable stratification on it. There must be some curve yielding a critical value for the Richardson number corresponding to each k for any given velocity profile.

Physically, instability becomes impossible when $Ri \leq -1$ because then there is not sufficient kinetic energy in the basic flow to overcome the potential energy created by a redistribution of the fluid. Seeing that instability may not always arise for homogeneous flows, we observe a certain inefficiency in the fluid instability processes. In fact, Howard has proven that if $Ri < -1/4$ throughout the fluid then there is no possibility for instability to normal modes.

Several general results which were valid for homogeneous fluids are applicable to the stratified case also. We are justified in considering only $k \geq 0$ and are guaranteed that any $c_i \neq 0$ implies instability. The semicircle theorem is valid for flows in which $Ri \leq 0$ and becomes slightly more restrictive. Consideration of the continuum is necessary for a complete analysis.

For monotonic shear flows with suitably restricted singularities Miles (1963) has proven that the curve traced out by singular neutral solutions is in fact a stability boundary. Thus, the result we would expect is justified in at least certain cases. Because the singularity is of a stronger nature than in the homogeneous case several of the results do no extend to the heterogeneous case.

Several problems have been solved, at least for the neutral curve. The Kelvin-Helmholtz problem has been solved completely for the growth speed and we obtain

$$c = \frac{\rho_1 U_1 + \rho_2 U_2}{(\rho_1+\rho_2)} \pm \sqrt{-\frac{(U_1-U_2)^2}{4} + 2\,\frac{(\rho_2-\rho_1)g}{(\rho_1+\rho_2)k}} \tag{1.15}$$

Thus, shorter wavelengths exhibit greater instability and there is a wavelength above which we get stable travelling waves. One of the best examples of the partial solution of a problem with a smooth velocity profile is the treatment by Drazin (1958) of a hyberbolic tangent velocity profile with $\bar{R}_i =$ const. When the equation is phrased in terms of velocity as the independent variable we obtain

$$\frac{d^2\chi}{dU^2} + \left\{\frac{\mu}{U} - \frac{2(\nu-1)U}{1-U^2}\right\}\frac{d\chi}{dU} - \frac{(2\nu+\mu+2)(2\nu+\mu-1)}{1-U^2}\chi = 0 \tag{1.16}$$

where $w = U^{\mu}(1-U^2)^{\nu}\chi$. Drazin observed that this impossible looking equation has the solution $\chi =$ const if

$$(2\nu+\mu+2)(2\nu+\mu-1) = 0 \tag{1.17}$$

where

$$\nu = {}^1\!/_2\sqrt{k^2+\bar{R}_i}$$

$$\mu = 1/2 + 1/2\sqrt{1 + 4\bar{R}_i}$$

This yields the result that

$$\bar{R}_i = -k^2(1-k^2) \tag{1.18}$$

Drazin and Howard (1961), using equation (1.14), expanded in powers of the wave number and the Richardson humber, both of which were taken to be small. They obtained the following rather simple formula for the wave speed for a certain class of unbounded flows.

$$k\left[(U_\infty - c)^2 + (U_{-\infty} - c)^2\right] - 2G + \int_{-\infty}^{\infty}\left[k\left\{(U-c)^2 - (U_\infty - c)^2\right\} + G(1-\lambda)\right]\left[k\left\{(U-c)^2 - (U_{-\infty} - c)^2\right\} + G(1+\lambda)\right]\frac{dz}{(U-c)^2} + \cdots = 0 \tag{1.19}$$

where $GD\lambda = -\bar{R}_i$ and G is the basic overall Richardson number. For both the Kelvin-Helmholtz problem and the neutral curve of a variation of Drazin's problem this formula gives the exact solution. Its success has merited extension to the magnetic problem.

The earliest works which included the effect of a magnetic field on stability problems resulted in the conclusion that the magnetic field acted as a stabilizing influence. In the Benard problem, the critical Rayleigh number increases as the vertical component of the magnetic field is increased. In the Kelvin-Helmholtz problem with an aligned magnetic field we obtain the formula

$$c = \frac{\rho_1 U_1 + \rho_2 U_2}{(\rho_1 + \rho_2)} \pm \sqrt{-\frac{(U_1 - U_2)^2}{4} + M^2 + \frac{2g}{k}\frac{(\rho_2 - \rho_1)}{(\rho_1 + \rho_2)}} \qquad (1.20)$$

for the wave speed. When $M \geq [1/2(U_1 - U_2)]^2$, all possibility of instability is ruled out.

The possibility that the magnetic field might act in a destabilizing manner was demonstrated by Drazin (1960) but his fluid model has finite conductivity. Axford (1960) has shown that so long as the magnetic field is not aligned with the velocity field we will get instability in the Kelvin-Helmholtz problem. A two-dimensional treatment will not be adequate for this problem because there will then be a wave component perpendicular to the basic current. The magnetic field does not act as a destabilizing agent for this problem, but only changes the direction at which an unstable wave will appear.

Two researchers have found that the presence of a magnetic field may destabilize the motion for ideal fluids. Both Stern (1963) and Kent (1966, 1968) have used the governing equation for a perfect homogeneous fluid with an aligned magnetic field

$$D\left\{\left[(U-c)^2 - M^2\right]DF\right\} - k^2\left[(U-c)^2 - M^2\right]F = 0 \qquad (1.21)$$

and investigated the behavior for small k.

Stern considered Couette flow and superposed a piecewise linear magnetic profile on it. Expanding in powers of k^2, he found that there were cases that had a c with an imaginary part. Kent (1966) showed that if the magnetic field, with symmetric Poisseuille flow satisfied the conditions

$$(D^2U_0)(DU_0) > (D^2M_0)(DM_0)$$
$$U_0 = 0 \;;\; M_0 = 0$$
$$U^2(z) \geq M^2(z)$$

then for small wave number, $c_i = O(k^4)$ and the problem is unstable.

In a more thorough analysis, Kent (1968) has considered general properties of (1. 21). As with (1. 3), the existence of any $c_i \neq 0$ implies instability and the study may be restricted to $k \geq 0$. Squire's theorem is valid and a stricter version of the semicircle theorem can be obtained (Stern 1963).

Finding neutral solutions becomes an even more difficult job and expansions around the neutral point are guaranteed valid only in the case that $c = U_0$ at some point where $M_0 = 0$ and

$$(D^2U_0)(DU_0) - (D^2M_0)(DM_0) = 0$$

This represents such a limited class of profiles as to be of relatively small value. There is one exception when the above conditions need not be met to provide a marginally stable solution. This occurs when $Kc = 0$ (Low 1961, etc) and , in fact, marginally stable solutions at $K = 0$ often do not conform to the above restrictions.

The Nyquist stability technique may be used for homogeneous fluids for the case of $K = 0$ and in the case of a fluid whose velocity profile satisfies Rayleigh's necessary condition but not Fjortoft's, Kent has shown that even a constant magnetic field may cause instability. In addition, it is important to study the continuum solution but Kent's conclusions are dependent on his use of a delta function amplitude disturbance and not any realistic form for the initial perturbation.

B. Summary

The purpose of this paper is to extend the theory of the stability characteristics of plane parallel flow of an ideal magnetic fluid. In the majority of the paper, the treatment includes the buoyancy effects due to the density stratification present in the fluid model. Because of the fact that for large scale motions dissipative forces play a lesser role in the dynamical processes, it is expected that there is some applicability to various geophysical and astrophysical phenomena. Emphasis is always placed on the theoretical aspect of the problems, however, and nowhere in the thesis is the strict applicability of any result stressed.

The brief historical background material which was presented in the previous section gives an idea of the basic approach taken in the body of the thesis. The remaining chapters are now outlined.

In chapter 2 a brief introduction to the magnetohydrodynamic approximation is presented. The basic equations for a Boussinesq liquid are perturbed, and the first order perturbation equations are combined into one governing equation, which is a second order ordinary differential equation when a normal modes solution is assumed. Integral theorems which place restrictions on the wave speed are then derived and Squire's theorem is proved. Finally, a scaling analysis is performed for a gaseous atmosphere to see for what range of parameters the equations are approximately Boussinesq.

In chapter 3, analytical solutions are obtained for two relatively simple problems. A three layer jet model is shown to have two modes (sinuous and varicose) through which instability may occur. A heuristic formula is developed for the sinuous mode of the long wavelength disturbances of a narrow jet in an unbounded fluid and is shown to agree with the long wave approximation to the solution of the sinuous wave of the three layer jet. The second model is the double shear layer and it is analyzed in some detail. Greater instability is often manifest in the magnetic case especially for long waves and even a constant magnetic field can destabilize the flow for small enough magnetic field values. A physical argument for the destabilization is presented.

In chapter 4, two general stability finding techniques are used. Noticing that the normal modes solution is often incomplete, the problem is reformulated by taking the Laplace transform and solving by an initial value approach in the case of a monotonically shearing fluid. Stability is established whenever the fluid has a gravitationally stable stratification.

The Nyquist stability criterion is then applied to several simple problems. This technique gives a graphical means of determing if there is any solution for which an unstable root exists. The technique is limited to the case of homogeneous fluid at zero wave number but may be used for arbitrary distributions of velocity and magnetic field. By continuity it is possible to extend these results to sufficiently small but nonzero wave number.

In chapter 5, we consider the long wave disturbances in an unbounded homogeneous fluid which has finite velocity and magnetic field limits. Two equivalent approaches can be taken: one, a series approach and the other an integral equation attack. Both give convergent eigenvalue relations for waves with a nonzero imaginary wave speed. The first two terms of the series are applied to the simple examples of chapter 3 and give excellent agreement. For the case of the sinuous mode for a jet flow, the eigenvalue relation agrees remarkably with the heuristic formula of chapter 3.

In chapter 6, the study of chapter 5 is extended to the case of a heterogeneous fluid. It is found to be profitable to use a double series expansion. The resultant eigenvalue relation is thus expressed in powers of the wave number and the overall Richardson number. Convergence is proven in the same manner as that of chapter 5, and the examples of chapter 3 are once again successfully applied. The case of marginal stability is investigated in greater detail for monotonically shearing flows and several examples are treated. In two cases, the indications are that the magnetic field serves to destabilize the flow by increasing the critical Richardson number.

In chapter 7, a brief critique is presented and some future research is suggested.

II. Delineation of the Problem

A. The Basic Equations

Magnetohydrodynamics is concerned with the behavior of an electrically conducting fluid which is characterized by velocities much smaller than the speed of light. Any conducting fluid obeys Maxwell's equations, which are

$$\vec{\nabla} \times \vec{B} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial E}{\partial t} \tag{2.1}$$

$$\nabla \cdot E = 4\pi\varepsilon \tag{2.2}$$

$$\nabla \times E = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} \tag{2.3}$$

$$\nabla \cdot B = 0 \tag{2.4}$$

We assume that phenomena are characterized by a length scale, L, and time scale, t, such that

$$\frac{L}{t} = V << c$$

where c is the speed of light. A dimensional analysis of Faraday's law of induction, (2. 3) states that

$$|\vec{\nabla} \times \vec{E}| \sim \frac{E}{L} \Longleftrightarrow \left|\frac{1}{c}\frac{\partial \vec{B}}{\partial t}\right| \sim \frac{B}{ct}$$

so that the electric field is related to the magnetic field by

$$E \sim \frac{V}{c} B$$

A dimensional analysis of Ampere's Law, (2.1) yields

$$|\vec{\nabla}\times\vec{B}| \sim \frac{B}{L} \Longleftrightarrow \left|\frac{4\pi}{c}\vec{j}\right| + \left|\frac{1}{c}\frac{\partial\vec{E}}{\partial t}\right| \sim$$

$$\sim \frac{j}{c} + \frac{E}{tc} \sim \frac{j}{c} + \frac{1}{tc}\frac{v}{c}B$$

The second term on the right is thus far lower than the left hand side so that

$$\frac{B}{L} \sim \frac{j}{c}$$

and we can write Ampere's Law as

$$\vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{j} \tag{2.5}$$

We have thus ruled out all effects of electromagnetic waves by considering that the electric field is basically an induced field. In our approximate form of Ampere's Law we have relinquished the strict consistency of Maxwell's equations for the simplification which we have obtained. We find further that when the fluid is fully ionized, Ohm's law reduces to

$$\frac{\vec{j}}{\sigma} = \vec{E} + \frac{\vec{v}}{c}\times\vec{B} \tag{2.6}$$

and since in this paper we always assume infinite conductivity ($\sigma = \infty$), we have

$$0 = \vec{E} + \frac{\vec{V}}{c} \times \vec{B} \tag{2.7}$$

The manner in which the electromagnetic effects enter the fluid equations of motion is through the Lorentz force,

$$\vec{\mathcal{F}} = \varepsilon \vec{E} + \frac{1}{c}(\vec{j} \times \vec{B}) \tag{2.8}$$

The order of magnitude of the charge distribution, ε , is determined from Coulomb's law, (2.2)

$$|\vec{\nabla} \cdot \vec{E}| \sim \frac{E}{L} \sim \frac{1}{ct} B \Longleftrightarrow 4\pi \varepsilon \sim \varepsilon$$

so that a comparison of the first and second terms of (2.8) yields

$$|\varepsilon E| \sim \frac{V}{c^2 t} B^2 \quad ; \quad \frac{1}{c} |\vec{j} \times \vec{B}| \sim \frac{B^2}{L}$$

so that

$$\frac{|\varepsilon \vec{E}|}{\frac{1}{c}|\vec{j} \times \vec{B}|} \sim \frac{V^2}{c^2} <\!<\!<\!< 1$$

and thus we can neglect the forces due to the charge distribution within the fluid. Here we see more clearly that we are considering fluids whose electric fields arise primarily as a result of the interaction of the fluid motion with magnetic fields; large local concentrations of charges are not considered in magnetohydrodynamics. The Lorentz force is thus approximated by

$$\vec{\mathcal{F}} = \frac{1}{c}\,\vec{j}\times\vec{B} \tag{2.9}$$

Now, it becomes possible to express all the electrical variables in terms of the magnetic field. Substituting for $\vec{j}$ in (2.9) from (2.5) we obtain

$$\vec{\mathcal{F}} = \frac{1}{4\pi}\,(\vec{\nabla}\times\vec{B})\times\vec{B} \tag{2.10}$$

Eliminating the electric field between (2.3) and (2.7) we obtain

$$\begin{aligned} -\frac{1}{c}\frac{\partial B}{\partial t} &= -\nabla\times\left(\frac{\vec{V}}{c}\times\vec{B}\right) \equiv \\ &= -\frac{1}{c}\Big[\vec{V}(\vec{\nabla}\cdot\vec{B}) + (\vec{B}\cdot\vec{\nabla})\vec{V} - \vec{B}(\vec{\nabla}\cdot\vec{V}) - \\ &\qquad -(\vec{V}\cdot\vec{\nabla})\vec{B}\Big] \end{aligned} \tag{2.11}$$

and we see that an incompressible fluid ($\nabla\cdot V = 0$) with use of (2.4) yields

$$\frac{\partial \vec{B}}{\partial t} + (\vec{V}\cdot\vec{\nabla})\vec{B} \equiv \frac{d\vec{B}}{dt} = (\vec{B}\cdot\vec{\nabla})\vec{V} \tag{2.12}$$

This equation takes the same form as the equation for the time rate of change of vorticity in an ideal incompressible fluid. We therefore find that magnetic lines, like vortex lines in a nonmagnetic fluid, move with the fluid. This equation depends heavily on the fact that we have considered a fluid with infinite conductivity and places several interesting restrictions on the possible fluid motions. Through (2.12), we shall be able to relate our boundary conditions on the magnetic field to those on the velocity field and shall also be able to obtain one governing equation.

We limit our initial consideration to those fluids which are incompressible and Boussinesq. This means we do not consider such phenomena as sound waves and neglect the inertial effects of the density variation since the latter is assumed to be small in comparison with the average density. The momentum equation appears in vector form as,

$$\rho_0 \frac{d\vec{V}}{dt} = -\vec{\nabla}p - \frac{1}{4\pi}\vec{B}\times(\vec{\nabla}\times\vec{B}) - \rho\vec{g} \tag{2.13}$$

The equation for the continuity of mass is

$$\vec{\nabla}\cdot\vec{V}=0 \tag{2.14}$$

The thermal equation is

$$\frac{dT}{dt}=0 \tag{2.15}$$

and the equation of state is

$$(\rho-\rho_0)=-\alpha\left[T-T_0\right] \tag{2.16}$$

The basic state is one of hydrostatic balance. The basic temperature, velocity and magnetic fields are all considered to be arbitrary functions of z . The equations for the disturbance quantities are approximated by neglecting squares of all the small terms. We obtain

$$\frac{\partial u}{\partial t}+U\frac{\partial u}{\partial x}+w\frac{\partial U}{\partial z}=-\frac{1}{\rho_0}\frac{\partial p}{\partial x}-\frac{1}{4\pi\rho_0}\frac{\partial}{\partial x}(Bh_x)+\frac{1}{4\pi\rho_0}\left\{B\frac{\partial h_x}{\partial x}+h_z\frac{\partial B}{\partial z}\right\} \tag{2.17}$$

$$\frac{\partial v}{\partial t}+U\frac{\partial v}{\partial x}=-\frac{1}{\rho_0}\frac{\partial p}{\partial y}-\frac{1}{4\pi\rho_0}\frac{\partial}{\partial y}(Bh_x)+\frac{1}{4\pi\rho_0}B\frac{\partial h_y}{\partial x}$$

$$\frac{\partial w}{\partial t}+U\frac{\partial w}{\partial x}=-\frac{1}{\rho_0}\frac{\partial p}{\partial z}-\frac{1}{4\pi\rho_0}\frac{\partial}{\partial z}(Bh_x)-\frac{\rho}{\rho_0}g+\frac{1}{4\pi\rho_0}B\frac{\partial h_z}{\partial x}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\frac{\partial \rho}{\partial t} + U\frac{\partial \rho}{\partial x} = -\beta \alpha W \quad ; \quad \beta = -\frac{\partial T_0}{\partial z}$$

$$\frac{\partial h_x}{\partial t} + U\frac{\partial h_x}{\partial x} + w\frac{\partial B}{\partial z} = B\frac{\partial u}{\partial x}$$

$$\frac{\partial h_y}{\partial t} + U\frac{\partial h_y}{\partial t} = B\frac{\partial v}{\partial x} \tag{2.17}$$

$$\frac{\partial h_z}{\partial t} + U\frac{\partial h_z}{\partial t} = B\frac{\partial w}{\partial x}$$

$$\frac{\partial h_x}{\partial x} + \frac{\partial h_y}{\partial y} + \frac{\partial h_z}{\partial z} = 0$$

These equations may now be nondimensionalized. We choose a time scale, t , given by the basic shear and a length scale, L , given by the width of the shearing region. Thus, we have

$$\frac{L}{t} = V$$

We shall thus scale the velocity so that

$$(U, u, v, w) = V(U^*, u^*, v^*, w^*)$$

$* \Rightarrow$ nondimensional

and we scale the magnetic field so that at any point in space the Alfven number, which gives the ratio of the Alfven wave speed to the velocity, can be obtained by dividing the nondimensional velocity U^* into M. Therefore, we have

$$(B, h_x, h_y, h_z) = \sqrt{4\pi\rho_0}\, V\,(M, h_x^*, h_y^*, h_z^*)$$

Furthermore, there is a part of the magnetic force which may be incorporated into the pressure term with no loss of generality. Thus,

$$\frac{1}{\rho_0}\nabla\left\{ P + \frac{1}{4\pi} B h_x \right\} = \frac{t^2}{L}\nabla p^*$$

nondimensionalizing β and g, we find that a local Richardson number, R_i, is expressed by

$$R_i(z) = \frac{\beta^* d\, g^*}{\rho_0\left(\frac{\partial U^*}{\partial z}\right)^2} = \frac{\bar{R}_i}{\left(\frac{\partial U^*}{\partial z}\right)^2}$$

The equations (2.17) then appear as (dropping stars)

$$\frac{\partial u}{\partial t} + U\frac{\partial u}{\partial x} + w\frac{\partial U}{\partial z} = -\frac{\partial p}{\partial x} + M\frac{\partial h_x}{\partial x} + h_z\frac{\partial M}{\partial z} \qquad (2.18)$$

$$\frac{\partial v}{\partial t} + U\frac{\partial v}{\partial x} = -\frac{\partial p}{\partial y} + M\frac{\partial h_y}{\partial x}$$

$$\frac{\partial w}{\partial t} + U\frac{\partial w}{\partial x} = -\frac{\partial p}{\partial z} + M\frac{\partial h_z}{\partial x} - \frac{\rho}{\rho_0} g$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\frac{\partial \rho}{\partial t} + U \frac{\partial \rho}{\partial x} = -\alpha\beta w$$

$$\frac{\partial h_x}{\partial t} + U \frac{\partial h_x}{\partial x} + w \frac{\partial M}{\partial z} = M \frac{\partial u}{\partial x}$$

$$\frac{\partial h_y}{\partial t} + U \frac{\partial h_y}{\partial x} = M \frac{\partial v}{\partial x} \tag{2.18}$$

$$\frac{\partial h_z}{\partial t} + U \frac{\partial h_z}{\partial t} = M \frac{\partial w}{\partial x}$$

$$\frac{\partial h_x}{\partial x} + \frac{\partial h_y}{\partial y} + \frac{\partial h_z}{\partial z} = 0$$

B. General Behavior of the Problem

It is now proven that in a stably stratified fluid two dimensional disturbances have the same nature as their three dimensional counterparts and are, in fact, more unstable than the latter. This analog of Squire's theorem assumes that the perturbations can represented by wavelike solutions of the form

$$f(x,y,z,t) = F(z)\, e^{i[k(x-ct)+\ell y]}$$

$$D \equiv \frac{\partial}{\partial z}$$

for all perturbation quantities. The equations of (2.18) then appear as

$$ik(U-c)u + wDU = -ikp + ikMh_x + h_z DM$$

$$ik(U-c)v = i\ell p + ikMh_y$$

$$ik(U-c)w = -Dp + ikMh_z - \rho/\rho_0\, g$$

$$ik(U-c)\rho = \alpha\beta w \tag{2.19}$$

$$iku + i\ell v + Dw = 0$$

$$ik(U-c)h_x + wDM = ikMu$$

$$ik(U-c)h_y = ikMv$$

$$ik(U-c)h_z = ikMw$$

$$ikh_x + i\ell y + Dh_z = 0$$

Multiplying the second equation of (2.19) by ℓ/k and adding it to the first, we have

$$i\tilde{k}(U-c)\tilde{u} + \tilde{w}DU = -i\tilde{k}\tilde{p} + i\tilde{k}M\tilde{h}_x + \tilde{h}_z DM$$

where we have used the following definitions

$$\begin{aligned}
\tilde{k} &= \sqrt{k^2+\ell^2} \\
\tilde{k}\tilde{u} &= ku + \ell v \\
\tilde{k}\tilde{h}_x &= kh_x + \ell h_y \\
\tilde{w} &= w \\
\tilde{p}/\tilde{k} &= p/k
\end{aligned}$$

Similarly, by adding the first magnetic equation to ℓ/k times the second, we obtain

$$i\tilde{k}(U-c)\tilde{h}_x + \tilde{w}DM = i\tilde{k}M\tilde{u}$$

Finally, defining

$$\begin{aligned}
\tilde{k}\tilde{\rho} &= k\rho \\
g/k^2 &= \tilde{g}/\tilde{k}^2
\end{aligned}$$

we have a two dimensional analog to (2.19).

$$i\tilde{k}(U-c)\tilde{u} + \tilde{w}DU = -i\tilde{k}\tilde{p} + i\tilde{k}M\tilde{h}_x + \tilde{h}_z DM$$

$$i\tilde{k}(U-c)\tilde{w} = -D\tilde{p} + i\tilde{k}M\tilde{h}_z - \tilde{\rho}(g/\rho_0)$$

$$i\tilde{k}(U-c)\tilde{\rho} = \alpha\beta\tilde{w}$$

$$i\tilde{k}\tilde{u} + D\tilde{w} = 0$$

$$i\tilde{k}(U-c)\tilde{h}_x + \tilde{w}DM = i\tilde{k}M\tilde{u} \tag{2.20}$$

$$i\tilde{k}(U-c)\tilde{h}_z = i\tilde{k}M\tilde{w}$$

$$i\tilde{k}\tilde{h}_x + D\tilde{h}_z = 0$$

We thus see that the two-dimensional, (2d), set is completely equivalent to the three-dimensional, (3d), set of equations. Furthermore, since $R_i = \alpha g\beta/(DU)^2$, we see that the 2d $\tilde{R}_i$ is larger (and thus more stable) than the 3d R_i by a factor $\tilde{k}^2/k^2$. We would expect the growth speed to be smaller for analogous 2d disturbances from a consideration of this factor alone. Nevertheless, since growth is given by Kc_i in the 3d case and kc_i in the 2d case we find that the 2d problem is, in fact more unstable and we are justified in limiting our consideration to the two-dimensional case. Finally, with no y dependence in (2.18) or (2.19) we find that the

y equations for v and h_y are formally independent of the other equations and need not be incorporated.

Now we may proceed to derive the basic governing equation from the 2d version of (2.18). Utilizing the operator

$$Q \equiv \left(\frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right)$$

and the fact that

$$\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x} \right) Q = Q \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x} \right)$$

$$\frac{\partial}{\partial z} Q = Q \frac{\partial}{\partial z} + \frac{\partial U}{\partial z} \frac{\partial}{\partial x}$$

when the x dependence is given by

$$f(x) \Rightarrow e^{ikx}$$

We ultimately obtain (Appendix A)

$$0 = \left[Q^4 - M^2 Q^2 \right] D^2 W + \left[-2M(DM)Q^2 + \right. \tag{2.21}$$

$$\left. + 2M^2(DU)Q \right] DW + \left[K^2 Q^4 - (D^2 U) Q^3 - \bar{R}_i Q^2 - \right.$$

$$- K^2M^2Q^2 + M^2(D^2U)Q + 2M(DM)(DU)Q - 2M^2(DU)^2\Big]W \tag{2.21}$$

This may be shortened to

$$D\left[\left(1-\frac{M^2}{Q^2}\right)DW\right] - \left[K^2\left(1-\frac{M^2}{Q^2}\right) + \right.$$

$$\left. + \frac{1}{Q}D\left[DQ\left(1-\frac{M^2}{Q^2}\right)\right] + \bar{R}_i\right]W = 0 \tag{2.22}$$

which assumes significance when $\partial/\partial t = -ikc$ and which reduces to Rayleigh's equation as $M \to 0$. By substituting so that $W \equiv QF$ we may write the equation in its most convenient form (Appendix B),

$$D\left[(Q^2-M^2)DF\right] - K^2(Q^2-M^2)F - \bar{R}_iF = 0 \tag{2.23}$$

Assuming that $\partial/\partial t = -ikc$, (2.23) becomes

$$D\left\{\left[(U-c)^2-M^2\right]DF\right\} - K^2\left[(U-c)^2-M^2\right]F - \bar{R}_iF = 0 \tag{2.24}$$

When the fluid is confined between two horizontal plates the boundary conditions at the plates demand that there is no vertical velocity, i.e.,

$$W(z_{\text{boundary}}) = 0$$

For unstable solutions this also means that $F=0$. When the basic current is symmetric about the central value of z we obtain two solutions: the symmetric wave (varicose) and the antisymmetric wave (sinuous). This enables us to use boundary conditions at the midpoint respectively of $W=0$ and $DW=0$. We need no further boundary condition for our equation. This simplicity is due to the non dissipative model we have chosen. Conditions on W are equivalent to conditions on h_z by the z component of equation (2.12). A condition of perfect conductivity at the walls implies that no tangential electric field can be tolerated and thus, no perturbation transverse magnetic field by Faraday's law.

Since our equation has only k^2 terms and the boundary conditions are symmetric in k we can thus limit consideration to the case $k \geq 0$ without any loss in generality. We shall now prove the property of this equation that if there is a solution, F_1 , corresponding to a wave speed, c_1 , then there also is a solution F_1^* corresponding to a wave speed c_1^*. In other words so long as the wave speed has an imaginary part, the situation is an unstable one. Splitting (2.24) into its real and imaginary parts, we have

Re:

$$\left[(U-c_r)^2 - c_i^2 - M^2\right]\left(D^2F_R - k^2F_R\right) +$$

$$+ 2(U-c_r)c_i\left(D^2F_I - k^2F_I\right) + 2(DU)(U-c_r)DF_R +$$

$$+2(DU)c_iDF_I - 2M(DM)DF_R = 0$$

Im:

$$\left[(U-c_r)^2 - c_i^2 - M^2\right]\left(D^2F_I - k^2F_I\right) -$$

$$-2(U-c_r)c_i\left(D^2F_R - k^2F_R\right) + 2(DU)(U-c_r)DF_I -$$

$$-2(DU)c_iDF_R - 2M(DM)DF_I$$

Replacing F_I by $-F_I$ and c_i by $-c_i$ leaves the real part the same and simply changes the sign of all terms in the imaginary part so that the equation if solved by (F, c) is also solved by (F^*, c^*).

Integral theorems are more difficult to derive in the magnetic case because they contain the unknown, c_r , in a non-positive definite manner. It is much the same difficulty as is introduced into the attempt to find integral theorems when density effects are added to the Rayleigh equation, but in the magnetic case the complication is more severe. Thus, we have not been able to find a theorem which places a limit on the stratification which will allow an unstable solution. An analog of Howard's semicircle theorem has been proven for the homogeneous case by Stern(1963) and extends somewhat more strongly to the heterogeneous case. It reveals the surprising fact that as the magnetic field increases, whatever instability appears does so at wave speeds closer to the average velocity. To prove the theorem, we multiply (2.24) by F^* and integrating across z, we obtain

$$\int_{z_1}^{z_2} \left[F^* D\left\{ \left[(U-c)^2 - M^2 \right] DF \right\} - K^2\left[(U-c)^2 - M^2 \right] |F|^2 - \bar{R}_i |F|^2 \right] dz = 0$$

To obtain a more convenient form, we integrate the first term by parts and then use the boundary condition that $F(z_{1,2}) = 0$. We then have

$$\int_{z_1}^{z_2} \left[-\left[(U-c)^2 - M^2 \right] |DF|^2 - K^2\left[(U-c)^2 - M^2 \right] |F|^2 - \bar{R}_i |F|^2 \right] dz = 0$$

We then separate this equation into its real and imaginary parts, each of which must equal zero. They are, respectively,

$$\int_{z_1}^{z_2} \left[\left[(U-c_r)^2 - c_i^2 - M^2 \right] \left\{ |DF|^2 + K^2 |F|^2 \right\} + \bar{R}_i |F|^2 \right] dz = 0$$

and

$$c_i \int_{z_1}^{z_2} (U - c_r) \left\{ |DF|^2 + k^2 |F|^2 \right\} dz = 0$$

We conclude from the imaginary part that if we are to have an unstable solution, c_r must be in the range of the basic current. Defining

$$|DF|^2 + k^2 |F|^2 = R$$

we observe that

$$\int_{z_1}^{z_2} U R \, dz = c_r \int_{z_1}^{z_2} R \, dz \tag{2.25}$$

Now, we are guaranteed that

$$\int_{z_1}^{z_2} (U - U_{min})(U - U_{max}) R \, dz \leq 0$$

The real part of our integrated equation then appears as

$$\int_{z_1}^{z_1} \left[U^2 - (U_{min} + U_{max}) U + U_{min} U_{max} \right] R \, dz = \int_{z_1}^{z_2} \left[M^2 + (c_r^2 + c_i^2) - \right.$$

$$\left. - (U_{min} + U_{max}) c_r + U_{min} U_{max} \right] R \, dz - \int_{z_1}^{z_2} \overline{R}_i |F|^2 dz \leq 0$$

where we have used the equation (2.25). Furthermore, since $\bar{R}_i \leq 0$ everywhere, we can drop the term it appears in and strengthen our inequality. The right hand side thus appears as

$$\int_{z_1}^{z_2} \left\{ \left[c_r - \tfrac{1}{2}(U_{min} + U_{max}) \right]^2 - \left[\tfrac{1}{2}(U_{max} - U_{min}) \right]^2 + M^2 \right\} R\,dz \leq 0$$

So that we are assured that whenever

$$\left[c_r - \frac{1}{2}(U_{min} + U_{max}) \right]^2 + c_i^2 \leq \left[\frac{1}{2}(U_{max} - U_{min}) \right]^2 - M_0^2 \qquad (2.26)$$

where M_0 is the lowest absolute value that the magnetic field assumes. When $M_0 = 0$ this reduces to Howard's result. If, however the magnetic field is never zero a more severe limit is placed on the maximum possible value that c_i may attain and more severely restricts the range of the phase speed, c_r. It should not be assumed that a nonvanishing magnetic field acts only as a stabilizing influence since examples have been found where the opposite is true. Certain instabilities will, nevertheless, be ruled out, as is discussed in chapter 3.

We see that all possibility of obtaining unstable solutions is ruled out whenever the magnetic energy exceeds the kinetic energy everywhere, i.e., whenever

$$M_0^2 \geq \left[\frac{1}{2} \left(U_{min} - U_{max} \right) \right]^2$$

Taking $U_{min} = - U_{max}$, we see that

$$c_r^2 + c_i^2 \leq U_{max}^2 - M_0^2$$

and thus

$$| c_r | \leq | U_{max} - M_0 |$$

which shows more clearly the restriction placed on the range of the wave speed.

C. Applicability to Gaseous Atmospheres, A Scaling Analysis

Because of the highly theoretical nature of the work in this thesis, no intention is made of emphasizing practical applications. It is appropriate, however, to mention two examples to which this thesis may bear relevance. The justification for suggesting a comparison between real phenomena and our nondissipative model can be illustrated by referring to the Benard problem. The stratification necessary to produce instability varies inversely as the fourth power of the depth of the fluid layer. Using typical values for viscosity and thermal conductivity, we find that by the time the dimensions reach thunderstorm size, the dissipative forces are virtually ineffective in restraining motion and virtually any lapse rate greater than adiabatic will produce instability. Our examples occur on large scale so that the ideal model is reasonable.

By tracing the motion of sunspots, and more recently by analyzing actual Doppler velocity measurements on the sun, observers have noticed that the sun does not rotate as a solid body at photospheric levels. The degree of this differential rotation is quite significant and many attempts have been made to explain its existence. The theories assume that the sun is in solid body rotation at some lower level and we are thus faced with a situation in which there is a zonal shearing current superimposed on whatever convective motions are occurring. If this differential rotation is confined to the photosphere, then the vertical component of the shear will be a rather strong. Regardless of the cause for this situation, it then becomes subject to a hydromagnetic stability analysis which, if the shear is strong enough, may occur on a time scale much shorter than that of solar rotation and so be virtually independent of rotation.

Boller and Stolov (1969) have attacked the problem of the semiannual variation of geomagnetic activity. They attributed this variation to the semiannual periodicity of the alignment of the earth's magnetic field with sun. The varying phase of the magnetic field at the magnetopause is related to the ease with which the Kelvin-Helmholtz instability may occur. Their theory fits the data with a good degree of reasonableness and lends support to a consideration of the stability characteristics of more general distributions of density, velocity and magnetic field.

Since both of these phenomena occur in gaseous atmospheres, it is important to see for what conditions, if any, our Boussinesq liquid may be representative of a gas. The main difficulty in attempting to equate the behavior of liquids and gases lies in the differences between the two equations of state. This is reflected in the added role which the pressure assumes in a gas. For an incompressible Boussinesq liquid, the pressure is passive and is invariably eliminated by taking the curl of the equations of motion. In a gas, the pressure has to adjust excessive density variations and is generally not eliminated by taking the curl except under rather restrictive assumptions on the scale of motions.

In somewhat unorthodox fashion, the approximations necessary to make the gas appear the same as the liquid equations, (2.18), will be made while scaling the equations and the restrictions which they place on the range of validity will be discussed afterwards. Particular attention is shown to the case of the solar atmosphere at photospheric levels.

Our equation of state becomes the ideal gas law

$$P = \rho R T \qquad ; \; R = c_p - c_v$$

Together with Poisson's law for adiabatic motions

$$\frac{T}{\theta} = \left(\frac{P}{1000} \right)^{R/c_p}$$

and the thermal equation expressing the conservation of potential temperature

$$\frac{d\theta}{dt} = 0$$

it forms the distinguishing aspects of a gas. Logarithmic differentiation of Poisson's equation leads to

$$\frac{1}{\rho_a} \frac{\partial \rho_a}{\partial z} + \frac{1}{\theta_a} \frac{\partial \theta_a}{\partial z} = \frac{c_v}{c_p} \frac{1}{p_a} \frac{\partial p_a}{\partial z} \qquad (2.27)$$

for the basic state which is denoted by the subscript, a. Since the basic state is in hydrostatic balance this becomes

$$\frac{1}{\rho_a} \frac{\partial \rho_a}{\partial z} = - \frac{c_v}{c_p} \frac{g}{R T_a} - \frac{1}{\theta_a} \frac{\partial \theta_a}{\partial z}$$

In many theoretical works the density profile is chosen so that

$$\frac{1}{\rho_a}\frac{\partial \rho_a}{\partial z} = \text{const}$$

for reasons of mathematical simplicity. This, however, does not give a constant stability factor as can be seen by a relatively simple example. Set

$$\frac{1}{\rho_a}\frac{\partial \rho_a}{\partial z} = -\frac{c_v}{c_p}\frac{g}{RT(z_0)}$$

We then see that above this height, where $T_a < T_0$, we will have

$$\frac{\partial \theta_a}{\partial z} < 0$$

and below it, where $T_a > T_0$,

$$\frac{\partial \theta_a}{\partial z} > 0$$

Since an atmosphere is stably stratified when the potential temperature increases with height and unstably stratified when the potential temperature decreases with height we see that our atmosphere has both stable and unstable regions. For this reason, the basic stratification has been defined in terms of the potential temperature and we note that because thermal conductivity has been neglected we need not impose a basic state that has a linear temperature profile.

We can now scale our equations. Defining a time scale, t , to be the inverse of the average shear, and a length scale, L , to be equal to the depth of the fluid we may write

$$\vec{V} = \frac{L}{t}\left(\bar{U} + \epsilon \vec{V}^* + \cdots\cdots\right) \qquad ; \; * \rightarrow \text{nondimensional}$$

where ϵ is the arbitrary but small amplitude of the velocity perturbation. Indicating an average magnitude for a variable by an overbar, we write

$$\begin{aligned} \bar{P} &= \bar{\rho} R \bar{T} \\ p_a &= \bar{P} p_s \\ \rho_a &= \bar{\rho} \rho_s \\ T_a &= \bar{T} T_s \\ \Theta_a &= \bar{\Theta} \Theta_s \end{aligned}$$

The variables are scaled in the following manner

$$p = \bar{P} p_s (1 + \tilde{P} p^* + \cdots\cdots$$

$$\rho = \bar{\rho} \rho_s (1 + \tilde{\rho} \rho^* + \cdots\cdots$$

$$\Theta = \bar{\Theta} \Theta_s (1 + \tilde{\Theta} \Theta^* + \cdots\cdots$$

where the symbol $\sim$ over the variable indicates the perturbation amplitude which is of order ϵ . Finally, we scale our magnetic field to be

$$B = \sqrt{4\pi\bar{\rho}}\ \frac{L}{t}\left(B^* + \epsilon b^* + \cdots\right.$$

We determine the value of $\tilde{p}$ by noticing that the pressure term has the same order of magnitude as the acceleration terms. Thus to first order in ϵ

$$\epsilon \frac{L}{t^2}\frac{\partial u^*}{\partial t} \sim \frac{1}{L}\frac{\bar{P}p_s}{\bar{\rho}\rho_s}\tilde{p}\frac{\partial p^*}{\partial x}$$

and the magnitude of $\tilde{p}$ is thus

$$\tilde{p} = \epsilon \frac{L^2}{t^2}\frac{\bar{\rho}}{\bar{P}} \qquad (2.28)$$

From Poisson's law we may write

$$\tilde{\rho}\rho^* = -\tilde{\Theta}\Theta^* + \frac{c_v}{c_p}\tilde{p}\,p^* \qquad (2.29)$$

So that the scaling for density and potential temperature is revealed as

$$\tilde{\rho} = \tilde{\Theta} = \frac{c_v}{c_p}\tilde{p}$$

The scaled continuity equation appears as (dropping stars)

$$\frac{\tilde{\rho}}{\epsilon}\left[\frac{\partial \rho}{\partial t} + U\frac{\partial \rho}{\partial x}\right] + \frac{w}{\rho_s}\frac{\partial \rho_s}{\partial z} = \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}$$

In order to have the continuity equation in the form

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$$

we must demand that

$$\frac{1}{\rho_s}\frac{\partial \rho_s}{\partial z} << 1 \quad ; \quad \frac{c_p}{c_v}\frac{L^2}{t^2}\frac{\bar{\rho}}{\bar{p}} << 1$$

The vertical equation of motion becomes

$$\frac{\partial w}{\partial t} + U\frac{\partial w}{\partial x} = -\frac{1}{\rho_s}\frac{\partial p p_s}{\partial z} - \frac{1}{\rho_s}\frac{\partial}{\partial z}(Bb_x) - \frac{\tilde{\rho}}{\epsilon}(\rho g) + \frac{B}{\rho_s}\frac{\partial b_z}{\partial x}$$

Since we wish the pressure term to appear as a pure gradient term and also since we don't want to see the ρ_s in connection with the magnetic terms, we rewrite our equation as

$$\frac{\partial w}{\partial t} + U\frac{\partial w}{\partial x} = -\frac{\partial}{\partial z}\left[\frac{p p_s}{\rho_s} + \frac{Bb_x}{\rho_s}\right] - \frac{\tilde{\rho}}{\epsilon}(\rho g) +$$

$$+ \frac{B}{\sqrt{\rho_s}}\frac{\partial}{\partial x}\left(\frac{b_z}{\sqrt{\rho_s}}\right) - \frac{p p_s}{\rho_s^2}\frac{\partial \rho_s}{\partial z} - \frac{Bb_x}{\rho_s^2}\frac{\partial \rho_s}{\partial z}$$

Defining a new magnetic field

$$M = \frac{B}{\sqrt{\rho_s}} \; ; \; \bar{h} = \frac{\bar{b}}{\sqrt{\rho_s}}$$

and a new pressure,

$$\pi = \frac{p p_s}{\rho_s} + M h_x$$

In order to render our set of equations analogous to (2.18), we should express the buoyancy in terms of potential temperature. From (2.29)

$$-\frac{\bar{\rho}}{\epsilon} \rho g = \left(\theta\tilde{\Theta} - \frac{c_v}{c_p} p\tilde{P}\right) g = \tilde{\Theta}\theta g - \frac{c_v}{c_p} p \frac{L^2}{t^2} \frac{\bar{\rho}}{\bar{P}} g$$

Since the term

$$-\frac{p p_s}{\rho_s^2} \frac{\partial \rho_s}{\partial z} = -\frac{c_v}{c_p} \frac{p}{\rho_s} \frac{\partial p_s}{\partial z} + \frac{p_s}{\rho_s \Theta_s} \beta p$$

which by the hydrostatic law becomes

$$= \frac{c_v}{c_p} p \frac{L^2}{t^2} \frac{\bar{\rho}}{\bar{P}} g + \frac{p_s}{\rho_s \Theta_s} \beta p$$

Our vertical equation thus becomes

$$\frac{\partial w}{\partial t} + U\frac{\partial w}{\partial x} = -\frac{\partial \pi}{\partial z} + \frac{\tilde{\theta}\theta g}{\epsilon} + M\frac{\partial h_x}{\partial x} + \qquad (2.30)$$

$$+ \frac{P_s}{\rho_s \theta_s}\beta p - M h_x \frac{\partial \rho_s}{\partial z}$$

To render this in the form of (2.18) we would like

$$\beta << 1 \quad ; \quad \frac{\partial \rho_s}{\partial z} << 1$$

These approximations cause the fourth and fifth terms on the right of (2.30) to be less than the first and third terms respectively. Our final adjustment must appear in the magnetic equations. The horizontal magnetic equation becomes

$$\frac{\partial h_x}{\partial t} + U\frac{\partial h_x}{\partial x} + w\frac{\partial M}{\partial z} = M\frac{\partial u}{\partial x} + \frac{wM}{\rho_s}\frac{\partial \rho_s}{\partial z}$$

and this reduces to the magnetic equation of (2.18) when

$$\frac{1}{\rho_s}\frac{\partial \rho_s}{\partial z} << 1$$

Our equations are now formally identical to the set (2.18). Let us investigate the meaning of the physical restrictions we have imposed. Requiring $\beta << 1$ simply means that the fractional range of the potential temperature must be much less than unity. Requiring that

$$\frac{1}{\rho_s}\frac{\partial \rho_s}{\partial z} << 1$$

means that

$$\frac{1}{\rho_s}\frac{\partial \rho_s}{\partial z} = \frac{c_v}{c_p}\frac{L^2\bar{\rho}}{t^2\bar{P}} g + \frac{\beta}{\theta_s} << 1$$

Now, we have already set $\beta << 1$ so that we want

$$\frac{L^2}{t^2 R\bar{T}} g \equiv \frac{L}{H} << 1$$

We thus want the depth of the layer to be but a fraction of the scale height, H . Finally, to remove sound waves and other effects of compressibility we have required that

$$\frac{L^2}{t^2}\frac{1}{R\bar{T}} << 1$$

In dimensional terms since $R\bar{T} = gH$ we now require that

$$t^2 \gtrsim L/g_{dim} \tag{2.31}$$

Let us now see when these approximations are valid for the sun. CGS units are used throughout. Typical values in the photosphere are

$$H = 2 \times 10^8 \qquad B = 10$$
$$g = 10^4 \qquad \Omega^{-1} = 2 \times 10^5$$
$$\rho = 10^{-6}$$

Assuming that the motion is due to the differential rotation, we have

$$U \leq 3 \times 10^4$$

In order to insure that $L/H << 1$, the study must be restricted to fluid layers with

$$L \leq 2 \times 10^7$$

In order to satisfy (2. 31), we must have

$$t^2 \geq 2 \times 10^3$$

but since we have neglected rotation, we must also have

$$t << 2 \times 10^5$$

When $t = 1/2 \times 10^2$, this means that the differential rotation occurs over a depth of 15 KM and stronger shears are not expected. When $t = 2 \times 10^4$, the differential rotation occurs over a depth of 6,000 KM and the existence of stronger shears seems quite likely.

Since the photosphere is a convective layer one might argue that there is only unstable stratification. Several studies (Veronis, Kuo) on nonlinear convection have shown that in the body of the convective layer the stratification is actually slightly stable.

Finally to show that the magnetic effects are important but not overwhelming, we note that using a velocity of $v = 10^4$, we find that a typical value for M^2 is

$$M^2 = \frac{B^2}{4\pi \rho v^2} = \frac{10^2}{10^{-5} \cdot 10^8} = \frac{10^2}{10^3} \sim 10^{-1}$$

III. Examples

The Kelvin-Helmholtz instability was the first problem in hydrodynamic stability to be solved. Several other problems have been completely solved since then, but all of these have one artificial characteristic in common. Each of these solved examples possesses either a number of homogeneous layers of distinct properties or, at best, a number of piecewise linear layers.

Despite this artificiality, such profiles are useful. The Kelvin-Helmholtz problem has been applied with considerable success to a variety of phenomena ever since it was initially used to explain the generation of surface water waves by a wind in the overlying air. Kelvin derived a critical wind speed necessary to produce waves when surface tension effects were included. Since waves form at much lower wind values, it is apparent that viscosity has an effect. Nevertheless, Munk (1947) has observed that the critical wind speed derived by Kelvin is accompanied by an increase in whitecaps and convection in the air above.

If we consider a fluid which extends across a rather broad expanse and in which there is a relatively narrow shear zone, then Howard and Drazin (1962) have shown by dimensional arguments that the fluid behaves like a Kelvin-Helmholtz fluid in the long wave length limit. Furthermore, these discontinuous models are more amenable to simple physical arguments than are the continuous models with any velocity and density profiles. Sufficient motivation therefore exists from both the mathematical and physical sides of the problem to warrant an investigation of the stability properties of discontinuous models.

The form that the equations take is slightly different from that in the continuous models. In addition to this, we must satisfy somewhat different boundary conditions. When we consider an unbounded fluid, we have one kinematic boundary condition at each of $z = \pm\infty$ and two boundary conditions, one kinematic and one dynamic, at each fluid interface. We assume that the perturbation velocity as $|z| \to \infty$ remains finite. We further assume that there is no discontinuity of the velocity normal to the interface at the interface. The dynamical boundary condition stipulates that the pressure must be continuous at an interface so as to avoid infinite accelerations.

Assuming that the height of the interface is given by η, then to first order we have

$$\frac{d\eta}{dt} = \frac{\partial\eta}{\partial t} + U\frac{\partial\eta}{\partial x} = W \tag{3.1}$$

When all the perturbation variables have solutions of the form

$$f(x,z,t) = F(z)\, e^{ik(x-ct)}$$

(3.1) becomes

$$ik(U-c)\eta = w \tag{3.2}$$

and the continuity of normal velocity at the interface is expressed as

$$\Delta_s \left[\frac{w}{U-c} \right]_{z_s} = 0 \tag{3.3}$$

where Δ_s indicates the jump in the term in brackets at the interface height approximated by $z = z_s$. Our governing equation can be derived from (2.20) and takes the form

$$\begin{aligned} &(U-c)D^2w + wD^2U + \frac{gD\rho}{\rho_0}\frac{w}{U-c} + k^2(U-c)w \\ &-k^2M^2\frac{w}{U-c} + M^2D^2\left[\frac{w}{U-c}\right] + 2M(DM)D\left[\frac{w}{U-c}\right] = 0 \end{aligned} \tag{3.4}$$

This is simply an alternate form of (2.21). For each region of constant U , M and ρ the governing equation reduces to

$$k(U-c)\left[1 - \frac{M^2}{(U-c)^2}\right]\left\{D^2w - k^2w\right\} = 0 \tag{3.5}$$

Whereas in the nonmagnetic case, piecewise linear velocity and density profiles are governed by a relatively simple equation (Goldstein (1932)), when $M \neq 0$ the resultant equation becomes prohibitive to solve analytically. It is for this reason that we are restricted to discontinuous models composed of several homogeneous layers.

Rather than phrasing the dynamic boundary condition in terms of pressure, we integrate the basic equation (3.4) across the interface from $z_s - \epsilon$ to $z_s + \epsilon$ and take the limit as $\epsilon \to 0$. This is mathematically equivalent to the condition that the pressure be constant. We arrive at the result

$$\Delta_s \left[-(U-c)DW + M^2 D\left(\frac{W}{U-c}\right) + \frac{g\rho}{\rho_0}\left(\frac{W}{U-c}\right)\right] = 0 \tag{3.6}$$

Our basic approach will be illustrated by first reproducing the solution to the standard Kelvin-Helmholtz problem in the presence of a parallel magnetic field (Figure 3). Equation (3.5) is valid for each layer and yields the solution

$$W \propto e^{\pm kz}$$

Satisfying the boundary conditions at $|z| = \infty$ and the kinematic boundary condition at $z = z_s = 0$ leads to

$$\begin{aligned} W &= (U_1 - c)e^{-kz} \qquad & z \geq 0 \\ W &= (U_2 - c)e^{kz} \qquad & z \leq 0 \end{aligned} \tag{3.7}$$

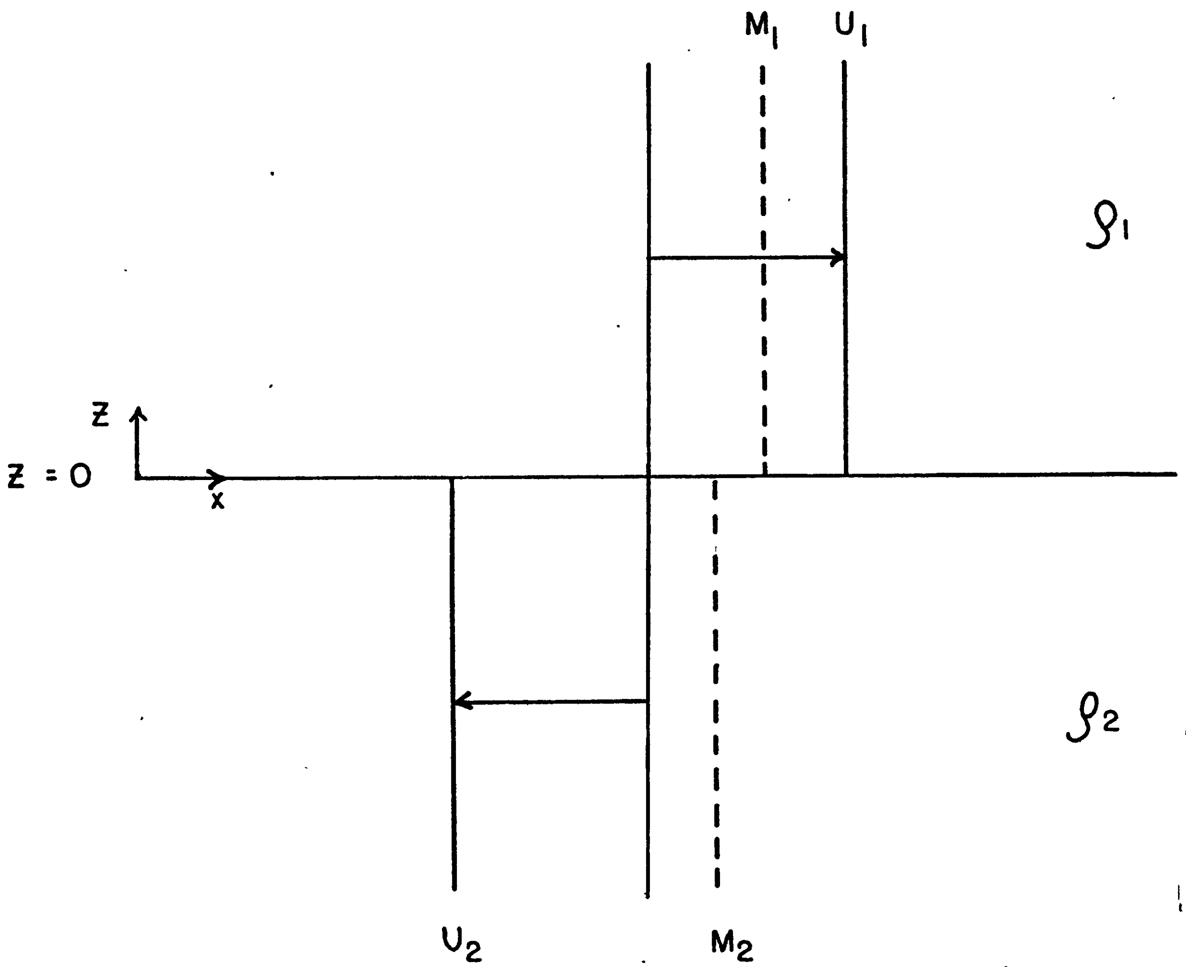

Figure 3. Two layer model.

We shall be able to determine **C** by satisfying the dynamical boundary condition, (3. 6) by substituting (3. 7) into the former. We obtain

$$(U_1+c)^2 k + M_1^2(-k) + (U_2-c)^2 k - M_2^2(k) + \frac{g}{\rho_0}(\rho_1-\rho_2) = 0$$

which results in the following solution for .

$$c = \frac{U_1+U_2}{2} \pm \sqrt{-\frac{(U_1-U_2)^2}{4} + \frac{g}{k\rho_0}(\rho_2-\rho_1) + \frac{(M_1^2+M_2^2)}{2}} \qquad (3.\ 8)$$

Equation (3. 8) appears in a slightly different form from (1. 20) because we have neglected the inertial effects of density here, but included the possibility that the magnetic field varies from one layer to the other. The two equations are identical otherwise and both indicate that the magnetic field acts solely as a stabilizing agent.

We now consider two slightly more complicated problems. These each consist of three fluid layers. The first problem represents a symmetric jet and the second, an antisymmetric double shear layer. The first problem has been considered by Axford (1960) when there is a magnetic field only in the two semi-infinite outer regions, and the second problem has been solved without any magnetic fields by Howard (1963). The method of solution for these problems is exactly the same as above, namely, obtain the general solution for each region and then solve for **c** by satisfying the kinematic and dynamic boundary

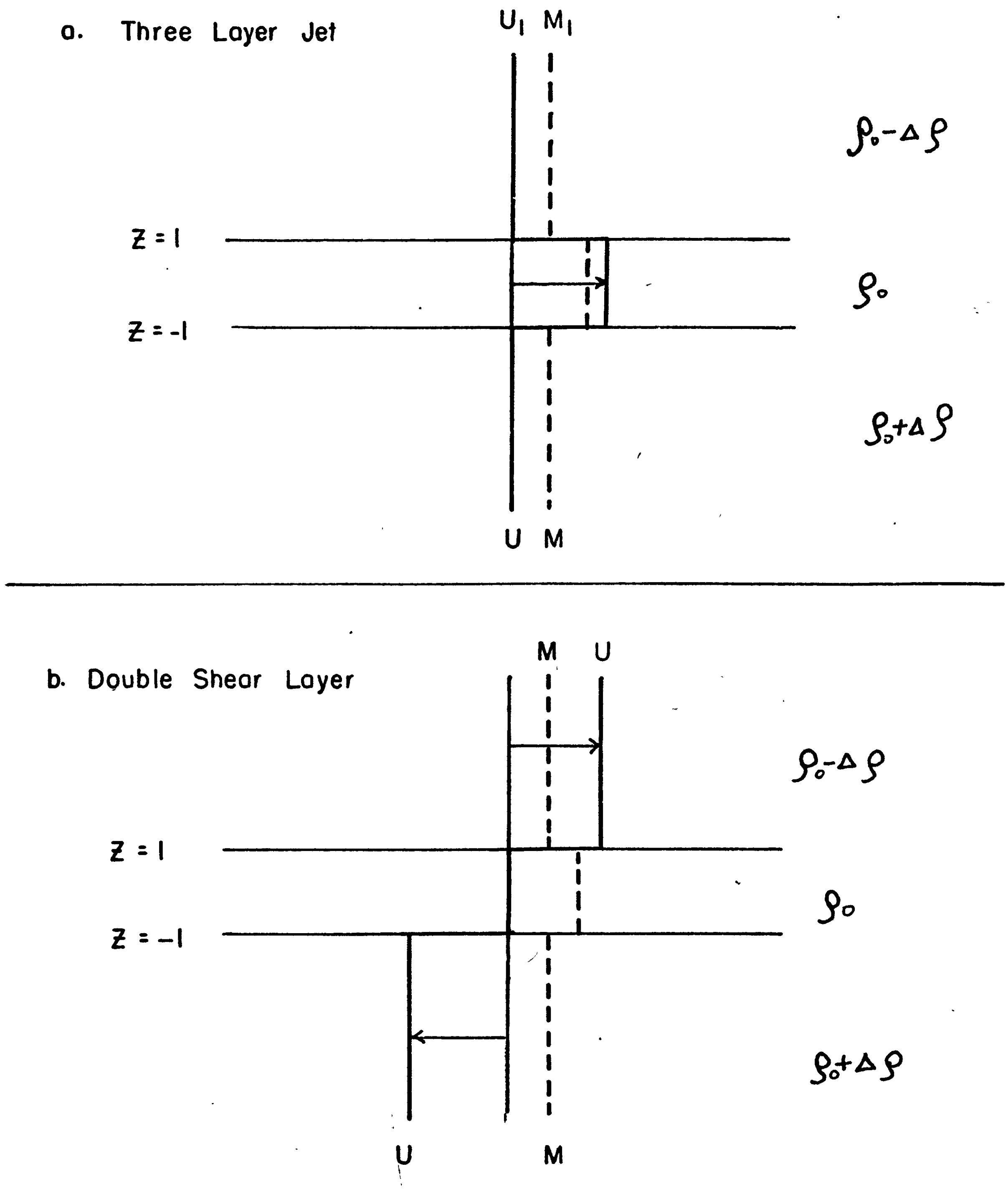

Figure 4. Three layer models.

conditions, respectively. The reason for choosing three layer models is justified in retrospect as the solutions exhibit several properties characteristic of the continuous profiles. In the remainder of the chapter, we normalize everywhere by considering $|U| = 1$ and we take our interfaces at $z = \pm 1$. All other variables are similarly normalized.

A. Three Layer Jet

Our governing equation for each of the three layers depicted in Figure 4a is once again (3. 5). By satisfying the kinematic boundary conditions at $z = \pm\infty$ and at the two interfaces, our solution for w is given by

$$
\begin{aligned}
w &= -\left(Ae^{-2k} + B\right) c\, e^{-k(z-1)} && z \geq 1 \\
w &= A(1-c)\, e^{-k(z+1)} + B(1-c)\, e^{k(z-1)} && |z| \leq 1 \qquad (3.9) \\
w &= -\left(A + Be^{-2k}\right) c\, e^{k(z+1)} && z \leq -1
\end{aligned}
$$

In order to eliminate one of the unknowns, A and B from (3. 9) and solve for the growth speed, c, we make use of the dynamical boundary conditions at $z = \pm 1$, which are, respectively

$$
\begin{aligned}
&\left[c^2 - (1-c)^2 - M_1^2 + M_0^2 - G/k\right] Ae^{-2k} + \\
&+ \left[c^2 + (1-c)^2 - M_1^2 - M_0^2 - G/k\right] B = 0
\end{aligned}
\qquad (3.10a)
$$

and

$$\left[(1-c)^2 + c^2 - M_0^2 - M_1^2 - G/k\right]A + \qquad (3.10b)$$

$$\left[c^2 + M_0^2 - (1-c)^2 - M_1^2 - G/k\right]Be^{-2k} = 0$$

where G is defined by

$$G \equiv -\frac{(\rho_1 - \rho_0)}{\rho_0} g = -\frac{(\rho_0 - \rho_{-1})}{\rho_0} g$$

and is negative when the stratification is gravitationally stable. We shall find it convenient to define

$$1 - M_0^2 \equiv p \qquad ; \qquad M_1^2 + G/k \equiv q$$

By substituting the conditions (3.10) into (3.9) we obtain

$$[2c - p - q]^2 e^{-4k} - [2c^2 - 2c + p - q]^2 = 0$$

which results in the following two equations for c .

$$2c^2 - 2c[1 + e^{-2k}] + p[1 + e^{-2k}] - q[1 - e^{-2k}] = 0$$

$$2c^2 - 2c[1 - e^{-2k}] + p[1 - e^{-2k}] - q[1 + e^{-2k}] = 0$$

By defining

$$\alpha^2 \equiv 1 - e^{-2k} \qquad\qquad 0 \le \alpha^2 \le 1$$

the two solutions for c become respectively

$$c_1 = \frac{1}{2}(2-\alpha^2) \pm \frac{1}{2}\sqrt{(2-\alpha^2)^2 - 2(2-\alpha^2)p + 2\alpha^2 q} \tag{3.11}$$

and

$$c_2 = \alpha^2 \pm \frac{1}{2}\sqrt{\alpha^4 - 2\alpha^2 p - 2(2-\alpha^2)q} \tag{3.12}$$

The appearance of two distinct solutions is characteristic of a fluid with symmetric velocity and magnetic field profiles and antisymmetric density profiles. The first solution represents the varicose disturbance, i. e., the name given by Rayleigh to waves which are symmetric about the midpoint of the channel. The second solution gives the sinuous disturbance, or the wave which is antisymmetric about the midpoint. In the long wave length limit $(\alpha^2 = 0)$ of the homogeneous nonmagnetic problem both these waves are marginally stable (unstable for $k > 0$). The varicose wave at $k = 0$ travels with the maximum value of the current and the sinuous wave at $k = 0$ has a wave speed equal to the minimum of the current.

Let us consider the homogeneous magnetic problem. At $k = 0$

the varicose wave has two solutions given by

$$C_1 = 1 \pm M_0$$

and we see it is independent of the surrounding magnetic field. We can see that neither of these waves is marginally stable and we rewrite (3.11) as

$$C_1 = 1 - \frac{a^2}{2} \pm \frac{1}{2}\sqrt{(2-a^2)\left[2M_0^2 - a^2\right] + 2a^2 M_1^2}$$

Any value of $M_0^2 \geq 1/2$ rules out all instability since then the discriminant can never be negative. In fact, increasing either M_1^2 or M_0^2 serves only to decrease instability wherever it exists and can never serve to produce instability if it does not already exist. This reasoning plainly extends to the stratified case as well. By investigating the discriminant further, we see that it can never be negative when $M_1^2 \geq 1.0$ for we then have

$$a^2\left[2M_1^2 - 2 + a^2\right] + 2(2-a^2)M_0^2 \geq 0$$

and thus a magnetic field of $M_1 \geq 1.0$ causes stability.

We can determine the maximum value of M_1^2 which can result in an unstable situation for any given M_0^2. This is accomplished by taking the derivative of M_1^2 determined from setting the discriminant equal to zero with respect to a^2 and setting that equal to zero. We thus have

$$\frac{d}{da^2} M_1^2 = - \frac{d}{da^2}\left[(2-a^2)\left[2M_0^2 - a^2\right]\frac{1}{2a^2}\right] =$$

$$= - \frac{(2a^4 - 8M_0^2)}{4a^4} = 0$$

Marginal stability thus occurs at a wave number for which

$$a^2 = 2M_0$$

and M_1^2 is then given by

$$M_1^2 = (1 - M_0)^2$$

This formula is valid for all $M_0^2 \leq 1/2$ since when $M_0^2 \geq 1/2$, marginal instability for the maximum M_1^2 occurs for $a^2 = 1$ and M_1^2 is given by

$$M_1^2 = 1/2\left[2M_0^2 - 1\right]$$

This procedure has also been followed for various values of C_L^2 and the results appear in Figure 5.

The sinuous wave for the homogeneous case has a wave speed given by the formula

$$C_2 = a^2 \pm \frac{1}{2}\sqrt{a^4 - 2a^2(1-M_0^2) + 2(2-a^2)M_1^2} \tag{3.13}$$

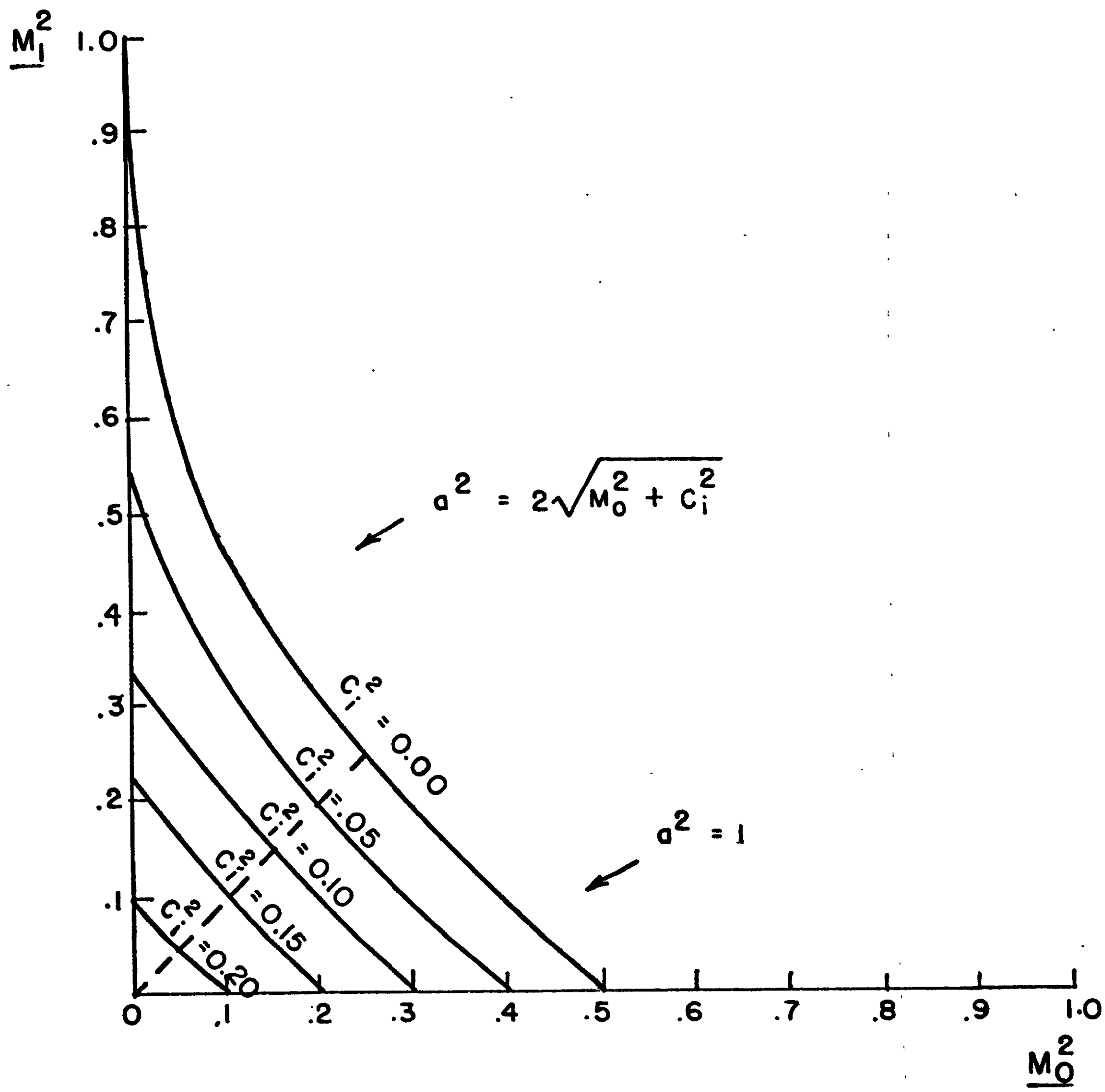

Figure 5. Maximum instability for (M_1, M_0) pair of varicose wave at three layer jet for homogeneous case.

which, in the long wave length limit is given by

$$C_2 = \pm M_1$$

and therefore is not marginally stable whenever $M_1 \neq 0$. Once again, increasing either M_1^2 or M_0^2 serves only to decrease instability whenever it is present. Instability now may occur for any value of $M_0^2 < 1$ so long as $M_1^2 = 0$. We can rewrite the discriminant of (3.13) as

$$(2-a^2)\left[2M_1^2 - a^2\right] + 2a^2M_0^2$$

which is non-negative for all $M_1^2 \geqslant 1/2$ so that $M_1^2 \geqslant 1/2$ implies stability. This, we see, is the reverse of the situation for the varicose wave. Using the same procedure as we did for the varicose wave, we determine the maximum value of M_1^2 for M_0^2 that may produce instability to be given by

$$M_1^2 = 1/2 - M_0^2$$

at $a^2 = 1$ for $M_0 \leq 1/2$ and by

$$M_1^2 = (1 - M_0)^2$$

at

$$a^2 = 2(1 - M_0)$$

for $1/2 \leq M_0 \leq 1$. More complete details including the maximum value of M_1^2 for each M_0^2 corresponding to given growth speeds, c_i , is presented in Figure 6. In Figure 7, a comparison of the stability characteristics of the varicose and sinuous waves is shown and we arrive at the simple result that if $M_1^2 < M_0^2$ we expect the sinuous wave to dominate and if $M_1^2 > M_0^2$ we expect the varicose wave to dominate.

Our mathematical result, at least in the case of the sinuous wave for long wave length disturbance, may be given physical significance by the extension of a rather ingenious heuristic argument, developed by Backus (1960) and refined by Drazin and Howard (1966), to the magnetic problem for finding the wave speed, c .

The logic involved is simply that for long wave length disturbances (when compared to the width of the jet) of the sinuous type, we can treat the jet essentially as a string. We have thus assumed that

$$\frac{2\pi}{k} >> L$$

where L is the width of the jet. Disturbances die out with a scale height of $1/k$.

We take the height of the disturbance to be

$$\eta = A(z) e^{ik(x-ct)}$$

There are three forces which will balance the acceleration term. One, due to the motion of the fluid, is a centrifugal force. The second, due to the heterogeneity of the fluid is a buoyancy force, and the third

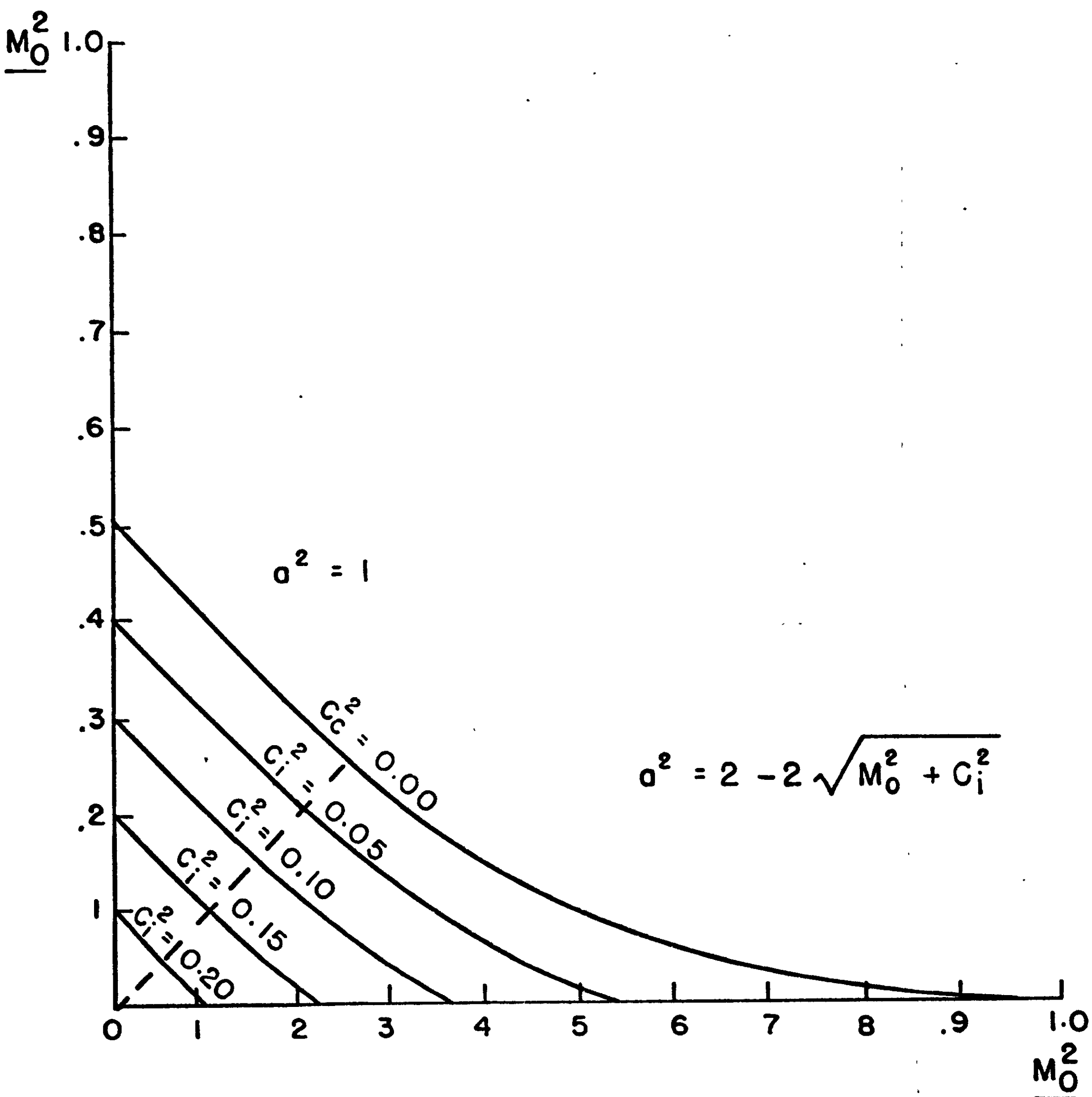

Figure 6. Maximum instability for (M_1, M_0) pair of sinuous wave of three layer jet for homogeneous case.

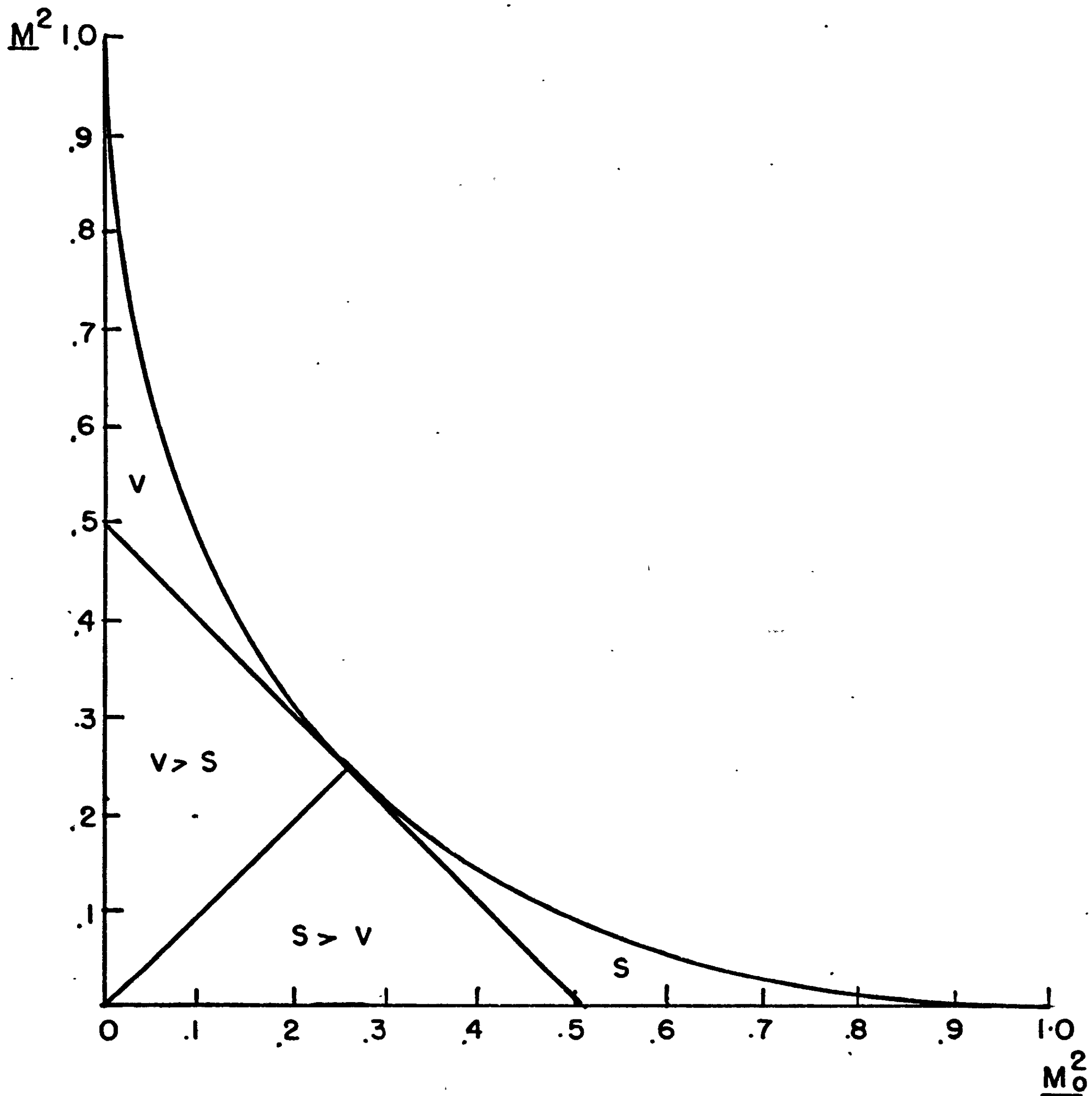

Figure 7. Comparison of maximum instability curves for sinuous and varicose waves.

is due to the tension which the magnetic field gives to the fluid. We proceed by considering each of these forces individually.

The centrifugal force is given by

$$CF = \int_{-\infty}^{\infty} \rho \frac{V^2}{r}\, dx\, dz \, .$$

Since we are considering only the vertical component of force which corresponds to the growth rate of the height amplitude, we have

$$CF = \int \rho \frac{U^2}{r}\, dx\, dz$$

To first order (since $A(z)$ is a small amplitude) we have

$$\frac{1}{r} \cong -\frac{\partial^2 \eta}{\partial x^2} = K^2 \eta$$

and thus

$$CF = \int K^2 \rho U^2 \eta \, dz\, dx$$

Because the large part of the contribution to this term comes in the jet area where $\eta \cong \eta(z=0) = \eta_0$, we can approximate

$$CF = k\rho_0 \int \eta_0 \int U^2 dz\, dx$$

We consider the wave disturbance from 0 to π so that

$$CF = K^2 \rho_0 \eta_0 \int U^2 dz$$

The buoyancy force for that part of the wave from 0 to π (where $\eta > 0$ as we have already assumed with the approximation for the curvature) is given by

$$BF = -g \int_{-\infty}^{\infty} \Delta\rho \, \eta \, dz$$

But

$$\Delta\rho \begin{array}{ll} \cong (\rho_{\infty} - \rho_{-\infty}) & \qquad 0 \leq z \leq \eta_0 \\ = \quad 0 & \qquad z \geq \eta_0 \end{array}$$

Therefore we simply have

$$BF = g(\rho_{\infty} - \rho_{-\infty})\eta_0$$

The magnetic force is composed of a pressure and a tension term. The pressure term is the integral of an exact differential so that we can neglect it and consider the tension term above. We thus have

$$MF = \int_{-\infty}^{\infty} \rho (M \cdot \nabla) M \, dz$$

and taking the vertical component this becomes

$$\int \rho \left(M \frac{\partial h_z}{\partial x} + h_z \frac{\partial h_z}{\partial z} \right) dz \cong \int \rho \, M \frac{\partial h_z}{\partial x} \, dz$$

Since in our ideal model the magnetic field lines are initially parallel to the flow, they remain so and we have

$$h_z = M(z)\frac{\partial \eta}{\partial x}$$

Thus, the magnetic force is given by

$$MF = -\rho \int M^2(z)\, k^2 \eta \, dz$$

Since we are considering a magnetic field which varies only in the jet region we may write

$$M = M_B + M_V$$

so that

$$MF = -\rho \int (M_B + M_V)^2 k^2 \eta \, dz$$

$$= -\rho \int M_B^2 k^2 \eta \, dz - \rho \int M_V (2M_B + M_V) k^2 \eta \, dz$$

Now, we assume that $\eta \propto \eta_0 e^{-k|z|}$ so that

$$\rho M_B^2 k^2 \int \eta \, dz = \rho M_B^2 k^2 \left\{ \int_0^\infty \eta_0 e^{-kz} dz + \right.$$

$$\left. + \int_{-\infty}^0 \eta_0 e^{kz} dz \right\} = 2 M_B^2 k \rho \eta_0$$

and thus

$$MF = -\rho k^2 \eta_0 \int_{-\infty}^{\infty} M_v (2M_B + M_v) dz - 2M_B^2 k \rho \eta_0$$

The sum of these three forces is equal to the density times the acceleration, which is given by

$$\int \rho \frac{\partial^2 \eta}{\partial t^2} dz = -\eta_0 k^2 c^2 \rho \left\{ \int_0^{\infty} e^{-kz} dz + \int_{-\infty}^{0} e^{kz} dz \right\}$$

$$= -2kc^2 \rho \eta_0 = MF + CF + BF$$

When we solve this for c our final result is then

$$c^2 = -\frac{k}{2} \int_{-\infty}^{\infty} \left[U^2 - M_v (2M_B + M_v) \right] dz + M_B^2 + \frac{g(\rho_{-\infty} - \rho_{\infty})}{2\bar{\rho} k} \tag{3.14}$$

Applying this to our three layer jet, we obtain

$$c^2 = -k\left[1 - M_0^2 + M_i^2\right] + G/2k + M_i^2$$

To show how closely related this result is to the exact solution, consider the discriminant of (3.12) for small k. Approximating to first order in k, we have

$$a^2 = 1 - e^{-2k} = 1 - 1 + 2k + \cdots \simeq 2k$$

and $a^4 = O(k^2)$. Inserting this into (3.12), we obtain

$$C^2 \cong \frac{1}{4}\left(0 - 4k(1-M_0^2) + 2\frac{G}{k} + 4M_1^2 - 4M_1^2k\right)$$

which is exactly equal to the result given by our heuristic theory. We will have further cause to refer to this argument when considering the expansions for long wave length. Let us note here that by this argument any basic magnetic field is always sufficient to stabilize the long wave sinuous disturbances simply because there is not sufficient kinetic energy available to be converted into the magnetic energy of the disturbances.

B. Double Shear Layer

We now consider the antisymmetric double shear layer depicted in Figure 4b. Because the flow indicates no preferred direction, it seems plausible that instability will set in as a wave with $c_r = 0$ so that the Principle of the Exchange of Stabilities would then be valid. Although it does seem as if this is often the case, it need no be so. Howard (1963) showed that the curve $c = 0$ in the G, k plane does not define the stability boundary for the antisymmetric double shear layer and that instability sets in as two waves travelling in opposite directions. While there are unstable solutions with $c_r = 0$, these lie entirely embedded in the unstable region.

When an aligned magnetic field is superposed on this pattern, several new interesting features arise. A small magnetic field actually destabilizes the problem in many instances. This example therefore proves to yield the first complete analytical solution for an ideal magnetic problem in which the magnetic field destabilizes the fluid. There are even ranges of the parameters for which this destabilization occurs when the magnetic field is constant. We shall now proceed to present an analysis of the problem.

In each of the three regions, (3.5) is seen to be the governing equation. After satisfying the kinematic boundary conditions, the solution for W is given by

$$\begin{aligned} W &= (A + Be^{2k})(1-c)\,e^{-k(z-1)}, & z &\geq 1 \\ W &= -Ace^{2k}e^{-k(z+1)} - Bce^{2k}e^{k(z-1)}, & |z| &\leq 1 \\ W &= -(Ae^{2k} + B)(1+c)\,e^{k(z+1)}, & z &\leq -1 \end{aligned} \tag{3.15}$$

The dynamical boundary conditions at $z \pm 1$ respectively, are given by

$$\begin{aligned} &\left[(1-c)^2 - c^2 - M_1^2 + M_0^2 - G/k\right]Ae^{-2k} \\ &\quad + \left[(1-c)^2 + c^2 - M_1^2 - M_0^2 - G/k\right]B = 0 \end{aligned} \tag{3.16a}$$

and

$$\begin{aligned} &\left[c^2 + (1+c)^2 - M_0^2 - M_1^2 - G/k\right]A \\ &\quad + \left[(c+1)^2 - c^2 + M_0^2 - M_1^2 - G/k\right]Be^{-2k} = 0 \end{aligned} \tag{3.16b}$$

We shall find it convenient to define

$$m \equiv 1 - M_1^2 - G/k$$
$$a^2 \equiv 1 - e^{-4k}$$

We again obtain the solution for c by substituting (3.15) into (3.16) and thus

$$c^2 = \frac{-(m - M_0^2 - a^2)}{2} \pm \frac{1}{2}\sqrt{a^4 - 2a^2(m - M_0^2) + (1-a^2)\left[m + M_0^2\right]^2} \qquad (3.17)$$

Instability can arise in any of three manners. Since (3.17) is of the form

$$c^2 = S \pm \sqrt{T}$$

we can see that c has an imaginary part if $S < 0$ or if $T > S^2$ or if $T < 0$. Furthermore, whenever we have a solution for c with an imaginary part we are assured that there is an unstable branch $(c_i > 0)$. An example of a stability diagram is presented in Figure 8 where we use the case of $M_0^2 = 0.2$. So long as we are within a region bounded by any one of the marginal stability curves, $T = 0$, $S = 0$, $T = S^2$, there is an instability.

Before analyzing the problem in completest generality, let us analyze the case where $G = 0$ and where we have $M_1^2 = 0$. Consider first the nonmagnetic problem. Equation (3.17) becomes

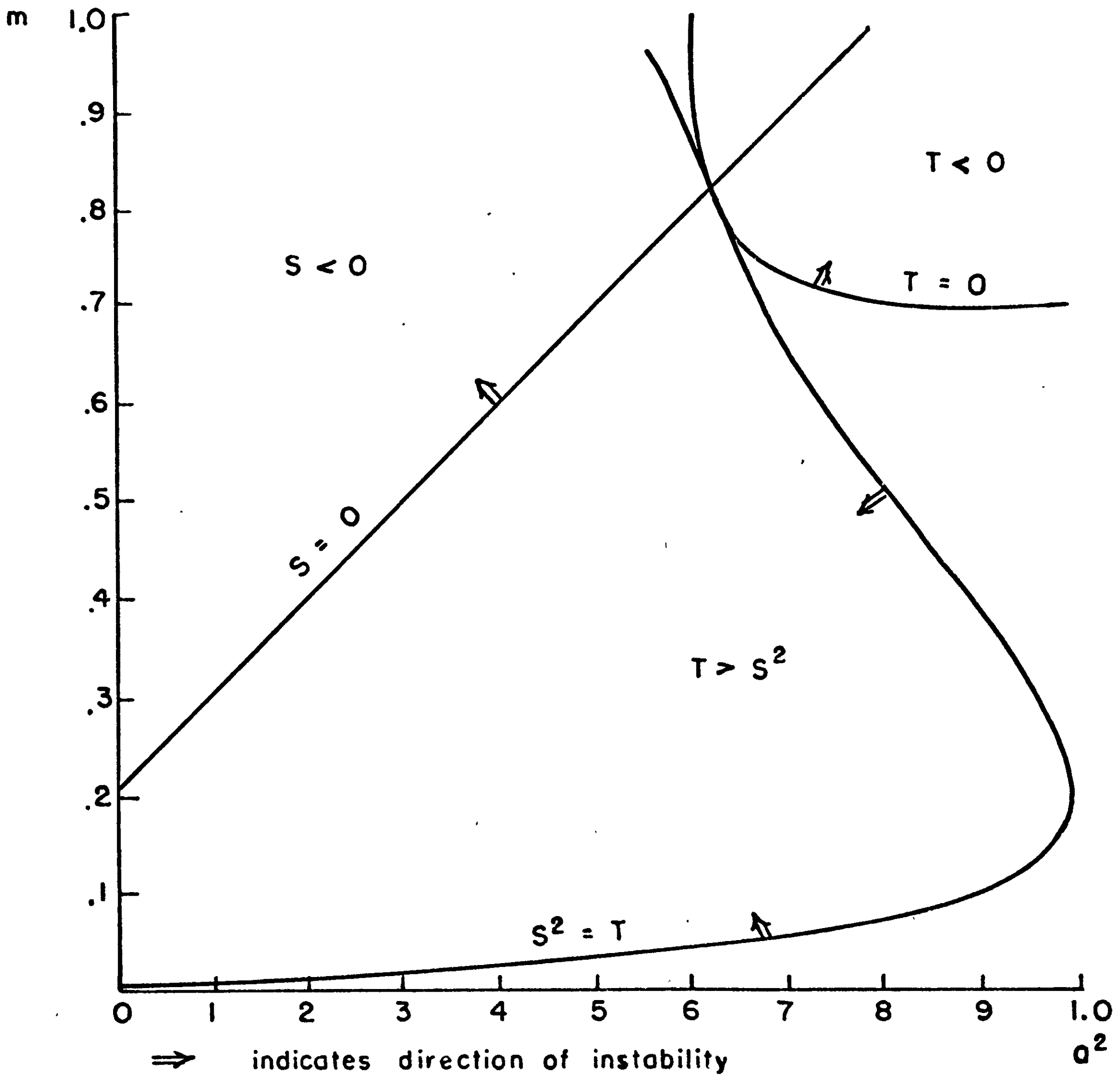

Figure 8. Schematic diagram for marginal stability curves of double shear layer.

$$c^2 = \frac{a^2-1}{2} \pm \frac{1}{2}\sqrt{a^4-3a^2+1} \tag{3.18}$$

The first term on the right is negative for all $a^2 < 1$ and the discriminant is negative for all $a^2 \geq 1/2\,(3-\sqrt{5}\,)$. so that we have instability for all values of a^2 . When $a^2 \leq 1/2(3-\sqrt{5}\,)$, instability of the type $S < 0$ occurs so that we have $c_r = 0$. For larger a^2 , we see that c^2 and thus c are complex.

When we include the effects of a magnetic field in the central region only, the problem becomes slightly more complicated. Formula (3.17) becomes

$$c^2 = \frac{a^2+M_0^2-1}{2} \pm \frac{1}{2}\sqrt{a^4-2a^2(1-M_0^2)+(1-a^2)[1+M_0^2]^2} \tag{3.19}$$

Any increase in M_0^2 increases the discriminant so that by substituting $M_0^2 = 1/2$ into the discriminant we obtain

$$T = a^4 - a^2 + \frac{9}{4}(1-a^2)$$

which is negative for $1 < a^2 < 9/4$ only, and thus for $M_0^2 \geq 1/2$, instability can arise through $c_r = 0$ only. Setting $S^2 = T$, we find that

$$a_0^2 = \frac{4M_0^2}{2M_0^2 + 1 + M_0^4}$$

Instability will occur through $T > S^2$ whenever $(M_0^2 + a^2 - 1) > 0$ and $a^2 < a_0^2$. The value for the magnetic field at which

$$M_0^2 + a_0^2 - 1 = 0$$

is $M_0^2 \cong .295$ and the significance of this point lies in the fact that for any value

$$0 \leq M_0^2 \lesssim .295$$

we have instability for all values of a^2 and for

$$.295 \lesssim M_0^2 < .5$$

there is an intermediate range of a^2 for which we have stability but there is instability at both the long and short wave length limits. As an example, consider that case of $M_0^2 = 0.4$. We find that $T<0$ for $a^2 \geq .845$ but $S<0$ for $a^2 \leq 0.6$ so that there is a region between $.60 \leq a^2 \leq .845$ in which $S^2 - T > 0$ and thus we have no unstable solution for certain intermediate k.

Instability may occur for virtually any value of M_0^2. Take the case of $a^2 = 0$. We then have

$$c^2 = \frac{1}{2}(M_0^2 - 1) \pm \frac{1}{2}(1 + M_0^2)$$

Taking the negative root we see that $c_i = 1$.

We now solve for the growth speed for several values of M_0^2. First, take the nonmagnetic problem. For

$$a^2 \leq \frac{3-\sqrt{5}}{2}$$

instability arises from $c_r = 0$ and we have

$$c_i^2 = \frac{1-a^2}{2} + \frac{\sqrt{a^4-3a^2+1}}{2}$$

When

$$a^2 > \frac{3-\sqrt{5}}{2}$$

instability is manifested by travelling waves and the growth speed is given by

$$c_i^2 = \frac{-a^2+a+1}{4}$$

When $M_0^2 = 1$, (3.19) becomes

$$\begin{aligned} 2c^2 &= a^2 \pm \sqrt{a^4-4a^2+4} \\ &= a^2 \pm (2-a^2) \end{aligned}$$

Taking the negative root, we have

$$c_i^2 = 1-a^2$$

A plot of the c_i^2 values for various values of M_0^2 appears in Figures 9 and 10. We see immediately that for the long wave length

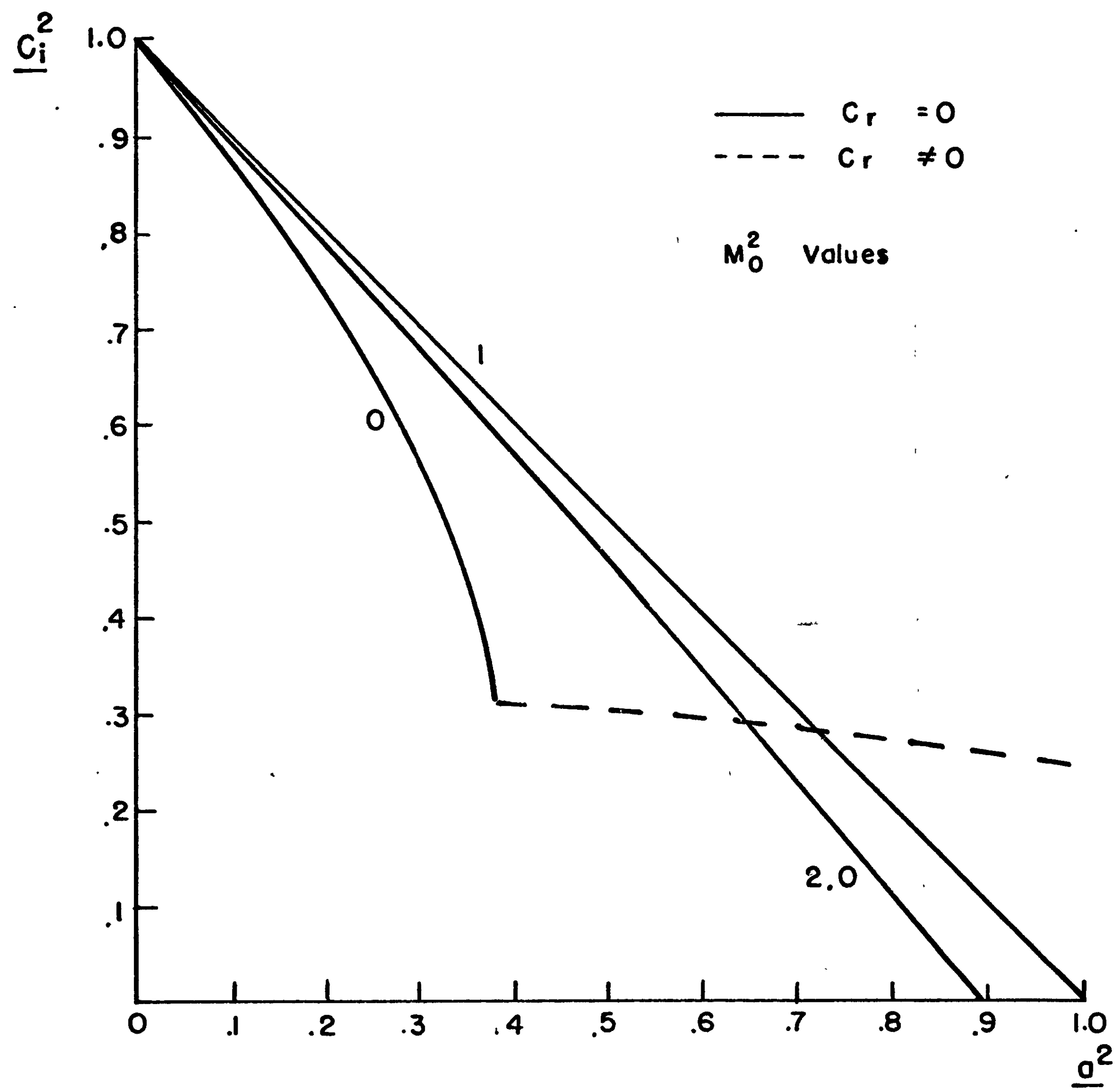

Figure 9. Growth curves for the homogeneous double shear layer with $M_1 = 0$, $M_0^2 = 0, 1, 2$.

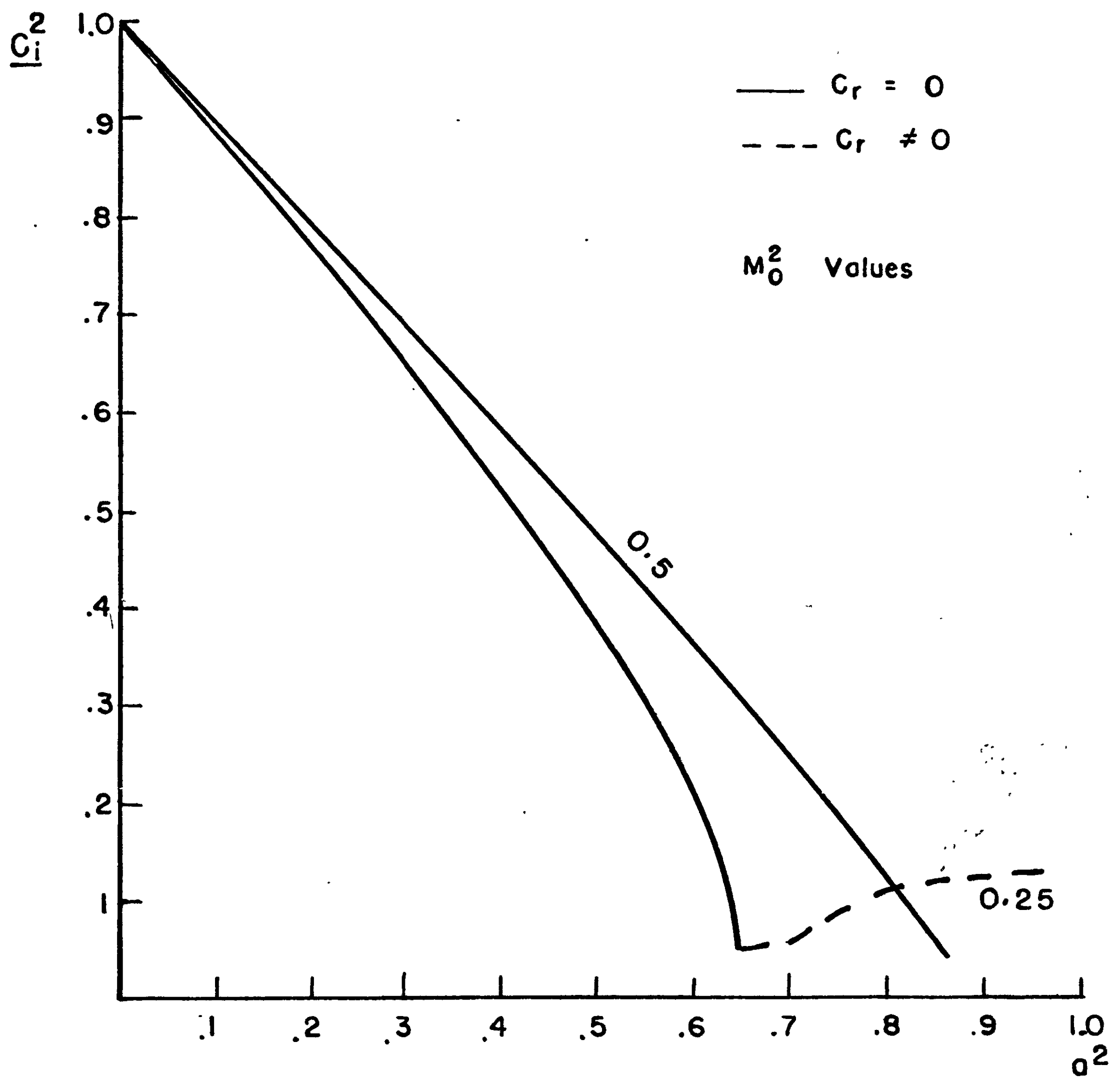

Figure 10. Growth curves for the homogeneous double shear layer with $M_1 = 0$, $M_0^2 = 0.5, 0.25$.

disturbances, any value for $M_0^2 \lesssim 5$ can produce an instability greater than for the nonmagnetic case at some wave number. Comparing $M_0^2 = 0$ with $M_0^2 = 1$, we see that for all $a^2 \lesssim .718$ $(k \lesssim .316)$ the magnetic problem is more unstable. The kink present in the curve for $M_0^2 = 0$ is physically meaningful. When $a^2 \cong 1$, we find that

$$c^2 = \frac{i}{2} \quad ; \quad c = \frac{1+i}{2}$$

Substituting this into (3.16b), we find that $A \sim Be^{-2k}$. Therefore, by (3.15), any solution of magnitude Be^{2k} at $z = 1$ has magnitude B at $z = -1$. Since k is large, any solution drops off rapidly from either interface. We essentially have, as Howard mentioned, two separate instabilities, one at each interface, travelling at the average velocity around that interface. For the magnetic problem, the situation is different. Taking the wave speed for the case $M_0^2 = 1$, we have

$$c^2 = -e^{-4k} \Rightarrow c = ie^{-2k}$$

and substituting this into (3.16), we obtain $A \sim B$ so that the disturbance maintains its amplitude throughout the intermediate layer. We thus see that there are two influences that the magnetic field exerts on a fluid. A magnetic field in a fluid of infinite conductivity adds a tension to the fluid, so that it becomes more difficult to produce an instability. But it also adds a cohesiveness to the fluid which it may not have possessed before. This cohesiveness may serve either to increase or produce an instabilityby making available an extra energy source which the nonmagnetic fluid may not be able to take advantage of. Thus, the two semi-infinite layers are always tied together in the magnetic problem so that there is more kinetic energy of the basic flow available for transformation to perturbation energy than in the nonmagnetic problem. It must be mentioned that this "cohesiveness" is, in fact, the tension of the magnetic field lines, so that the two influences spring from the same source, but it is clear that the behavior of this one force may manifest itself in a variety of ways.

We now generalize the discussion to include the effects of a background magnetic field, M_1, and stratification G. For this problem, we find that $m<1$. Consider first the case $M_0^2 = 0$. We find that $T<S^2$ for all m and so instability may arise by only two means $(S<0, T<0)$ now so that if $S>0$ instability may occur only when $T<0$. To see for what range of m instability may occur, set

$$(1-a^2)m^2 - 2a^2 m + a^4 < 0$$

or

$$m^2 - 2a^2 m + a^4 = (m-a^2)^2 > a^2 m^2$$

so that instability occurs for

$$m > \frac{a^2}{a+1}$$

and the stability boundary is given by

$$m = \frac{a^2}{a+1}$$

When we include the effects of M_0^2 we find that T may be greater than S^2. In fact, the value of m for which $S^2 = T$ is found by solving the equation

$$a^2\left[m^2 + M_0^4 - 2M_0^2 m\right] - 4M_0^2 m = 0$$

so that

$$m = M_0^2 \, \frac{(2-a^2) \pm 2\sqrt{1-a^2}}{a^2} \tag{3.20}$$

For all m with values between these two roots, we have instability. At $M_0^2 = 0$ this range of m degenerates to zero so that, as has been mentioned above, no instability may occur in this way.

We are assured that instability will occur for $S < 0$, or for

$$m > a^2 + M_0^2$$

Finally, when $T < 0$, we also have instability. This mode of instability may occur only for $M_0^2 < 1/2$, beyond that point, instability occurs through $C_r = 0$.

By inspecting (3.17), we see that instability may occur for any m arbitrarily close to zero for the proper M_0^2. The range of M_0^2 which will produce instability is given by

$$\frac{m(2-a^2)-2m\sqrt{1-a^2}}{a^2} < M_0^2 < \frac{m(2-a^2)+2m\sqrt{1-a^2}}{a^2}$$

This range shrinks to zero for nonzero a^2 as m approaches zero so that at $m=0$ we no longer have instability.

A comparison of the marginal stability curves for various M_0^2 values is given in Figure 11. The apparently unreasonable result that as M_0^2 decreases the bottom line departs further from the line $M_0^2 = 0$ is resolved by noticing that in the limit $M_0^2=0$ the curves designated with the asterik ($*$) coincide with the $M_0^2=0$ marginal stability curve. We see that the presence of a magnetic field in the central layer may destabilize a stable configuration for the nonmagnetic problem at virtually any wave number and any value of G/K up to one.

That the instability is manifest with $C_r = 0$ might be anticipated by looking at Howard's semicircle theorem. As the magnetic field increases, if we still have an instability, then the range of possible C_r becomes restricted to values closer to the average velocity. The fact that we may expect instability at $C_r=0$ might also be inferred from the coherency which the magnetic field imparts to the fluid so that the waving at either interface is closely tied up with the waving motions at the other interface.

If $M_1^2 = 0$, we can compare the G/K values for various values of M_0^2 . In Figure 12, we plot G vs K and find that for

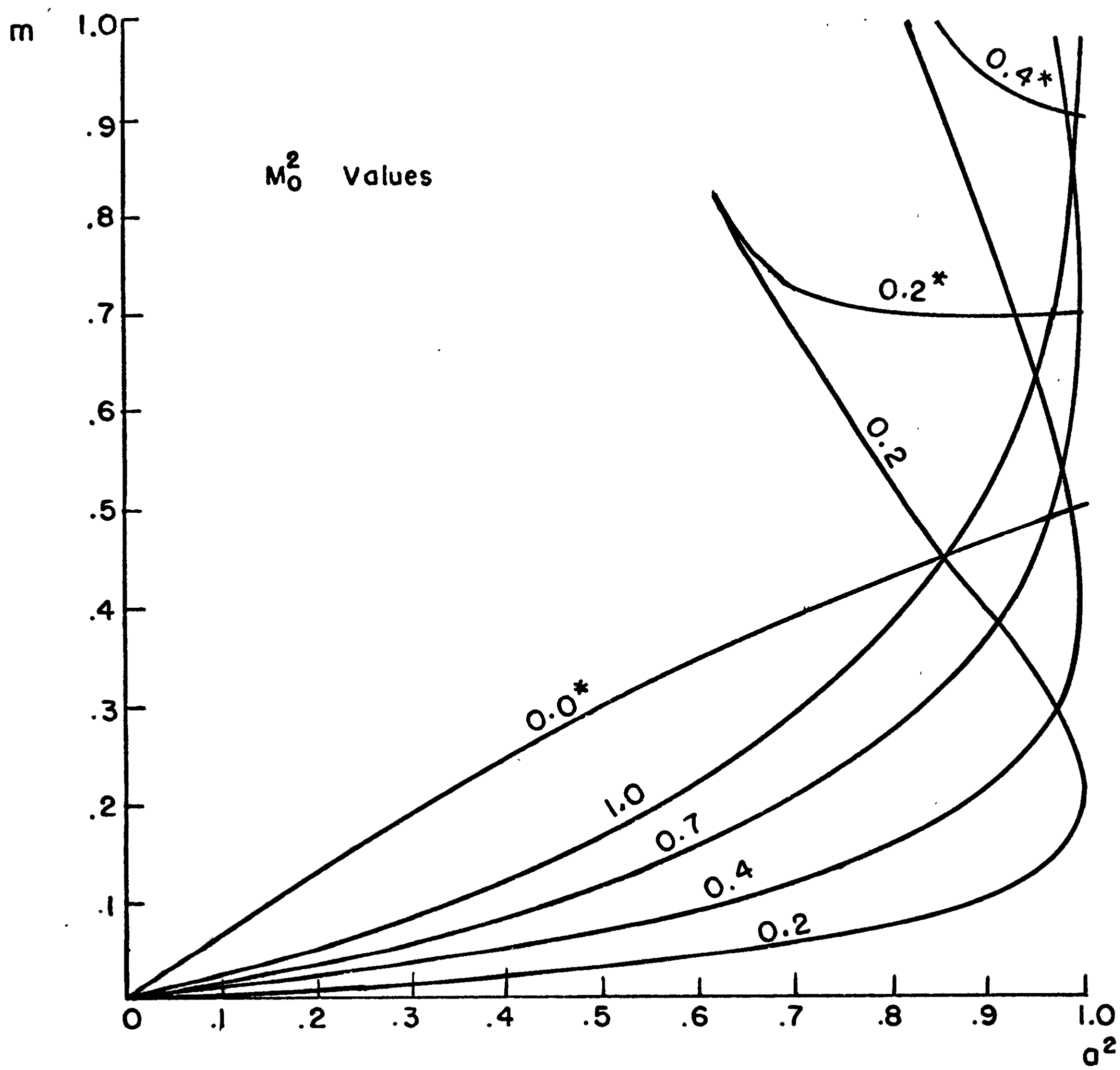

Figure 11. Marginal stability curves for various M_0^2 values of double shear layer.

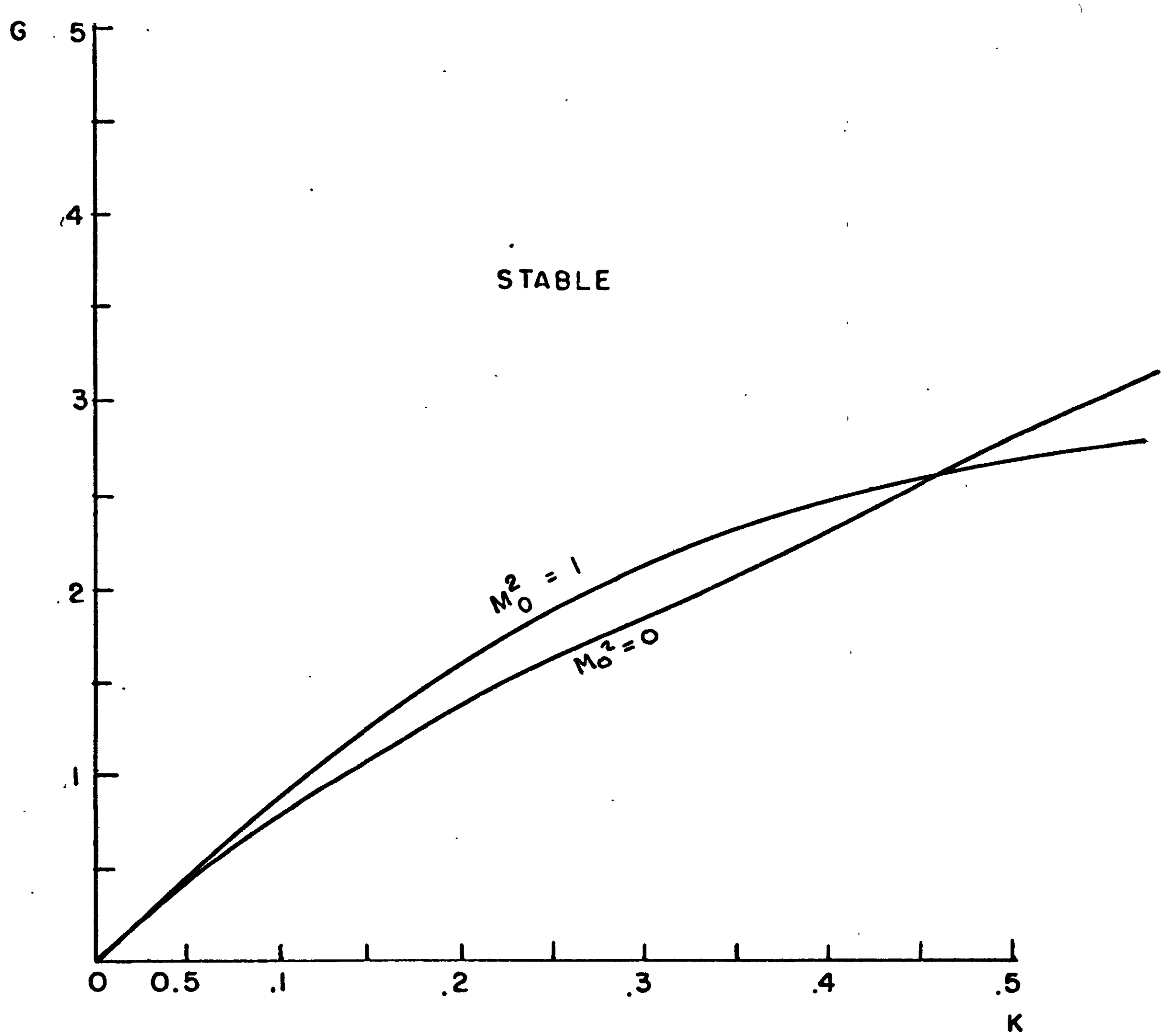

Figure 12. Marginal stability curves for $M_1 = 0$ given in terms of G and k.

$K \lesssim .475$ the problem with $M_0^2 = 1$ is unstable at larger stratification than the nonmagnetic problem.

The application of a constant magnetic field may also destabilize the double shear problem for a considerable range of wave numbers. Consider the situation depicted in Figure 13. We see that the heutral curve for $M_0^2 = M_1^2 = 0.0$ lies within the unstable region for , for $M_0^2 = M_1^2 = 0.1$ for a considerable range of wave number $(15 \lesssim a^2 \lesssim 75)$. By appropriate choice of G/K, we can destabilize all wave numbers up to $a^2 = 1$ $(K = \infty)$. As the magnetic field increases, this property is valid for an increasingly restricted range of wave numbers, until at about $M_0^2 = 0.3$, where the destabilizing effect vanishes.

We can see that there is a range of G/K for which the magnetic problem will be stable while the nonmagnetic problem will be unstable. To be able to tell by physical means exactly what combination of magnetic field and stratification have destabilizing effect and which are stabilizing is not a simple matter. Energy considerations lead to no new insights simply because of the artificiality of the model. Energy exchanges are expressed in terms of pressure and the mathematical formulation reveals no new information, because all terms are proportional to c_i and a Reynold's stress gives no contribution.

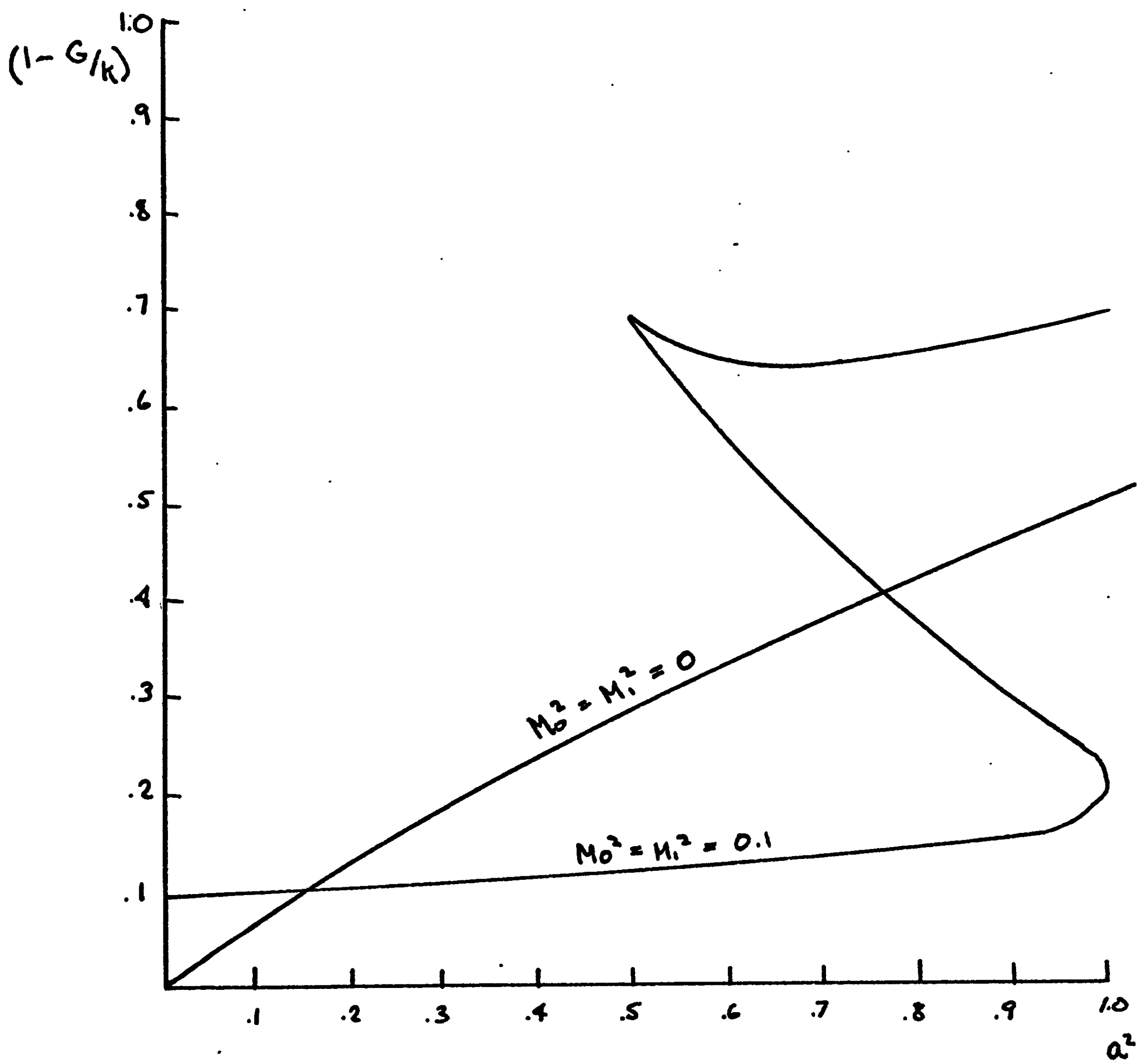

Figure 13. Marginal curve for $M_1^2 = M_0^2 = 0.0$; $M_1^2 = M_0^2 = 0.1$.

IV. Generalized Stability Revealing Techniques

After succesfully presenting complete solutions to two problems, it is disappointing to recall that we generally do not proceed with such impunity in these problems. We are very often satisfied when we possess sufficient information to tell if a problem will exhibit instability. For this reason we do not trouble ourselves with the attempt to obtain a complete solution in every case, but rather, utilize mathematical techniques which reveal information about the time behavior of the problem. In this chapter, two general stabilty criteria are reviewed and applied to our problem.

The normal mode solutions, as we have seen from one example presented in the Introduction, will not always give a full description of the time behavior. One must thenapproach the problem as one of initial values taking into account the nature of the initial perturbations by Laplace transforming (2. 23) with respect to time. The initial value approach yields whatever distinct normal mode solutions exist and includes the effects of the continuum solution. Emphasis will be placed on determining the time dependence of the continuum in the first section of this chapter.

Whenever we know the form of the general solution of (2. 23) we may make use of a stability technique long known to electrical engineers. The Nyquist Stability test was first applied to hydrodynamic stability problems by Rosenbluth and Simon (1964). As we shall see, the requirement that we know the general form of the solution restricts the applicability of this technique to the homogeneous problem at zero wave number, but we may consider arbitrary distributions of velocity and magnetic field.

A. Initial Value Treatment

We now solve the initial value problem for general distributions of velocity, density and magnetic field profiles which are subject to the conditions that the quantity

$$X(z) \equiv (U-c)^2 - M^2$$

has, at worst, zeros of second order in z, but that nowhere does

$$(DU)^2 = (DM)^2$$

We are, thus, restricting the discussion to shearing flows and shall follow much the same procedure as Case (1960).

It is necessary to begin by investigating the singularity properties of (2.24) and giving a solution valid in the region of the singular points. Equation (2.24) may be written as

$$D^2F + \frac{DX}{X} DF - K^2F - \frac{\bar{R}_i}{X} F = 0 \tag{4.1}$$

where X has a n^{th} order zero at z_0. For the homogeneous case, X may have a zero of any order because the equation is always amenable to solutions by the Frobenius technique, but for the stratified problem we are limited to, at most, second order zeros of X because otherwise there is no assurance that the Frobenius technique is valid.

We expand around the singular point, z_0 , and define

$$\eta = z - z_0$$

Then when X has an n^{th} order zero at z_0 , we may write

$$\frac{DX}{X} = \frac{n}{\eta} + \sum_{s=0}^{\infty} \alpha_s \eta^s \qquad \alpha_0 = \frac{1}{n+1} \frac{X_0^{(n+1)}}{X_0^{(n)}}$$

$$\frac{\bar{R}_i}{X} = \frac{n!\,\bar{R}_{i_0}}{\eta^n X_0^{(n)}} + \sum_{-n+1=s}^{\infty} \beta_s \eta^s \tag{4.2}$$

We now proceed to solve (4.1) in the neighborhood of z_0 . Assuming a solution of the form

$$F_1 = \sum_{s=0}^{\infty} A_s \eta^{s+r} \tag{4.3}$$

we consider the case where $n = 1$ first. Substituting (4.2) into (4.1) and then using (4.3) we obtain

$$\begin{aligned} &\sum_{s=0} A_s \eta^{s+r-2}(s+r)(s+r-1) + \sum_0 A_s \eta^{s+r-2}(s+r) + \\ &+ \sum_1 A_{s-1}(s+r-1)\,\eta^{s+r-2}\Big(\sum_{s'=0}^{\infty} \alpha_{s'} \eta^{s'}\Big) - \sum_2 A_{s-2}\eta^{s+r-2} k^2 \\ &+ \sum_1 A_{s-1}\eta^{s+r-2}\, \bar{R}_{i_0}/DX_0 - \sum_2 A_{s-2}\eta^{s+r-2}\Big(\sum_{s'=0} \beta_{s'} \eta^{s'}\Big) = 0 \end{aligned} \tag{4.4}$$

The coefficient of each power of η must equal zero so that by taking the case $s=0$ we obtain

$$A_0(r)(r-1) + A_0(r) = 0$$

and thus $r^2 = 0$. In the case that the two r values are identical we are guaranteed that there are two solutions given by

$$F_1 \sum A_s \eta^s$$

and

$$F_2 = \left(\sum A_s \eta^s\right)\log\eta + \sum\left(\frac{\partial}{\partial r} A_s\right)_{r=0}\eta^s \tag{4.5}$$

Setting $A_0 = 1$, we determine from (4.4) that

$$\partial/\partial r\, A_0 = 0$$

$$A_1 = \frac{-r\alpha_0 + \bar{R}_{i_0}/DX_0}{(1+r)^2} = \frac{\bar{R}_{i_0}}{DX_0}\,; \quad \frac{\partial}{\partial r} A_1 = -\alpha_0 + 2\frac{\bar{R}_{i_0}}{DX_0}$$

and the recursion relation is given by

$$A_s = \frac{k^2 A_{s-2} + A_{s-1}\bar{R}_{i_0}/DX_0 - d_{s-1} + f_{s-2}}{s^2}$$

where

$$d_{s-1} = \sum_{s'+s''=s} (s'+r-1) A_{s'-1}\, \alpha_{s''}$$

$$f_{s-2} \sum_{s'+s''=s} A_{s'-2}\, \beta_{s''}$$

The thing of importance for our purposes is knowledge of the lowest power of η , for that is what is connected ultimately with the time behavior. For the case $n=1$ our solutions are similar to those obtained by Kent (1968). Since the nature of the solutions is the same in our case, we are assured of stability because Kent obtained a stable time dependence even when using a delta function initial amplitude disturbance. Thus, we arrive at the rather surprising result that the density configuration has no effect on the time dependence of the continuum solution so long as X has zeros of, at worst, first order. Instability must then be manifest by distinct modes.

We now consider the case when X has second order zeros. This means that at the point where

$$(U_0-c_0)^2 - M_0^2 = 0$$

we also have

$$DU_0(U_0-c_0) - (DM_0)M_0 = 0$$

When $M_0 \neq 0$, we must have $(DU_0)^2 = (DM_0)^2$ and when $M_0=0$ we must have $(DU_0)^2 \neq (DM_0)^2$ or X_0 will have a zero of at least third order. We restrict this discussion to problems in which $(DU_0)^2 - (DM_0)^2$ is always of one sign for a reason which will be made clear shortly. Upon substituting (4.3) into (4.1) with $n=2$ we obtain

$$\sum_{s=0} A_s \eta^{s+r-2}(s+r)(s+r-1) + \sum_{s=0} A_s \eta^{s+r-2}(s+r)\,2 +$$

$$+ \sum_{1} A_{s-1}\eta^{s+r-2}(s+r-1)\left(\sum_{s'=0} \alpha_{s'}\eta^{s'}\right) - \sum_{2} A_{s-2} k^2 \eta^{s+r-2} -$$

$$- \sum_{0} A_s \eta^{s+r-2} \frac{2\bar{R}_{i_0}}{DX_0} - \sum_{1} A_{s-1}\eta^{s+r-2}\left(\sum_{s'=-1} \beta_{s'}\eta^{s'+1}\right) = 0$$

Equating powers of , we obtain for the case

$$A_0\left[r(r-1) + 2r - \frac{2\bar{R}_{i_0}}{DX_0}\right] = 0$$

Solving for r, we obtain

$$r = \frac{-1 \pm \sqrt{1 + 8\bar{R}_{i_0}/D^2X_0}}{2} = -\frac{1}{2} \pm \nu \qquad (4.6)$$

where

$$\nu = \sqrt{\frac{1}{4} + \frac{\bar{R}_{i_0}}{(DU_0)^2 - (DM_0)^2}}$$

We can see that the two r values will generally not differ by an integer, so that both solutions are of the form (4.3). Now that we have knowledge of the solutions of F (determination of the recursion re relation for the coefficients is not necessary for our purposes), we may proceed to solve the initial value problem.

We return to (2.23) as our governing equation. We take the Laplace transform with respect to time The Laplace transform pair of F is given by

$$F_p = \int_0^\infty e^{-pt} F(t)\, dt$$

$$F = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} e^{pt} F_p(p)\, dp$$

The Laplace transforms of $\partial F/\partial t$ and $\partial^2 F/\partial t^2$ may be redefined in terms of the Laplace transform of F . They are respectively

$$\int_0^\infty \frac{\partial F}{\partial t} e^{-pt} dt = F(t) e^{-pt} \Big|_0^\infty + p \int_0^\infty F(t) e^{-pt} dt$$

$$= F(t=0) + pF_p$$

and

$$\int_0^\infty \frac{\partial^2 F}{\partial t^2} e^{-pt} dt = \frac{\partial}{\partial t} F(t=0) - pF(t=0) + p^2 F_p$$

Using these operations in (2. 23), it becomes

$$D\left\{\left[(U - p/ik)^2 - M^2\right] DF_p\right\} - k^2\left[(U - p/ik)^2 - M^2\right] F_p -$$
$$- \bar{R}_i F_p = \frac{1}{ik}\left[\left(2U + \frac{p}{ik} + \frac{1}{ik}\frac{\partial}{\partial t}\right)\left[D^2 - k^2\right] F + 2(DU) DF\right]_{t=0} \tag{4. 7}$$

$F(t=0)$ is the initial amplitude perturbation and may be any function of z subject to the boundary conditions

$$(U-c)F(z_0) = \left(U - \frac{p}{ik}\right)F_p(z_1) = \left(U - \frac{p}{ik}\right)F_p(z_2) = 0$$

The right hand side of (4. 7) contains all the inhomogeneous terms. We shall solve (4. 7) by the method of Green's Functions and obtain the time dependence by inverting the Laplace Transform.

The solution to the homogeneous problem in the vicinity of are given by

$$F_1 = \eta^{\nu - 1/2}\varphi_1$$

and

$$F_2 = \eta^{-(\nu + 1/2)}\varphi_2$$

where φ_1 and φ_2 are analytic in the vicinity of z_0 . The amplitude, F , is given by

$$F = AF_1 + BF_2$$

and using our boundary conditions (dropping the subscript p where $c = P/ik$) , we obtain

$$AF_1(\eta_1) + BF_2(\eta_1) = 0$$

$$AF_1(\eta_2) + BF_2(\eta_2) = 0$$

In searching for a Green's Function we use two solutions, ψ_1 and ψ_2 such that

$$\psi_1(\eta_1) = 0 = \psi_2(\eta_2)$$

The ψ's are then defined by

$$\psi_1 = F_1(\eta) F_2(\eta_2) - F_2(\eta) F_1(\eta_2)$$

$$\psi_2 = F_1(\eta) F_2(\eta_1) - F_2(\eta) F_1(\eta_1)$$

The Green's Function is then given by the formula

$$G(\eta, \bar{\eta}; p) = -\frac{1}{\Delta}\left[\psi_1(\eta_>; p)\,\psi_2(\eta_<; p)\right]$$

where:

$$\eta_{\gtrless} = \begin{matrix} \eta ; & \eta > \bar{\eta} ; & \bar{\eta}, & \bar{\eta} > \eta \\ \eta , & \bar{\eta} > \eta & \bar{\eta}, & \eta > \bar{\eta} \end{matrix}$$

and

$$\Delta = F_1(\eta_1) F_2(\eta_2) - F_1(\eta_2) F_2(\eta_1)$$

Values for which $\Delta = 0$ give the distinct mode solutions. We neglect this part of the solution and consider only the continuum. The colution to (4. 7) is given by

$$F_p = \int_{\eta_1}^{\eta_1} G(\eta, \bar{\eta}; p)\left[r.h.s.\right] d\bar{\eta} \tag{4. 8}$$

where $[r.h.s.]$ is the right hand side of (4. 7).

We first invert the Laplace transform. The major contribution will come in the vicinity of z_0 . A small change in c corresponds to a linear change in η so long as $M_0 = 0$ at the point where χ has a second order zero. If $M_0 \neq 0$ at a second order zero of χ , then the change in z corresponding to a change in c is given by

$$\Delta c \sim \eta^2$$

Since we want to expand in terms of $(z-c)$ in the neighborhood of the second order zero of χ, we will consider the case when $M_D^2=0$ at z_2. Since a change in c by some real quantity equals a change in p by an imaginary quantity, such a change does not affect the value of the Laplace transform. We can thus write

$$\bar{\eta} = \bar{z} - c$$

Since the largest contribution comes when

$$\bar{z} \cong c$$

we also write

$$\eta = z - \bar{z}$$

The Greeh's function then takes the explicit form

$$G(z,\bar{z};p) = -\frac{1}{\Delta}\left[F_1(\bar{z}-z)F_2(z_2) - F_2(\bar{z}-z)F_1(z_2)\right] \cdot$$

$$\cdot\left[F_1(\bar{z}-c)F_2(z_1) - F_2(\bar{z}-c)F_1(z_1)\right] \qquad \bar{z} > z$$

$$G(z,\bar{z};p) = -\frac{1}{\Delta}\left[F_1(\bar{z}-c)F_2(z_2) - F_2(\bar{z}-c)F_1(z_2)\right] \cdot$$

$$\cdot\left[F_1(\bar{z}-z)F_2(z_1) - F_2(\bar{z}-z)F_1(z_1)\right] \qquad \bar{z} < z$$

The essential behavior of G is then given by

$$G_{\bar{z}} \sim \left[\left(\frac{ip}{k} + \bar{z} \right)^{\nu - 1/2} - \left(\frac{ip}{k} + \bar{z} \right)^{-\nu - 1/2} \right] \cdot \tag{4.9}$$

$$\cdot \left[(z - \bar{z})^{\nu - 1/2} - (z - \bar{z})^{-(\nu + 1/2)} \right] \Phi$$

where Φ is a regular function for each region of G. We now invert the Laplace transform.

$$\int e^{pt} \left(\frac{ip}{k} + \bar{z} \right)^{-\lambda} dp = \left(\frac{i}{k} \right)^{\lambda} \int (p - ik\bar{z})^{-\lambda} e^{pt} dp$$

Setting

$$p - ik\bar{z} = p'$$

we have

$$\left(\frac{i}{k} \right)^{-\lambda} e^{ik\bar{z}t} \int \frac{e^{pt}}{(p')^{\lambda}} dp \sim e^{ik\bar{z}t} \, t^{\lambda - 1}$$

Therefore, from (4.9) we see that the dominant behavior is given by the term which has

$$(\bar{z} - c)^{-(\nu + 1/2)}$$

We now integrate over $\bar{z}$. The integral is approximated for large values of time, since the mode which eventually dominates emerges mathematically only at large time. This does not upset our assumption of linearity since we can choose the amplitude of the initial disturbance to be as small as is necessary. The general form of the equation (4. 8) then becomes

$$- \int_{z_1}^{z} \frac{1}{\Delta}\left[e^{ik\bar{z}t}\, t^{\nu-1/2}\right]\left[(z-\bar{z})^{-(\nu+1/2)} + (z-\bar{z})^{\nu-1/2}\right]\bar{\Phi}\,[r.h.s]\,d\bar{z}$$

$$- \int_{z}^{z_2} \frac{1}{\Delta}\left[e^{ik\bar{z}t}\, t^{\nu-1/2}\right]\left[(z-\bar{z})^{-(\nu+1/2)} + (z-\bar{z})^{\nu-1/2}\right]\bar{\Phi}\,[r.h.s.]\,d\bar{z} \qquad (4.10)$$

For large time integrals of this form may be represented approximately as

$$\int_{z} e^{ik\bar{z}t}\,(z-\bar{z})^{\lambda-1}\,\bar{\Phi}[r.h.s] = B_N(t) + O(t^{-N})$$

where

$$B_N(t) = \sum_{0}^{N-1} \frac{\Gamma(n+\lambda)}{n!}\, e^{\pi i(n-\lambda)/2}\, \frac{D^n}{Dz^n}\left[\bar{\Phi}\,[r.h.s]\, t^{-n-\lambda}\, e^{ikzt}\right]$$

$$\text{for} \qquad 0 \leq \lambda \leq 1$$

The term giving the dominant contribution to the time behavior of (4. 10) has the factor

$$(z-\bar{z})^{-(\nu+1/2)}$$

the dominant terms of (4. 10) thus give a time dependence of

$$F \propto t^{2\nu-1} = t^{2\left[\sqrt{1/4 - \bar{R}_{i_0}/[(DU_0)^2-(DM_0)^2]} \; - 1/2\right]} \qquad (4.11)$$

So long as $(DU_0)^2 - (DM_0)^2 > 0$ this answer seems reasonable. It is analogous to the result for nonmagnetic flows, to which it reduces as the magnetic field approaches zero. The continuum is stable for all values of $\bar{R}i_0 \leq 0$ with stability increasing as the stratification becomes more gravitationally stable. (4.11) does not agree with the result Kent derived, simply because Kent used a delta function disturbance for the amplitude, a disturbance which is physically questionable.

Equation (4.11) is not universally applicable. When $(DU_0)^2 > (DM_0)^2$ the magnetic energy exceeds the kinetic energy and it would thus seem that instability should be ruled out for the continuum solution so long as $\bar{R}i_0 \leq 0$. (4.11) indicates instability increasing as the stratification becomes more stable. The fact that our result is undesirable in this case does not imply that the technique is illegitimate. There is one loophole. The Green's Function technique guarantees a unique solution only whenever X is always of one sign. For our problem, X necessarily has a zero so that our result is not necessarily unique and may be rejected if it contradicts common sense. Whereas it is possible to accept a result which states that the stratification has no effect on a certain part of the solution, as when X has at most a first order zero, it is difficult to accept a result which indicates that the effect of the density is exactly opposite that which is physically reasonable for any part of the solution.

The Nyquist Stability Criterion

The guiding principle of the Nyquist Stability Criterion is so simple that I regret not having thought of it independently. . Consider an analytic function,* $\phi(c)$, of the complex wave speed, c. For every curve that c traces out in the complex plane, there is a corresponding curve in the complex plane that ϕ traces out. For every value of c interior to the first curve, there will be a value of ϕ in terior to the ϕ curve (Figure 14). The interior must, of course, be consistently defined with respect to the orientation of the curve.

We now consider our example. For a homogeneous fluid at zero wave number, (2. 24) reduces to

$$\frac{d}{dz}\left[\left\{(U-c)^2-M^2\right\}\frac{dF}{dz}\right]=0 \tag{4. 12}$$

The general solution to this equation is given by

$$F(z)=\int_{z_1}^{z}\frac{dz_0}{(U-c)^2-M^2} \tag{4. 13}$$

and (4. 13) satisfies the boundary condition at z_1 and z_2, namely

$$F(z_2)=\int_{z_1}^{z_2}\frac{dz}{(U-c)^2-M^2}$$

Clearly, this is true only for special values of c. We consider the function

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* Actually, $\phi(c)$ need only be single valued.

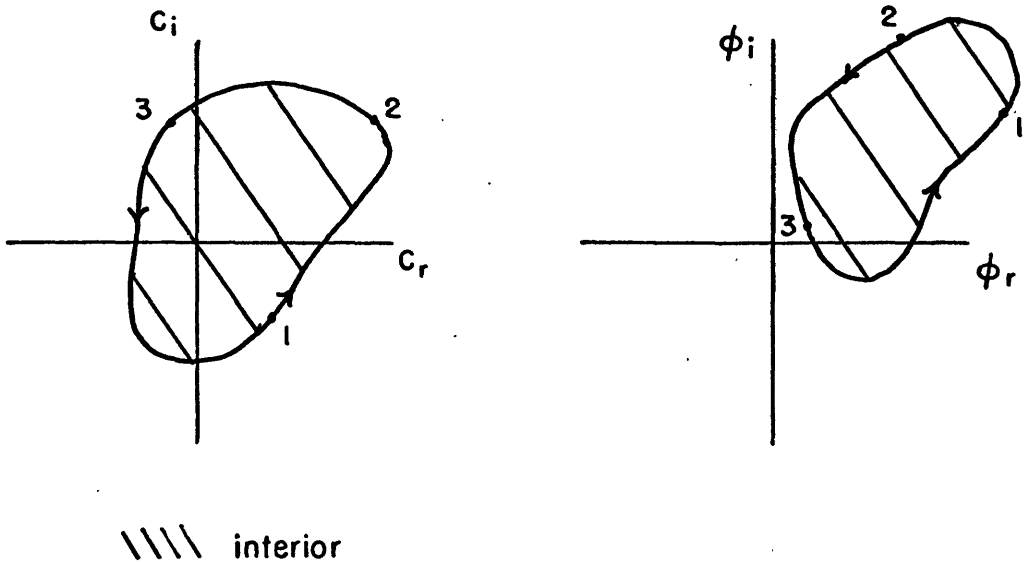

Figure 14. Mapping from c plane to φ plane.

$$\varphi(c) = \int_{z_1}^{z_2} \frac{dz}{(U-c)^2 - M^2} \tag{4.14}$$

When c is complex $\varphi(c)$ is integrable and single valued in c so that the basic principle of the Nyquist Stability Criterion applies. Consider the curve in the c plane which includes the entire upper half plane $(c_i > 0)$. The points where χ is zero on the real axis are excluded so that when we compute φ along the real axis we are taking the principle value of the integral in (4.14). If the corresponding $\varphi(c)$ curve then circles the origin, there is a root for which $\varphi(c) = 0$ or $F(z_i) = 0$ for some value of $c_i > 0$. For the fluid problem it is not necessary that the orientation of the curves in the c and φ planes be the same. The extra degree of freedom is due to the fact that if the φ curve circles the origin in the opposite direction, then there is a root with $c_i < 0$ and we are guaranteed by one of the fundamental properties of (4.12) that there is also a solution with $c_i > 0$. Thus, all that is needed for instability is that the φ curves encircle the origin a nonzero net number of times in either direction. Thus, without locating the eigenvalue, it is possible to determine when there is an instability. This may be done for arbitrary distributions of velocity and magnetic fields.

Kent (1968) has used this technique to show that a constant magnetic field may destabilize an otherwise stable flow. He chose a velocity profile which does not satisfy Fjortoft's necessary condition and produced an instability. Stern (1963) found the first instance in which a magnetic field destabilized a perfect fluid. Using Couette flow with a piecewise linear magnetic field (Figure 15) he used expansion techniques to establish the existence of a solution with a positive c_i. Stern's problem will now be solved by using the Nyquist technique. More complicated profiles may be handled by computer.

The expression

$$\frac{d}{dz}\log\frac{[(U-c)+M]}{[(U-c)-M]} = +\frac{-2(DU)M+2(DM)(U-c)}{[(U-c)^2-M^2]}$$

so that (4.14) may be rewritten as

$$\phi(c) = \log\frac{[(U-c)+M]}{[(U-c)-M]}\cdot\left\{\frac{1}{-2M(DU)+2(U-c)(DM)}\right\}\Bigg|_{z_1}^{z_2} \tag{4.15}$$
$$-\int_{z_1}^{z_2}\log\frac{[(U-c)+M]}{[(U-c)-M]}\cdot\left\{\frac{-(D^2U)M+(D^2M)(U-c)}{-(DU)M+(DM)(U-c)}\right\}dz$$

and this form will prove convenient for computing the real part of $\phi(c)$ for the examples. The polar plot for c is given in Figure 16 and the ϕ curve corresponding to the c values from points $8\to1\to2\to3\to4$ is independent of the specific details of the velocity and magnetic profiles. For $c \sim Re^{i\theta}_{R\to\infty}$, ϕ is given by

$$\phi \sim |z_2-z_1|R^{-2}e^{-2i\theta}$$

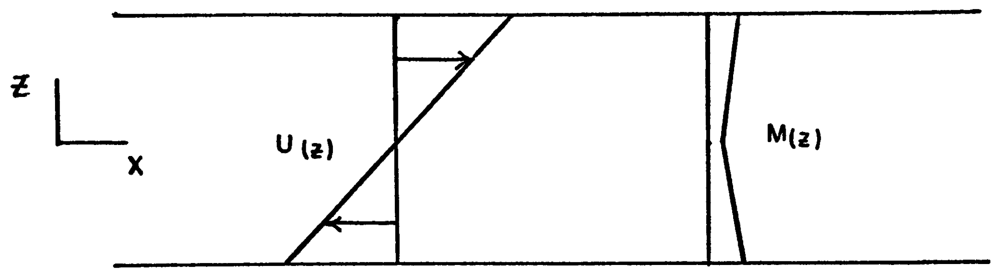

Figure 15. Stern's example.

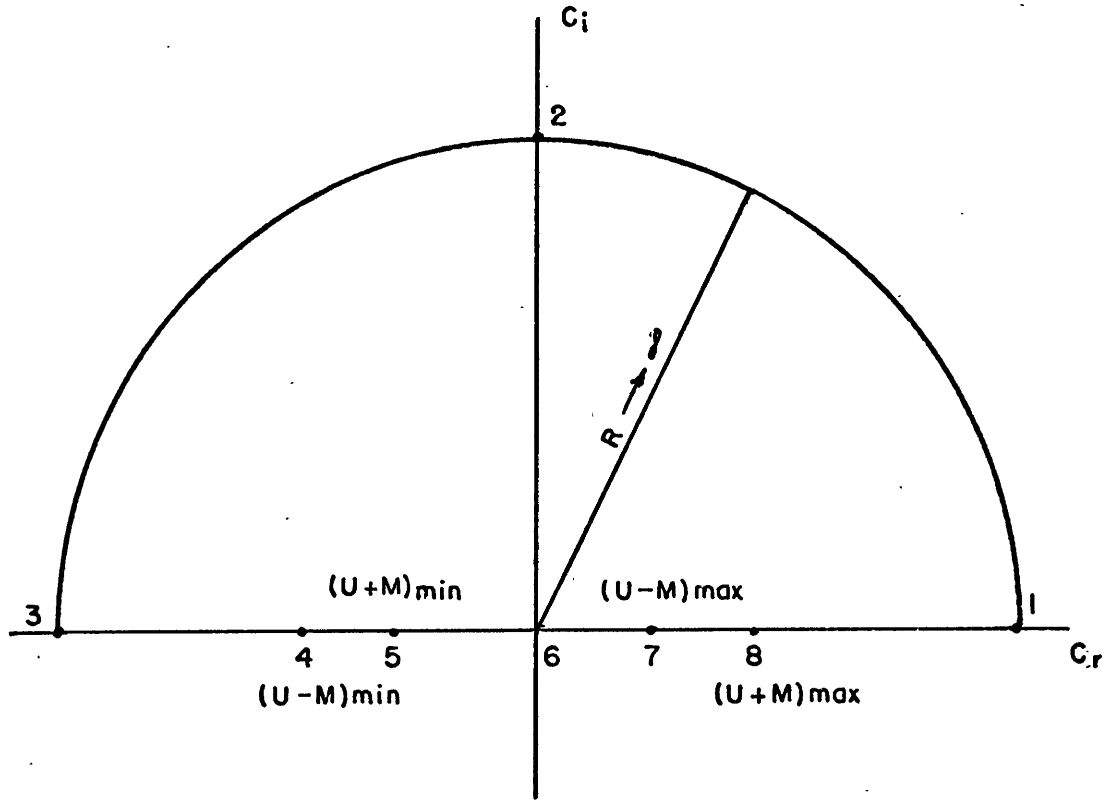

Figure 16. Nyquist diagram in c plane.

so that over $1 \to 2 \to 3$ the origin is circled once in the clockwise direction.

When c lies between 3 and 4 or 8 and 1, the integral for φ has no singularities and is a positive real quantity in both regions. This is all we can determine in general, and for the behavior of φ when

$$(U-M)_{min} \le c \le (U+M)_{max}$$

we must look at the individual profiles.

Consider first the magnetic field and velocity profiles given by

$$U = z \qquad -1 \le z \le 1$$

$$M = \begin{cases} M_0 - az & -1 \le z \le 0 \qquad a > 0 \\ M_0 + bz & 0 \le z \le 1 \qquad b > 0 \end{cases}$$

where

$$(M_0 + a) < 1/2 \; ; \qquad M_0 >> a$$

$$(M_0 + b) < 1/2 \; ; \qquad M_0 >> b$$

Substituting this into (4.15), we obtain

$$- Re.\,\varphi = \log\left|\frac{1+M_0+b-c}{1-M_0-b-c}\right| \cdot \frac{1}{2(M_0+bc)} + \log\left|\frac{-1+M_0+a-c}{-1-M_0-a-c}\right| \cdot$$

$$\cdot \frac{1}{2(ac-M_0)} + \log\left|\frac{M_0-c}{-M_0-c}\right| \frac{(b+a)c}{M_0} = (1) + (2) + (3) \qquad (4.16)$$

For the purposes of the technique, the only concern is to have knowledge of the sign of the real and imaginary parts of φ for each value of c. Each of the three terms will dominate in some part of the region

$$-1-M_0-a \leq c \leq 1+M_0+b$$

and we now consider the following subregions

$$-1-M_0-a \leq c \leq -1+M_0+a \qquad (i)$$

$$-1+M_0+a \leq c \leq -M_0 \qquad (ii)$$

$$-M_0 \leq c \leq M_0 \qquad (iii)$$

$$M_0 \leq c \leq 1-M_0-b \qquad (iv)$$

$$1-M_0-b \leq c \leq 1+M_0+b \qquad (v)$$

When

$$c = -1-M_0-a+\epsilon$$

it is clear that the term giving the largest contribution is (2) which is a large negative number. When

$$c = -1+M_0+a-\epsilon$$

term (2) again gives the largest contribution but here it is positive. Thus in subregion (i), $Re\,\varphi$ changes from positive to negative. This procedure is followed throughout the five subregions and the manner in which the real part of φ behaves is depicted in Figure 17.

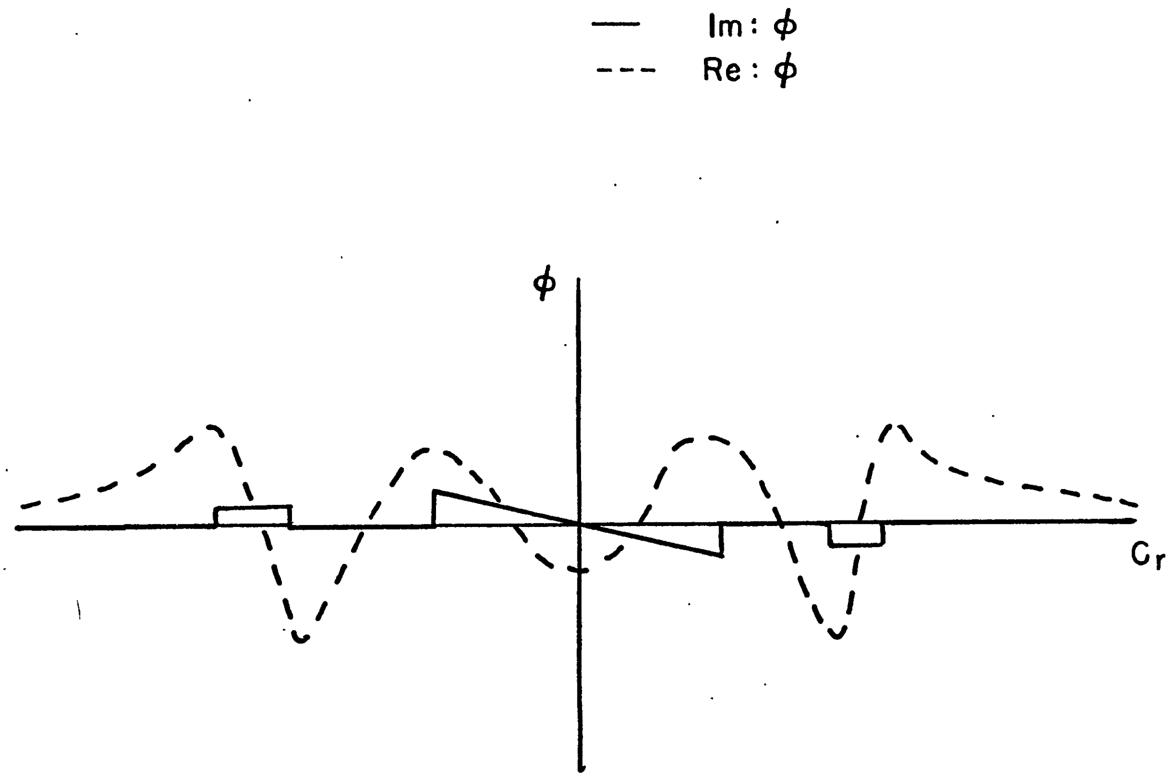

Figure 17. Trace of $\varphi(c)$ for $M < \frac{1}{2} U_{max}$.

The imaginary part of φ is determined from

$$\text{Im}:\varphi = P\int_{-1}^{0}\frac{dz}{\left[(z-c)^2-(M_0-az)^2\right]} + P\int_{0}^{1}\frac{dz}{\left[(z-c)^2-(M_0+bz)^2\right]} \tag{4.17}$$

The only contributions to this integral come from the places where there is a zero in the denominator of the integrand and for this profile the zeros are simple and of first order. Analyzing the zeros in each subregion, we find one zero for $[(U-c)-M]$ in (i) which yields

$$P\int_{-1}^{0}\frac{1}{\left[(z-c)-(M_0-az)\right]}\frac{1}{\left[(z-c)+(M_0-az)\right]}\,dz$$

$$= \pi i\,\frac{1}{1+a}\left\{\frac{1}{-2(az-M_0)}\right.$$

and thus the imaginary part of φ is positive. In (ii), there are two zeros, both of which occur for negative values of z. The contributions cancel so that in this subregion the imaginary part of φ is zero. The behavior in each region is depicted in Figure 17.

When the Nyquist plot is traced out (Figure 18) it is seen that there is an unstable root since the origin is circled a net total of one time. If we had chosen $M_0 \geq 1/2$, our subregions would have been

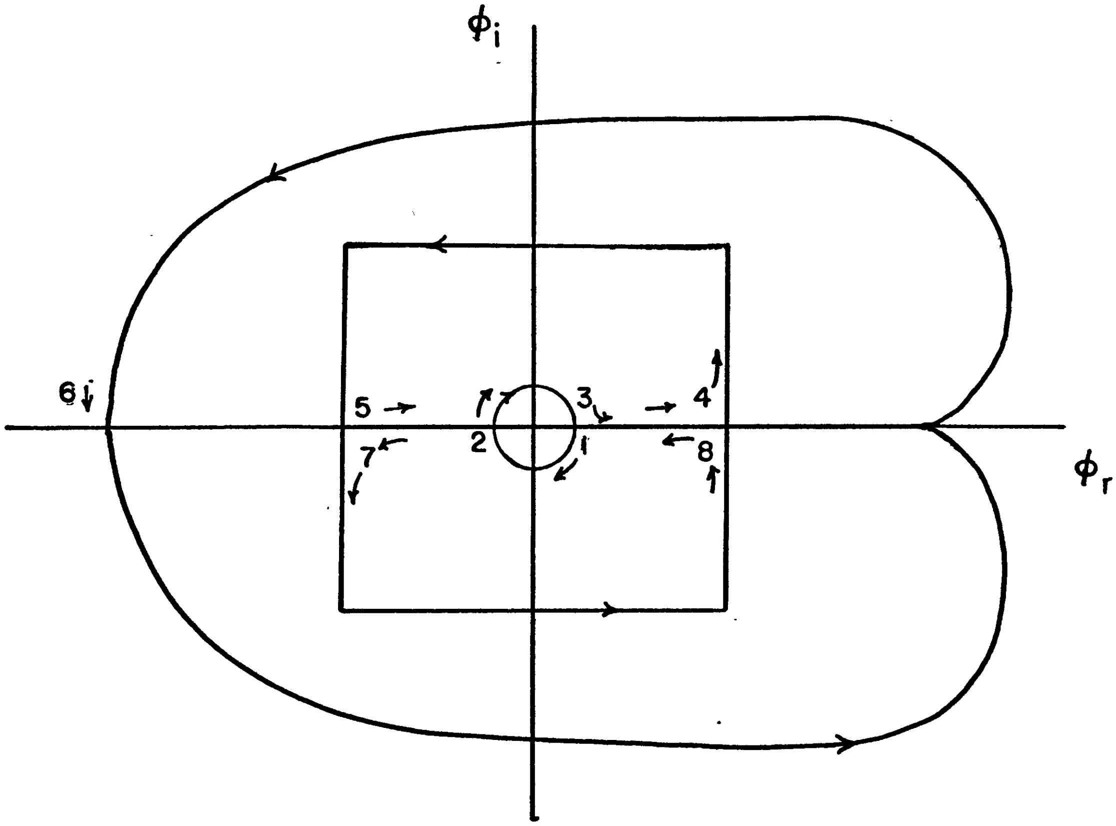

Figure 18. Nyquist diagram in φ plane.

$$-1 - M_0 - a \leq c \leq -M_0$$
$$-M_0 \leq c \leq -1 + M_0 + a$$
$$-1 + M_0 + a \leq c \leq 1 - M_0 - b$$
$$1 - M_0 - b \leq c \leq M_0$$
$$M_0 \leq c \leq 1 + M_0 + b$$

and the behavior of φ is shown in Figure 19 and since there are no net circlings of the origin, no instability arises. Here, the coherency effect of the magnetic field is dominated by either the stabilizing tension producing effect or boundary effects. Taking the case of a constant magnetic field impressed on Couette flow leads to stability for all values of the magnetic field.

The usefulness of this technique is not limited to the case of zero wave number. When $c_i \neq 0$, c is continuously dependent on k so that if $c_i \neq 0$ for $k = 0$, $c_i \neq 0$ for some region of $k > 0$ and thus instability is manifested for sufficiently long waves.

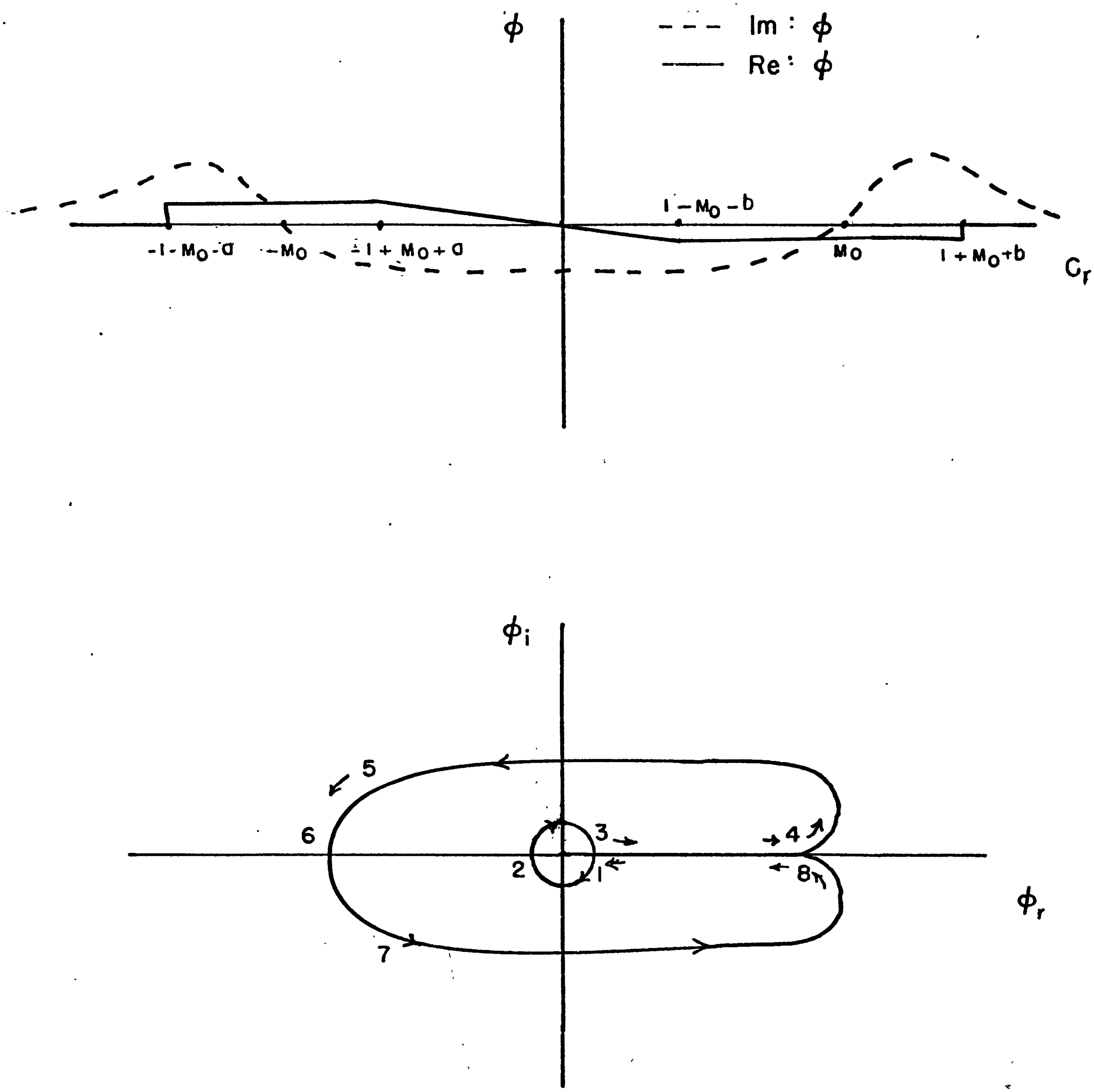

Figure 19. Trace of $\phi(c)$ and Nyquist diagram for ϕ

$M_{max} > \frac{1}{2} U_{max}$.

V. Long Wave Theory for Unbounded Homogeneous Flow

Lateral boundaries place a constraint on the motions in a fluid and therfore generally exert a stabilizing influence for plane parallel flow. When the boundaries are sufficiently close, it has been shown by Tollmien (see Drazin and Howard 1966) for certain nonmagnetic flows that all instability may be suppressed. On the other hand, boundaries can occasionally destabilize a fluid as can be seen by considering Poiseuille flow. In this case, the boundaries communicate the instability to the interior of the fluid through viscous effects. It is therefore of interest to investigate those stability characteristics which are due entirely to the fluid acting on itself and divorced from any boundary effect. This omission does not restrict the applicability of the material severely for the effect of boundaries is small in many physical situations where there are relatively narrow jets or shear layers present.

When considering ideal models which posses discontinuous profiles, we find that instability occurs for all wave lengths and, in fact, the greatest instability often occurs for the shortest waves. If we include surface tension or viscosity, the short wave length end of the spectrum is immediately stabilized. Models which posses continuous velocity profiles are always stable to short wave length disturbances for a homogeneous nonmagnetic fluid, as has been shown by Drazin and Howard (1962). Unfortunately, a similar result has not been demonstrated for the magnetic case. Eventually, it is hoped, a proof will be developed which will show exactly how a magnetic profile changes the lowest unstable wave length corresponding to a given velocity profile.

When the zones in which shear is present is relatively narrow in comparison to the lateral extent of the fluid, the differences between different velocity profiles is not so important in the long wave length limit because a shearing fluid may be approximated by a Kelvin-Helmholtz fluid and (if momentum flux is conserved) a jet may be approximated by a delta function flow.

It is of interest, then, to consider long wave length disturbances in an unbounded fluid. There are two procedures by which this may be done. One can expand in powers of the wave number, κ, or use an integral equation approach. Both give equivalent results but the integral equation approach leads to a somewhat simpler means for arriving at an explicit eigenvalue relation.

The work done in this and the following chapters represents an extension of the treatments of Drazin and Howard (1962, 1961) respectively to a magnetic fluid. The approach is more rigorous for the homogeneous case which is considered in this chapter. In Chapter 6, we will include the effects of density stratification.

A. Series Approach

Heisenberg (1924) was the first to use a wave number expansion scheme to attempt the solution of a hydrodynamic stability problem.

Writing the solution as

$$F_j = F_{0j} + k^2 F_{1j} + k^4 F_{2j} + \cdots$$

we can determine each of the terms F_{ij} by substitution into the governing equation, (1.10). For example, to zeroth order in k^2 (1.10) becomes

$$D\left[(U-c)^2 D F_{0j}\right] = 0$$

which, upon integrating twice yields two independent solutions

$$F_{01} = \int_{z_1}^{z_2} \frac{1}{(U-c)^2}\, dz \quad ; \quad F_{02} = 1$$

The higher order terms are then evaluated rather easily and take the form

$$F_{ij} = \int_{z_1}^{z_2} \frac{1}{(U-c)^2}\, dz \int (U-c)^2 F_{i-1\,j}\, dz \qquad i = 1,2,3\cdots$$

When the fluid has infinite lateral extent, this expansion scheme does not converge uniformly because the limiting F is dependent on the order in which we take the limits $z \to \infty$, $k \to 0$. Restricting our velocity profile to be of the form such that $U \to \text{const}$ and

$$\int_{-\infty}^{\infty} \left| \frac{D^2 U}{U} \right| dz \qquad\qquad \text{converges}$$

in the limit $k \to 0$, $F \to 0$ for all z values larger than the z value at which $U = c$. On the other hand, when $z \to \infty$, $F \sim e^{-kz}$ so that the results are not in accord and besides,

e^{-kz} cannot be expanded in a power series in k^2 when z is sufficiently large. Some other expansion scheme is necessary.

In the magnetic case, because of the formal similarity of (1.21) and (1.10), the situation is virtually the same. Equation (1.21) may, by using the transformation

$$Q = X^{1/2} F$$

be written as

$$D^2 Q - \left(k^2 + \frac{2X D^2 X - (DX)^2}{X^2} \right) Q = 0 \qquad (5.1)$$

Whenever the integral

$$\int_{-\infty}^{\infty} \left| \frac{2X D^2 X - (DX)^2}{X^2} \right| dz$$

converges, and $X \longrightarrow \text{const}$, we are guaranteed that we can write (4.1) as $|z| \rightarrow \infty$ as

$$D^2 Q - k^2 Q = 0$$

so that

$$Q \sim e^{-k|z|}$$

and, making use of the fact that $X \longrightarrow \text{const}$, this enables us to write

$$F_{\pm} \sim e^{\mp kz} \qquad \text{as} \qquad z \rightarrow \pm\infty$$

also.

Making use of the asymptotic form of F, it is now possible to obtain a series expansion. The factor $e^{-k|z|}$ has caused the difficulty in the first place so here we write

$$F_+ = e^{-kz}\,\Theta(z)$$
$$F_- = e^{kz}\,\Psi(z) \tag{5.2}$$

and expand the variables Ψ and Θ in terms of k rather than doing so for F.. Thus

$$\Theta = \sum_{n=0}^{\infty} k^n \Theta_n$$
$$\Psi = \sum_{n=0}^{\infty} k^n \Psi_n \tag{5.3}$$

This is the series expansion we want. The eigenvalue relation is obtained by matching these two solutions at the origin. For boundary conditions, therefore, we require

$$F_+(0) = \text{const} \cdot F_-(0)$$
$$DF_+(0) = \text{const} \cdot DF_-(0)$$

and this may be written without the constant as

$$F_+ DF_- - F_- DF_+ \Big|_{z=0} = 0 \tag{5.4}$$

Substituting the first equation (5.2) into (1.21),

$$D[\chi DF] - k^2 \chi F = 0$$

We obtain

$$D[e^{-kz}\chi(-k\Theta + D\Theta)] - k^2 \chi e^{-kz}\Theta = 0$$

or

$$-2kXD\Theta + DX(D\Theta - k\Theta) + XD^2\Theta = 0 \tag{5.5}$$

and by substituting in the first of (5. 3) and equating powers of k ,

(5. 5) then gives us the recursion relation which appears as

$$D[XD\Theta_0] = 0 \tag{5.6}$$

$$D[XD\Theta_{n+1}] = D[X\Theta_n] + XD\Theta_n \quad ; n = 0, 1, 2 \cdots \tag{5.7}$$

Solving for Θ_0 by integrating twice yields

$$\Theta_0 = \int_\infty^z \frac{c_1}{X} dz_1 + c_2$$

The constants must be chosen so that $\Theta_0(0)$ is finite and since we normalize to $\Theta(\infty) = 1$, we must have $\Theta_0(\infty) = 1$. Therefore, $c_1 = 0$ and we have

$$\Theta_0 = 1 \tag{5.8a}$$

Similarly for Θ_1 , we have

$$D\Theta_1 = \Theta_0 + \frac{c_1}{X}$$

and

$$\Theta_1 = \int_\infty^z \left(1 + \frac{c_1}{X}\right) dz_1 + c_2$$

$\Theta_n(\infty)$, $n \neq 0$, is equal to zero and again requiring that

$\Theta_1(0)$ is finite requires that

$$\Theta_1 = \int_\infty^z \left(1 - \frac{X_\infty}{X}\right) dz_1 \tag{5.8b}$$

where the subscript on X denoted the z value at which it is evaluated. The next term is given by

$$D\Theta_2 = \int_{\infty}^{z}\left(1 - \frac{X_\infty}{X}\right)dz_1 + \frac{1}{X}\int_{\infty}^{z}(X - X_\infty)\,dz_1$$

so that

$$\Theta_2 = \int_{\infty}^{z}\left[\int_{\infty}^{z_1}\left(1 - \frac{X_\infty}{X}\right)dz_2 + \frac{1}{X}\int_{\infty}^{z_1}(X - X_\infty)\,dz_2\right]dz_1$$

In the same manner we can evaluate the Ψ_n . Their recursion relation is given by

$$D[XD\Psi_0] = 0$$

$$D[XD\Psi_{n+1}] = -D[X\Psi_n] - XD\Psi_n \; ; n = 0, 1, 2 \cdots \tag{5.9}$$

and this leads to

$$\Psi_0 = 1 \tag{5.10a}$$

$$\Psi_1 = \int_{-\infty}^{z}\left(-1 + \frac{X_{-\infty}}{X}\right)dz_1 \tag{5.10b}$$

$$\Psi_2 = \int_{-\infty}^{z}\left[\int_{-\infty}^{z_1}\left(1 - \frac{X_{-\infty}}{X}\right)dz_2 + \frac{1}{X}\int_{-\infty}^{z_1}(X - X_{-\infty})\,dz_2\right]dz_1 \tag{5.10c}$$

The eigenvalue relation results from (5.4) which becomes

$$\Psi(0)\, D\Theta(0) - \Theta(0)\, D\Psi(0) - 2k\,\Theta(0)\,\Psi(0) = 0 \tag{5.11}$$

when written in terms of Θ and Ψ. Substituting (5.3) into (5.1) and equating powers of k, the eigenvalue relation becomes

$$\begin{aligned} &k\left[\Psi_1 D\Theta_0 + \Psi_0 D\Theta_1 - \Theta_0 D\Psi_1 - \Theta_1 D\Psi_0 - 2\Theta_0\Psi_0\right] + \\ &+ k^2\left[\Psi_1 D\Theta_1 + \Psi_2 D\Theta_0 + \Psi_0 D\Theta_2 - \Theta_1 D\Psi_1 - \Theta_2 D\Psi_0 - \right. \\ &\left. - \Theta_0 D\Psi_2 - 2\Theta_0\Psi_1 - 2\Psi_0\Theta_1\right] + \cdots = 0 \end{aligned} \tag{5.12}$$

After substituting the equations of (5.8) and (5.10) into (5.12), we obtain

$$\begin{aligned} &k\left[-2 + \left(1 - \frac{X_\infty}{X_0}\right) - \left(-1 + \frac{X_{-\infty}}{X_0}\right)\right] + k^2\Big[\Big\{- \\ &- 2\int_{-\infty}^{0}\left(-1 + \frac{X_{-\infty}}{X}\right)dz + \int_{\infty}^{0}\left(1 - \frac{X_\infty}{X}\right)dz\Big\} + \\ &+ \int_{\infty}^{0}\left(1 - \frac{X_\infty}{X}\right)dz + \frac{1}{X_0}\int_{\infty}^{0}(X - X_\infty)\,dz + \left(1 - \frac{X_\infty}{X_0}\right)\int_{-\infty}^{0} \cdot \\ &\cdot\left(-1 + \frac{X_{-\infty}}{X}\right)dz - \int_{-\infty}^{0}\left(1 - \frac{X_{-\infty}}{X}\right)dz - \frac{1}{X}\int_{-\infty}^{0}(X - X_{-\infty})\,dz \\ &+ \left(1 - \frac{X_{-\infty}}{X_0}\right)\int_{\infty}^{0}\left(1 - \frac{X_{-\infty}}{X}\right)dz + \cdots = 0 \end{aligned}$$

which, after manipulation (the third term in this series is almost proportionally longer), reduces to a surprisingly simple form

$$k\left[\chi_{\infty}+\chi_{-\infty}\right]+k^2\int_{-\infty}^{\infty}\frac{(\chi_{\infty}-\chi)(\chi_{-\infty}-\chi)}{\chi}\,dz+\cdots=0 \tag{5.13}$$

We consider a normalized velocity so that if we are dealing with a shear layer

$$U_{\infty}=1 \quad , \quad U_{-\infty}=-1$$

and if we have jet flow

$$U_{\infty}=U_{-\infty}=0 \quad ; \quad U_0=1$$

The first order approximation to the wave speed is then given by

$$2c^2=-\left\{U_{\infty}^2+U_{-\infty}^2-\left(M_{\infty}^2+M_{-\infty}^2\right)\right\}$$

which for shearing flow is

$$2c^2=-2+M_{\infty}^2+M_{-\infty}^2$$

We therefore have instability whenever the average magnetic energy is smaller than the average kinetic energy. In the jet case, the first approximation to the wave speed is given by

$$2c^2=\left(M_{\infty}^2+M_{-\infty}^2\right)$$

and thus the background magnetic field is a stabilizing agent. When there is no background field the next approximation for the wave speed is given by

$$2c^2 = -k\int_{-\infty}^{\infty}(U^2 - M^2)\,dz \tag{5.14}$$

This is seen to coincide exactly with the result of the heuristic argument as given by (3.14) which describes the long wave sinuous disturbances. We therefore see that (5.14) gives the correct second order approximation for the sinuous wave disturbance of the three layer jet model of Chapter 3. This agreement is all the more pleasing when we consider that if (5.13) should given any trouble, it would much sooner be expected to do so for a model with discontinuous velocity and magnetic field profiles than otherwise.

But, we may ask, does (5.13) ignore the existence of the varicose wave? To this question, we may answer a cautious no. Considering the three layer jet model fo Chapter 3, (5.13) becomes

$$c^2 - M_i^2 + k\left[\frac{(-1 - M_i^2 + M_0^2 + 2c)^2}{(1-c)^2 - M_0^2}\right] + \cdots = 0$$

Multiplying by $[(1-c)^2 - M_0^2]$, we obtain

$$[(1-c)^2 - M_0^2][c^2 - M_i^2] + k[-1 - M_i^2 + M_0^2 + 2c]^2 = 0 \tag{5.15}$$

Taking the limit as $k \to 0$ gives us

$$[(1-c)^2 - M_0^2][c^2 - M_i^2] = 0$$

When the term inside the second set of brackets is set equal to zero, the wave speed represents the sinuous disturbance, but if the term in the first set of brackets equals zero, the wave speed then represents the varicose disturbance. The second order approximation to the varicose mode is determined by an iterative process where to first order

$$C = 1 \pm M_0$$

and then we write in (5.15)

$$\left[(1-c)^2 - M_0^2\right]\left[1 \pm 2M_0 + M_0^2 - M_1^2\right] +$$

$$+ k\left[-1 - M_1^2 + M_0^2 + 2(1 \pm M_0)\right]^2 + \cdots = 0$$

The solution for C is then

$$C = 1 \pm \frac{1}{2}\sqrt{4M_0^2 - 4k\left[1 \pm 2M_0 + M_0^2 - M_1^2\right]} \qquad (5.16)$$

When $M_0 = 0$, $k = 0$ represents marginal stability and (4.16) becomes

$$C = 1 \pm \frac{1}{2}\sqrt{-4k\left[1 - M_1^2\right]} \qquad (5.17)$$

Referring back to (3.11), a^2 may be approximated by

$$1 - e^{-2k} = a^2 \cong 2k$$

for small k and (3.11) becomes

$$C = 1 - k \pm \frac{1}{2}\sqrt{(2-k)\left[2-k-2\right] + 4k M_1^2}$$

which is essentially the same as (5.17).

For the Kelvin-Helmholtz fluid the first two terms of (5.13) give

$$C^2 = -1 + \frac{M_\infty^2 + M_{-\infty}^2}{2}$$

which is exact. For the double shear model considered in Chapter 3 when we approximate a^2 as

$$a^2 = 1 - e^{-4k} \cong 4k$$

(3.17) becomes

$$C^2 \cong -\frac{1}{2}\left(1 - M_1^2 - M_0^2 - 4k\right) \pm \sqrt{16k^2 - 8k\left[1 - M_0^2 - M_1^2\right] + (1-4k)\left[1 - M_0^2 - M_1^2\right]^2} \tag{5.18}$$

Formula (5.13) leads to

$$0 \cong 2\left(1 + c - M_1^2\right) + \int_{-1}^{1} \frac{\left[1 - 2c - M_1^2 + M_0^2\right]\left[1 + 2c - M_1^2 + M_0^2\right]}{C^2 - M_0^2}$$

or

$$C^4 + C^2\left[1 - M_1^2 - M_0^2 - 4k\right] + k\left(1 - M_1^2 + M_0^2\right)^2 - M_0^2\left(1 - M_1^2\right) = 0$$

Solving for C^2 , we obtain

$$C^2 = -1/2\left(1 - M_0^2 - M_1^2 - 4k\right) \pm \frac{1}{2}\sqrt{\left(1 - M_1^2 - M_0^2 - 4k\right)^2 - 4k\left(1 - M_1^2 + M_0^2\right) + 4M_0^2\left(1 - M_1^2\right)}$$

which coincides exactly with (5.18).

B. Convergence of the Eigenvalue Relation

The remarkable agreement between the series solution and those two examples for which analytic solution do exist is quite encouraging. It is possible to show that (5.13) represents a convergent series so long as the imaginary part of the wave speed is nonzero. Such a result lends considerable weight to the validity of the entire procedure. We now prove convergence of the series for Θ and remark here that the proof for Ψ is entirely analogous.

Using the recursion relation (5.7) for Θ in the form

$$D[XD\Theta_{n+1}] = \Theta_n DX + 2XD\Theta_n \tag{5.19}$$

We find it convenient to define a new independent variable, η, such that

$$\eta = e^{-z} \quad ; \quad 0 \leq \eta \leq 1$$

and therefore

$$-\eta \frac{d}{d\eta} = \frac{d}{dz} \equiv D$$

Equation (5.19) then becomes

$$\eta \frac{d}{d\eta}\left[X\eta \frac{d\Theta_{n+1}}{d\eta}\right] = -\eta\Theta_n \frac{dX}{d\eta} - 2\eta X \frac{d\Theta_n}{d\eta}$$

or, upon integrating,

$$\frac{d\Theta_{n+1}}{d\eta} = -\frac{1}{\eta X}\left[\int_0^\eta \Theta_n \frac{dX}{d\eta_1} d\eta_1 + \int_0^\eta 2X \frac{d\Theta_n}{d\eta_1} d\eta_1\right] \tag{5.20}$$

The restrictions which we place on the velocity and magnetic field profiles are such that

$$\left| \frac{d}{d\eta} X \right| < A\eta^{a-1} \quad ; \qquad a-1>0$$

We are also guaranteed that

$$|X^{-1}| \leq c_i^{-2}$$

Making use of these inequalities, we see that (5.20) may be written as

$$\frac{d\theta_n}{d\eta} \leq \frac{1}{c_i^2\eta}\left[\int_0^\eta \theta_{n-1} \left\{ A\,\eta_1^{a-1} \right\} d\eta_1 + \int_0^\eta \left\{ 2\,\frac{A}{a}\,\eta_1^{a} \right\} \frac{d\theta_{n-1}}{d\eta_1} d\eta_1 \right]$$

This leads to

$$\frac{d\theta_1}{d\eta} \leq \frac{1}{c_i^2\eta}\,\underline{A}$$

and

$$\theta_1 \leq \frac{A}{c_i^2 a^2}\,\eta^a$$

Similarly,

$$\frac{d\theta_2}{d\eta} \leq \frac{1}{c_i^2\eta}\left[\int_0^\eta \frac{A}{c_i^2 a^2} \left\{ A + \frac{2A}{a} \right\} \eta_1^{2a-1}\, d\eta_1 \right]$$

$$\leq \frac{A^2}{c_i^4 a^3} \left\{ 1+2 \right\} \frac{1}{2}\, \eta^{2a-1}$$

and

$$\theta_2 \leq \frac{A^2}{c_i^4 a^4} \left\{ 1+2 \right\} \left(\frac{1}{2}\right)^2 \eta^{2a}$$

We compute one more term in order to obtain an idea of the general form of the series

$$\frac{d\theta_3}{d\eta} \leq \frac{1}{c_i^2\eta}\left[\int_0^\eta \frac{A^2}{c_i^4 a^4}\left\{1+2\right\}\frac{1}{2}\left\{A + \frac{2A}{a}\cdot 2a\right\}\eta_1^{3a-1}\,d\eta_1\right.$$

$$\leq \frac{A^3}{c_i^6 a^5}\left\{1+2\right\}\left\{1+2\cdot 2\right\}\left(\frac{1}{2}\right)^2\left(\frac{1}{3}\right)\eta^{3a-1}$$

so that

$$\theta_3 \leq \frac{A^3}{c_i^6 a^6}\left\{1+2\right\}\left\{1+2.2\right\}\left(1/2\right)^2\left(1/3\right)^2\eta^{3a}$$

and now, by mathematical induction we can say

$$\theta_n \leq \frac{A^n\eta^{an}}{c_i^{2n}a^{2n}}\prod_{r=1}^{n}\left(1+2(r-1)\right)\frac{1}{r^2}$$

Since $\eta \leq 1$ we can write this as

$$\theta_n \leq \left(\frac{A}{c_i^2 a^2}\right)^n \prod_{r=1}^{n}\frac{2}{r}$$

The series will converge for every value of k for which the problem supports a $c_i \neq 0$ by the ratio test, since

$$\frac{k^{n+1}}{k^n}\frac{\theta_{n+1}}{\theta_n} \sim \frac{2}{n+1}\frac{Ak}{c_i^2 a^2} \tag{5.21}$$

is necessarily less than unity for n sufficiently large. We should expect that our series will thus give a good estimate for the shortest wave which will produce instability, so long as enough terms are included. This is, of course, no simple matter and we have not been guaranteed that such a lower wave length limit does in fact exist.

For a jet velocity profile we know that $c_i \rightarrow 0$ as $k \rightarrow 0$ so that we must look more closely at the above result. Our first approximation for c_i gives $c_i \sim k^{1/2}$ so that by (5.21) we again expect convergence, but it is desirable when possible to prove convergence without first approximating c_i. This can be done by using an integral equation approach.

C. Integral Equation Approach

The integral equation approach actually has two advantages. The convergence proof for the eigenvalue relation is more rigorous, and also the eigenvalue relation itself is more easily obtainable in a compact form than is the virtually equivalent relationship obtained by the series expansion in powers of the wave number.

Once again, we utilize the governing equation in the form (1.21) and find it advantageous to write it as

$$D[XDF] - k^2 \frac{X_\infty^2}{X} F = \frac{k^2}{X}[X^2 - X_\infty^2]F \qquad (5.22)$$

We now solve (5.22) by the method of Green's Functions and, with the left hand side of (5.22) as the homogeneous part, we can write

$$XD[XDF_H] - A^2F_H = 0 \quad ; \; A^2 = k^2X_\infty^2 \qquad (5.23)$$

Using a new independent variable such that

$$XD \equiv X\frac{d}{dz} = \frac{d}{dy}$$

and

$$y = \int^z \frac{dz}{X}$$

(5.23) becomes

$$\frac{d^2F_H}{dy^2} - A^2F_H = 0$$

which has solutions

$$F_{H^\pm} = \text{const } e^{\mp A\int_{\pm\infty}^{z} \frac{dz}{X}} = \begin{matrix} v \\ u \end{matrix}$$

u and v are two solutions which satisfy the boundary conditions at $-\infty$ and ∞, respectively, and then the Green's Function is given by

$$G(z,z_1) = \begin{cases} -u(z_1)v(z)B^{-1} & z > z_1 \\ -u(z)v(z_1)B^{-1} & z_1 > z \end{cases}$$

The constant B, is determined from the equation

$$u(z_1)Dv(z_1) - v(z_1)Du(z_1) = \frac{B}{X(z_1)}$$

and is given by

$$B = e^{A\int_{-\infty}^{\infty} \frac{dz}{X}} \; 2\frac{A}{X}(X) = 2Ae^{A\int_{-\infty}^{\infty} \frac{dz}{X}}$$

Substituting this into the expression for the Green's Function, we write

$$G(z,z_1) = -[2kX_\infty]^{-1}\exp\left[-kX_\infty \operatorname{sgn}(z-z_1)\int_{z_1}^{z}\frac{dz_2}{X}\right]$$

and the integral equation corresponding to (5. 22) is then given by

$$F(z) = -\frac{k}{2X_\infty}\int_{-\infty}^{\infty} K(z,z_1)F(z_1)\,dz_1 \tag{5.24}$$

where

$$K(z_i,z_j) \equiv \frac{1}{X(z_j)}\left[X^2(z_j) - X_\infty^2\right]\cdot\exp\left[-\right.$$

$$\left.-kX_\infty \operatorname{sgn}(z_i-z_j)\int_{z_j}^{z_i}\frac{dz}{X}\right]$$

The eigenvalue relation is determined by the vanishing of a quantity called the Fredholm determinant, which is given by

$$\mathcal{D} = 1 + \frac{k}{2X_\infty}\int_{-\infty}^{\infty}K(z,z)\,dz + \sum_{n=1}^{\infty}\frac{1}{n!}\frac{k^n}{2^n X_\infty^n}\cdot$$

$$\cdot\int dz_1 \cdots \int dz_n \det\left|K(z_i,z_j)\right| \tag{5.25}$$

$\mathcal{D} \leq N < \infty$ is ordinarily guaranteed for problems with finite functions in finite intervals, but since we are dealing with an unbounded fluid, a new proof of convergence must be given. Convergence depends on the restrictions which we place on the asymptotic behavior of the velocity and magentic fields. It is sufficient to require, as before, that as $z \to \infty$

$$|X - X_\infty| < \text{const}\ e^{(-a|z|)} \qquad a > 0$$

and we also want

$$\frac{|X_\infty|}{c_i^2} \leq N < \infty$$

which necessitates that we restrict our consideration tothe case where $M_\infty = 0$ for jet flows. This latter restriction is reasonable, since we are investigating the case of marginal stability but we are limited to the case of the sinuous wave.

The argument for convergence now proceeds as follows. For $1 \leq i, j \leq n$ we write

$$\det |K(z_i, z_j)| \equiv \prod_{i=1}^{n} \frac{1}{X(z_i)} \left[X(z_i) - X_\infty \right] \det |k(z_i, z_j)|$$

where

$$k(z_i, z_j) = \exp\left[-kc^2 \operatorname{sgn}(z_i - z_j) \int_{z_j}^{z_i} \frac{dz}{X} \right.$$

Since the determinant is symmetric in all variables, we are guaranteed that

$$\int_0^{z_n} \int_0^{z_{n-1}} \cdots \int_0^{z_1} \det K \, dz\, dz_1 \cdots dz_{n-1} = \frac{1}{n!} \int_0^{z_n} \int_0^{z_n} \cdots \int_0^{z_n} \det K \, dz \cdots dz_{n-1}$$

It is advantageous to consider the interval $z_1 \leq z_2 \leq z_3 \leq \cdots \leq z_n$ and then the determinant for $k(z_i, z_j)$ may be written schematically as

$$\Delta_n = \begin{vmatrix} 1 & e^{-\int_{z_1}^{z_2}} & \cdots & e^{-\int_{z_1}^{z_n}} \\ e^{-\int_{z_1}^{z_2}} & 1 & \cdots & e^{-\int_{z_2}^{z_n}} \\ e^{-\int_{z_1}^{z_3}} & e^{-\int_{z_2}^{z_3}} & \cdots & \cdot \\ \vdots & \vdots & \ddots & \vdots \\ e^{-\int_{z_1}^{z_n}} & e^{-\int_{z_2}^{z_n}} & \cdots & 1 \end{vmatrix}$$

In order to facilitate computation of the determinant, we multiply the i^{th} row by the i^{th} term of the $(i+1)st$ row and subtract these products from each term of the $(i+1)st$ row. Since what is being done is to subtract a constant multiple of one row from another, the value of the determinant is unchanged. But in the new term for Δ_n, all the terms below the principal diagonal vanish and the determinant is evaluated by taking the product of the terms of the principal diagonal so that

$$\Delta_n = \left[1 - \exp\left(-2kc^2 \int_{z_1}^{z_2} \frac{dz}{X}\right)\right] \cdots$$

$$\cdots \left[1 - \exp\left(-2kc \int_{z_{n-1}}^{z_n} \frac{dz}{X}\right)\right]$$

We now show that $\mathcal{D}$ is convergent under the restrictions mentioned.

above. Since, for all z ,

$$\frac{C^2}{X} \leq \frac{|C|^2}{C_i^2}$$

we have

$$\Delta_n \leq \left[1 + \exp\left(2k \frac{|C|^2}{C_i^2}(z_2 - z_1)\right)\right] \cdots \left[1 + \exp\left(2k \frac{|C|^2}{C_i^2}(z_n - z_{n-1})\right)\right]$$

$$\leq 2^{n-1} \exp\left[2k \frac{|C|^2}{C_i^2}(z_n - z_1)\right]$$

or, if we abandon the requirement that $z_1 \leq z_2 \leq z_3 \leq \cdots \leq z_n$, the last line may be rewritten

$$\leq 2^{n+1} \exp\left[2k \frac{|C|^2}{C_i^2}\left\{|z_1| + |z_2| + \cdots + |z_n|\right\}\right]$$

Therefore

$$\det|K(z_i, z_j)|_n \leq \prod_{i=1}^{n} \frac{1}{X(z_i)}\left(X^2(z_i) - X_\infty^2\right) \exp\left(2k \frac{|C|^2}{C_i^2}|z_i|\right) \qquad (5.26)$$

and (5.25) converges whenever the integral of (5.26) is finite. To show that this is so we observe that

$$\int_{-\infty}^{\infty} \frac{1}{X}(X^2 - X_\infty^2) \exp\left(2k \frac{|C|^2}{C_i^2}|z|\right) dz$$

$$\leq \int_{-\infty}^{\infty} \left|1 + \frac{|C|^2}{C_i^2}\right| |X - X_\infty| \exp\left(2k \frac{|C|^2}{C_i^2}|z|\right) dz$$

which by our restriction on $|X - X_\infty|$

$$\leq \int_{-\infty}^{\infty} \left| 1 + \frac{|c|^2}{c_i^2} \right| \text{const} \exp(-a|z|) \exp\left(2k \frac{|c|^2}{c_i^2} |z|\right) dz$$

which converges so long as

$$k < \frac{a c_i^2}{2|c|^2}$$

We can thus see that for k sufficiently small, $\mathcal{D}$ converges and the first two terms of the eigenvalue relation for a jet flow are given by

$$1 + \frac{k}{2c_i^2} \int_{-\infty}^{\infty} \frac{1}{X} \left[X^2 - X_\infty^2 \right] + \cdots = 0 \qquad (5.27)$$

Further terms are easy to write down formally, although they may not be simple to evaluate analytically.

D. Completeness of the Scheme

One may ask whether there are solutions which our perturbation scheme (5.13) leaves out. To this we cannot give a complete answer for the magnetic problem. It is, however, possible to prove that the limiting value $c_i^2 \to 1 - M_\infty^2$ as $k \to 0$ in the shear case represents the only solution with a wave speed which is not real in the limit $k = 0$. This we prove by referring to (1.21). When $k = 0$,

$$XDF = \text{const}$$

and in order to avoid divergence as $|z| \to \infty$, this constant must equal zero. Thus, so long as $X \neq 0$, which must be the case for all z when $c_i \neq 0$, F is constant. At points where $X = 0$,

F may have a jump. Differentiating (1.21), we have

$$D^2(XDF) - k^2(XDF) = k^2 F DX$$

whose corresponding integral equation is

$$XDF = -\frac{1}{2}k\int_{-\infty}^{\infty} \exp(-k|z-z_1|)\, DX \cdot F\, dz_1$$

Differentiating this and making use of (1.21), we have;

$$XF = k^{-2}D(XDF) = \frac{1}{2}\int_{-\infty}^{\infty} \exp(-k|z-z_1|)\, \mathrm{sgn}(z-z_1)\, DX F\, dz_1$$

Assuming that F has no jumps and normalizing it to one, this becomes

$$X(z) = \frac{1}{2}\left[2X - X_{\infty} - X_{-\infty}\right]$$

in the limit, $k = 0$.

In the shear layer case we have

$$c_i = \left(1 - \frac{M_{\infty}^2 + M_{-\infty}^2}{2}\right) i$$

and in the jet case $c = 0$. F is therefore continuous only for the unstable shear layer wave, which is the only wave which has a nonzero imaginary part to the wave speed at $k=0$, and F is continuous for the sinuous wave disturbance of a jet. There may be other marginal modes, as our examples clearly show, but they do not have a continuous F at $k=0$. In the nonmagnetic case it has been proven that at $k=0$ the only other possible marginal modes occur for c values equal to the velocity at points where $DU = 0$ (Drazin and Howard 1962). The case $k=0$ is seen to be exceptional to the rule that a neutral solution must occur at a point $U = c$ where

$D^2U = 0$. The case $k=0$ is exceptional for the magnetic problem also so that the conditions found by Kent (1968) and mentioned in the introductory chapter need not be obeyed. Unfortunately no alternate condition has been found in the magnetic problem. In fact, Kent's conditions are violated by the marginal mode of the double shear layer model since marginal stability occurs at a $(U-c)$ where $M_0 \neq 0$. For the jet case, though, it seems that the magnetic field must vanish at the maximum value of U in order to support marginal stability.

VI. Long Wave Theory for Unbounded Heterogeneous Flow

The stability theory for long wave length disturbances in an unbounded fluid is now extended to the case of a heterogeneous fluid. As before, we consider a fluid in which the buoyancy effects completely overshadow the inertial effects of the heterogeneity and thus the governing equation is given by

$$D[XDF] - k^2XF - \bar{R}_iF = 0 \tag{2.24}$$

One of the restrictions we must place on the density profile is that its percentual change is small. Indeed, we shall also demand that the main variation of density occurs in the same region in which the velocity and magnetic field also vary. It proves convenient to define

$$\bar{R}_i = -GD\lambda$$

where

$$G = \frac{\rho_{-\infty} - \rho_{\infty}}{\rho_{\infty} + \rho_{-\infty}} g$$

is the same as that of Chapter 3, and is defined so that λ is a normalized variable. Thus

$$\lambda(\infty) = 1$$
$$\lambda(-\infty) = -1$$

The governing equation thus takes the form

$$D[XDF] - k^2XF + GD\lambda F = 0 \tag{6.1}$$

In order to find the correct expansion scheme, we would like to see the behavior of (6.1) as $|z| \to \infty$. It is clear that if we retain the restrictions placed on X and notice that the integral of $D\lambda$ easily converges (it in fact equals two), the asymptotic form of (6.1) is given by

$$X D^2 F - X k^2 F = 0$$

and thus

$$F_{\pm} \propto e^{\mp kz} \quad \text{as} \quad z \to \pm\infty$$

A. Series Approach

Since the emphasis in this study is placed on the stability characteristics as they are affected by the stratification, it is desirable to form an expansion in terms of two parameters: the wave number, k, and a stratification parameter, G. The double expansion is necessary because as $k \to 0$, any finite value of G guarantees stability. It is of interest to find the critical value of G, which is an overall Richardson number, for a given k, and this procedure proves ideal. As with the homogeneous case, we first factor out the asymptotic behavior of F before expanding and define two new variables, φ and ξ, such that

$$\begin{aligned} F_+ &= e^{-kz}\, \xi \\ F_- &= e^{kz}\, \varphi \end{aligned} \qquad (6.2)$$

These solutions are matched at $z = 0$ by (5.4) and are phrased in terms of the new variables

$$\xi(0)\, D\varphi(0) - \varphi(0)\, D\xi(0) - 2k\, \xi(0)\, \varphi(0) = 0 \tag{6.3}$$

The expanded forms of our new variables appear as

$$\begin{aligned} \xi(z) &= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \xi_{i,j}\, k^i G^j \\ \varphi(z) &= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \varphi_{i,j}\, k^i G^j \end{aligned} \tag{6.4}$$

We solve for φ in some detail and then present the results for ξ, since the latter is computed in similar fashion. By substituting (6.2) into (6.1) and equating powers of k and G by use of the second equation of (6.4) we arrive at the recursion relation

$$D[X D\varphi_{i,j}] = -D[X \varphi_{i-1,j}] - X D\varphi_{i-1,j} + D\lambda\, \varphi_{i,j-1} \tag{6.5}$$

where

$$\varphi_{i,j} = 0$$

whenever one of the subscripts is less than zero.

φ is normalized so that at $z = -\infty$ it is equal to one. At $z = 0$, we require only that it be finite. Integrating

$$D[X D\varphi_{00}] = 0$$

twice we obtain

$$\varphi_{00} = \int_{-\infty}^{z} \frac{c_1}{X}\, dz_1 + c_2$$

$c_1 = 0$ to keep $\varphi_{00}(0)$ finite and $c_2 = 1$ because $\varphi(-\infty) = 1$.
Therefore

$$\varphi_{00} = 1 \tag{6.6a}$$

The equation for φ_{10} is given by

$$D[X D\varphi_{10}] = -DX$$

and upon integrating twice this becomes

$$\varphi_{10} = \int_{-\infty}^{z} \frac{c_1 - X}{X}\, dz_1 + c_2$$

Since $\varphi(-\infty) = 1$, all terms other than φ_{00} equal zero at $z = -\infty$. Therefore $c_2 = 0$ and in the next few computations we shall not bother inserting it at all, In order to render the integral finite, we must have

$$c_1 = X_{-\infty}$$

where the subscript indicates the z value at which X is evaluated. φ_{10} is then given by

$$\varphi_{10} = \int_{-\infty}^{z} \frac{X_{-\infty} - X}{X}\, dz_1 \tag{6.6b}$$

For φ_{01}, (6.5) gives

$$D[X D\varphi_{01}] + D\lambda = 0$$

which becomes

$$\varphi_{01} = \int_{-\infty}^{z} \frac{c_1 - \lambda}{X}\, dz_1$$

after integrating twice. To avoid an invinite answer, we choose

$$C_1 = \lambda_{-\infty} = -1$$

so that

$$\varphi_{01} = -\int_{-\infty}^{z} \frac{\lambda+1}{X}\, dz \tag{6.6c}$$

In like manner, the second order terms are found to be

$$\varphi_{20} = \int_{-\infty}^{z} \left[\frac{1}{X} \int_{-\infty}^{z_1} (X - X_{-\infty})\, dz_2 - \int_{-\infty}^{z_1} \left(\frac{X_{-\infty} - X}{X} \right) dz_2 \right] dz_1 \tag{6.6d}$$

$$\varphi_{11} = \int_{-\infty}^{z} \left[-\frac{1+\lambda}{X} \int_{-\infty}^{z_1} \left(\frac{X_{-\infty} - X}{X} \right) dz_2 + 2\frac{X_{-\infty}}{X} \int_{-\infty}^{z_1} \frac{\lambda+1}{X}\, dz_2 \right] dz_1 \tag{6.6e}$$

$$\varphi_{02} = \int_{-\infty}^{z} \left[\frac{\lambda+1}{X} \int_{-\infty}^{z_1} \frac{\lambda+1}{X}\, dz_2 - \frac{1}{X} \int_{-\infty}^{z_1} \frac{(\lambda+1)^2}{X}\, dz_2 \right] dz_1 \tag{6.6f}$$

For $\xi(z)$ the recuresion relation becomes

$$D[XD\,\xi_{ij}] = D[X\xi_{i-1,j}] + XD\xi_{i-1,j} + D\lambda\,\xi_{i,j-1} \tag{6.7}$$

and the terms up to order $i+j=2$ are given by

$$\xi_{00} = 1 \tag{6.8a}$$

$$\xi_{10} = \int_{\infty}^{z} \left(\frac{X_{\infty} - X}{X} \right) dz_1 \tag{6.8b}$$

$$\xi_{01} = \int_{\infty}^{z} \frac{\lambda - 1}{X}\, dz_1 \tag{6.8c}$$

$$\xi_{20} = \int_{\infty}^{z}\left[\frac{1}{X}\int_{\infty}^{z_1}(X - X_{\infty})\,dz_2 - \int_{\infty}^{z_1}\left(\frac{X_{\infty} - X}{X}\right)dz_2\right]dz_1 \tag{6.8d}$$

$$\xi_{11} = \int_{\infty}^{z}\left[\frac{\lambda - 1}{X}\int_{\infty}^{z_1}\left(\frac{X_{-\infty} - X}{X}\right)dz_2 - 2\frac{X_{\infty}}{X}\int_{\infty}^{z_1}\frac{\lambda - 1}{X}\,dz_2\right]dz_1 \tag{6.8e}$$

$$\xi_{02} = \int_{\infty}^{z}\left[\frac{\lambda - 1}{X}\int_{\infty}^{z_1}\frac{\lambda - 1}{X}\,dz_2 - \frac{1}{X}\int_{\infty}^{z_1}\frac{(\lambda - 1)^2}{X}\,dz_2\right]dz_1 \tag{6.8f}$$

When these relations are substituted into the boundary condition, (6.3), we have, up to second order in $(i+j)$,

$$\begin{aligned}
&X(0)\Big[k(-2 + D\xi_{10} - D\varphi_{10}) + G(D\xi_{01} + D\varphi_{01}) + \\
&+ k^2(-2\varphi_{10} - 2\xi_{10} + D\xi_{10}\varphi_{10} - \xi_{10}D\varphi_{10} + \\
&+ D\xi_{20} - D\varphi_{20}) + kG(-2\xi_{01} - 2\varphi_{01} + \\
&+ \varphi_{01}D\xi_{10} + \varphi_{10}D\xi_{01} - \xi_{01}D\varphi_{10} - \xi_{10}D\varphi_{01} + \\
&+ D\xi_{11} - D\varphi_{11}) + G^2(\varphi_{01}D\xi_{01} - \xi_{01}D\varphi_{01} \\
&+ D\xi_{02} - D\varphi_{02})\Big] = 0
\end{aligned} \tag{6.9}$$

After a finite amount of manipulation on the second order terms, (6.9) reduces to the surprisingly simple form

$$\begin{aligned}
&k\left[X_{\infty} + X_{-\infty}\right] - 2G + \int_{-\infty}^{\infty}\left[k(X - X_{\infty}) + G(1 - \lambda)\right]\cdot \\
&\cdot\left[k(X - X_{-\infty}) + G(1 + \lambda)\right]\frac{dz}{X} + \cdots = 0
\end{aligned} \tag{6.10}$$

This is our eigenvalue equation. For a shear layer the first order approximation gives

$$k\left[2c^2 - M_{-\infty}^2 - M_{\infty}^2\right] - 2G = 0$$

which is exact for the Kelvin-Helmholtz problem. Marginal stability occurs for

$$\frac{G}{k} = 1 - \frac{M_{\infty}^2 + M_{-\infty}^2}{2}$$

For the double shear layer model of Chapter 3, (6. 10) becomes

$$0 \cong \left(1 + c^2 - M_i^2 - \frac{G}{k} - 4k\right) + \frac{\left[1 - 2c - M_i^2 + M_o^2 - G/k\right]}{c^2 - M_o^2} \cdot$$

$$\cdot \left[1 + 2c - M_i^2 + M_o^2 - G/k\right]$$

Solving for c^2 , we get

$$c^2 = -\frac{1}{2}\left(1 - M_o^2 - M_i^2 - \frac{G}{k} - 4k\right) \pm \frac{1}{2}\sqrt{\left(1 - M_o^2 - M_i^2 - \frac{G}{k} - 4k\right) - 4k\left(1 - M_i^2 - \frac{G}{k} + M_o^2\right)^2 + 4M_o^2\left(1 - M_i^2 - \frac{G}{k}\right)^2}$$

which is exactly equal to (3. 17) when the latter is approximated for long waves by writing

$$a^2 \equiv 1 - e^{-4k} \cong 4k$$

In the jet case, approximating (6. 10) to first order leads to the conclusion that

$$G/k \rightarrow 0 \quad \text{as} \quad k \rightarrow 0$$

When the background field is neglected we may have marginal stability at $k=0$ for G less than the critical value given by

$$G \cong \frac{k^2}{2}\int_{-\infty}^{\infty}(U^2-M^2)\,dz \tag{6.11}$$

We mention once again that this describes the sinuous wave, and (6.11) agrees with the heuristic formula, (3.14). The critical G is thus somewhat smaller than for the corresponding nonmagnetic problem.

Once again, if used properly, (6.10) yields information about the varicose wave. For the three layer jet, (6.10) may be written to second order as

$$\left[c^2-M_1^2-\frac{G}{k}\right]\left[(1-c)^2-M_0^2\right]+k\left[1-2c-M_0^2+M_1^2+G\right]^2 \cong 0 \tag{6.12}$$

One of the roots at $k=0$ is given by

$$c = 1 \pm M_0$$

which is marginally stable when $M_0 = 0$. To second order then, (6.13) may be written approximately as

$$(1-c)^2\left[1-M_1^2-G/k\right] + k\left(-1+M_1^2+G\right)^2 = 0$$

to which the solution for c is given by

$$c = 1 \pm \frac{1}{2}\sqrt{-4k\left(1-M_1^2-G/k\right)}$$

The agreement with the long wave approximation of (3.12) is good and captures the essence of this mode. The critical G is then given by

$$\frac{G}{k} = 1 - M_1^2 \tag{6.14}$$

so that we can see that the varicose mode is more unstable to long wave length disturbances for this example. A more general statement will not be made although it would seem as if the varicose mode is probably marginally stable at $k=0$ when $M=0$ and $DM=0$ at the point where U has its maximum, and exhibits a larger critical Richardson number than does the sinuous mode

B. Convergence of the Series and the Integral Equation Approach

Before attempting to solve for an approximate value for the overall Richarson number, G, for several shear velocity profiles, it is desirable to investigate the convergence of the series. In Chapter 5, convergence was established for the case $G=0$ but now as we increase G towards its critical value c_i approaches zero and convergence does not follow immediately from the homogeneous case. Since we are interested in obtaining the marginal stability curves, convergence of the scheme is essential to the validity of the results.

The approach is essentially that of Chapter 5 where we first prove the series converges and then reformulate the integral equation method and show that, here too, it leads to a simple means for obtaining an explicit form for the eigenvalue relation. Convergence is proven for ξ and extends without any essential changes to φ.

Retaining the restrictions on X and subjecting the density field to the restriction that

$$\left| \frac{d\lambda}{d\eta} \right| < A\eta^{a-1}$$

where, as before, we use a new independent variable, η, such that

$$\eta = e^{-z} \qquad 0 \leq \eta \leq 1$$

and

$$-\eta \frac{d}{d\eta} = \frac{d}{dz} \equiv D$$

Writing the integral of the recursion relation (6.7) leads to

$$\frac{d\xi_{i+1,j+1}}{d\eta} = -\frac{1}{\eta X}\left[\int_0^\eta \xi_{i,j+1} \frac{dX}{d\eta_1} d\eta_1 + \right. \tag{6.15}$$

$$\left. + \int_0^\eta 2X \frac{d\xi_{i,j+1}}{d\eta_1} d\eta_1 - \int_0^\eta G \frac{d\lambda}{d\eta_1} \xi_{i+1,j} d\eta_1 \right]$$

Observing that

$$\left| \frac{1}{X} \right| \leq \frac{1}{c_i^2}$$

(6.15) becomes

$$\frac{d\xi_{i+1,j+1}}{d\eta} \le \frac{1}{\eta c_i^2}\left[\int_0^\eta \xi_{i,j+1}\frac{A}{a}\eta_1^{a-1}d\eta_1 + \right.$$

$$\left. + \int_0^\eta 2A\eta_1^a \frac{d\xi_{i,j+1}}{d\eta_1}d\eta_1 + \int_0^\eta GA\frac{\eta_1^{a-1}}{a}\xi_{i+1,j}\,d\eta_1\right]$$

Notice that we have already computed $\xi_{i,0}$ in Chapter 5 since

$$\xi_{i,0} = \Theta_i$$

Convergence of $\xi_{0,j}$ follows simply. (6. 7) gives

$$\frac{d\xi_{0,j}}{d\eta} \le \frac{1}{\eta c_i^2}\int_0^\eta GA\,\frac{\eta_1^{a-1}}{a}\,\xi_{0,j-1}\,d\eta_1$$

and by mathematical induction

$$\xi_{0,j} \le \left(\frac{GA}{c_i^2 a^2}\right)^j \eta^{aj}\prod_{r=1}^{j}\frac{1}{r^2} \le \left(\frac{GA}{c_i^2 a^2}\right)^j \prod_{r=1}^{j}\frac{1}{r^2}$$

These two cases form the framework by which the convergence argument is completed. It follows immediately that

$$\xi_{i,j} \le \left(\frac{A}{c_i^2 a^2}\right)^{i+j}\prod_{r=1}^{i+j}(2r-1+G)\frac{1}{r^2}$$

$$\le \left(\frac{A}{c_i^2 a^2}\right)^{i+j}\prod \frac{2}{r}$$

and therefore with $G < k$

$$\ell k^{\ell} \xi_{\ell} \geq \sum_{i+j=\ell} \xi_{i,j} k^{i} G^{j}$$

we see that the series converges for all k and G such that $c_i \neq 0$.

The change in the integral equation due to the indroduction of density variations is minor. The Fredholm determinant, $\mathcal{D}$, given by (5. 25) retatins the same functional form but the kernel $K(z_i, z_j)$ is now defined as

$$K(z_i, z_j) = \left\{ \frac{1}{X(z_j)} \left[X^2(z_j) - X_\infty^2 \right] - \frac{G}{k^2} D\lambda \right\}$$

When all the restrictions of the integral equation approach of Chapter 5 are retained and we now demand that

$$|D\lambda| < \text{const} \exp(-a|z|) \text{ as } |z| \to \infty$$

for a positive a , convergence follows immediately for sufficiently small k so long as G/k^2 is finite as $k \to 0$. The first two terms of the eigenvalue relationship as given by , is then

$$1 + \frac{k}{2c^2} \int_{-\infty}^{\infty} \left(\frac{X^2 - X_\infty^2}{X} - \frac{G}{k^2} D\lambda \right) dz + \cdots = 0 \qquad (6.16)$$

We recall that this argument has been used for the sinuous wave of the jet.

C. Stability Boundaries for Shear Layers

Since the series expansions for ξ and φ converge, our eigenvalue relation also converges for all k, G such that $c_i \neq 0$ and this suggests that we may look more closely at (6.10) for a more accurate estimate of the critical overall Richardson number. We shall restrict consideration to the case when U and λ are antisymmetric and M is symmetric about the point $z = 0$. Physically, we can expect instability being manifest at $c_r = 0$. For reasons mentioned in Chapter 3, the double shear layer is an example of a flow which may, for small enough magnetic field, exhibit instability by two waves travelling at equal but opposite velocity. Generally, $c_r = 0$ when $c_i \neq 0$ and although this result will not be proven, it will be used. Taking the limit $c_i \to 0$, (6.10) becomes

$$2k - (M_\infty^2 + M_{-\infty}^2) - 2G + \int_{-\infty}^{\infty} \left[k(U^2 - M^2 - 1 + M_\infty^2) + \right.$$

$$\left. + G(1-\lambda) \right] \cdot \left[k(U^2 - M^2 - 1 + M_{-\infty}^2) + G(1+\lambda) \right] \frac{dz}{U^2 - M^2} \tag{6.17}$$

Whenever $U = \pm M$ the integrand is singular and thus (6.17) is strictly speaking not integrable. This difficulty can be circumvented by a method used by Drazin and Howard (1961). What we do is to subtract a value from the integrand equal to the integrand at the singular point and; expressing the ;subtracted quantity as an exact differential, add it outside the integral. Noting that

$$\frac{1}{U^2-M^2} = \frac{d}{dz}\left[\log \frac{U-M}{U+M}\right] \frac{1}{2(MDU-UDM)}$$

and making use of the symmetry properties, (6.17) appears as

$$2K - (M_\infty^2 + M_{-\infty}^2) - 2G - 2\int_0^\infty \Big\{ \big[K(U^2-M^2-1+M_\infty^2) +$$

$$+ G(1-\lambda)\big]\cdot\big[K(U^2-M^2-1+M_{-\infty}^2) + G(1+\lambda)\big] - \big[K(-1+M_\infty^2) +$$

$$+ G(1-\lambda_{U=M})\big]\big[K(-1+M_{-\infty}^2) + G(1+\lambda_{U=M})\big]\left\{\frac{MDU-UDM}{(MDU-UDM)_{U=M}}\right\} \tag{6.18}$$

$$\Big\} \frac{dz}{U^2-M^2} + \frac{1}{(MDU-UDM)_{U=M}} \big[K(-1+M_\infty^2) + G(1-\lambda_{0=M})\big]\cdot$$

$$\big[K(-1+M_\infty^2) + G(1+\lambda_{0=M})\big]\left(\log\left(\frac{U-M}{U+M}\right)\Big|_{-\infty}^{\infty}\right) = 0$$

An equivalent formulation can be derived by using

$$\frac{1}{U^2-M^2} = \frac{d}{dz}\left[\log(U^2-M^2)\right] \frac{1}{2(UDU-MDM)}$$

When the background magnetic field is zero, (6.18) reduces to

$$G \cong k + k^2 \int_0^\infty \left\{ \left(\frac{X^2 - \lambda^2}{X} \right) + \frac{\lambda^2_{U=M} [MDU - UDM]}{[MDU - UDM]_{U=M}} \frac{1}{X} \right\} dz \qquad (6.19)$$

or equivalently

$$G \cong k + k^2 \int_0^\infty \left\{ \left(\frac{X^2 - \lambda^2}{X} \right) + \frac{\lambda^2_{U=M} [UDU - MDM]}{X [UDU - MDM]_{U=M}} \right\} dz$$
$$+ k^2 \frac{\lambda^2_{U=M}}{[UDU - MDM]_{U=M}} \log M(0) \qquad (6.20)$$

where, because $c^2 = 0$

$$X \equiv U^2 - M^2$$

To see the initial effect of the magnetic field on the stability, we define

$$M^2(z) = M_0^2 f^2(z) \qquad f(0) = 1 \,, \; Df(0) = 0$$

and take the derivative of (6.19) which respect to M_0^2 and evaluate this at $M_0^2 = 0$. It is first necessary to find

$$\lambda^2_{U=M}$$
$$(MDU - UDM)_{U=M}$$

as functions of M_0 when M_0 is small. The point $U = M$ occurs close to the origin for M_0 small, so that we can expand in a Taylor series and write

$$U = 0 + z\,DU(0)$$
$$M = M_0 + 0 + \tfrac{z^2}{2} M_0 D^2 f(0)$$
$$\lambda = 0 + z\,D\lambda(0)$$

Therefore, the z value at which $U = M$ is given by

$$z_{U=M} = \frac{M_0}{DU(0)}$$

Therefore, we may write

$$\lambda_{U=M} = M_0 \frac{D\lambda(0)}{DU(0)}$$

and

$$[MDU - UDM]_{U=M} = M_0\left[1 - M_0^2 \frac{D^2 f(0)}{DU(0)}\right]$$

so that

$$\left.\frac{dG}{dM_0^2}\right)_{M_0=0} \cong k^2 \int_0^\infty \left\{ f^2(z)\left[-1 - \frac{\lambda^2}{U^4}\right] - \left(\frac{D\lambda(0)}{DU(0)}\right) D\left(\frac{f(z)}{U}\right)\right\} dz \tag{6.21}$$

By expanding near $z = 0$ the two parts of the integrand which become singular cancel, so that the integral converges. When (6.21) is positive, the magnetic field acts as a destabilizing agent since it serves to increase the critical Richardson number.

We now consider two problems, Our formulas will not work for the discontinuous models because they involve derivative terms but this is not overly disappointing because by direct use of (6.10) their solutions have been depicted rather well. The two problems that are chosen are among the small number of examples for which the stability boundary is known in the nonmagnetic case. Since no analytic solutions exist for any magnetic problem which fits our model there can be no comparison of results. Nevertheless agreement is observed in the nonmagnetic verson between the expansion results and the actual solution

For Goldstein's problem

$$U = \lambda = \begin{cases} -1 , & z \le -1 \\ z , & -1 \le z \le 1 \\ 1 , & 1 \le z \end{cases}$$

the stability boundary is given by

$$G = k - \frac{2}{3}k^2 - \frac{4}{9}k^3 - \frac{16}{45}k^4$$

and in the limit $M = 0$ (6. 19) gives

$$G = k - \frac{2}{3}k^2$$

which is exact to second order. For Holmboe's problem

$$U = \lambda = \tanh(z)$$

the nonmagnetic limit of (6. 19) gives

$$G = k(1-k)$$

which is exact. With this somewhat encouraging information in view, we now proceed.

In Goldstein's problem consider a magnetic field given by

$$M = \begin{cases} 0 , & |z| \ge 1 \\ M_0(1-z) , & 0 \le z \le 1 \\ M_0(1+z) , & -1 \le z \le 0 \end{cases}$$

(6.19) and (6.20) both lead to

$$G = k + \frac{k^2}{3} - k^2 \left\{ \frac{M_0^2}{3} + \frac{1}{1-M_0^2} + \frac{M_0^2}{(1-M_0^2)^2} \log M_0^2 \right\} \qquad (6.22)$$

Taking the derivative with respect to M_0^2 we see

$$\frac{DG}{DM_0^2} = -\left\{ \frac{k^2}{3} + \frac{k^2}{(1-M_0^2)^2} + \frac{k^2(\log M_0^2 + 1)(1-M_0^2)^2}{(1-M_0^2)^4} \right.$$

$$\left. + \frac{2M_0^2 k^2 \log M_0^2}{(1-M_0^2)^3} \right\}$$

The dominant term for small M_0^2 is given by

$$\frac{-k^2 \log M_0^2}{(1-M_0^2)^2}$$

and is positive so that the magnetic field acts as a destabilizing influence for small values. A plot of (6.22) (Figure 20) indicates that the magnetic field is destabilizing for all values of $M_0^2 \lesssim 1.7$ and sta-bilizing for $M_0^2 \gtrsim 1.7$

Consider Holmboe's example with $M = M_0 \operatorname{sech} z$. Equation (6.19) then takes the form

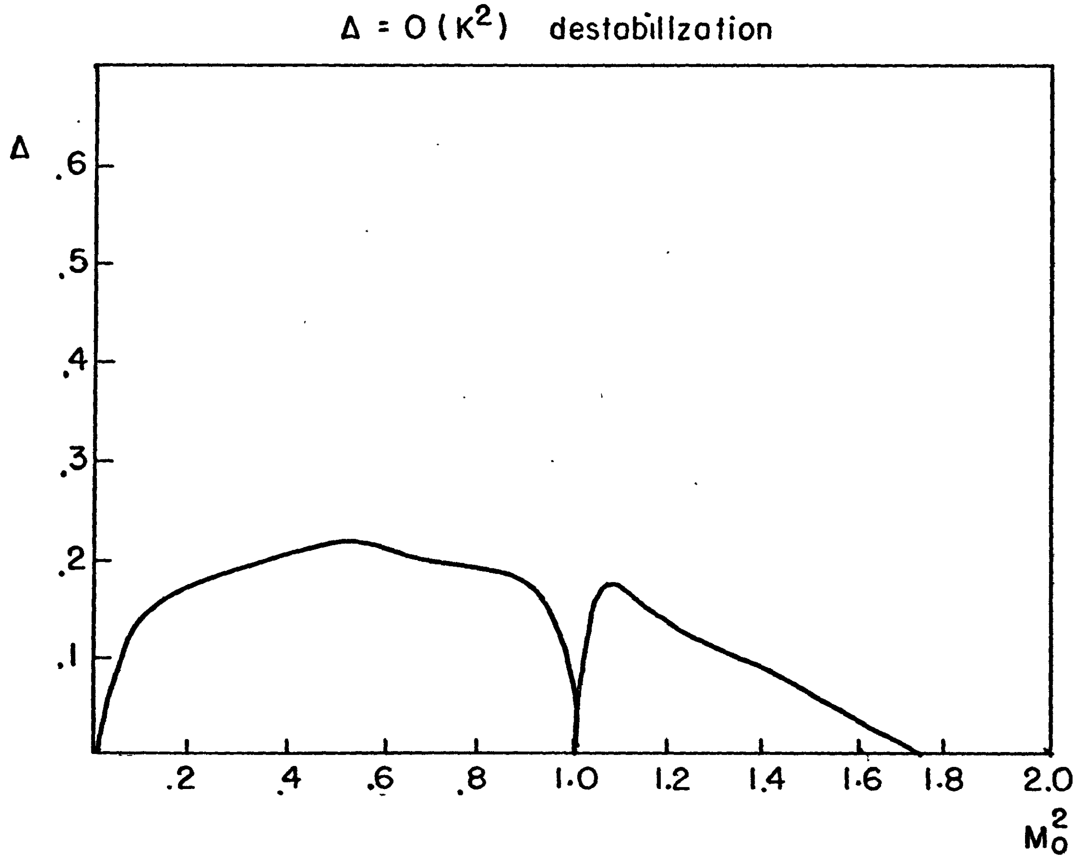

Figure 20. Destabilization to $O(k^2)$ for Goldstein's problem.

$$G = k + k^2 \int_0^\infty \left[\tanh^2 z - M_0^2 \operatorname{sech}^2 z - \frac{\tanh^2 z}{\tanh^2 z - M_0 \operatorname{sech}^2 z} \right.$$

$$\left. + \frac{M_0^2}{\sqrt{1+M_0^2}} \frac{\operatorname{sech} z}{\tanh^2 z - M_0^2 \operatorname{sech}^2 z} \right] dz$$

After several pages of maniputation this becomes

$$G = k - k^2 \left[1 + M_0^2 - \frac{M_0}{2\sqrt{1+M_0^2}} \log\left(\frac{\sqrt{1+M_0^2} + M_0}{\sqrt{1+M_0^2} - M_0} \right) \right]$$

and it is relatively simple to show that this produces greater stability than in the nonmagnetic case for all M_0 . The proof rests on the fact that

$$M_0 \sqrt{1+M_0^2} \geq \log\left(\frac{\sqrt{1+M_0^2} + M_0}{\sqrt{1+M_0^2} - M_0} \right)$$

At $M_0 = 0$ there is equality and beyond it we take

$$\frac{d}{dM} \left\{ M_0 \sqrt{1+M_0^2} - \log\left(\frac{\sqrt{1+M_0^2} + M_0}{\sqrt{1+M_0^2} - M_0} \right) \right\} = 4 \frac{M_0^2}{\sqrt{1+M_0^2}} > 0$$

so that the nonmagnetic problem is more stable.

If, however, we choose

$$M = M_0 \operatorname{sech}^2 z$$

for the Holmboe problem, (6.21) becomes

$$\left.\frac{dG}{dM_0^2}\right)_{M_0=0} = k^2 \int_0^\infty \left\{ \operatorname{sech}^4 z \left[-1 - \frac{1}{\tanh^2 z} \right] - \right.$$

$$\left. - D\left(\frac{\operatorname{sech}^2 z}{\tanh z} \right) \right\} dz$$

$$= k^2 \int_0^\infty \left(2\operatorname{sech}^2 z - \operatorname{sech}^4 z \right) dz > 0$$

and since the integrand is always positive, a small magnetic field causes greater instability.

VII. Critique

Roughly one hundred years of experience with hydrodynamic stability studies of ideal parallel flow has shown that the problem is, indeed, quite difficult. When a parallel magnetic field is added, the situation becomes even more complex. The methodology employed in the creation of this dissertation has been to attempt to apply the techniques used in the hydrodynamic studies to the magnetic version of the problem. Some of these techniques have proven to be fruitful but many have not.

Most of the early work on magnetic fluids has shown the magnetic field to act as a stabilizing agent. In fact when the magnetic energy is larger than the kinetic energy everywhere all possibility of instability is ruled out for normal mode disturbances (see also, eg., Gilman 1966). Several researchers, notably Kent, have been able to demonstrate, without obtaining complete solutions, that magnetic configurations which destabilize otherwise stable flows (of a homogeneous fluid) can frequently be found. The technique most successfully used in this connection is to expand for small wave number around the known (but somewhat artificial) solution at zero wave number. All that is determined from this procedure is knowledge of whether the wave speed has an imaginary part.

The double shear layer model presented in this dissertation represents the first analytically solved problem for parallel flow of an ideal magnetofluid which shows there are magnetic field values which cause destabilization of the fluid, both with and without stratification. Because of the simplicity of the model, a tentative physical explanation for the destabilizing effect that the magnetic field may produce was offered. It seems quite possible that this mechanism may extend to fluids with continuous distributions of velocity and density.

The value of this example suffers from one main drawback. The distributions of velocity and density have discontinuities and when the magnetic field varies at all, it too does so discontinuously. The degree to which discontinuous models are representative of the physical processes of continuous models falls somewhat short of completeness.

No complete solution for a continuous velocity and density profile exists, even in the nonmagnetic case. Reliance is placed on general theorems such as Rayleigh's theorem for homogeneous flows and Howard's result for heterogeneous fluids that no instability can exist when the Richardson number is less than minus one quarter. In the magnetic case a result analogous to Rayleigh's theorem has been proven, but for a heterogeneous fluid no result limiting the stratification for instability could be proved. For a homogeneous nonmagnetic flow there is stability for sufficiently short waves and although a similar result is expected for the magnetic problem, none has been found.

When dealing with a heterogeneous fluid, one often tries to locate a stability boundary. This consists of a curve of values of the

Richardson number as a function of wave number for which the real part of the wave speed lies within the range of values of the velocity and the imaginary part of the wave speed equals zero. The physical situation is supposed to reflect the fact that any decrease of the Richardson number at a given wave length below the "critical" value leads to a situation where there is now sufficient energy to cause the system to become unstable. Miles has provided mathematical support for the conclusion for a certain class of shear flows. The same assumption seems reasonable for magnetic flows but is, so far, without an equivalent proof.

This shortcoming is not a pressing question at present, if only because so far no marginally stable solutions have been obtained for magnetic problems. This is due to the fact that whereas the singularities of the nonmagnetic problem are generally algebraic, in the magnetic problem we have logarithmic singularities to contend with in most cases. In several of the formulas of Chapters 4, 5 and 6, we see how the logarithmic terms enter and complicate the situation. This difficulty is further reflected in the fact that whereas the nonmagnetic problem has a governing equation which may be written in at least two highly useful forms, in the magnetic case only one of the forms yields much information readily. In any case, the obtaining of a marginally stable solution is a very desirable goal and would give a better indication of the accuracy of the formulas at the end of Chapter 6 which purport to describe the stability boundary for unbounded shear flow.

No numerical techniques have been developed or used in this work. It was felt that there was large enough ground to make progress by more elemental methods, which would afford, at least in the early stages of research, more insight into the nature of the problem. Although it would be desirable to have some analytic result to which the numerical results may be compared, the computer study might still lend support to the expansion schemes of Chapters 5 and especially 6. It is anticipated that this will provide an open avenue for future research in this area.

Appendix A.

We take the y component of the curl of the equations of motion and then differentiate with respect to x. Using the fact that the velocity and magnetic field are divergenceless, the result is expressed as

$$Q\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial z^2}\right)-\frac{\partial w}{\partial x}\frac{\partial^2 U}{\partial z^2}=\alpha g\frac{\partial^2 T}{\partial x^2}+M\frac{\partial}{\partial x}\left(\frac{\partial^2 h_z}{\partial x^2}+\frac{\partial^2 h_z}{\partial z^2}\right)-\frac{\partial h_z}{\partial x}\frac{\partial^2 M}{\partial z^2} \tag{A.1}$$

Taking Q (A.1) enables us to eliminate reference to T by use of the thermal equation. We must take Q^4 (A.1) in order to eliminate reference to the perturbation magnetic field by use of the z component of the magnetic equation in (2.18). Care must be taken since

$$Q\frac{\partial}{\partial z}f \neq \frac{\partial}{\partial z}Qf$$

In fact using the operations

$$Q\frac{\partial}{\partial z}f=\frac{\partial}{\partial z}Qf-\frac{\partial U}{\partial z}\frac{\partial f}{\partial x}$$

$$Q\frac{\partial^2}{\partial z^2}f=\frac{\partial^2}{\partial z^2}Qf-\frac{\partial^2 U}{\partial z^2}\frac{\partial f}{\partial x}-2\frac{\partial U}{\partial z}\frac{\partial}{\partial x}\frac{\partial f}{\partial z}$$

repeatedly on the magnetic terms of Q^4 (A.1) leads directly to (2.21).

Appendix B.

Equation (2. 23) is now derived from (2. 22). Since

$$W = QF$$

we can write (2. 22) as

$$D\left[\left(1-\frac{M^2}{Q^2}\right)D(QF)\right]-\left\{k^2\left[1-\frac{M^2}{Q^2}\right]+\right.$$
$$\left.+\frac{1}{Q}D\left(DQ\left[1-\frac{M^2}{Q^2}\right]\right)+\bar{R}_i\right\}QF=0 \qquad \text{(B. 1)}$$

Now

$$D(QF) = QDF + DQF$$
$$D^2(QF) = 2DQDF + QD^2F + D^2QF$$

so that (B. 1) can be written

$$\left[Q-\frac{M^2}{Q}\right]D^2F+\left[-\left(\frac{DM^2}{Q^2}-\frac{DQ^2}{Q^4}M^2\right)Q+\left(1-\frac{M^2}{Q^2}\right)\cdot\right.$$
$$\left.\cdot 2DQ\right]DF+\left[\left(1-\frac{M^2}{Q^2}\right)D^2Q-\frac{1}{2}\left(\frac{DM^2}{Q^2}-\frac{DQ^2}{Q^4}M^2\right)DQ^2-\right.$$
$$\left.-k^2\left(1-\frac{M^2}{Q^2}\right)-D\left[DQ\left(1-\frac{M^2}{Q^2}\right)\right]-\bar{R}_iQ\right]F=0$$

Cancelling where appropriate and multiplying by Q leads directly to equation (2. 23). Other general forms for (2. 23) may be derived simply by substituting

$$F = Q^{-n}(Q^2-M^2)^{-m}R$$

and when $n=0$; $m=1/2$ we obtain another useful form of the equation, given for the homogeneous case by (5. 1).

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