THEORY OF TRANSIENT HEAT TRANSFER

IN LAMINAR FLOW

Applied to the entrance region of tubes with heat capacity

by

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M.S., Technical University of Istanbul, Turkey
(1952)

SUBMITTED IN PARTIAL FULFILLMENT OF THE

REQUIREMENTS FOR THE DEGREE OF

DOCTOR OF SCIENCE

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

May, 1958

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May 19, 1958

Professor Warren M. Rohsenow
Chairman of the Departmental Committee
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Department of Mechanical Engineering
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Cambridge 39, Massachusetts

Dear Professor Rohsenow:

I submit herewith my thesis entitled "Theory of Transient Heat Transfer in Laminar Flow" (Applied to the Entrance Region of Tubes with Heat Capacity), in partial fulfillment of the requirements for the degree of Doctor of Science.

Very truly yours,

VEDAT S. ARPACI

To My Mother

Aliye Fatma ARPACI

"I believe that the engineer needs primarily the fundamentals of mathematical analysis and sound methods of approximation."

Theodore von Kármán*

---

* "Some Remarks on Mathematics from the Engineer's Viewpoint", Mechanical Engineering, April 1940.

CONTENTS

ERRATA

b_0, b_1, b_2 as defined on pp. 53 are respectively equal to c_0, c_1, c_2 which are defined on pp. 60. Therefore, the second notations should be discarded.

Chapters I through VII, the equations are numbered between two parenthesis. The same notation is also used to indicate the references referred to in the Bibliography.

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ABSTRACT

The transient heat transfer phenomenon in laminar incompressible flow at the entrance region of tubes having small length-to-diameter ratios has been investigated.

To introduce the effect of heat capacity of the tube-wall, the principle of conservation of energy for the tube-wall was considered in addition to the usual fluid continuity, momentum and energy equations.

Assuming that the velocity profiles at the entrance region of the tube can be approximated by velocities of the laminar incompressible flow over a flat plate, the solution of the fluid flow problem was taken from previous work. The remaining two partial differential equations, the fluid and tube-wall energy equations, have been solved by using the method of successive approximations. The first two approximations were considered. Because of the analytical difficulties encountered in the solution of the second approximations, these approximations have been evaluated only for small values of the dimensionless number $MFo/X^{1/2}$, that is $MFo/X^{1/2} \leq 0.4$.

The theory applies to viscous oils, water and air in the range of δ_v/R less than 0.3, and to the liquid metals where δ_v/R is less than 0.1.

A modified Nusselt number based on the step temperature input was defined as

$$Nu(x,t) = \frac{q(x,t)}{k\Delta T_e/D} .$$

The use of this number in a one-dimensional analysis to predict the mean temperature variations of the fluid and tube-wall, uncouples the two energy equations. This method is most convenient for analytical solutions.

The theory has been checked by some experimental measurements. The first approximation of the tube-wall temperature has been compared with a quasi-steady theory. The result is that, for very small and very large values of $MFo/X^{1/2}$, the quasi-steady theory closely approximates the present theory.

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ACKNOWLEDGEMENTS

The author is greatly indebted to Professor W. M. Rohsenow for the opportunity to work on this problem and for his continuous encouragement.

Thanks are due to the members of the thesis committee, Professors S. H. Crandall, F. A. McClintock, T. Y. Toong and R. J. Nickerson and Professors J. A. Clark and P. Griffith for their criticism, suggestions and for many other ways.

Mr. A. J. Shashaty and Mr. M. M. Chen have been helpful in every respect. Mr. A. J. Shashaty must be separately credited for reading the text and correcting its English.

The author wishes to express his appreciation to the U. S. Air Force, Wright Air Development Center, Aeronautical Research Laboratory, and Dr. Erich Soehngen of that organization for their financial support throughout this work. He also expresses his gratitude to M.I.T. Computation Center who have generously given him their advise and help during the numerical evaluation of some functions which have appeared in the last stage of this study.

Finally, but by no means least, thanks are due to Mr. E. Hartell for his help in the experimental work and to Miss H. Nicholson for her excellent typing.

I. INTRODUCTION

1. General Problem

Consider a fluid flowing over a fixed solid S with a boundary surface σ (Fig. 1), and suppose that either fluid or solid is subjected to a transient effect throughout its volume or at one of the boundaries. This effect can be accomplished by a time-dependent heat generation within the solid or fluid, by a time-dependent temperature change at one of the fluid boundaries, or by a time-dependent fluid velocity. The temperature field in the fluid and solid, and the velocity field in the fluid, are desired.

The relevant equations are those of the conservation of mass, momentum, and energy, and the equation of state. The energy equation is written for the solid as well as for the fluid. The continuity and momentum equations are not independent of the energy equations and the equations of state, being related through the physical properties of the fluid and wall. At the same time, the fluid energy equation will also be connected with the solid energy equation through the boundary condition on the fluid-solid interface. These two types of dependence, or coupling, which are basically different, often result from the application of fundamental laws to physical problems.

Owing to the mathematical difficulties encountered in solving the general problem, very little attention in the past has been devoted to solutions of the problem in which the second type of coupling is included. The major effort to obtain information about transient heat transfer phenomena has been confined to problems with certain simplified boundary conditions. These conditions are those of constant or prescribed temperature or heat flux at the fluid-solid interface. In this way, the second type of coupling described above is avoided. As a first step in the solution of the general problem it is possible to neglect the first type of coupling and to consider the second type in detail.

The second problem is actually analogous to one encountered in many mechanical problems, that of determining the speed of rotation of a system which is composed of a driven element which is connected with a driver by means of a real coupling (Fig. 2). For a stable system, the answer is simply that steady angular velocity ω_0 and torque C_0 for which the $C-\omega$ characteristics of the driven element and driver intersect. Thus, in order to predict the speed and torque in advance of an experiment it is necessary to know the speed of each element at any torque load.

Similarly, in the heat transfer problem, the temperature and the heat flux at the fluid-solid interface for all values of this temperature and flux must be known. In analogy to the mechanical illustration, the solution to the problem would require that the boundary conditions agree, that is, that the temperature and heat flux of the fluid and solid be equal at every point of σ .

Before the second world war, the problem described so far--even for steady-state cases--was too complicated for analytical solution. The usual steady-state procedure was to assume either the conductivity of the solid so high that the temperature throughout is approximately uniform or an approximate temperature or heat flux based on experiments on similar bodies.

2. Present Problem

In view of the importance of circular tubes to many applications of engineering, the above mentioned second problem is considered for the entrance region of tubes having small length-to-diameter ratios. In such short tubes, the fluid and heat flows are found to be greatly influenced by the rapid change of the velocity and temperature profiles. As a result of these rapid changes, the use of a steady or transient but lengthwise one-dimensional analysis to predict the pressure drop of the fluid and the mean temperatures of the fluid and tube-wall requires specific knowledge of the variations of the apparent friction factor and the heat flux along the fluid-tube interface. Therefore, radial variations of the velocity and temperature profiles must be taken into account. For the present problem, a time-independent developing laminar velocity boundary layer is considered and necessary information is taken from previous works. Then the transient heat transfer phenomena for the fluid and tube, for a step temperature change at the entrance of the tube, are investigated.

3. Previous Work

Original interest in this problem, arising in connection with the study of transient heat exchange systems, is found in the German literature dating back to thirty years ago. Generally speaking, there are two types of systems by which this is effected: heat exchangers and heat regenerators. For the present discussion, double fluid heat exchange systems are omitted. In heat regenerators the hot and cold fluids are passed cyclically over a solid wall. The storage of heat in the wall is fundamentally important and must be taken into account.

For a slug flow of an incompressible fluid, theoretical analyses of length and time-dependent fluid and wall temperatures are given by Anzelius (1), Nusselt (2), Schumann (3), Hausen (4) (5), Carslaw and Jeager (6) and Rizika (7) (8). In each of these cases, a step temperature change or a temperature sinusoidal in time is considered.

The same problem has recently been extended to transient power variations within the tube wall by Clark, Arpaci and Treadwell (9) and Arpaci and Clark (10) (11) (12). Allowing radial variations in fluid temperature, another extension resulting from the combination of (1) with the first Graetz problem (13) has also been made by Arpaci (14). In all the above analyses, excluding the last, a constant heat transfer coefficient was assumed.

The recent appearance of papers by Bryson and Edwards (15), Emmons (16) and Yoshihara (17) in laminar boundary layer flow, deal with quasi-steady heat transfer phenomena over a flat plate, including the effect of the heat capacity of the wall.

4. Method of Present Study

As previously indicated, the fluid flow part of the problem has already been solved. The solution obtained by Blasius (18) using a similarity variable is well known. With respect to the heat flow, the existence of the terms $\frac{\partial T}{\partial t}$ and $\frac{1}{r}\frac{\partial T}{\partial r}$ in the fluid energy equation preclude a similar type of solution. Therefore, the calculation of the commencement of the temperature boundary layer will be carried out by successive approximations.

Most of the starting flat plate velocity boundary layer problems assume (after Blasius), on the basis of physical reasoning, that at the beginning of the motion, the boundary layer is very thin and the viscous term $\nu \frac{\partial^2 u}{\partial y^2}$ is very large, whereas the convective terms retain their normal values. For a first approximation, neglecting the convective terms $u\frac{\partial u}{\partial x}$ and $v\frac{\partial u}{\partial y}$, the momentum equation is reduced to the heat conduction equation. Then the convective terms of the second approximation are calculated from the first approximation, and so on.

Therefore, for a non-similar velocity or temperature boundary layer problem, the general method should be to seek a first approximate solution which gives the closest answer to the problem. Then the successive approximations are iterated from this first approximation.

This logic has been followed for the first approximation of the present problem by simply taking a slug velocity U, which means a convective term $U\frac{\partial T}{\partial x}$ instead of $u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial r}$. This reduction of the fluid energy equation appears to correspond to the well-known Oseen approximation for the fluid momentum equation. However, there is a fundamental difference between these two cases. For fluid flow, the change is mathematical and is the approximation of the momentum equations through a linearization. But the physical character of the problem is not altered. For heat flow, on the other hand, the physical character of the fluid flow is basically changed by taking a slug (perfect fluid) flow instead of the proper viscous profile.

Now, suppose T_0 and θ_0 are the first approximations of the fluid and the tube-wall temperatures. A set of exact solutions may be written in the form

$$T = T_0 + T_1(T_0) + T_2(T_1) + \cdots = \sum_{k=0}^{\infty} T_k(T_{k-1})$$

$$\theta = \theta_0 + \theta_1(\theta_0) + \theta_2(\theta_1) + \cdots = \sum_{k=0}^{\infty} \theta_k(\theta_{k-1})$$

For a sufficiently large number of terms, the solutions do not depend on the first approximations. But as a result of the increasing complexity of the higher order approximations, only the first two approximations will be considered. Therefore, the accuracy of the solutions will depend primarily on the first approximations.

II. ANALYSIS

The system under consideration is shown in Fig. 4 and consists of the entrance region of a constant-diameter circular tube in which a fluid is flowing.

The following assumptions are made:

a) The fluid flow is steady and laminar,

b) The velocity profile along the tube is approximated by the laminar flow over a flat plate,

c) The outer surface of the tube is adiabatic,

d) Axial heat conduction is negligible, both in the fluid and the tube-wall,

e) Radial heat conduction is infinite in the tube-wall. Therefore, the tube temperature is not a function of radial distance,

f) The physical properties $(\rho_w, C_{p_w}, \rho, C_p, k)$ of the fluid and tube-wall are constant,

g) Initially both fluid and tube-wall have the same constant temperature, which may be taken equal to zero,

h) Kinetic energy and dissipation terms in the fluid energy equation are negligible compared to the others,

i) Boundary-layer assumptions are valid for fluid momentum and energy equations,

j) The transient effect is introduced by a step change in the fluid temperature from an initial condition.

A system of co-ordinates fixed to the stationary tube is chosen. The equations of conservation of mass and momentum are omitted here because the fluid flow problem is taken to be already solved in previous assumption (b). The application of the first law of thermodynamics (energy equation) to a tube element (a closed system) and fluid control volume results in the following two partial differential equations. Details of this derivation and the integration of the differential equations are outlined in Appendix A.

For the fluid

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial r} = \alpha\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r}\right) \qquad (1)$$

For the tube-wall

$$\frac{\partial \theta}{\partial t} + \epsilon\left(\frac{\partial T}{\partial r}\right)_{r=R} = 0 \qquad (2)$$

where

$$\epsilon = 2\,\frac{\alpha}{R}\cdot\frac{(\rho C_p)}{(\rho C_p)_w}\cdot\frac{1}{(R_0/R)^2 - 1} \qquad (3)$$

The initial and boundary conditions of the problem are:

Initial

$$T(r,x,0) = 0 \qquad (4)$$

$$\theta(x,0) = 0 \qquad (5)$$

Boundary

$$T(r,0,t) = \Delta T_e \qquad (6)$$

$$\frac{\partial T(0,x,t)}{\partial r} = 0 \qquad (7)$$

$$T(R,x,t) = \theta(x,t) \qquad (8)$$

As indicated in part I.4, the problem is solved by a method of successive approximations. The first two approximations are obtained.

In terms of these approximations, the fluid and tube-wall temperatures, respectively, are

$$T = T_0 + T_1 \tag{9}$$

$$\theta = \theta_0 + \theta_1 \tag{10}$$

Then the first approximations satisfy the following two simultaneous partial differential equations, and proper initial and boundary conditions

$$\frac{\partial T_0}{\partial t} + U\frac{\partial T_0}{\partial x} = \alpha\left(\frac{\partial^2 T_0}{\partial r^2} + \frac{1}{r}\frac{\partial T_0}{\partial r}\right) \tag{11}$$

$$\frac{\partial \theta_0}{\partial t} + \epsilon\left(\frac{\partial T_0}{\partial r}\right)_{r=R} = 0 \tag{12}$$

$$T_0(r,x,0) = 0 \tag{13}$$

$$\theta_0(x,0) = 0 \tag{14}$$

$$T_0(r,0,t) = \Delta T_e \tag{15}$$

$$\frac{\partial T_0(0,x,t)}{\partial r} = 0 \tag{16}$$

$$T_0(R,x,t) = \theta_0(x,t) \tag{17}$$

If these equations are subtracted from equations (1), (2), (3), (4), (5), (6), (7) and (8), and the convective terms $U\frac{\partial T}{\partial x}$ and $V\frac{\partial T}{\partial r}$ are approximated by $U\frac{\partial T_0}{\partial x}$ and $V\frac{\partial T_0}{\partial r}$, the following system is obtained for the second approximations

$$\frac{\partial T_1}{\partial t} + U\frac{\partial T_1}{\partial x} = \alpha\left(\frac{\partial^2 T_1}{\partial r^2} + \frac{1}{r}\frac{\partial T_1}{\partial r}\right) + U\left[\left(1-\frac{u}{U}\right)\frac{\partial T_0}{\partial x} - \frac{v}{U}\frac{\partial T_0}{\partial r}\right] \qquad (18)$$

$$\frac{\partial \theta_1}{\partial t} + \epsilon\left(\frac{\partial T_1}{\partial r}\right)_{r=R} = 0 \qquad (19)$$

$$T_1(r,x,0) = 0 \qquad (20)$$

$$\theta_1(x,0) = 0 \qquad (21)$$

$$T_1(r,0,t) = 0 \qquad (22)$$

$$\frac{\partial T_1(0,x,t)}{\partial r} = 0 \qquad (23)$$

$$T_1(R,x,t) = \theta_1(x,t) \qquad (24)$$

The method of solution described below is general and applies to both of the approximations.

Let $\Lambda_i(x,t)$, $(i=1,2)$ be any arbitrary function which is zero when $t < \frac{x}{U}$, where U is the core velocity of the tube flow. Suppose boundary conditions (17) and (24) are equal to this function. Then each approximation of the fluid energy equation can be solved in terms of this arbitrary function. For convenience, the simple Laplace transform in the time variable is employed first. Next a transformation $\bar{T}_i(r,x,p) = \bar{\Pi}_i(r,x,p)e^{-\frac{x}{U}p}$ for the dependent variable is used to simplify the problem. Then the use of finite Hankel transforms in the radial direction and the inverse transformation results in the following expression for the Laplace transformed fluid temperature

$$\bar{\Pi}_i(r,x,p) = f_i\left[\bar{\Lambda}_i(x,p)\right] \qquad (25)$$

If this equation is substituted into the tube-wall energy equation, a Volterra integral equation of the second kind

$$\bar{\Lambda}_i(x,p) = \frac{1}{p^2} G_i(x) + \frac{1}{p}\int_0^x K(x,\xi)\bar{\Lambda}_i(\xi,p)d\xi \qquad (26)$$

is obtained. In this equation $\bar{\Lambda}_i(x,p)$ is the unknown function to be determined.

Since in the present case the tube-wall temperature is taken to be independent of the radius, the solutions of the Volterra equation above give the successive approximations of the tube-wall temperature. Then substitution of these temperatures into $\bar{\Pi}_i(r,x,p)$ followed by inverse transformation gives the successive approximations of the fluid temperature. In the solution of the Volterra equation, the kernel $K(x,\xi)$, and the function $G_i(x)$ involve summations over one or two indices. Although a theoretical solution is possible and has been obtained in Appendix C, numerical application is considerably difficult. To avoid this difficulty, $K(x,\xi)$ and $G_i(x)$ are first calculated and plotted. Then simple curves are used to represent the results so obtained. The solution is carried out with these approximations.

Inspection reveals that, for successive terms of the tube-wall temperature expansion, the kernel $K(x,\xi)$ is the same. The reason for this is readily seen after investigating the mathematical behaviour of (A-40) and (A-99). The exact and approximated curves for $K(x,\xi)$ are shown in Fig. 5. Since the theory applies within the range $0 \leq \frac{x/D}{Pe} \leq 10^{-2}$ the maximum error introduced, in the approximation of $K(x,\xi)$, is 1.52% occurring at the extreme value, $\frac{x/D}{Pe} = 10^{-2}$.

The function $G_i(x)$ is different for each value of i. The curves for $G_1(x)$ are shown in Fig. 5. The analysis described above gives the following equation for the first approximation of the Laplace transformed fluid temperature in terms of the unknown function $\bar{\Lambda}_0(x,p)$.

$$\bar{\Pi}_0(r,x,p) = 2\frac{\Delta T_e}{p}\sum_{k=1}^{\infty} e^{-4\lambda_k^2 \frac{x/D}{Pe}} \frac{J_0(\lambda_k \frac{r}{R})}{\lambda_k J_1(\lambda_k)} + 2\left(\frac{4}{DPe}\right)\sum_{k=1}^{\infty} \frac{\lambda_k J_0(\lambda_k \frac{r}{R})}{J_1(\lambda_k)} \int_0^x e^{-4\lambda_k^2 \frac{(x-\xi)/D}{Pe}} \bar{\Lambda}_0(\xi,p)\,d\xi \tag{27}$$

By means of this, and the first approximation of the Laplace transformed form of the tube-wall energy equation (12), the following integral equation results

$$\bar{\Lambda}_0(x,p) = \frac{\Delta T_e}{p^2}\left(\frac{2\epsilon}{R}\right)\sum_{k=1}^{\infty} e^{-4\lambda_k^2 \frac{x/D}{Pe}} + \frac{1}{p}\left(\frac{2\epsilon}{R}\right)\left(\frac{4}{DPe}\right)\int_0^x \sum_{k=1}^{\infty} \lambda_k^2 e^{-4\lambda_k^2 \frac{(x-\xi)/D}{Pe}} \bar{\Lambda}_0(\xi,p)\,d\xi \tag{28}$$

This is the first approximation of the Laplace transformed tube-wall temperature. In this equation, $K(x,\xi)$ and $G_1(x)$ are approximated by curves in Fig. 5.

Then the solution for the first approximation of the tube-wall temperature is

$$\frac{\theta_0(x,t)}{\Delta T_e} = \begin{cases} 0\ , & t < \frac{x}{U} \\ erf\left[\dfrac{MF_0}{\left(\frac{x/D}{Pe}\right)^{1/2}}\right]\ , & t \geq \frac{x}{U} \end{cases} \tag{29}$$

Substitution of this result into $T_0(r,x,t)$ gives the first approximation of the fluid temperature

$$T_0(r,x,t) = \begin{cases} 0, & t < \frac{x}{U} \\ 2\sum_{k=1}^{\infty}\left[1 + 4\lambda_k^2 A_k\left(\lambda_k, \frac{x/D}{P\acute{e}}, MFo\right)\right] e^{-4\lambda_k^2 \frac{x/D}{P\acute{e}}} \frac{J_0(\lambda_k r/R)}{\lambda_k J_1(\lambda_k)}, & t \geq \frac{x}{U} \end{cases} \tag{30}$$

where

$$A_k\left(\lambda_k, \frac{x/D}{P\acute{e}}, MFo\right) = \int_0^{\frac{x/D}{P\acute{e}}} e^{4\lambda_k^2 \frac{\xi/D}{P\acute{e}}} \operatorname{erf}\left[\frac{MFo}{\left(\frac{\xi/D}{P\acute{e}}\right)^{1/2}}\right] d\left(\frac{\xi/D}{P\acute{e}}\right) \tag{31}$$

The use of equations (12) and (29) gives the following expression for the modified Nusselt number

$$Nu_0(x,t) = \frac{1}{\pi^{1/2}\left(\frac{x/D}{P\acute{e}}\right)^{1/2}} e^{-\left[\frac{MFo}{\left(\frac{x/D}{P\acute{e}}\right)^{1/2}}\right]^2} \tag{32}$$

Following the same procedure, the second approximation of the Laplace transformed fluid temperature in terms of the unknown function $\bar{\Lambda}_1(x,p)$ is obtained as follows

$$\bar{\Pi}_1(r,x,p) = 2\sum_{n=1}^{\infty}\frac{J_0(\lambda_n r/R)}{J_1^2(\lambda_n)}\int_0^x\int_0^1 e^{-4\lambda_n^2\frac{(x-\xi)/D}{P\acute{e}}}(r/R)\,\bar{F}(r/R,\xi,p)\,J_0(\lambda_n r/R)\,d(r/R)\,d\xi$$
$$+ 2\left(\frac{4}{DP\acute{e}}\right)\sum_{n=1}^{\infty}\frac{\lambda_n J_0(\lambda_n r/R)}{J_1(\lambda_n)}\int_0^x e^{-4\lambda_n^2\frac{(x-\xi)/D}{P\acute{e}}}\bar{\Lambda}_1(\xi,p)\,d\xi \tag{33}$$

where

$$\bar{F}(r/R,\xi,p) = U\left(1-\frac{u}{U}\right)\frac{\partial\bar{\Pi}_0}{\partial\xi} - v\frac{\partial\bar{\Pi}_0}{\partial r} \tag{34}$$

Again, by substituting this equation into the second approximation of the Laplace transformed form of the tube-wall energy equation (19), a Volterra equation of the second kind, involving the second approximation of the Laplace transformed tube-wall temperature is obtained.

$$\bar{\Lambda}_1(x,p) = \left(\frac{2\epsilon}{R}\right)\frac{1}{p}\sum_{n=1}^{\infty}\frac{\lambda_n}{J_1(\lambda_n)}\int_0^{\frac{x/D}{P\acute{e}}}\int_0^1 e^{-4\lambda_n^2\frac{(x-\xi)/D}{P\acute{e}}}(r/R)(DP\acute{e})\bar{F}(r/R,\Xi,p)J_0(\lambda_n r/R)\,d(r/R)\,d\Xi$$

$$+\frac{1}{p}\left(\frac{2\epsilon}{R}\right)\left(\frac{4}{DP\acute{e}}\right)\int_0^x\sum_{n=1}^{\infty}\lambda_n^2 e^{-4\lambda_n^2\frac{(x-\xi)/D}{P\acute{e}}}\bar{\Lambda}_1(\xi,p)\,d\xi \qquad (35)$$

Because of the complicated form of (35), only the first term of $\bar{F}(r/R,\Xi,p)$ is considered.

For large values of $MFo/\left(\frac{x/D}{P\acute{e}}\right)^{1/2}$, the tube-wall temperature asymptotically approaches unity. For small values of $MFo/\left(\frac{x/D}{P\acute{e}}\right)^{1/2}$, in the expansion of $\Lambda_1(x,t)$, the first term is proportional to $MFo/\left(\frac{x/D}{P\acute{e}}\right)^{1/2}$, the second one to $1/2!\left[MFo/\left(\frac{x/D}{P\acute{e}}\right)^{1/2}\right]^2$, etc. Thus, for small values of this argument, the first term of (35) is alone satisfactory in determining the second approximation of the tube-wall temperature. Therefore, for small values of $MFo/\left(\frac{x/D}{P\acute{e}}\right)^{1/2}$, the second approximation can be written as

$$\frac{\theta_1(x,t,Pr)}{\Delta T_e} = \begin{cases} 0, & t < \frac{x}{U} \\ -MFo\displaystyle\int_0^X\int_{1-\delta/R}^1 (r/R)\left(\frac{DP\acute{e}}{\Delta T_e}\right)I(r/R,\Xi,Pr)\,8\sum_{n=1}^{\infty}\lambda_n e^{-4\lambda_n^2(X-\Xi)}\frac{J_0(\lambda_n r/R)}{J_1(\lambda_n)}\,d(r/R)\,d\Xi, & t \geq \frac{x}{U} \end{cases} \qquad (36)$$

where $\left(\frac{DP\acute{e}}{\Delta T_e}\right)I(r/R,\Xi,Pr)$ is defined with (A-119). $\left(\frac{DP\acute{e}}{\Delta T_e}\right)I(r/R,\Xi,Pr)$ and $\sum_{n=1}^{\infty}\lambda_n e^{-4\lambda_n^2 X}\frac{J_0(\lambda_n r/R)}{J_1(\lambda_n)}$ have been evaluated by a high speed digital computer. If the approximated forms of these functions are substituted into (36), the following is obtained

$$\frac{\theta_1(x,t,Pr)}{\Delta T_e} = \begin{cases} 0, & t < \frac{x}{U} \\ -0.2P(Pr)\left[1 - 5\left(\frac{Pr}{\pi}\right)^{1/2} e^{-\frac{25}{4}Pr} - \left(1 - \frac{25}{2}Pr\right) \operatorname{erfc}\left(\frac{5}{2}Pr^{1/2}\right)\right] \frac{MFo}{\left(\frac{x/D}{Pe}\right)^{1/2}}, & t \geq \frac{x}{U} \end{cases} \quad (37)$$

In Appendix A, $P(Pr)$ is given by (A-124). The terms in the square brackets have appreciable effect on the temperature only for nuclear metals.

Then the use of (19) and (37) gives the following expression for the second approximation of the modified Nusselt number

$$Nu_1(x,t,Pr) = \begin{cases} 0, & t < \frac{x}{U} \\ -0.1\frac{P(Pr)}{\left(\frac{x/D}{Pe}\right)^{1/2}}\left[1 - 5\left(\frac{Pr}{\pi}\right)^{1/2} e^{-\frac{25}{4}Pr} - \left(1 - \frac{25}{2}Pr\right) \operatorname{erfc}\left(\frac{5}{2}Pr^{1/2}\right)\right], & t \geq \frac{x}{U} \end{cases} \quad (38)$$

Finally, because of the linear nature of the problem, the tube-wall temperature and the modified Nusselt number can be written in terms of the first and second approximations as

$$\theta = \theta_0 + \theta_1 \quad (39)$$

$$Nu = Nu_0 + Nu_1 \quad (40)$$

or, explicitly

$$\frac{\theta(x,t,Pr)}{\Delta T_e} = \begin{cases} 0, & t < \frac{x}{U} \\ \operatorname{erf}\left[\frac{MFo}{\left(\frac{x/D}{Pe}\right)^{1/2}}\right] - 0.2P(Pr)\left[1 - 5\left(\frac{Pr}{\pi}\right)^{1/2} e^{-\frac{25}{4}Pr} \right. \\ \left. \quad -\left(1 - \frac{25}{2}Pr\right)\operatorname{erfc}\left(\frac{5}{2}Pr^{1/2}\right)\right]\frac{MFo}{\left(\frac{x/D}{Pe}\right)^{1/2}}, & t \geq \frac{x}{U} \end{cases} \quad (41)$$

and

$$Nu(x,t,Pr) = \begin{cases} 0\,, & t < \frac{x}{U} \\ \frac{1}{\pi^{1/2}\left(\frac{x/D}{P\acute{e}}\right)^{1/2}} \left[e^{-\left[\frac{MFO}{\left(\frac{x/D}{P\acute{e}}\right)^{1/2}}\right]^2} - 0.1\,\pi^{1/2} p(Pr)\left[1 - 5\left(\frac{Pr}{\pi}\right)^{1/2} e^{-\frac{25}{4}Pr} - \left(1 - \frac{25}{2}Pr\right) erfc\left(\frac{5}{2}Pr^{1/2}\right)\right]\right]\,, & t \geq \frac{x}{U} \end{cases} \qquad (42)$$

III. EXPERIMENT

1. Description of the Test Apparatus

Ordinarily, experimental verification of analytical boundary layer theories requires delicate and expensive experiments involving much time and instrumentation. This necessity arises from the fact that the condition of a constant or prescribed temperature or heat flux, is usually imposed at the fluid-solid interface. These theories, therefore, can be checked only by taking measurements in fluid stream. The advantage of the present theory is that it can be checked by making only tube-wall or fluid temperature measurements, or both. In this report, experimental work based on tube-wall temperature measurements was carried out. A schematic representation of this work is shown in Fig. 10 (see also Plate 1). For simplicity, a once-through water system was used. Water drawn from the central supply mains was conducted to a tank of constant temperature and head through standard 1/2 in. pipe lines. To obtain a temperature level above room temperature in the tank, two hot water heaters (each having two elements rated at 240 V., 7 KW.) were connected to the input of the tank. One element of one of the heaters was directly connected to the circuit. Two elements were controlled by two on and off switches, and the remaining one was adjusted by means of a variac type autotransformer (50-60 cycles, 10 A., 240 V.). The desired temperature was obtained and kept constant by the proper adjustment of heater elements.

The test section was constructed from a 1/2 in. (0.625 I.D., 0.840 O.D.) copper tube 24 in. long and had a 0.1075 in. wall thickness. To insure laminar flow, a boundary layer suction slot (Fig. 11) was provided at the entrance of the test section in addition to two small heads, 1 in. and 2.5 in. Up to 15% of the flow could be removed from this slot by means of an ordinary valve. Four pairs of 30 gauge Iron-Constantan thermocouples (Fig. 12) were mounted to the test section. These thermocouples were chosen because of their availability at the Laboratory and high EMF output per degree of temperature difference. In the theory of this study the tube-wall temperature was assumed independent of the radial distance. To check the validity of this assumption, two thermocouples were planned to fix at each location, one on the outside of the tube-wall, the other on the inside. However, the finite thickness of thermocouples located at the fluid-solid interface, would prevent exact measurement of the inside wall temperature. For this reason a vertical-tangential saw cut was made (Fig. 12) at four locations on the tube in order to measure the inside wall temperature, assuming that the remaining wall thickness has negligible transient effect compared with the entire tube-wall thickness. For small Reynolds numbers, because of free convection effects, the temperature profiles in tubes become slightly non-symmetric with respect to the tube axis. To eliminate the possible influence of this fact, thermocouples were located vertically at both sides of the tube. For convenience, each thermocouple was connected to two cold-junctions. A rotary switch was included in the circuit to connect the desired pair of thermocouples to the automatic recorder. Two thermometers were suspended in constant level tanks as a check of the average water temperatures.

To obtain a constant pressure difference, another tank of constant head was connected to the end of the test section. In addition to this pressure difference, the undesired velocity transient also depends on the valve opening as a function of time. To keep this transient as small as possible, two quick opening valves were attached to the entrance and exit of the test section. The flow rates were measured by weighing the drained water over a period of time.

The test section was insulated with 1 in. of 85% Magnesia. Two layers of 1/2 in. glass fibre blanket were used to cover the tank of constant temperature and heaters.

A Sanborn 150 recording oscillograph with a 150-1500 Preamplifier and a 152-100 B Recorder was used to measure the time dependent EMF produced by the thermocouples in the test section. The recorder was calibrated against a known voltage before each run.

Some characteristics of the recorder used are listed below:

Sensitivity - 100 microvolts to 0.1 volts per centimeter,

Rise time - 0.03 seconds, which is the time response of the recorder to a unit step input,

Calibration error - ± 0.25%,

Zero suppression error - ± 0.05 millivolt on low range.

2. Test Procedure

The flow rate of the tank of constant temperature was adjusted to keep a constant head during the experiment. Approximately two hours were necessary to bring the tank temperature 50°F above room temperature. During this time, all electric heater units were loaded at full power. Later on, by proper use of these units, tank temperature equilibrium was maintained. In addition to the axial insulation at the suction slot, reverse flow through this slot was used to decrease the axial conduction along the tube-wall. This reverse flow, also, decreases the cooling time of the test section between two runs.

With the recorder in operation, the quick opening valve at the entrance was suddenly opened and the transient EMF was recorded. This EMF was converted to a temperature by the use of the proper calibration curve.

IV. RESULTS

The method described in $I.4$ has been used to obtain the heat flux on the fluid-tube interface, and the fluid and tube-wall temperatures. The first two approximations were considered. The resulting solutions may be written implicitly as follows

$$\frac{T(r,x,t)}{\Delta T_e} = \phi_1\left[r/R, \frac{x/D}{Pe}, MFO, Pr\right] \tag{43}$$

$$\frac{\theta(x,t)}{\Delta T_e} = \phi_2\left[\frac{x/D}{Pe}, MFO, Pr\right] \tag{44}$$

$$Nu(x,t) = \phi_3\left[\frac{x/D}{Pe}, MFO, Pr\right] \tag{45}$$

Because of the analytical difficulties encountered, the second approximations were obtained only for small values of $MFO/\left(\frac{x/D}{Pe}\right)^{1/2}$, less than 0.4. The theory applies within the range $0 \leq \frac{x/D}{Pe} \leq 10^{-2}$. This includes Viscous Oils, Water, Air and partially covers Nuclear metals. In this range, $\frac{x/D}{Pe}$ and MFO appear in a single dimensionless group, namely $MFO/\left(\frac{x/D}{Pe}\right)^{1/2}$.

As indicated by Toong and Shapiro (25), the assumption that the growth of the boundary layer for tube flow may be approximated by that for plate flow is valid when $\delta_V/R \leq 0.3$. Therefore, using the Blasius solution for flat plate, the upper limit of validity of the present theory may be summarized in the following table:

Fluid	Liquid Metals		Air	Water		Viscous Oils	
Pr	10^{-3}	10^{-2}	0.7	1	10	10^{2}	10^{4}
$\frac{x/D}{Pe} \leq$	1	10^{-1}		10^{-3}	10^{-4}	10^{-5}	10^{-7}

Each approximation of the tube-wall temperature has been reduced to the solution of a Volterra equation of the second kind which may be written in a transformed form as

$$\bar{\Lambda}_i(x,p) = \frac{1}{p^2}\bar{G}_i + \frac{1}{p}\int_0^x K(x,\xi)\bar{\Lambda}_i(\xi,p)d\xi \qquad (46)$$

The kernel of this function is the same for all approximations, and, if $\frac{x/D}{Pe} \leq 10^{-2}$, it can be approximated within 1.52% error by the upper line of Fig. 5. This maximum error occurs at the upper limit. However, $\bar{G}_i$ is different in form for each value of i . $\bar{G}_1 = G_1(x)$, which corresponds to the first approximation, has been calculated by the use of a desk calculator and plotted. The resulting plot was approximated by the lower line in Fig. 5. The maximum error is 2.34% when $\frac{x/D}{Pe} = 10^{-2}$. Complicated form of $\bar{G}_2 = \bar{G}_2(x,p,Pr)$ requires the use of computer. The two functions indicated in (A-126) and (A-127) in the first term $G_{21}(x, Pr)$ of $\bar{G}_2(x,p,Pr)$ have been evaluated in this manner.

For constant values of specific heat, viscosity and thermal conductivity, as previously assumed, the first two approximations of the tube-wall temperature and heat flux at the fluid-tube interface versus $MFo/\left(\frac{x/D}{Pe}\right)^{1/2}$ were obtained for values of $\frac{x/D}{Pe} \leq 10^{-2}$, as shown in Figs. 7 and 9a.

The second approximations have been evaluated only for small values of $MFO/\left(\frac{x/D}{P\acute{e}}\right)^{1/2}$. Two experimental runs based on the tube-wall temperature measurements are presented versus $MFO/\left(\frac{x/D}{P\acute{e}}\right)^{1/2}$ in Figs. 9b and 9c. In 9d is shown the dimensionless tube-wall temperature versus the length Reynolds number.

The first approximation for the tube-wall temperature of the present theory has been compared with the first approximation of a quasi-steady theory, which is based on the constant surface temperature assumption. The results are shown versus $MFO/\left(\frac{x/D}{P\acute{e}}\right)^{1/2}$ in Figs. 6 and 8.

In Fig. 9 is shown the first approximation of the Modified Nusselt number versus $MFO/\left(\frac{x/D}{P\acute{e}}\right)^{1/2}$. In the same figure, the results of the Graetz problem for slug flow which hold when $MFO/\left(\frac{x/D}{P\acute{e}}\right)^{1/2} = 0$, are indicated.

In Fig. 8a, there are shown the first approximations to the three Nusselt numbers Nu_0, Nu_0^{*}, Nu_0^{**} which are respectively functions of the differences between, (a) - the entrance temperature and the initial tube-wall temperature, (b) - the entrance temperature and the instantaneous tube-wall temperature, (c) - instantaneous mean fluid temperature and the instantaneous tube-wall temperature.

The present theory is compared with two other theories (7) and (27), at one location and for one flow condition. The results are shown in Fig. 8b.

The first approximation to the tube-wall temperatures is plotted in Fig. 8c for two locations and at various times.

V. DISCUSSION OF RESULTS

The present theory shows that through an approximate theory the modified Nusselt number (Fig. 9a) may be represented by

$$Nu(x,t,Pr) = \begin{cases} 0, & t < \frac{x}{U} \\ \frac{1}{\pi^{1/2}\left(\frac{x/D}{P\acute{e}}\right)^{1/2}} \left[e^{-\left[\frac{MFo}{\left(\frac{x/D}{P\acute{e}}\right)^{1/2}}\right]^2} - 0.1\,\pi^{1/2} P(Pr) \left[1 - 5\left(\frac{Pr}{\pi}\right)^{1/2} e^{-\frac{25}{4}Pr} - \left(1 - \frac{25}{2}Pr\right) \mathrm{erfc}\left(\frac{5}{2}Pr^{1/2}\right)\right]\right], & t \geq \frac{x}{U} \end{cases} \tag{42}$$

provided that the dimensionless number $MFo/\left(\frac{x/D}{P\acute{e}}\right)^{1/2}$ is small enough, that is $MFo/\left(\frac{x/D}{P\acute{e}}\right)^{1/2} \leq 0.4$.

Initially $FO = 0$, this equation may implicitly be written in the following form

$$Nu(x, Pr) = \psi_1 \left[\frac{x/D}{Re.Pr}, Pr\right] \tag{47}$$

On the other hand, if the problem of free convection on a vertical plate is considered, the Nusselt number is

$$Nu(x, Pr) = \psi_2 \left[\frac{x/L}{Gr.Pr}, Pr\right] \tag{48}$$

Noting that, resulting from the dimensional analysis, the Reynolds number of a forced convection problem is analogous to the Grashof number of any free convection problem, the analogy between the above two problems may be clearly seen. Moreover, instead of taking a free convection problem, a forced convection problem can also be taken for a direct analogy.

In Figs. 6 and 8, the first approximations to the tube-wall temperature and modified Nusselt number of the present study are compared with a quasi-steady theory developed in Appendix E. Both theories, for small values of $MFO/\left(\frac{x/D}{Pe}\right)^{1/2}$, agree closely because of the assumption made in the calculation of the quasi-steady theory. For large values of this argument (the steady-state case or in the neighborhood of the entrance) good agreement is again obtained. Therefore, for very small or very large values of $MFO/\left(\frac{x/D}{Pe}\right)^{1/2}$, the quasi-steady theory can be conveniently used in place of the present theory. For intermediate values of this ratio, a greater difference exists. For example, the discrepancy between the two theories assumes a maximum 17% for the tube-wall temperatures near $MFO/\left(\frac{x/D}{Pe}\right)^{1/2} = 1$, and a maximum of 20% in the heat fluxes near $MFO/\left(\frac{x/D}{Pe}\right)^{1/2} = 0.5$.

In Fig. 8a are shown, the first approximation of the three Nusselt numbers which are respectively based on the differences between, (a) - the step temperature and the initial tube-wall temperature, (b) - the step temperature and the instantaneous tube-wall temperature and (c) - instantaneous mean fluid temperature and the instantaneous tube-wall temperature. Nu_0 decreases with increasing time as expected. On the other hand, Nu_0^* and Nu_0^{**} increase without limit. At first, this seems rather surprising. However, if a fluid which flows through a tube having the same temperature as the tube, is subjected to an infinitely small temperature decrease (or increase), the conditions are physically identical to the entrance conditions for which Nu_0^* and

Nu_D^{**} become infinite.

Dusinberre's (27) and Rizika's (7) simplified analyses are compared with the present theory for the tube-wall temperatures in Fig. 8b. Both of these works give somewhat higher (10-20%) tube-wall temperature responses compared to the present theory. A steady heat transfer coefficient at the entrance of the tube was used for these analyses. However, in the actual case, the conventional heat transfer coefficient increases with increasing time. Therefore, in these works the value of the heat transfer coefficient was less than its true value. As Fig. 8a shows, smaller heat transfer coefficients as conventionally defined, correspond to higher heat transfer coefficients as defined in this study. For that reason, both of the simplified analyses give higher values for the tube-wall temperature response. Therefore, the conventional heat transfer coefficient, for the present type of transient problems is irrelevant and should not be used.

The first approximation of the present theory initially gives the same solution as the Graetz problem (13) for slug flow. Therefore, in Figs. 6 and 9, the ordinate corresponds to the solution of the Graetz problem.

The theory of this study, assuming a time independent, developing velocity field, was obtained for a step temperature change at the entrance of the tube. However, for the experimental work, it was necessary to take a fluid which was suddenly subjected to pressure and temperature differences. In this way a velocity transient is introduced as well as a temperature transient (Appendix D). Therefore, the experimental work is meaningful only for $t_T \gg$ than the other transients.

Consider the experimental data when $H \cong 1$ in., $l \cong 1$ in., $\nu = 0.906 \times 10^{-5}$ ft^2/sec., $^{h}/r_0 \cong 1/5$ (this value may even be smaller in the actual case).

Then from (D-17)

$$t_{v_p} \cong 2.2 \frac{1/5 \times 1/12}{(2 \times 32.16 \times 1/12)^{1/2}} \cong 0.016 \text{ sec.} \qquad (49)$$

and from (D-20)

$$(t_{v_b})_{max} \cong \frac{(0.625/12)^2}{144 \times 0.906 \times 10^{-5}} \cong 2.08 \text{ sec.} \qquad (50)$$

This transient depends on the thickness of the velocity boundary layer, and at the fluid-wall interface is equal to zero. On the other hand, the effect of this transient on the temperature transient increases as the fluid-wall interface is approached. As an average value, if $\delta_v/R = 0.15$ is taken instead of $\delta_v/R = 0.30$

$$t_{v_b} \cong \frac{1}{4} \times 2.08 = 0.52 \text{ sec.} \qquad (51)$$

results.

For the first location $(x = 5 \text{ in.})$, from (D-21)

$$t_T^* = t_T + \frac{x}{U} = 0.9 \frac{(0.625/12)^2 \times 3600 \times 1.68 \times 10^{-2}}{3 \times 5.69 \times 10^{-3}} + \frac{5}{12 \times 0.858} = 9.13 \text{ sec.} \qquad (52)$$

Actually, by the use of the second approximation, this transient time is approximately increased by an amount of $4/3$ as

$$t^{*} \cong 1.2\,\frac{(0.625/12)^2 \times 3600 \times 1.68 \times 10^{-2}}{3 \times 5.69 \times 10^{-3}} + \frac{5}{12 \times 0.858} \cong 12.00\ \text{sec.} \qquad (53)$$

Another simplification was made in the theory neglecting the axial conduction within the fluid and the tube-wall. This axial conduction effect in the tube-wall may be found simply by taking a pipe insulated at the inner and outer surfaces and at one end, and subjected to a sudden temperature change at the other end. The solution of this problem (28) can be written as

$$\frac{\theta}{2\Delta T_e} = \operatorname{erfc}\frac{1}{2}\frac{L}{(\alpha t)^{1/2}} - \operatorname{erfc}\frac{3}{2}\frac{L}{(\alpha t)^{1/2}} + \operatorname{erfc}\frac{5}{2}\frac{L}{(\alpha t)^{1/2}} - \cdots \qquad (54)$$

This series converges quite rapidly except small values of $L/(\alpha t)^{1/2}$.
As was done in Appendix D, if a transient time is defined according to
$\theta/\Delta T_e = 0.8$, by trial-and-error

$$L^2/\alpha t \cong 0.756 \qquad (55)$$

is found.

In the experimental work, α (copper) = $4.35\ ft^2/hr$, $L = 22$ in. (fourth location) were used. Then from (55)

$$\left(\frac{22}{12}\right)^2 \frac{1}{4.35 \times t} = 0.756 \qquad (56)$$

$$t = 0.585\ \text{hour} \qquad (57)$$

is obtained. This value is very small compared to the temperature transient. The same calculation can be made for the fluid. In this case an even greater transient time should be obtained.

Due to the acceleration, the hot fluid moves with the velocity V, which increases in the flow direction, and is different from U. The ratio of these velocities (26) is

$$\frac{V}{U} = \frac{1}{1 - \frac{3}{4}(\delta/R) + \frac{17}{60}(\delta/R)^2 - \dots} \qquad (58)$$

Therefore, the time required for the hot flow to travel from the entrance to the location considered is

$$t' = \frac{L}{U_m} = \frac{L}{1/L \int_0^L V(x)dx} \cong \frac{L}{\frac{1}{2}(U+V)} \quad \text{but not} \quad t = \frac{L}{U} \qquad (59\text{-}60)$$

The difference is

$$\Delta t = t - t' = \frac{L}{U}\,\frac{V/U - 1}{V/U + 1} \qquad (61)$$

The worst case occurs when $\delta/R = 0.3$ for which

$$\frac{V}{U} = \frac{1}{1 - \frac{3}{4} \times 0.3 + \frac{17}{60} \times 0.3^2} \cong 1.25 \qquad \text{and} \qquad (62)$$

$$\Delta t = \frac{0.25}{2.25} \cdot \frac{L}{U} \cong 0.11 \frac{L}{U} \qquad (63)$$

For the fourth location, $L = 22/12$ ft. and $U = 0.858$ ft/sec.

$$\Delta t = 0.11 \times \frac{22}{12 \times 0.858} = 0.235 \text{ sec.} \qquad (64)$$

Again Δt is negligible compared to the temperature transient. Therefore, the theory may be safely used to check the experimental work.

Experiments were run at two velocities. Figs. 9b and 9c show the dimensionless tube-wall temperature versus $MFo/\left(\frac{x/D}{Pe}\right)^{1/2}$ for a fixed Prandtl number. Temperature measurements were made at four locations along the tube (see Fig. 12). As these figures show, experimental points are spread around the theory. It seems rather difficult to make any interpretation. On the other hand, for a fixed $MFO/\left(\frac{x/D}{P\acute{e}}\right)^{1/2}$, for example 0.5, the same dimensionless tube-wall temperature $\theta/\Delta T_e$ may be plotted versus length Reynolds number (Fig. 9c). For $Re_D = 4.93 \times 10^3$, the first station for the temperature measurements is in the laminar region, and for $Re_D = 8.17 \times 10^3$, the first point is at the beginning of the transition region. In the laminar region, mean experimental tube-wall temperature is a little higher than the theory predicts. This may be due to the fact that experimental points were taken as the arithmetic mean of the inner and outer tube-wall temperatures. In the actual case, the mean temperature is much less than the arithmetic mean, being nearer to the inner surface temperature.

To simplify the theory, radial variations of the tube-wall temperature were neglected in this analysis. To check the validity of this assumption, the inner and outer tube-wall temperatures were measured. At the first location, the difference between these two temperatures was at most 37% of the mean temperature. This maximum value occurred in the neighborhood of $MFO/\left(\frac{x/D}{P\acute{e}}\right)^{1/2} = 0.5$. As expected this temperature difference decreased with distance downstream (Fig. 9d). The runs for lower velocities must be discarded, since δ_v/R becomes appreciably greater than 0.3. Also, temperature measurements at shorter distances could not be made due to the large temperature gradient in the tube-wall.

The separation point moves upstream with increasing diameter Reynolds number as expected. For $Re_D = 8.17 \times 10^3$, the separation occurs when $Re_x = 2.7 \times 10^4$, and for $Re_D = 4.93 \times 10^3$ it takes place at $Re_x = 3.3 \times 10^4$. These values are very low compared to the case of steady flow in smooth tubes (26). This probably was due to the disturbance caused by the quick opening valve at the entrance of the test section.

VI. CONCLUSIONS

The problem has been solved for a step temperature change at the entrance region of tubes. A method of successive approximations was used. It is readily seen, from the expressions for tube-wall temperature $\theta(x,t)$, fluid temperature $T(r,x,t)$ and modified Nusselt number $Nu(x,t)$, that these are linear with respect to the input step temperature. Therefore, by the use of the principle of superposition, the results can be applied to any type transient input at the entrance of the tube.

In order to use a lengthwise one-dimensional analysis, the necessary heat flux at the fluid-tube interface has been obtained as a function of axial distance and time. If the conventional Nusselt number is used, the resulting one-dimensional equations become

Fluid

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + ah(T-\theta) = 0 \qquad (65)$$

Tube-wall

$$\frac{\partial \theta}{\partial t} - bh(T-\theta) = 0 \qquad (66)$$

where a, b are related to the geometry and thermal properties of the fluid and tube-wall, and h is the conventional heat transfer coefficient.

Complication in the solution of above equations arises from the coupling between them. For this reason a modified Nusselt number $Nu(x,t) = \dfrac{q(x,t)}{k\Delta T_e/D}$, based on the step-temperature input ΔT_e, has been defined. With this definition, the above differential equations may be written as

Fluid

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + a\,q(x,t) = 0 \qquad (67)$$

Tube-wall

$$\frac{\partial \theta}{\partial t} - b\,q(x,t) = 0 \qquad (68)$$

These equations are uncoupled and can be solved separately. The required analysis is considerably simplified in comparison to the previous problem. Therefore, instead of the conventional Nusselt number, the modified Nusselt number has been used throughout.

The theory presented applies to viscous oils, water and air in the range of δ_v/R less than 0.3, and to liquid metals when δ_v/R is less than 0.1.

VII. RECOMMENDATIONS

Further analytical and experimental investigations are needed. At the entrance region of tubes, when $10^{-3} \leq \frac{x/D}{Pe} \leq 10^{-2}$ the theory applies approximately, and when $10^{-2} \leq \frac{x/D}{Pe} \leq 10^{-1}$ it gives appreciable error. The latter may be important in nuclear metal applications. Better approximation is necessary in this range for $G_1(x)$ (Fig. 5). However, a primary calculation indicated that even the addition of a constant to the previous approximation results in three additional terms in the tube-wall temperature function.

For small values of time, the first terms in each of the expansions for $G_1, G_2, \ldots$ are much larger than the succeeding terms. Due to the complexity of G_2 relative to G_1, therefore, only the first term of G_2 was evaluated, instead of obtaining the complete function as with G_1. In future work, the remaining terms of G_2 should be numerically calculated by computer for some characteristic values of the Prandtl number $(e.g., 10^{-2}, 0.7, 1, 10, 10^2, 10^4)$. Then the approximate form of these results can be used in the solution of the Volterra equation (A-108), giving the successive terms of the second approximations.

By the use of the present apparatus, it is difficult to obtain better experimental results. For more points in the laminar region two methods are suggested. First, additional temperature measurements between the entrance and the first location may be used. However, the temperature difference between the inner and the outer surfaces for which the present theory is no longer valid, increases appreciably. Secondly, the velocity may be decreased. For this case, the entrance region defined by $\delta_v/R \leq 0.3$ becomes very small.

The experimental work, instead of using the present quick opening valve should be repeated with the use of another quick opening system which results in less disturbance. Under these circumstances, for the same diameter Reynolds number, it would be possible to delay the beginning of separation.

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NOMENCLATURE

A cross-sectional area of tube,

A_e instantaneous cross-sectional area of tube exit,

A_k (k)th term of function defined in Appendix A,

a shown in Part VI and related to fluid geometry and physical properties,

B_0, B_1, B_2 functions defined in Appendix A,

b_0, b_1, b_2 defined in Appendix A,

b shown in Part VI and related to tube-wall geometry and physical properties,

C torque,

C_0 steady torque,

c_p specific heat of fluid at constant pressure,

c_{p_w} specific heat of tube-wall at constant pressure,

c_0, c_1, c_2 defined in Appendix A,

D inside diameter of tube,

f dimensionless stream function,

F defined in Appendix A,

Fo Fourier number of fluid,

Gr Grashof number of fluid,

$\bar{G}, G_1, G_{21}$ functions defined in Appendix A,

H function defined in Appendix A,

h heat transfer coefficient,

h constant fluid level,

I_0 modified Bessel function of the first kind, of order zero,

J_0 Bessel function of the first kind, of order zero,

J_1 Bessel function of the first kind, of order one,

I function defined in Appendix A,

k thermal conductivity of fluid,

k (k)th term of series,

K_0 modified Bessel function of the second kind, of order zero,

K function defined in Appendix A,

$\mathcal{K}_x$ function defined in Appendix A,

$\mathcal{K}_x^N$ function defined in Appendix A,

l approximated length of fluid flow before test section,

L length of test section,

M dimensionless number defined in Appendix A,

N (N)th term of series,

Nu Nusselt number based on the difference between the entrance fluid temperature and initial tube-wall temperature,

Nu^* Nusselt number based on the difference between the entrance fluid temperature and instantaneous tube-wall temperature,

Nu^{**} Nusselt number based on the difference between the instantaneous mean fluid temperature and the instantaneous tube-wall temperature,

p static pressure,

p Laplace transform variable,

P Total pressure

Pr Prandtl number of fluid,

Pe' Peclet number of fluid,

$p(Pr)$ defined in Appendix A,

q Laplace transform variable,

q instantaneous heat flux at fluid-tube wall interface,

R inside diameter of tube,

R_0 outside diameter of tube,

r radial distance from tube center,

r_0 radius of fluid stream as shown in Fig. 16,

r_e radius of fluid stream as shown in Fig. 16,

Re_D diameter Reynolds number,

Re_x length Reynolds number,

S defined in Appendix D,

s curvilinear coordinate along fluid flow,

t time

t axial conduction transient time,

t_T temperature transient time,

t_T^* $t_T + \frac{x}{U}$

t_{v_p} potential flow transient time,

t_{v_b} boundary layer transient time,

Δt defined in part V,

T_k (k)th approximation of fluid temperature,

T_s surface temperature,

ΔT_e step temperature input,

u axial velocity in boundary layer,

v	radial velocity in boundary layer,
U	entrance velocity for tube flow or free stream velocity for plate flow,
V	core velocity for tube flow,
V	potential velocity for tube flow,
V_e	exit velocity for potential flow,
$V_{e\infty}$	steady-state value of exit velocity,
x	axial distance from entrance of tube or leading edge of flat plate,
X	dimensionless axial distance,
y	radial distance from plate wall,
Y	$1-r/R$
Y_0	Bessel function of the second kind, of order zero,
z	dummy variable,
z	height of potential flow,
α	thermal conductivity of fluid,
γ	defined in Appendix A,
Γ_1	defined in Appendix A,
Γ_2	defined in Appendix A,
δ_v	dimensionless velocity boundary layer thickness,
ϵ	defined in Appendix A,
θ_k	(k)th approximation of tube-wall temperature,
λ_k	(k)th term of eigenvalues,

$\bar{\Lambda}_k$ (k)th approximation of Laplace transformed tube-wall temperature,

ν kinematic viscosity of fluid,

$\bar{\Pi}_k$ (k)th term of Laplace transformed fluid temperature,

$\bar{\bar{\Pi}}_{kj}$ (k)th approximation of Laplace and Hankel transformed fluid temperature,

$\bar{\Gamma}_0$ defined in Appendix B,

ρ density of fluid,

ρ_w density of tube-wall,

σ fluid-tube wall interface,

ψ_1, ψ_2 implicit function notations,

ϕ_1, ϕ_2, ϕ_3 implicit function notations,

ξ dummy variable,

Ξ dimensionless dummy variable,

η dimensionless variable defined in Appendix A,

ζ dimensionless variable defined in Appendix D,

ω angular velocity,

ω_0 steady angular velocity,

Ω_1 defined in Appendix A,

Ω_2 defined in Appendix A,

Ω_3 defined in Appendix A,

Ω_3' defined in Appendix A.

VIII. APPENDICES

APPENDIX A

Derivation and Integration of the Differential Equations

Consider Fig. 3 and the assumptions made in part II. Then between the sections x and $x+dx$, for a tube element (a closed system) and for a control volume of an incompressible flow, the terms of the energy equations may be written as shown in Fig. 4.

The transient energy equation for the tube-wall expresses the fact that the time rate of change of internal energy within the tube element must be equal to the net rate of heat transfer (by conduction) to the surface of the system. The transient energy equation for the flow through the control volume states that the time rate of change of internal energy within the control volume plus the net efflux of enthalpy is equal to the net rate of heat transfer (by conduction) from the surface of the control volume.

Using the foregoing figure and definitions, the energy equations of the fluid and tube-wall may be written in the form

Fluid

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial r} = \alpha\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r}\right) \qquad (A\text{-}1)$$

Tube-wall

$$\frac{\partial \theta}{\partial t} + \epsilon\left(\frac{\partial T}{\partial r}\right)_{r=R} = 0 \qquad (A\text{-}2)$$

where

$$\epsilon = 2\frac{\alpha}{R}\frac{(\rho c_p)}{(\rho c_p)_w}\frac{1}{(R_0/R)^2-1} \qquad (A\text{-}3)$$

The initial and boundary conditions to be taken according to the outline of this problem are

Initial conditions

$$T(r,x,0) = 0 \qquad (A\text{-}4)$$

$$\theta(x,0) = 0 \qquad (A\text{-}5)$$

Boundary conditions

$$T(r,0,t) = \Delta T_e \qquad (A\text{-}6)$$

$$\partial T(0,x,t)/\partial r = 0 \qquad (A\text{-}7)$$

$$T(R,x,t) = \theta(x,t) \qquad (A\text{-}8)$$

Let the first and second approximations for the fluid and tube-wall temperatures, respectively, be

$$T = T_0 + T_1 \qquad (A\text{-}9)$$

$$\theta = \theta_0 + \theta_1 \qquad (A\text{-}10)$$

Within this assumption the first approximation of the fluid and tube-wall temperatures T_0 and θ_0 satisfy the following partial differential equations and the proper initial and boundary conditions:

Fluid

$$\frac{\partial T_0}{\partial t} + U\frac{\partial T_0}{\partial x} = \alpha\left(\frac{\partial^2 T_0}{\partial r^2} + \frac{1}{r}\frac{\partial T_0}{\partial r}\right) \qquad (A\text{-}11)$$

Tube-wall

$$\frac{\partial \theta_0}{\partial t} + \epsilon\left(\frac{\partial T_0}{\partial r}\right)_{r=R} = 0 \qquad (A\text{-}12)$$

$$T_0(r,x,0) = 0 \qquad (A\text{-}13)$$

$$\theta_0(x,0) = 0 \qquad (A\text{-}14)$$

$$T_0(r,0,t) = \Delta T_e \qquad (A\text{-}15)$$

$$\frac{\partial T_0(0,x,t)}{\partial r} = 0 \qquad (A\text{-}16)$$

$$T_0(R,x,t) = \theta_0(x,t) = \begin{cases} 0, & t < \frac{x}{U} \\ \Lambda_0(x,t), & t \geq \frac{x}{U} \end{cases} \qquad (A\text{-}17)$$

where $\Lambda_0(x,t)$ is a function to be determined and, as it will be seen later on, for only mathematical convenience it is introduced here.

Before giving the details of derivation of the differential equations and initial-boundary conditions satisfied by the second approximations of the fluid and tube-wall temperatures T_1 and θ_1 , it may be convenient to solve the above problem.

From the theory of Multi Laplace transforms (19), it is known that partial differential equations in which the several independent variables are in the domain $(0, \infty)$ may be converted to transform differential equations by simultaneous Laplace transformations in these variables. Therefore, the present problem can be considered as one of partial differential equations of $T(r, x, t)$ and $\theta(x, t)$ defined in the region $0 \leq x \leq \infty$, $0 \leq t \leq \infty$, with r as a parameter which varies between $-R$ and $+R$. However, in the present study the most convenient and shortest solution, the use of one dimensional Laplace transforms in the t-variable and finite Hankel transforms in the r-variable are chosen.

From the definition of the Laplace transforms

$$\bar{T}_0(r, x, p) = \int_0^\infty e^{-pt} T_0(r, x, t)\, dt \qquad (A-18)$$

$$\bar{\theta}_0(x, p) = \int_0^\infty e^{-pt} \theta_0(x, t)\, dt \qquad (A-19)$$

it follows that

$$\int_0^\infty e^{-pt} \frac{\partial T_0(r,x,t)}{\partial t} dt = p\bar{T}_0(r,x,p) - T_0(r,x,0) \qquad (A-20)$$

$$\int_0^\infty e^{-pt} \frac{\partial \theta_0(x,t)}{\partial t} dt = p\bar{\theta}_0(x,p) - \theta_0(x,0) \qquad (A-21)$$

$$\int_0^\infty e^{-pt} \frac{\partial T_0(r,x,t)}{\partial x} dt = \frac{\partial \bar{T}_0(r,x,p)}{\partial x} \qquad (A-22)$$

$$\int_0^\infty e^{-pt} \frac{\partial T_0(r,x,t)}{\partial r} dt = \frac{\partial \bar{T}_0(r,x,p)}{\partial r} \qquad (A-23)$$

$$\int_0^\infty e^{-pt} \frac{\partial^2 T_0(r,x,t)}{\partial r^2} dt = \frac{\partial^2 \bar{T}_0(r,x,p)}{\partial r^2} \qquad (A-24)$$

so that multiplying both sides of equations (A-11), (A-12), (A-15), (A-16) and (A-17) by e^{-pt}, integrating with respect to t from 0 to ∞, and using (A-13), (A-14), (A-20), (A-21), (A-22), (A-23) and (A-24) gives

$$p\bar{T}_0 + U\frac{\partial \bar{T}_0}{\partial x} = \alpha\left(\frac{\partial^2 \bar{T}_0}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{T}_0}{\partial r}\right) \tag{A-25}$$

$$p\bar{\theta}_0 + \epsilon\left(\frac{\partial \bar{T}_0}{\partial r}\right)_{r=R} = 0 \tag{A-26}$$

$$\bar{T}_0(r,0,p) = \frac{\Delta T_e}{p} \tag{A-27}$$

$$\frac{\partial \bar{T}_0(0,x,p)}{\partial r} = 0 \tag{A-28}$$

$$\bar{T}_0(R,x,p) = \bar{\theta}_0(x,p) = \bar{\Lambda}_0(x,p)\, e^{-\frac{x}{U}p} \tag{A-29}$$

The above equations may be put into more convenient form with the substitution

$$\bar{T}_0(r,x,p) = \bar{\Pi}_0(r,x,p)\, e^{-\frac{x}{U}p} \tag{A-30}$$

The result is

$$U\frac{\partial \bar{\Pi}_0}{\partial x} = \alpha\left(\frac{\partial^2 \bar{\Pi}_0}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{\Pi}_0}{\partial r}\right) \tag{A-31}$$

$$p\bar{\Lambda}_0(x,p) + \epsilon\left(\frac{\partial \bar{\Pi}_0}{\partial r}\right)_{r=R} = 0 \tag{A-32}$$

$$\bar{\Pi}_0(r,0,p) = \frac{\Delta T_e}{p} \tag{A-33}$$

$$\frac{\partial \bar{\Pi}_0(0,x,p)}{\partial r} = 0 \tag{A-34}$$

$$\bar{\Pi}_0(R,x,p) = \bar{\Lambda}_0(x,p) \tag{A-35}$$

where (A-32) is obtained by the combination of (A-26) and (A-29).

This problem, defining a new function $\bar{\mathcal{T}}_0(r,x,\xi,p)$ which satisfies (A-31), (A-33), (A-34) and

$$\bar{\mathcal{T}}_0(R,x,\xi,p) = \bar{\Lambda}_0(\xi,p) \tag{A-36}$$

where ξ and p are parameters, can be solved by using the method of separation of variables, or another application of the Laplace transforms in the x- variable (Appendix B).

However, the use of finite Hankel transforms is most convenient. For a radially symmetric problem, (A-34) is identically satisfied. Denoting by $\bar{\bar{\Pi}}_{0j}(\lambda_k, x, p)$ the finite Hankel transform of order zero of the function $\bar{\Pi}_0(r/R, x, p)$, then

$$\bar{\bar{\Pi}}_{0j}(\lambda_k, x, p) = R^2\int_0^1 \frac{r}{R}\,\bar{\Pi}_0\left(\frac{r}{R}, x, p\right) J_0\left(\lambda_k \frac{r}{R}\right) d\left(\frac{r}{R}\right) \qquad (A\text{-}37)$$

After two integrations by parts → this becomes

$$\int_0^1 \frac{r}{R}\left[\frac{\partial^2\bar{\Pi}_0}{\partial r_R^2} + \frac{1}{r/R}\frac{\partial\bar{\Pi}_0}{\partial(r/R)}\right] J_0\left(\lambda_k\frac{r}{R}\right) d\left(\frac{r}{R}\right) = \lambda_k J_1(\lambda_k)\bar{\Lambda}_0(x,p) - \frac{\lambda_k^2}{R^2}\bar{\bar{\Pi}}_{0j} \qquad (A\text{-}38)$$

Consider the radial dimensionless form of (A-31) in terms of (r/R)

$$U\frac{\partial\bar{\Pi}_0}{\partial x} = \alpha\left[\frac{\partial^2\bar{\Pi}_0}{\partial(r/R)^2} + \frac{1}{r/R}\frac{\partial\bar{\Pi}_0}{\partial(r/R)}\right]\frac{1}{R^2} \qquad (A\text{-}39)$$

Multiplying (A-39) and (A-33) throughout by $R^2\left(\frac{r}{R}\right)J_0\left(\lambda_k\frac{r}{R}\right)$ and integrating with respect to (r/R) over the range $(0,1)$ it is found that $\bar{\bar{\Pi}}_{0j}(\lambda_k, x, p)$ is determined by the solution of the first-order linear differential equation

$$\frac{d\bar{\bar{\Pi}}_{0j}}{dx} + \left(\frac{4\lambda_k^2}{DP\acute{e}}\right)\bar{\bar{\Pi}}_{0j} = \frac{\alpha}{U}\lambda_k J_1(\lambda_k)\bar{\Lambda}_0(x,p) \qquad (A\text{-}40)$$

with the boundary condition

$$\bar{\bar{\Pi}}_{0j}(\lambda_k, 0, p) = \frac{\Delta T_e}{p} R^2 \frac{J_1(\lambda_k)}{\lambda_k} \qquad (A\text{-}41)$$

where

$$\frac{4}{DP\acute{e}} = \frac{\alpha}{U}\cdot\frac{1}{R^2}$$

The solution of this differential equation is in the form

$$\bar{\bar{\Pi}}_{0j}(\lambda_k,x,p) = \frac{\Delta T_e}{p} R^2 \frac{J_1(\lambda_k)}{\lambda_k} e^{-4\lambda_k^2 \frac{x/D}{Pe}} + \frac{\alpha}{U}\lambda_k J_1(\lambda_k)\int_0^x e^{-4\lambda_k^2 \frac{(x-\xi)/D}{Pe}} \bar{\Lambda}_0(\xi,p)d\xi \tag{A-42}$$

Inverting this equation by means of the following theorem (20)

$$\bar{\Pi}_0(r,x,p) = \frac{2}{R^2}\sum_{k=1}^{\infty} \bar{\bar{\Pi}}_{0j}(\lambda_k,x,p)\frac{J_0(\lambda_k \frac{r}{R})}{J_1^2(\lambda_k)} \tag{A-43}$$

gives

$$\bar{\Pi}_0(r,x,p) = 2\frac{\Delta T_e}{p}\sum_{k=1}^{\infty} e^{-4\lambda_k^2 \frac{x/D}{Pe}} \frac{J_0(\lambda_k \frac{r}{R})}{\lambda_k J_1(\lambda_k)} + 2\left(\frac{4}{DPe}\right)\sum_{k=1}^{\infty}\frac{\lambda_k J_0(\lambda_k \frac{r}{R})}{J_1(\lambda_k)}\int_0^x e^{-4\lambda_k^2 \frac{(x-\xi)/D}{Pe}} \bar{\Lambda}_0(\xi,p)d\xi \tag{A-44}$$

which is the first approximation of the Laplace transformed (in variable t) fluid temperature in terms of the unknown function $\bar{\Lambda}_0(x,p)$. Now, by using (A-32) and (A-44), a Volterra integral equation of the second kind is obtained in which $\bar{\Lambda}_0(x,p)$ is the only unknown function. This integral equation is of the form

$$\bar{\Lambda}_0(x,p) = \frac{\Delta T_e}{p^2}\left(\frac{2\epsilon}{R}\right)\sum_{k=1}^{\infty} e^{-4\lambda_k^2 \frac{x/D}{Pe}} + \frac{1}{p}\left(\frac{4}{DPe}\right)\left(\frac{2\epsilon}{R}\right)\int_0^x \sum_{k=1}^{\infty}\lambda_k^2 e^{-4\lambda_k^2 \frac{(x-\xi)/D}{Pe}} \bar{\Lambda}_0(\xi,p)d\xi \tag{A-45}$$

The exact solution of this equation is given in Appendix C. The solution is rather complicated and is not practical for the calculation of second approximations. Here, an approximate method is chosen for convenience.

Consider the above integral equation (A-45) in the following form

$$\bar{\Lambda}_0(x,p) = \frac{\Gamma_1}{p^2} G_1\left(\frac{x/D}{P\acute{e}}\right) + \frac{\Gamma_2}{p}\int_0^x K\left[\frac{(x-\xi)/D}{P\acute{e}}\right]\bar{\Lambda}_0(\xi,p)\,d\xi \tag{A-46}$$

where

$$G_1\left(\frac{x/D}{P\acute{e}}\right) = \sum_{k=1}^{\infty} e^{-4\lambda_k^2 \frac{x/D}{P\acute{e}}}, \tag{A-47}$$

$$K\left[\frac{(x-\xi)/D}{P\acute{e}}\right] = \sum_{k=1}^{\infty} \lambda_k^2 e^{-4\lambda_k^2\frac{(x-\xi)/D}{P\acute{e}}}, \tag{A-48}$$

$$\Gamma_1 = \left(\frac{2\epsilon}{R}\right)\Delta T_e , \tag{A-49}$$

$$\Gamma_2 = \left(\frac{2\epsilon}{R}\right)\left(\frac{4}{DP\acute{e}}\right). \tag{A-50}$$

This Volterra equation includes an ordinary convolution integral on its right-hand side. If use is made of the fact that the Laplace transform of the convolution of two functions is equal to the product of the transforms of these functions, the problem of solving the above Volterra equation is reduced to the problem of determining an inverse transformation.

By taking the transforms of (A-46) in the x-variable, and carrying out the calculation in terms of $\bar{G}_1\left(\frac{q/D}{P\acute{e}}\right)$ and $\bar{K}\left(\frac{q/D}{P\acute{e}}\right)$, the following form is obtained

$$\bar{\bar{\Lambda}}_0(q,p) = \frac{\Gamma_1}{p^2}\bar{G}_1\left(\frac{q/D}{P\acute{e}}\right) + \frac{\Gamma_2}{p}\bar{K}\left(\frac{q/D}{P\acute{e}}\right)\bar{\bar{\Lambda}}_0(q,p) \tag{A-51}$$

Solving this equation for $\bar{\bar{\Lambda}}_0(q,p)$ and rearranging gives

$$\bar{\bar{\Lambda}}_0(q,p) = \Gamma_1 G_1\left(\frac{q/D}{P\acute{e}}\right)\frac{1}{p}\cdot\frac{1}{p-\Gamma_2 K\left(\frac{q/D}{P\acute{e}}\right)} \tag{A-52}$$

After an inverse transformation from p to t, this becomes

$$\bar{\Lambda}_0(q,t) = \Gamma_1 \bar{G}_1\left(\frac{q/D}{P\acute{e}}\right) \int_0^t e^{\Gamma_2 \bar{K}\left(\frac{q/D}{P\acute{e}}\right) s} ds , \qquad (A.53)$$

and the integration with respect to s gives

$$\bar{\Lambda}_0(q,t) = \frac{\Gamma_1}{\Gamma_2} \frac{\bar{G}_1\left(\frac{q/D}{P\acute{e}}\right)}{\bar{K}\left(\frac{q/D}{P\acute{e}}\right)} \left[e^{\Gamma_2 \bar{G}_2\left(\frac{q/D}{P\acute{e}}\right) s} - 1 \right] \qquad (A.54)$$

At this stage, it is necessary to specify the functions $\bar{G}_1\left(\frac{q/D}{P\acute{e}}\right)$ and $\bar{K}\left(\frac{q/D}{P\acute{e}}\right)$. If use is made of Fig. 5, these two functions may be approximated by the following functions

$$G_1\left(\frac{x/D}{P\acute{e}}\right) = \frac{\Omega_1}{x^{1/2}} - \Omega_2(x) \qquad (A.55)$$

$$K\left(\frac{x/D}{P\acute{e}}\right) = \frac{\Omega_3}{x^{3/2}} + \frac{1}{4}\Omega_2'(x) \qquad (A.56)$$

where

$$G_1'\left(\frac{x/D}{P\acute{e}}\right) = -4K\left(\frac{x/D}{P\acute{e}}\right) \qquad (A.57)$$

$$\Omega_1 = 0.1410616\,(DP\acute{e})^{1/2} \qquad (A.58)$$

$$\Omega_3 = 0.0176327\,(DP\acute{e})^{3/2} \qquad (A.59)$$

If $\frac{x/D}{P\acute{e}} \leq 10^{-2}$, the effect of $\Omega_2'(x)$ on $K\left(\frac{x/D}{P\acute{e}}\right)$ is at most 1.52%, assuming this latter value when the equality holds. Therefore, in the approximation of $K\left(\frac{x/D}{P\acute{e}}\right)$, $\Omega_2'(x)$ can be safely neglected. For the values of $\frac{x/D}{P\acute{e}} \leq 10^{-3}$, the maximum effect of $\Omega_2(x)$ on $G_1\left(\frac{x/D}{P\acute{e}}\right)$ is 6.07%, which occurs when $\frac{x/D}{P\acute{e}} = 10^{-3}$. For that reason, in the following procedure a theory is developed which is based on the assumption that when $\frac{x/D}{P\acute{e}} \leq 10^{-3}$, the terms $\Omega_2(x)$ and $\Omega_2'(x)$ are of negligible effect on the calculation of the tube-wall temperature. The theory is also approximately true for the values of $10^{-3} \leq \frac{x/D}{P\acute{e}} \leq 10^{-2}$. Maximum error occurs when $\frac{x/D}{P\acute{e}} = 10^{-2}$, giving about 5% error in temperatures and 20% in heat fluxes.

The Laplace transforms in the x-direction of the above two functions may be written in the form

$$\bar{G}_1\left(\frac{q/D}{P\acute{e}}\right) = \Omega_1 \frac{\pi^{1/2}}{q^{1/2}} \tag{A.60}$$

$$\bar{K}\left(\frac{q/D}{P\acute{e}}\right) = -2\pi^{1/2}\Omega_3 q^{1/2} \tag{A.61}$$

Substituting these transformed functions into (A-53),

$$\bar{\Lambda}_0(q,t) = -\frac{1}{2}\frac{\Gamma_1\Omega_1}{\Gamma_2\Omega_3}\left[\frac{e^{-2\pi^{1/2}\Gamma_2\Omega_3 t\, q^{1/2}}}{q} - \frac{1}{q}\right] \tag{A.62}$$

is obtained. The use of transform pair No. 803 (pp. 92) in reference (21) results in the first approximation for the tube-wall temperature

$$\frac{\theta_0(x,t)}{\Delta T_e} = \begin{cases} 0, & t < \frac{x}{U} \\ \operatorname{erf}\left[\frac{MFo}{\left(\frac{x/D}{P\acute{e}}\right)^{1/2}}\right], & t \geq \frac{x}{U} \end{cases} \tag{A.63}$$

where

$$M = 2\frac{(\rho C_p)}{(\rho C_p)_w}\cdot\frac{1}{(R_0/R)^2 - 1} \tag{A.64}$$

Then, inverse transformation of (A-44) and (A-57) gives the first approximation of the fluid temperature

$$\frac{T_0(r,x,t)}{\Delta T_e} = \begin{cases} 0, & t < \frac{x}{U} \\ 2\sum_{k=1}^{\infty}\left[1 + 4\lambda_k^2 A_k(\lambda_k,x,t)\right] e^{-4\lambda_k^2 \frac{x/D}{Pe}} \frac{J_0(\lambda_k \frac{r}{R})}{\lambda_k J_1(\lambda_k)}, & t \geq \frac{x}{U} \end{cases} \qquad (A\text{-}65)$$

where

$$A_k(\lambda_k,x,t) = \int_0^{\frac{x/D}{Pe}} e^{4\lambda_k^2 \frac{\xi/D}{Pe}} \operatorname{erf}\left[\frac{MFo}{\left(\frac{\xi/D}{Pe}\right)^{1/2}}\right] d\left(\frac{\xi/D}{Pe}\right) \qquad (A\text{-}66)$$

For the first approximation of the fluid temperature, only $A_k(\lambda_k,x,t)$ remains to be determined. Indeed, the integral involved can be numerically calculated for different values of λ_k and MFo. However, for the second approximations of the fluid and tube-wall temperatures, the explicit form of $A_k(\lambda_k,x,t)$ is convenient. The expansion of $\operatorname{erf}\left[MFo/\left(\frac{x/D}{Pe}\right)^{1/2}\right]$ for small values of $MFo/\left(\frac{x/D}{Pe}\right)^{1/2}$ cannot be used owing to the indefinite forms of the resulting integrals. This difficulty is avoided if the following expansion

$$H(z) = \frac{2}{\pi^{1/2}} z - \sum_{i=0}^{\infty} a_i z^{\frac{2i+3}{i+2}} \qquad (A\text{-}67)$$

is used. This function, taking a sufficient number of terms, can be approximated by $\operatorname{erf}(z)$. Since the range $0.00 \leq MFo/\left(\frac{x/D}{Pe}\right)^{1/2} \leq 1.00$ includes 85% of the transient phenomena, a two point approximation valid for this range and based on the equality of the two functions at

$MFo/\left(\frac{x/D}{Pe}\right)^{1/2} = 0.4$ and 0.8 gives a maximum error of 2.47% when $MFo/\left(\frac{x/D}{Pe}\right)^{1/2} = 0.2$. If $\Delta T_e = 100^\circ F$ is taken as the step temperature, the discrepancy due to the above error is $22.82 - 22.27 = 0.55^\circ F$. With this approximation, the error function may be written as

$$erf\, z = \frac{2}{\pi^{1/2}} z + 1.00096 z^{3/2} - 1.27184 z^{5/3} \qquad (A.68)$$

By using this expansion, (A-66) gives

$$A_k(\lambda_k, x, t) = \int_0^X e^{4\lambda_k^2 \Xi} \left(\frac{b_0}{\Xi^{1/2}} + \frac{b_1}{\Xi^{3/4}} - \frac{b_2}{\Xi^{5/6}} \right) d\Xi \qquad (A.69)$$

or

$$A_k(\lambda_k, x, t) = 2b_0 X^{1/2} e^{4\lambda_k^2 X} B_0(\lambda_k, X) + 4b_1 X^{1/4} e^{4\lambda_k^2 X} B_1(\lambda_k, X) + 6b_2 X^{1/6} e^{4\lambda_k^2 X} B_2(\lambda_k, X) + \cdots \qquad (A.70)$$

where

$$b_0 = \frac{2}{\pi^{1/2}} (MFo) \qquad (A.71)$$

$$b_1 = 1.00096 (MFo)^{3/2} \qquad (A.72)$$

$$b_2 = 1.27184 (MFo)^{5/3} \qquad (A.73)$$

$$B_0(\lambda_k, X) = \sum_{m=0}^{\infty} (-1)^m \frac{(8\lambda_k^2 X)^m}{\prod_{j=0}^{m} (2j+1)} \qquad (A.74)$$

$$B_1(\lambda_k, X) = \sum_{m=0}^{\infty} (-1)^m \frac{(16\lambda_k^2 X)^m}{\prod_{j=0}^{m} (4j+1)} \qquad (A.75)$$

$$B_2(\lambda_k, X) = \sum_{m=0}^{\infty} (-1)^m \frac{(24\lambda_k^2 X)^m}{\prod_{j=0}^{m} (6j+1)} \qquad (A.76)$$

with $X = \frac{x/D}{Pe}$.

Equations (A-63) and (A-65) give in final form the first approximations for the fluid and tube-wall temperatures.

The differential equations for the second approximations, T_1 and θ_1, are obtained by the use of equations (A-1), (A-2), (A-11) and (A-12) which are in the form

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial r} = \alpha\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r}\right) \qquad (A\text{-}1)$$

$$\frac{\partial T_0}{\partial t} + U\frac{\partial T_0}{\partial x} = \alpha\left(\frac{\partial^2 T_0}{\partial r^2} + \frac{1}{r}\frac{\partial T_0}{\partial r}\right) \qquad (A\text{-}11)$$

$$\frac{\partial \theta}{\partial t} + \epsilon\left(\frac{\partial T}{\partial r}\right)_{r=R} = 0 \qquad (A\text{-}2)$$

$$\frac{\partial \theta_0}{\partial t} + \epsilon\left(\frac{\partial T_0}{\partial r}\right)_{r=R} = 0 \qquad (A\text{-}12)$$

Adding $U\frac{\partial T}{\partial x} - U\frac{\partial T}{\partial x}$ to (A-1) and subtracting (A-2) from (A-1), the following equation is obtained

$$\frac{\partial T_1}{\partial t} + U\frac{\partial T_1}{\partial x} = \alpha\left(\frac{\partial^2 T_1}{\partial r^2} + \frac{1}{r}\frac{\partial T_1}{\partial r}\right) + U\left(1-\frac{u}{U}\right)\frac{\partial T}{\partial x} - v\frac{\partial T}{\partial r} \qquad (A\text{-}77)$$

In this differential equation, the convective terms are to be estimated from the first approximation $T_0(r,x,t)$.

Defining $F(r,x,t)$ as

$$F(r,x,t) = \begin{cases} 0, & t < \frac{x}{U} \\ \left(1-\frac{u}{U}\right)\frac{\partial T_0}{\partial x} - \frac{v}{U}\frac{\partial T_0}{\partial r}, & t \geq \frac{x}{U} \end{cases} \qquad (A\text{-}78)$$

the differential equations satisfied by the second approximations of the temperatures, T_1 and θ_1, may be written as follows:

$$\frac{\partial T_1}{\partial t} + U\frac{\partial T_1}{\partial x} = \alpha\left(\frac{\partial^2 T_1}{\partial r^2} + \frac{1}{r}\frac{\partial T_1}{\partial r}\right) + UF(r,x,t) \qquad (A.79)$$

$$\frac{\partial \theta_1}{\partial t} + \epsilon\left(\frac{\partial T_1}{\partial r}\right)_{r=R} = 0 \qquad (A.80)$$

$$T_1(r,x,0) = 0 \qquad (A.81)$$

$$\theta_1(x,0) = 0 \qquad (A.82)$$

$$T_1(r,0,t) = 0 \qquad (A.83)$$

$$\frac{\partial T_1(0,x,t)}{\partial r} = 0 \qquad (A.84)$$

$$T_1(R,x,t) = \theta_1(x,t) = \begin{cases} 0, & t < \frac{x}{U} \\ \Lambda_1(x,t), & t \geq \frac{x}{U} \end{cases} \qquad (A.85)$$

where again $\Lambda_1(x,t)$ is defined only for mathematical convenience, as will be seen in the following procedure.

By the use of the method previously outlined in the calculation of first approximations, that is, by first taking the Laplace transforms of the above equations with respect to t, there is obtained

$$p\bar{T}_1 + U\frac{\partial \bar{T}_1}{\partial x} = \alpha\left(\frac{\partial^2 \bar{T}_1}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{T}_1}{\partial r}\right) + U\bar{F}(r,x,p)\,e^{-\frac{x}{U}p} \qquad (A.86)$$

$$p\bar{\theta}_1 + \epsilon\left(\frac{\partial \bar{T}_1}{\partial r}\right)_{r=R} = 0 \qquad (A.87)$$

$$\bar{T}_1(r,0,p) = 0 \qquad (A.88)$$

$$\frac{\partial \bar{T}_1(0,x,p)}{\partial r} = 0 \qquad (A.89)$$

$$\bar{T}_1(R,x,p) = \bar{\theta}_1(x,p) = \Lambda_1(x,p)\,e^{-\frac{x}{U}p} \qquad (A.90)$$

Making use of the same transformation

$$\bar{T}_1(r,x,p) = \bar{\Pi}_1(r,x,p)\, e^{-\frac{x}{U}p} \qquad (A.91)$$

the above equations are converted into the form

$$U\frac{\partial\bar{\Pi}_1}{\partial x} = \alpha\left(\frac{\partial^2\bar{\Pi}_1}{\partial r^2} + \frac{1}{r}\frac{\partial\bar{\Pi}_1}{\partial r}\right) + U\bar{F}(r,x,p) \qquad (A.92)$$

$$p\bar{\Lambda}_1(x,p) + \epsilon\left(\frac{\partial\bar{\Pi}_1}{\partial r}\right)_{r=R} = 0 \qquad (A.93)$$

$$\bar{\Pi}_1(r,0,p) = 0 \qquad (A.94)$$

$$\frac{\partial\bar{\Pi}_1(0,x,p)}{\partial r} = 0 \qquad (A.95)$$

$$\bar{\Pi}_1(R,x,p) = \bar{\Lambda}_1(x,p) \qquad (A.96)$$

Equation (A-92) can also be written in the following dimensionless form in the r- variable in terms of (r/R)

$$U\frac{\partial\bar{\Pi}_1}{\partial x} = \alpha\left[\frac{\partial^2\bar{\Pi}_1}{\partial(r/R)^2} + \frac{1}{(r/R)}\frac{\partial\bar{\Pi}_1}{\partial(r/R)^2}\right]\frac{1}{R^2} + U\bar{F}\left(\frac{r}{R},x,p\right) \qquad (A.97)$$

Denoting by $\bar{\bar{\Pi}}_{1j}(\lambda_n,x,p)$ the finite Hankel transform of order zero of the function $\bar{\Pi}_1(r/R,x,p)$

$$\bar{\bar{\Pi}}_{1j}(\lambda_n,x,p) = R^2\int_0^1 (r/R)\,\bar{\Pi}_1(r/R,x,p)\,J_0(\lambda_n r/R)\,d(r/R) \qquad (A.98)$$

by definition. Multiplying equations (A-94), (A-95), (A-96) and (A-97) throughout by $R^2(r/R)J_0(\lambda_n r/R)$ and integrating with respect to over the range $(0,1)$, it is found that $\bar{\bar{\Pi}}_{1j}(\lambda_n,x,p)$ is determined by the solution of the following first-order linear differential equation

$$\frac{d\bar{\bar{\Pi}}_{1j}}{dx} + \left(\frac{4\lambda_n^2}{DP\acute{e}}\right)\bar{\bar{\Pi}}_{1j} = \frac{\alpha}{U}\lambda_n J_1(\lambda_n)\bar{\Lambda}_1(x,p) + R^2\int_0^1 (r/R)\bar{F}(r/R,x,p)J_0(\lambda_n r/R)\,d(r/R) \tag{A-99}$$

with the boundary condition

$$\bar{\bar{\Pi}}_{1j}(\lambda_n, 0, p) = 0 \tag{A-100}$$

The solution of this equation is in the form

$$\bar{\bar{\Pi}}_{1j}(\lambda_n, x, p) = R^2\int_0^x\int_0^1 e^{-4\lambda_n^2\frac{(x-\xi)/D}{P\acute{e}}}(r/R)\bar{F}(r/R,\xi,p)J_0(\lambda_n r/R)\,d(r/R)\,d\xi + \frac{\alpha}{U}\lambda_n J_1(\lambda_n)\int_0^x e^{-4\lambda_n^2\frac{(x-\xi)/D}{P\acute{e}}}\bar{\Lambda}_1(\xi,p)\,d\xi \tag{A-101}$$

Again inverting this equation by means of the same inversion theorem (A-43) results in

$$\bar{\Pi}_1(r/R, x, p) = 2\sum_{n=1}^{\infty}\frac{J_0(\lambda_n r/R)}{J_1^2(\lambda_n)}\int_0^x\int_0^1 e^{-4\lambda_n^2\frac{(x-\xi)/D}{P\acute{e}}}(r/R)\bar{F}(r/R,\xi,p)J_0(\lambda_n r/R)\,d(r/R)\,d\xi + \left(\frac{8}{DP\acute{e}}\right)\sum_{n=1}^{\infty}\frac{\lambda_n J_0(\lambda_n r/R)}{J_1(\lambda_n)}\int_0^x e^{-4\lambda_n^2\frac{(x-\xi)/D}{P\acute{e}}}\bar{\Lambda}_1(\xi,p)\,d\xi \tag{A-102}$$

as the second approximation of the transformed (in time) fluid temperature in terms of the unknown function $\bar{\Lambda}_1(x,p)$. If use is made of (A-93) and (A-102), there follows

$$\bar{\Lambda}_1(x,p) = \frac{1}{p}\left(\frac{2\epsilon}{R}\right)\sum_{n=1}^{\infty}\frac{\lambda_n}{J_1(\lambda_n)}\int_0^{\frac{x/D}{P\acute{e}}}\int_0^1 e^{-4\lambda_n^2\frac{(x-\xi)/D}{P\acute{e}}}(DP\acute{e})\left(\frac{r}{R}\right)\bar{F}(r/R,\xi,p)J_0(\lambda_n r/R)\,d(r/R)\,d\left(\frac{\xi/D}{P\acute{e}}\right) + \frac{1}{p}\left(\frac{2\epsilon}{R}\right)\left(\frac{4}{DP\acute{e}}\right)\int_0^x\sum_{n=1}^{\infty}\lambda_n^2 e^{-4\lambda_n^2\frac{(x-\xi)/D}{P\acute{e}}}\bar{\Lambda}_1(\xi,p)\,d\xi \tag{A-103}$$

This equation gives the second approximation of the tube-wall temperature, again in the form of a Volterra equation of the second kind. Here $\bar{F}(r/R, \Xi, p)$, which is the transformed form of (A-78) may be written as

$$\bar{F}(r/R, \Xi, p) = \left(1 - \frac{u}{U}\right)\frac{\partial \bar{\Pi}_0}{\partial x} - \frac{v}{U}\frac{\partial \bar{\Pi}_0}{\partial r} \qquad (A.104)$$

As previously indicated in part II (pp. 7), the growth of the velocity boundary layer at the entrance region of a tube is approximated by that for laminar flow over a flat plate. This latter problem is well-known and, using the concept of similarity of velocity profiles, was first solved by Blasius (18). The flat plate solution, which includes the usual boundary layer approximations, gives

$$(DP\acute{e})\bar{F}(r/R, \Xi, p) = (1-f')\frac{\partial \bar{\Pi}_0}{\partial \Xi} - \left(\frac{Pr}{\Xi}\right)^{1/2}(\eta f' - f)\frac{\partial \bar{\Pi}_0}{\partial (r/R)} \qquad (A.105)$$

where

$$\eta = \frac{y/D}{(\Xi Pr)^{1/2}}, \qquad (A.106)$$

and $f(\eta)$ (A.107)

is the dimensionless stream function.

In addition to r/R and X, another dimensionless parameter, the Prandtl number Pr is introduced. Considering this dependence and defining a new function $\bar{G}_2(X, p, Pr)$, the above integral equation may be written in the following form

$$\bar{\Lambda}_1(x,p,Pr) = \frac{1}{p}\left(\frac{2\epsilon}{R}\right)\Delta T_e\, \bar{G}_2(X,p,Pr) + \frac{1}{p}\left(\frac{4}{DP\acute{e}}\right)\left(\frac{2\epsilon}{R}\right)\int_0^X \sum_{n=1}^{\infty} \lambda_n^2 e^{-4\lambda_n^2 \frac{(x-\xi)/D}{P\acute{e}}}\bar{\Lambda}_1(\xi,p)\,d\xi \qquad (A.108)$$

where

$$\bar{G}_2(X,p,Pr) = \int_0^X\int_0^1 \left(\frac{r}{R}\right)\left(\frac{DP\acute{e}}{\Delta T_e}\right)\bar{F}(r/R,\Xi,p,Pr)\sum_{n=1}^{\infty} \lambda_n e^{-4\lambda_n^2(X-\Xi)} \frac{J_0(\lambda_n r/R)}{J_1(\lambda_n)}\, d(r/R)\, d\Xi \qquad (A\text{-}109)$$

Comparison of the integral equations (A-45) and (A-108), corresponding to the first and second approximations of the transformed tube-wall temperature, shows that both have the same kernel. Therefore, the approximation made by (A-56) for this kernel can be used again. For the approximation of $\bar{G}_2(X,p,Pr)$, first (A-65) is considered for the radial and axial fluid temperature gradients. Taking the Laplace transforms in time and combining the result with (A-30) gives

$$\frac{\bar{\Pi}_0(r,x,p)}{\Delta T_e} = 2\sum_{k=1}^{\infty}\left[\frac{1}{p} + 4\lambda_k^2 \bar{A}_k(\lambda_k,x,p)\right] e^{-4\lambda_k^2 \frac{x/D}{P\acute{e}}} \frac{J_0(\lambda_k r/R)}{\lambda_k J_1(\lambda_k)} \qquad (A\text{-}110)$$

Now, calculating the transformed temperature gradients $\frac{\partial\bar{\Pi}_0}{\partial\Xi}$ and $\frac{\partial\bar{\Pi}_0}{\partial(r/R)}$ from the above equation and substituting into (A-104) results in

$$\begin{aligned}\left(\frac{DP\acute{e}}{\Delta T_e}\right)\bar{F}(r/R,\Xi,p,Pr) = {} & \frac{1}{p}\, 2\sum_{k=1}^{\infty} \frac{e^{-4\lambda_k^2\Xi}}{J_1(\lambda_k)}\Delta_1 \\ & + \frac{1}{p^2}\,\frac{8C_0}{\Xi^{1/2}}\sum_{k=1}^{\infty}\frac{\lambda_k^2}{J_1(\lambda_k)}(2\Xi B_0\Delta_1 + \Delta_2) \\ & + \frac{1}{p^{3/2}}\,\frac{8C_1}{\Xi^{3/4}}\sum_{k=1}^{\infty}\frac{\lambda_k^2}{J_1(\lambda_k)}(4\Xi B_1\Delta_1 + \Delta_2) \\ & - \frac{1}{p^{8/3}}\,\frac{8C_2}{\Xi^{5/6}}\sum_{k=1}^{\infty}\frac{\lambda_k^2}{J_1(\lambda_k)}(6\Xi B_2\Delta_1 + \Delta_2) + \cdots\end{aligned} \qquad (A\text{-}111)$$

where

$$\Delta_1(\lambda_k, r/R, \Xi, Pr) = -\left(\frac{Pr}{\Xi}\right)^{1/2}(\eta f' - f)J_1(\lambda_k r/R) - 4(1-f')\lambda_k J_0(\lambda_k r/R), \quad (A\text{-}112)$$

$$\Delta_2(\lambda_k, r/R, \Xi) = (1-f')\,\frac{J_0(\lambda_k r/R)}{\lambda_k} \qquad (A\text{-}113)$$

$$C_0 = 2/\pi^{1/2}(MFO) \qquad (A\text{-}114)$$

$$C_1 = 1.00096\,(MFO)^{3/2} \qquad (A\text{-}115)$$

$$C_2 = 1.27184\,(MFO)^{5/3} \qquad (A\text{-}116)$$

In the calculation of $\bar{G}(X,p,Pr)$, because of the complicated form of (A-111), numerical integration is necessary. In this study the first term of (A-111) was taken into account. Since the first term is proportional to $MFO/\left(\frac{x/D}{Pe}\right)^{1/2}$ while the second varies with $\frac{1}{2!}\left[MFO/\left(\frac{x/D}{Pe}\right)^{1/2}\right]^2$ the error made is about 10%, for values of $MFO/\left(\frac{x/D}{Pe}\right)^{1/2} \leq 0.2$. However, if the second approximation is assumed to be 30% of the tube-wall temperature, 10% error in the second approximation gives approximately 3% error in the final solution.

The integral equation (A-108) is linear with respect to $\bar{G}(X,p,Pr)$. Therefore, it may be separately solved for each term of $\bar{G}(X,p,Pr)$. If the first two terms of this function are considered instead of first term alone, it would be sufficient to solve the integral equation for the second term and superimpose the solution on the solution obtained from the consideration of the first term alone.

The first term of the second approximation of the transformed tube-wall temperature may be obtained from the following integral equation

$$\bar{\Lambda}_1(x,p,Pr) = 2\frac{\Gamma_1}{p^2} G_{21}(x,Pr) + \Omega_3 \frac{\Gamma_2}{p}\int_0^x \frac{\bar{\Lambda}_1(\xi,p,Pr)}{(x-\xi)^{3/2}}\,d\xi \qquad (A\text{-}117)$$

where

$$G_{21}(x,Pr) = -\int_0^X\int_{1-\delta/R}^1 (r/R)\left(\frac{DP\acute{e}}{\Delta T_e}\right) I(r/R,\Xi,Pr) \sum_{n=1}^{\infty} \lambda_n e^{-4\lambda_n^2(X-\Xi)} \frac{J_0(\lambda_n r/R)}{J_1(\lambda_n)}\, d(r/R)\, d\Xi \qquad (A\text{-}118)$$

and

$$\left(\frac{DP\acute{e}}{\Delta T_e}\right) I(r/R,\Xi,Pr) = 4(1-f')\sum_{k=1}^{\infty}\lambda_k e^{-4\lambda_k^2\Xi}\frac{J_0(\lambda_k r/R)}{J_1(\lambda_k)} + \left(\frac{Pr}{\Xi}\right)^{1/2}(\eta f'-f)\sum_{k=1}^{\infty} e^{-4\lambda_k^2\Xi}\frac{J_1(\lambda_k r/R)}{J_1(\lambda_k)} \qquad (A\text{-}119)$$

As indicated previously, the second approximations are investigated for small values of $MFo/\left(\frac{x/D}{P\acute{e}}\right)^{1/2}$ owing to their complicated form. These values allow the use of the first term alone, that is $G_{21}(x,Pr)$, for the second approximation of the transformed tube-wall temperature.

$$\bar{\Lambda}_1(x,p,Pr) = 2\frac{\Gamma_1}{p^2} G_{21}(x,Pr) \qquad (A\text{-}120)$$

Then, the first term of the second approximation of the tube-wall temperature may be written as

$$\frac{\theta_1(x,t,Pr)}{\Delta T_e} = -MFo\int_0^X\int_{1-\delta/R}^1 (r/R)\left(\frac{DP\acute{e}}{\Delta T_e}\right) I(r/R,\Xi,Pr)\, 8\sum_{n=1}^{\infty}\lambda_n e^{-4\lambda_n^2(X-\Xi)}\frac{J_0(\lambda_n r/R)}{J_1(\lambda_n)}\, d(r/R)\, d\Xi \qquad (A\text{-}121)$$

where $\sum_{n=1}^{\infty} \lambda_n e^{-4\lambda_n^2 X} \frac{J_0(\lambda_n r/R)}{J_1(\lambda_n)}$ has been evaluated for different values of r/R and X, and $\left(\frac{DP\acute{e}}{\Delta Te}\right) I(r/R, \Xi, Pr)$ for different values of r/R, X and Pr, by high speed digital computer. The results may be approximated by the following functions

$$\sum_{n=1}^{\infty} \lambda_n e^{-4\lambda_n^2 X} \frac{J_0(\lambda_n r/R)}{J_1(\lambda_n)} \cong \frac{1}{32\pi^{1/2}} \cdot \frac{1-r/R}{(r/R)^{1/2}} \cdot \frac{e^{-\frac{(1-r/R)^2}{16X}}}{X^{3/2}} \qquad (A\text{-}122)$$

$$\left(\frac{DP\acute{e}}{\Delta Te}\right) I(r/R, X, Pr) \cong \frac{P(Pr)}{20} \cdot \frac{1-r/R}{(r/R)^{1/2}} \cdot \frac{e^{-\frac{(1-r/R)^2}{16X}}}{X^{3/2}} \qquad (A\text{-}123)$$

where

Pr	0.01	1	10	100
$P(Pr)$	0.3	1	1.3	1.5

(A-124)

Substituting the approximate forms given above into (A-121)

$$\frac{\theta_1(x,t,Pr)}{\Delta Te} = -\frac{(MFo)}{80\pi^{1/2}} P(Pr) \int_0^X \int_{1-\delta/R}^{1} (1-r/R)^2 \frac{e^{-\frac{X}{16\Xi(X-\Xi)}(1-r/R)^2}}{\Xi^{3/2}(X-\Xi)^{3/2}} d(r/R)\, d\Xi \qquad (A\text{-}125)$$

is obtained.

The above equation may be put into more convenient form with the substitution

$$Y = 1 - r/R \qquad (A\text{-}126)$$

The result is

$$\frac{\theta_1(x,t,Pr)}{\Delta T_e} = -\frac{0.2P(Pr)(MFO)}{16\pi^{1/2}}\int_0^X \frac{1}{\Xi^{3/2}(X-\Xi)^{3/2}}\int_0^{\delta/R} Y^2 e^{-\gamma Y^2}\, dY\, d\Xi \qquad (A.127)$$

where

$$\gamma = \frac{X}{16\Xi(X-\Xi)} \qquad (A.128)$$

Integrating (A-127) by parts in the Y -variable gives

$$\frac{\theta_1(x,t,Pr)}{\Delta T_e} = -\frac{0.2P(Pr)MFO}{16\pi^{1/2}}\left\{\frac{8}{X}\int_0^X \frac{1}{\Xi^{1/2}(X-\Xi)^{1/2}}\left[\frac{1}{2}\left(\frac{\pi}{\gamma}\right)^{1/2} erf\left(\gamma^{1/2}\delta/R\right) - \delta/R\, e^{-\gamma(\delta/R)^2}\right]\right\} \qquad (A.129)$$

or, explicitly

$$\frac{\theta_1(x,t,Pr)}{\Delta T_e} = -0.2P(Pr)\frac{MFO}{X^{3/2}}\left[\int_0^X erf\left[\frac{10(XPr)^{1/2}}{4(X-\Xi)^{1/2}}\right]d\Xi - 5\left(\frac{XPr}{\pi}\right)^{1/2}\int_0^X \frac{e^{-\frac{25XPr}{4(X-\Xi)}}}{(X-\Xi)^{1/2}}d\Xi\right] \qquad (A.130)$$

Using the convolutive property of the transform calculus and taking the Laplace transforms of (A-130) in the X -variable, the following form is obtained

$$\frac{\bar{\theta}_1(q,t,Pr)}{\Delta T_e} = -0.2P(Pr)\frac{MFO}{X^{3/2}}\left[-\left(\frac{e^{-5(XPr)^{1/2}q^{1/2}}}{q^2} - \frac{1}{q^2}\right) - 5(XPr)^{1/2}\frac{e^{-5(XPr)^{1/2}q^{1/2}}}{q^{1/2}}\right] \qquad (A.131)$$

The use of transform pairs *No. 804.1* (pp. 92) and *No. 808.1* (pp. 93) in reference (21) results in the second approximation for the tube-wall temperature

$$\frac{\theta_1(x,t,Pr)}{\Delta T_e} = \begin{cases} 0\,, & t < \frac{x}{U} \\ -0.2\,P(Pr)\left[1 - 5\left(\frac{Pr}{\pi}\right)^{1/2} e^{-\frac{25}{4}Pr} - \left(1 - \frac{25}{2}Pr\right)\mathrm{erfc}\left(\frac{5}{2}Pr^{1/2}\right)\right]\dfrac{MFo}{\left(\frac{x/D}{Pe}\right)^{1/2}} & t \geq \frac{x}{U} \end{cases} \quad (A\text{-}132)$$

which holds for small values of $MFo/\left(\frac{x/D}{Pe}\right)^{1/2}$.

Excluding nuclear metals, the terms in the brackets have negligible effect on the second approximations.

By the use of equations (A-80) and (A-132), the following expression for the second approximation of the modified Nusselt number is obtained

$$Nu_1(x,t,Pr) = \begin{cases} 0\,, & t < \frac{x}{U} \\ -0.1\,\dfrac{P(Pr)}{\left(\frac{x/D}{Pe}\right)^{1/2}}\left[1 - 5\left(\frac{Pr}{\pi}\right)^{1/2} e^{-\frac{25}{4}Pr} - \left(1 - \frac{25}{2}Pr\right)\mathrm{erfc}\left(\frac{5}{2}Pr^{1/2}\right)\right], & t \geq \frac{x}{U} \end{cases} \quad (A\text{-}133)$$

APPENDIX B

Alternate Solutions of $\bar{\Pi}_0(r,x,p)$

The purpose of this appendix is to give alternate solutions to $\bar{\Pi}_0(r,x,p)$ by means of more familiar but longer methods.

The differential equation (A-31), assuming t instead of x, can be considered as a transient heat conduction equation with variable surface temperature. When surface temperature varies with time in a transient heat conduction problem, Duhamel's theorem (22), can be used to reduce that problem to one of constant surface temperature.

Suppose now that the transformed temperature function $\bar{T}_0(r,x,\xi,p)$ depending on the fixed parameters ξ and p is a solution of (A-31), (A-33), (A-34) and (A-35) in which case the surface temperature $\bar{\Lambda}_0$ is a function of the parameters ξ and p, but not of the axial distance x. Then the function $\bar{T}_0(r,x,\xi,p)$ satisfies the following equation and boundary conditions

$$U\frac{\partial \bar{T}_0}{\partial x} = \alpha\left(\frac{\partial^2 \bar{T}_0}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{T}_0}{\partial r}\right) \qquad (B\text{-}1)$$

$$\bar{T}_0(r,0,\xi,p) = \frac{\Delta T_0}{p} \qquad (B\text{-}2)$$

$$\frac{\partial \bar{T}_0(0,x,\xi,p)}{\partial r} = 0 \qquad (B\text{-}3)$$

$$\bar{T}_0(R,x,\xi,p) = \bar{\Lambda}_0(\xi,p) \qquad (B\text{-}4)$$

Two different methods may be used in the solution of the above boundary value problem:

a) Separation of variables

Assuming $\bar{T}_0 = \bar{T}_s + \bar{T}_x$, the problem is divided into the following two separate problems:

$$\frac{\partial^2 \bar{T}_s}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{T}_s}{\partial r} = 0 \qquad (B\text{-}5)$$

$$\frac{\partial \bar{T}_s(0,\xi,p)}{\partial r} = 0 \qquad (B\text{-}6)$$

$$\bar{T}_s(R,\xi,p) = \bar{\Lambda}_0(\xi,p) \qquad (B\text{-}7)$$

and

$$U\frac{\partial \bar{T}_x}{\partial x} = \alpha\left(\frac{\partial^2 \bar{T}_x}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{T}_x}{\partial r}\right) \qquad (B\text{-}8)$$

$$\bar{T}_x(r,0,\xi,p) = \frac{\Delta T_e}{p} - \bar{T}_s(r,\xi,p) \qquad (B\text{-}9)$$

$$\frac{\partial \bar{T}_x(0,x,\xi,p)}{\partial r} = 0 \qquad (B\text{-}10)$$

$$\bar{T}_x(R,x,\xi,p) = 0 \qquad (B\text{-}11)$$

For the first problem, (B-5) has a general solution

$$\bar{T}_s(r,\xi,p) = C_1 \ln r + C_2 \qquad (B\text{-}12)$$

and with (B-6) and (B-7) becomes

$$\bar{T}_s(r,\xi,p) = \bar{\Lambda}_0(\xi,p) \qquad (B\text{-}13)$$

For the second problem, if use is made of the well-known methods of separation of variables technique, a general solution of (B-8) is obtained in the form

$$\bar{T}_x(r,x,\xi,p) = e^{-4\lambda_k^2 \frac{x/D}{Pe}}\left[C_3 J_0\left(\lambda_k \frac{r}{R}\right) + C_4 Y_0\left(\lambda_k \frac{r}{R}\right)\right] \qquad (B\text{-}14)$$

which with (B-9), (B-10) and (B-11) gives the following solution

$$\bar{T}_x(r,x,\xi,p) = 2\left[\frac{\Delta T_e}{p} - \bar{\Lambda}_0(\xi,p)\right] \sum_{k=1}^{\infty} e^{-4\lambda_k^2 \frac{x/b}{Pe}} \frac{J_0(\lambda_k \frac{r}{R})}{\lambda_k J_1(\lambda_k)} \qquad (B\text{-}15)$$

Therefore, the solution of $\bar{T}_0$ is in the form

$$\bar{T}_0(r,x,\xi,p) = \bar{\Lambda}_0(\xi,p) + 2\left[\frac{\Delta T_e}{p} - \bar{\Lambda}_0(\xi,p)\right] \sum_{k=1}^{\infty} e^{-4\lambda_k^2 \frac{x/b}{Pe}} \frac{J_0(\lambda_k \frac{r}{R})}{\lambda_k J_1(\lambda_k)} \qquad (B\text{-}16)$$

Before proceeding further, it may be shown that the application of Laplace transforms in the x-variable leads to the same result.

b) The use of Laplace transforms in the x-variable

Multiplying both sides of equations (B-1), (B-3) and (B-4) by e^{-qx} and integrating with respect to x from 0 to ∞, it is found that

$$\frac{d^2\bar{\bar{T}}_0}{dr^2} + \frac{1}{r}\frac{d\bar{\bar{T}}_0}{dr} - \left(\frac{U}{\alpha}q\right)\bar{\bar{T}}_0 = -\frac{\Delta T_e}{p}\cdot\frac{U}{\alpha} \qquad (B\text{-}17)$$

with

$$\frac{d\bar{\bar{T}}_0(0,q,\xi,p)}{dr} = 0 \qquad (B\text{-}18)$$

$$\bar{\bar{T}}_0(R,q,\xi,p) = \frac{1}{q}\bar{\Lambda}_0(\xi,p) \qquad (B\text{-}19)$$

The differential equation (B-17) has a general solution in the form

$$\bar{\bar{T}}_0(r,q,\xi,p) = C_5 I_0\left[\left(\frac{U}{\alpha}q\right)^{1/2} r\right] + C_6 K_0\left[\left(\frac{U}{\alpha}q\right)^{1/2} r\right] + \frac{\Delta T_e}{pq} \qquad (B\text{-}20)$$

If use is made of (B-18) and (B-19), there follows

$$\bar{\bar{\Upsilon}}_0(r,q,\xi,p) = \frac{\Delta T_e}{pq} + \left[\bar{\Lambda}_0(\xi,p) - \frac{\Delta T_e}{p}\right] \frac{I_0[(\frac{U}{\alpha}q)^{\frac{1}{2}}r]}{q\, I_0[(\frac{U}{\alpha}q)^{\frac{1}{2}}R]} . \qquad (B\text{-}21)$$

By the use of the following Inversion theorem for the Laplace transformation (23) in the x-variable

$$f(x) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{zx}\bar{f}(z)dz \ , \qquad (B\text{-}22)$$

the same result with (B-16) is obtained.

Now, according to Duhamel's theorem, the solution for $\bar{\Pi}_0(r,x,p)$ in the boundary value problem with variable surface temperature (A-31), (A-33), (A-34) and (A-35) is given in terms of the solution $\bar{\Upsilon}_0(r,x,\xi,p)$ of the boundary value problem with constant surface temperature (B-1), (B-2), (B-3) and (B-4) by the formula

$$\bar{\Pi}_0(r,x,p) = \frac{\partial}{\partial x}\int_0^x \bar{\Upsilon}_0(r,x-\xi,\xi,p)d\xi . \qquad (B\text{-}23)$$

The use of the inversion theorem (A-43) results in the following identity for the transform of unity

$$2\sum_{k=1}^{\infty}\frac{J_0(\lambda_k\frac{r}{R})}{\lambda_k J_1(\lambda_k)} \equiv 1 \qquad (B\text{-}24)$$

Substituting (B-16) into (B-23) and then using (B-24) gives, finally

$$\bar{\Pi}_0(r,x,p) = 2\frac{\Delta T_e}{p}\sum_{k=1}^{\infty} e^{-4\lambda_k^2\frac{x/D}{Pe}}\frac{J_0(\lambda_k\frac{r}{R})}{\lambda_k J_1(\lambda_k)} + 2\left(\frac{4}{DPe}\right)\sum_{k=1}^{\infty}\frac{\lambda_k J_0(\lambda_k\frac{r}{R})}{J_1(\lambda_k)}\int_0^x e^{-4\lambda_k^2\frac{(x-\xi)/D}{Pe}}\bar{\Lambda}_0(\xi,p)d\xi$$
(B-25)

which is identical with (A-44).

APPENDIX C

The Exact Solution of $\bar{\Lambda}_0(x,p)$

Equation (A-45) may be written as

$$\bar{\Lambda}_0(x,p)=\frac{1}{p^2}\left(\frac{2\epsilon}{R}\right)\Delta T_e\sum_{k=1}^{\infty}e^{-4\lambda_k^2\frac{x/D}{P\acute{e}}}+\frac{1}{p}\left(\frac{2\epsilon}{R}\right)\left(\frac{4}{DP\acute{e}}\right)\int_0^x\sum_{k=1}^{\infty}\lambda_k^2 e^{-4\lambda_k^2\frac{(x-\xi)/D}{P\acute{e}}}\bar{\Lambda}_0(\xi,p)d\xi \quad (A\text{-}45)$$

The second term of the right hand side is an ordinary convolution integral, and therefore the above integral equation may be solved by using Laplace transforms technique (24). However, for the present case, the problem is solved by a method of successive approximations.

The analysis is considerably reduced if an integral operator $\mathcal{K}_x(x)$ defined by the equation

$$\mathcal{K}_x(x)=\int_0^x K(x,\xi)f(\xi)d\xi \quad (C\text{-}1)$$

is introduced. Then the solution of (A-45) can be expressed in terms of the following finite series

$$\frac{\bar{\Lambda}_0(x,p)}{\Delta T_e}=\frac{1}{p^2}\left(\frac{2\epsilon}{R}\right)\sum_{k=1}^{\infty}e^{-4\lambda_k^2\frac{x/D}{P\acute{e}}}+\sum_{N=1}^{\infty}\left(\frac{4}{DP\acute{e}}\right)^N\left(\frac{2\epsilon}{R}\right)^{N+1}\frac{1}{p^{N+2}}\mathcal{K}_x^N(\overbrace{i,j,k,\dots}^{N+1}) \quad (C\text{-}2)$$

where

$$\left.\begin{aligned}
K(x,\overset{j}{\hat{}}\xi) &= \sum_{j=1}^{\infty}\lambda_j^2 e^{-4\lambda_j^2\frac{(x-\xi)/D}{P\acute{e}}}\\
\mathcal{K}_x(i,j) &= \int_0^x K(x,\overset{j}{\hat{}}\xi)\sum_{i=1}^{\infty}e^{-4\lambda_i^2\frac{\xi/D}{P\acute{e}}}d\xi\\
\mathcal{K}_x^2(i,j,k) &= \int_0^x K(x,\overset{k}{\hat{}}\xi)\,\mathcal{K}_x(i,j)\,d\xi\\
&\dots\dots\dots\dots\dots\dots
\end{aligned}\right\} \quad (C\text{-}3)$$

It remains to determine the conditions under which the last term of the above series converges. However, it may be formally proved (24) that the series (C-2) converges to a unique and continuous solution of the Volterra equation (A-45) for all values of $\frac{1}{p}\left(\frac{2\epsilon}{R}\right)\left(\frac{4}{DP\acute{e}}\right)$, in any interval (a,b) in which $K(x,\xi)$ is continuous.

Inverse transformation of the above equation from p to t and use of (A-17) gives the zeroth approximation of the wall temperature, which may be written in the following form

$$\frac{\theta_0(x,t)}{\Delta T_e}=\begin{cases}0, & t<\frac{x}{U}\\ \left(\frac{2\epsilon}{R}t\right)\sum_{k=1}^{\infty}e^{-4\lambda_k^2\frac{x/D}{P\acute{e}}}+\sum_{N=1}^{\infty}\left(\frac{4}{DP\acute{e}}\right)^N\frac{\left(\frac{2\epsilon}{R}t\right)^{N+1}}{(N+1)!}\mathcal{K}_x^N(\overline{i,j,k,\ldots}^{\,N+1}), & t\doteq\frac{x}{U}\end{cases}\tag{C-4}$$

Equation (C-2) may be used also to obtain the zeroth approximation of the fluid temperature. Taking the value of $\bar{\Lambda}_0(x,p)$ from (C-2) and substituting into the equation (A-44) results in

$$\frac{\bar{\Pi}_0(r,x,p)}{\Delta T_e}=\frac{2}{p}\sum_{k=1}^{\infty}e^{-4\lambda_k^2\frac{x/D}{P\acute{e}}}\frac{J_0(\lambda_k\frac{r}{R})}{\lambda_k J_1(\lambda_k)}+2\sum_{N=1}^{\infty}\left(\frac{4}{DP\acute{e}}\right)^N\left(\frac{2\epsilon}{R}\right)^N\frac{1}{p^{N+1}}\mathcal{K}_x^N(\overline{i,j,k,\ldots}^{\,N+1})\frac{J_0(\lambda_k\frac{r}{R})}{\lambda_k J_1(\lambda_k)}\tag{C-5}$$

Referring to the transformation (A-30) gives the zeroth approximation of the fluid temperature, Laplace transformed in variable t . Then, by an inverse transformation, the zeroth approximation of the fluid temperature may be written as

$$\frac{T_0(r,x,t)}{\Delta T_e}=\begin{cases} 0, & t<\frac{x}{U} \\ 2\sum_{k=1}^{\infty} e^{-4\lambda_k^2\frac{x/D}{Pe}}\frac{J_0(\lambda_k\frac{r}{R})}{\lambda_k J_1(\lambda_k)}+2\sum_{N=1}^{\infty}\left(\frac{4}{DPe}\right)^N\frac{\left(\frac{2\epsilon}{R}t\right)^N}{N!}\mathcal{K}_x^N(i,j,k,\ldots)\frac{J_0(\lambda_k r/R)}{\lambda_k J_1(\lambda_k)} & , t\geq\frac{x}{U} \end{cases} \quad (C\text{-}6)$$

The following recurrence formula may be given if this is to be used for digital computation.

$$\left.\begin{aligned} \mathcal{K}_x(i,j) &= \sum_{j=1}^{\infty}\sum_{i=1}^{\infty}\frac{1}{1-\left(\frac{\lambda_i}{\lambda_j}\right)^2}\begin{bmatrix} e^{-4\lambda_i^2\frac{x/D}{Pe}} & 1 \\ e^{-4\lambda_j^2\frac{x/D}{Pe}} & 1 \end{bmatrix}, \\ \mathcal{K}_x^2(i,j,k) &= \mathcal{K}_x(i,k)\sum_{j=1}^{\infty}\frac{1}{1-\left(\frac{\lambda_i}{\lambda_j}\right)^2}-\mathcal{K}_x(j,k)\sum_{i=1}^{\infty}\frac{1}{1-\left(\frac{\lambda_i}{\lambda_j}\right)^2}, \\ \mathcal{K}_x^3(i,j,k,l) &= \mathcal{K}_x^2(i,k,l)\sum_{j=1}^{\infty}\frac{1}{1-\left(\frac{\lambda_i}{\lambda_j}\right)^2}-\mathcal{K}_x^2(j,k,l)\sum_{i=1}^{\infty}\frac{1}{1-\left(\frac{\lambda_i}{\lambda_j}\right)^2}, \\ &\text{------------------------------------} \\ \mathcal{K}_x^N(i,j,k,l,\ldots) &= \mathcal{K}_x^{N-1}(i,k,l,\ldots)\sum_{j=1}^{\infty}\frac{1}{1-\left(\frac{\lambda_i}{\lambda_j}\right)^2}-\mathcal{K}_x^{N-1}(j,k,l,\ldots)\sum_{i=1}^{\infty}\frac{1}{1-\left(\frac{\lambda_i}{\lambda_j}\right)^2} \end{aligned}\right\} (C.7)$$

where $N \geq 1$, and

$$\mathcal{K}_x^0(i)=\sum_{i=1}^{\infty}e^{-4\lambda_i^2\frac{x/D}{Pe}} \quad (C.8)$$

$$\mathcal{K}_x^0(j)=\sum_{j=1}^{\infty}e^{-4\lambda_j^2\frac{x/D}{Pe}} \quad (C.9)$$

are taken by definition.

Care must be given to the indefinite forms of the above recurrence formula, determining each indefinite case separately by the well-known methods for these forms.

APPENDIX D

The Physical Problem

The present transient theory, assuming a time independent, developing velocity field, was introduced by a step temperature change at the entrance region of the tube. However, if a fluid at rest, and in the test section having a temperature equal to the tube-wall, is subjected to a sudden pressure difference, and the new fluid enters the test section at a temperature ΔT_e above the tube-wall temperature, both temperature and velocity transients are introduced. The temperature transient depends primarily on the heat capacity of tube-wall material. ~~which is independent of the effect of velocity transient.~~

The purpose of this appendix is to derive some approximate formulas which make possible a comparison of the temperature and velocity transients. Thus, it would be possible to find cases in which t_T (temperature transient time) $\gg$ t_V (velocity transient time). For these cases, the velocity transient effect on the temperature transient phenomena may be neglected.

Consider Fig. 14 with a constant-level tank connected to a closed-end tube having a small length-to-diameter ratio. The end of the tube is now opened in a time-specified manner, and the resulting velocity transient is desired.

For the transient, incompressible, potential flow, for a control volume between s and $s+ds$, the terms of the continuity and momentum equations may be written as shown in Fig. 13.

The transient continuity equation for an incompressible flow through a control volume states that the net efflux of mass from the control surface is equal to zero. The transient momentum equation in streamline direction expresses the fact that the resultant of all external forces must be equal to the time rate of change of momentum within the control volume plus the net efflux of momentum from the control surface.

Using these fundamental laws and Fig. 13, the combination of the continuity and momentum equations may be written in the form

$$\frac{\partial P}{\partial s} + \rho \frac{\partial V}{\partial t} = 0 \qquad (D\text{-}1)$$

where

$$P = p + \rho g z + \rho \frac{V^2}{2} \qquad (D\text{-}2)$$

If this equation is integrated along a streamline between the points 1 and 2, the result obtained is

$$\frac{P_2 - P_1}{\rho} + \int_1^2 \frac{\partial V}{\partial t} ds = 0 \qquad (D\text{-}3)$$

With incompressibility, the continuity equation may be written also in the form

$$A.V = A_e \cdot V_e \qquad (D\text{-}4)$$

Differentiating with respect to time and rearranging gives

$$\frac{\partial V}{\partial t} = \frac{1}{A}\left(A_e \frac{\partial V_e}{\partial t} + V_e \frac{\partial A_e}{\partial t}\right) \qquad (D\text{-}5)$$

For simplicity, sudden opening of the valve is assumed (Fig. 15).

$$\frac{\partial A_e}{\partial t} = 0 \quad , \quad \text{when} \quad t \geq 0$$

which reduces (D-5) to

$$\frac{\partial V}{\partial t} = \frac{A_e}{A} \cdot \frac{\partial V_e}{\partial t} \tag{D-6}$$

Substituting (D-6) into (D-3) and considering $V_e = V_e(t)$ results in

$$\frac{P_2 - P_1}{\rho} + \frac{dV_e}{dt} S = 0 \tag{D-7}$$

where

$$S = \int_1^2 \frac{A_e}{A} ds \tag{D-8}$$

From the definition of (D-2)

$$\frac{P_2 - P_1}{\rho} = \frac{p_2 - p_1}{\rho} + g(0 - h) + \frac{V_e^2 - 0}{2} \tag{D-9}$$

Since

$p_2 = p_1$, and for the steady state case

$$V_{e\infty}^2 = 2gh \tag{D-10}$$

(D-9) may be written

$$\frac{P_2 - P_1}{\rho} = \frac{V_e^2 - V_{e\infty}^2}{2} \tag{D-11}$$

Combining (D-7) with (D-11) and integrating gives

$$\frac{V_e}{V_{e\infty}} = \tanh \frac{V_{e\infty} t}{2S} \tag{D-12}$$

or

$$\frac{V_{e\infty} t}{S} = \ln \frac{1 + \frac{V_e}{V_{e\infty}}}{1 - \frac{V_e}{V_{e\infty}}} \tag{D-13}$$

In the calculation of S , the actual stream tube may be approximated by the straight tube (Fig. 16). Then substituting

$$\frac{A_e}{A} = \frac{r_e^2}{r^2} = \frac{r_e^2}{r_0^2\left[1 - \left(1 - \frac{r_e}{r_0}\right)\frac{x}{l}\right]^2} \tag{D-14}$$

and

$$ds \cong dx \tag{D-15}$$

into (D-8) results in

$$S \cong \frac{r_e}{r_0} l \tag{D-16}$$

If a characteristic time for the potential velocity transient is defined as that time for which $V_e/V_{e\infty} = 0.8$, then from (D-10), (D-13) and (D-16)

$$(t_v)_p \cong 2.2 \frac{(r_e/r_0)\, l}{(2gh)^{1/2}} \tag{D-17}$$

is obtained.

Now, if it were possible to find cases for which $(t_v)_b \gg (t_v)_p$, where $(t_v)_b$ is the characteristic time for the boundary layer, the potential transient effect could be neglected relative to the boundary layer transient effect. In this way, the problem would be considerably simplified. For the flat plate, the problem was solved by Blasius (18). Since the velocity profile of the present problem is taken to be the same as the flat plate velocity, the solution is

$$u(x, y, t) = U \operatorname{erf} \zeta \tag{D-18}$$

where

$$\zeta = y / 2(\nu t_{v_b})^{1/2} \tag{D-19}$$

Defining a transient time corresponding to $u(x,y,t)/U = 0.8$, $\zeta = 0.9$ is found. If use is made of $y_{max} = 0.3R$,

$$(t_v)_{b_{max}} = D^2/144\,\nu \qquad (D.20)$$

results for the boundary layer transient time.

A solution to the temperature transient may be found from the first approximation of the present theory. If a transient time is defined according to $\vartheta_0(x,t)/\Delta T_e = 0.8$, from Fig. 6, $MFo/\left(\frac{x/D}{Pe}\right)^{1/2} = 0.9$ results. Thus

$$t_T = 0.9\,\frac{D^2}{M\alpha}\left(\frac{x/D}{Pe}\right)^{1/2} \qquad (D.21)$$

is obtained.

APPENDIX E

The Quasi-steady Theory

To investigate the validity of well-known quasi-steady theories, a simple quasi-steady analysis based on a constant surface temperature is given. For simplicity, only first approximations of the quasi-steady theory are compared to the more exact theory of this report.

As previously formulated (A-12), the transient energy equation of the tube-wall is

$$\frac{\partial \theta_0}{\partial t} + \epsilon \left(\frac{\partial T_0}{\partial r} \right)_{r=R} = 0 \qquad \text{(A-12)}$$

in which an infinite conductivity has been assumed in the radial direction.

The basic assumption of the quasi-steady theory is to take a steady temperature distribution on the fluid-solid interface and to solve the fluid temperature problem in terms of this steady interface temperature. In the final solution, the interface temperature is assumed to be dependent on time, and the fluid temperature gradient found in this way is substituted into (A-12). Then an integration in time gives the approximate, unsteady tube-wall temperature.

For the first approximation of the fluid temperature, the differential equation resulting from the first law of thermodynamics (energy equation) and the boundary conditions to be satisfied are:

$$U\frac{\partial T_0}{\partial x} = \alpha\left(\frac{\partial^2 T_0}{\partial r^2} + \frac{1}{r}\frac{\partial T_0}{\partial r}\right) \tag{E.1}$$

$$T_0(r,0) = T_e \tag{E-2}$$

$$T_0(R,x) = T_S \tag{E-3}$$

$$\frac{\partial T_0(0,x)}{\partial r} = 0 \tag{E.4}$$

This problem is well-known and was first solved by Graetz (13). One of the convenient ways of solving the above differential equation with the proper boundary conditions is the use of finite Hankel transforms in the radial direction, which are frequently employed in this study.

The resulting solution is

$$T_0(r,x) = 2(T_e - T_s)\sum_{k=1}^{\infty} e^{-4\lambda_k^2 \frac{x/D}{Pe}} \frac{J_0(\lambda_k \frac{r}{R})}{\lambda_k J_1(\lambda_k)} \tag{E.5}$$

Now if the fluid-solid interface temperature is assumed to depend on the axial distance and time, i.e., $T_S = \theta_0(x,t)$, the above equation may be written as

$$T_0(r,x,t) = 2[T_e - \theta_0(x,t)]\sum_{k=1}^{\infty} e^{-4\lambda_k^2 \frac{x/D}{Pe}} \frac{J_0(\lambda_k \frac{r}{R})}{\lambda_k J_1(\lambda_k)} \tag{E.6}$$

Calculating the fluid temperature gradient on the fluid-solid interface from (E-6) and substituting into (A-12) results in

$$\frac{\partial \theta_0}{\partial t} + \frac{4\epsilon}{D}\left[T_e - \theta_0(x,t)\right]\sum_{k=1}^{\infty} e^{-4\lambda_k^2 \frac{x/D}{Pe}} = 0 \tag{E.7}$$

By the use of the same approximation (Fig. 5) for the sum of the exponential function

$$\frac{\partial \theta_0}{\partial t} + \frac{\epsilon}{D} \cdot \frac{1}{\pi^{1/2}\left(\frac{x/D}{Pe}\right)^{1/2}} \theta_0 = \frac{\epsilon}{D} \cdot \frac{1}{\pi^{1/2}\left(\frac{x/D}{Pe}\right)^{1/2}} T_e \qquad (E\text{-}8)$$

is obtained. The initial condition is

$$\theta_0(x,0) = T_s \qquad (E\text{-}9)$$

The solution of this equation may be obtained in the form

$$\frac{\theta_0(x,t) - T_s}{\Delta T_e} = 1 - e^{-\frac{2}{\pi^{1/2}} \cdot \frac{MFo}{\left(\frac{x/D}{Pe}\right)^{1/2}}} \qquad (E\text{-}10)$$

where $\Delta T_e = T_e - T_s$

To calculate the heat flux from the fluid-solid interface, again (A-12) may be used but in the following form

$$q(x,t) = -k\left(\frac{\partial T_0}{\partial r}\right)_{r=R} = -\frac{k}{\epsilon}\frac{\partial \theta_0}{\partial t} \qquad (E\text{-}11)$$

Combining (E-10) and (E-11), and rearranging gives

$$\frac{q(x,t)}{k\Delta T_e/D} = Nu_0(x,t) = \frac{1}{\pi^{1/2}\left(\frac{x/D}{Pe}\right)^{1/2}} e^{-\frac{2}{\pi^{1/2}} \cdot \frac{MFo}{\left(\frac{x/D}{Pe}\right)^{1/2}}} \qquad (E\text{-}12)$$

The functions (E-10) and (E-12) are given in Figures 6 and 8.

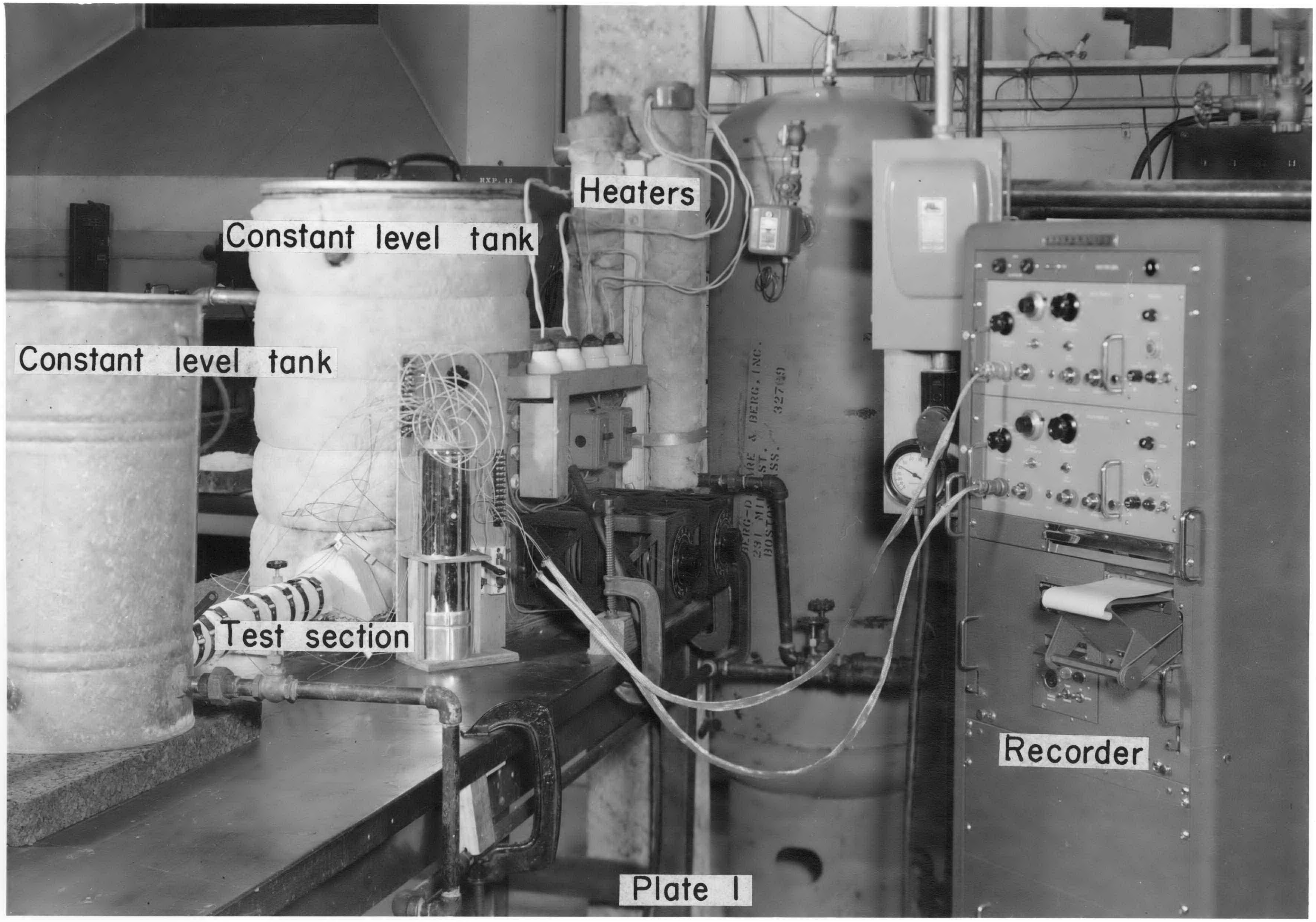

Heaters
Constant level tank
Constant level tank
Test section
Recorder

Plate I

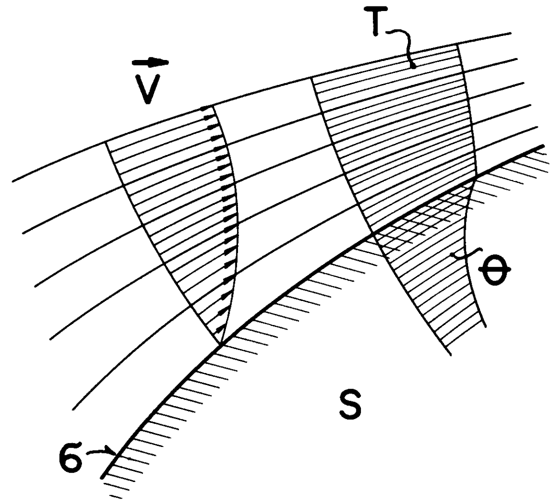

T
V
θ
S
σ

FIG. 1

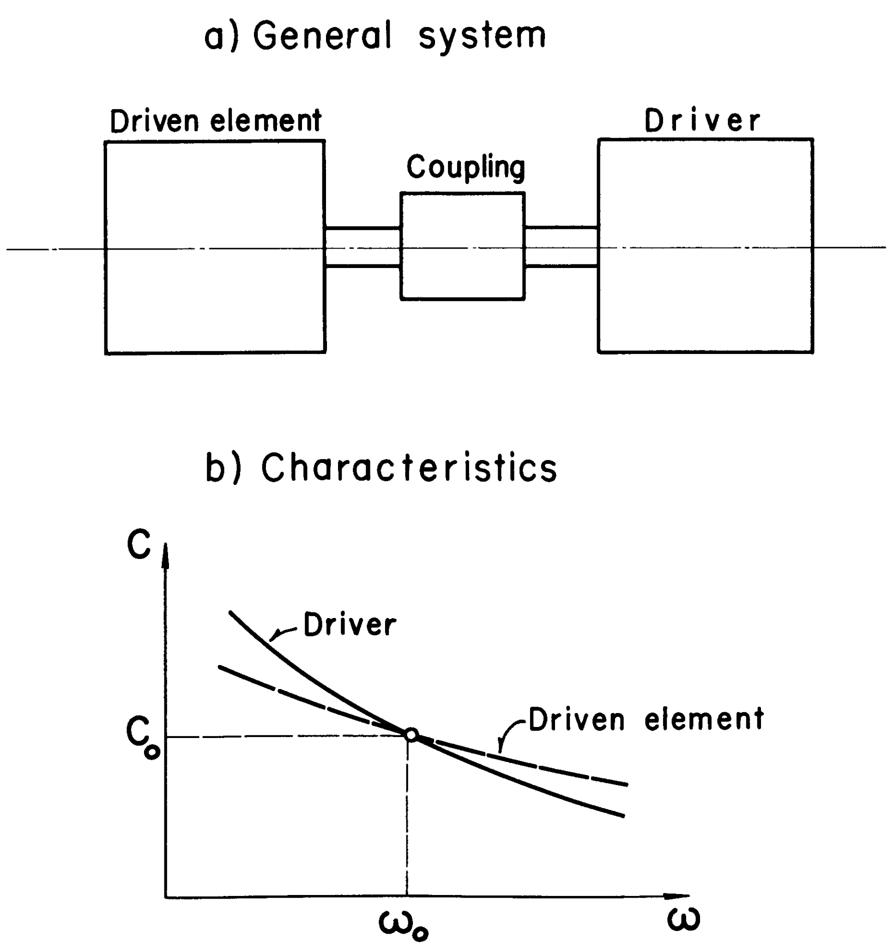

a) General system
Driven element
Coupling
Driver
b) Characteristics
C
Driver
C_o
Driven element
ω_o
ω

FIG. 2

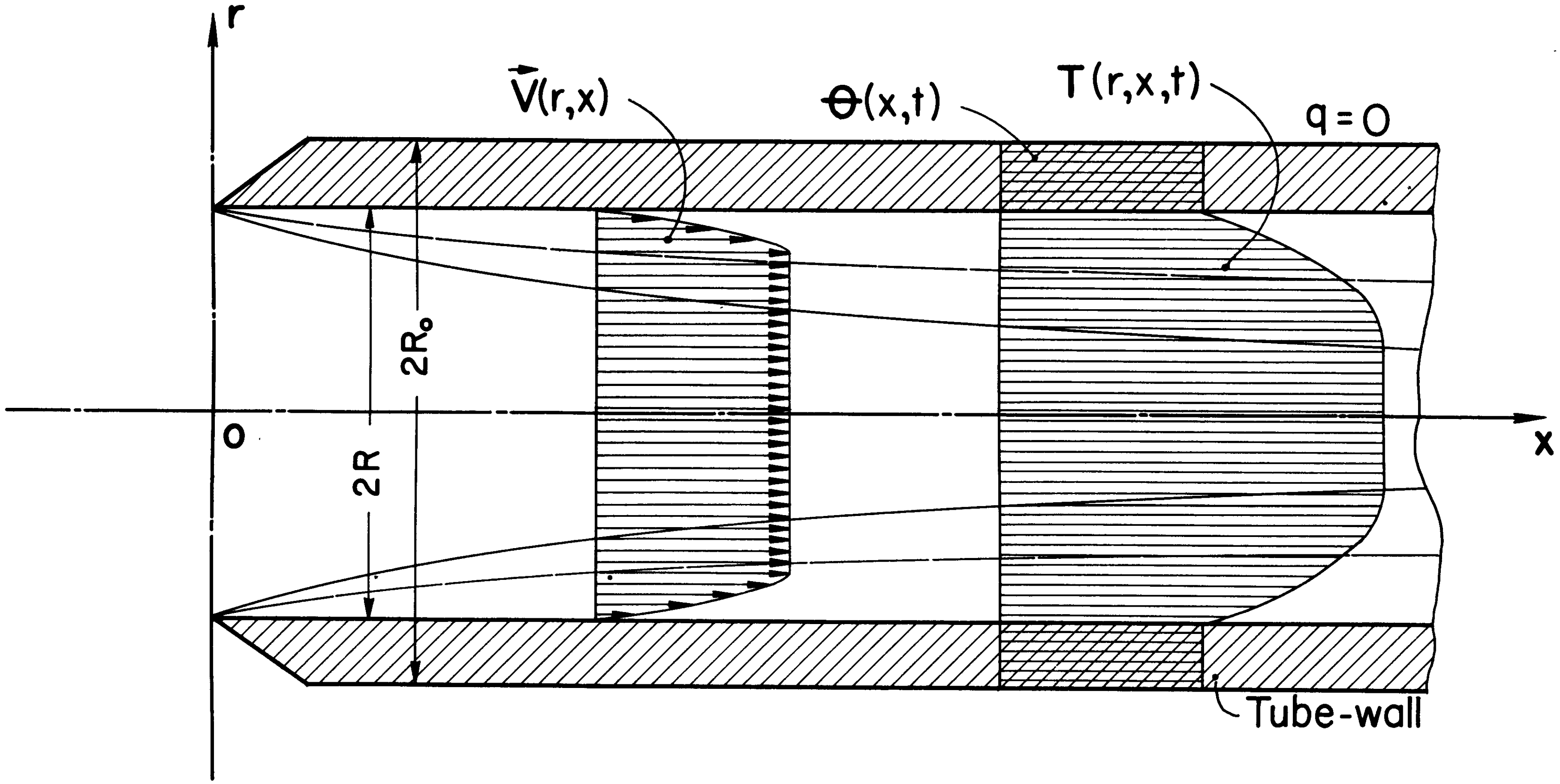
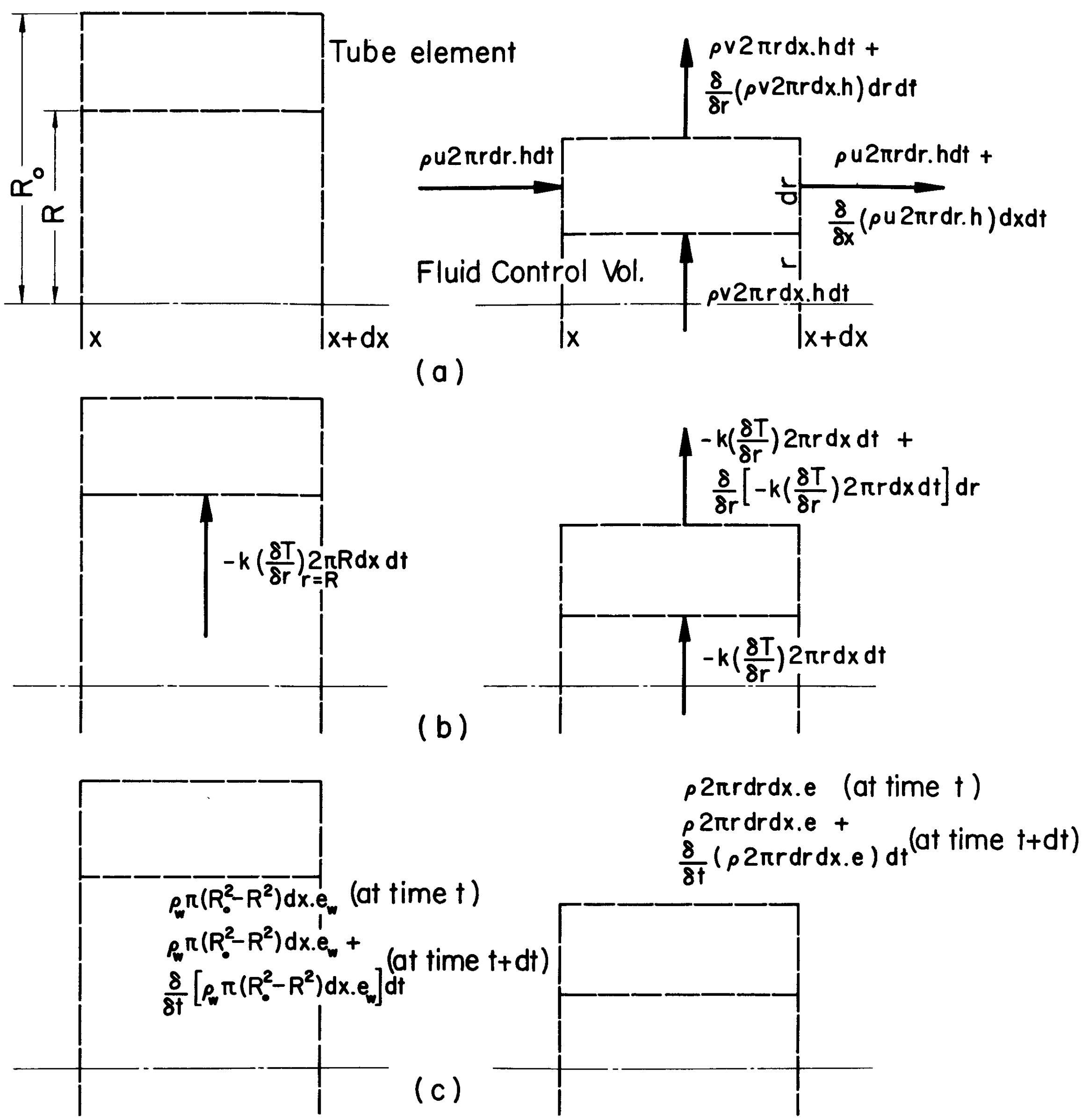
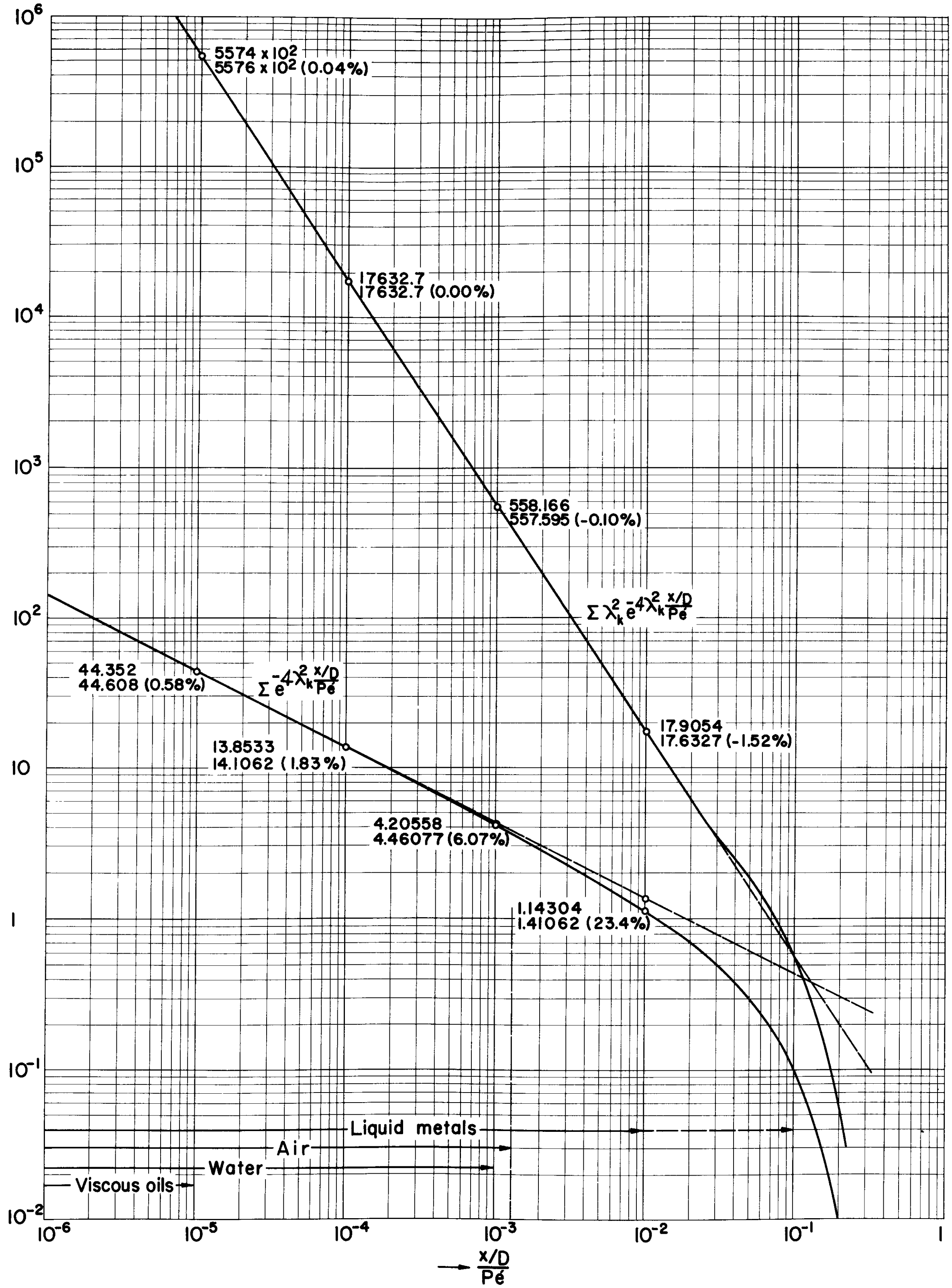


FIG. 3



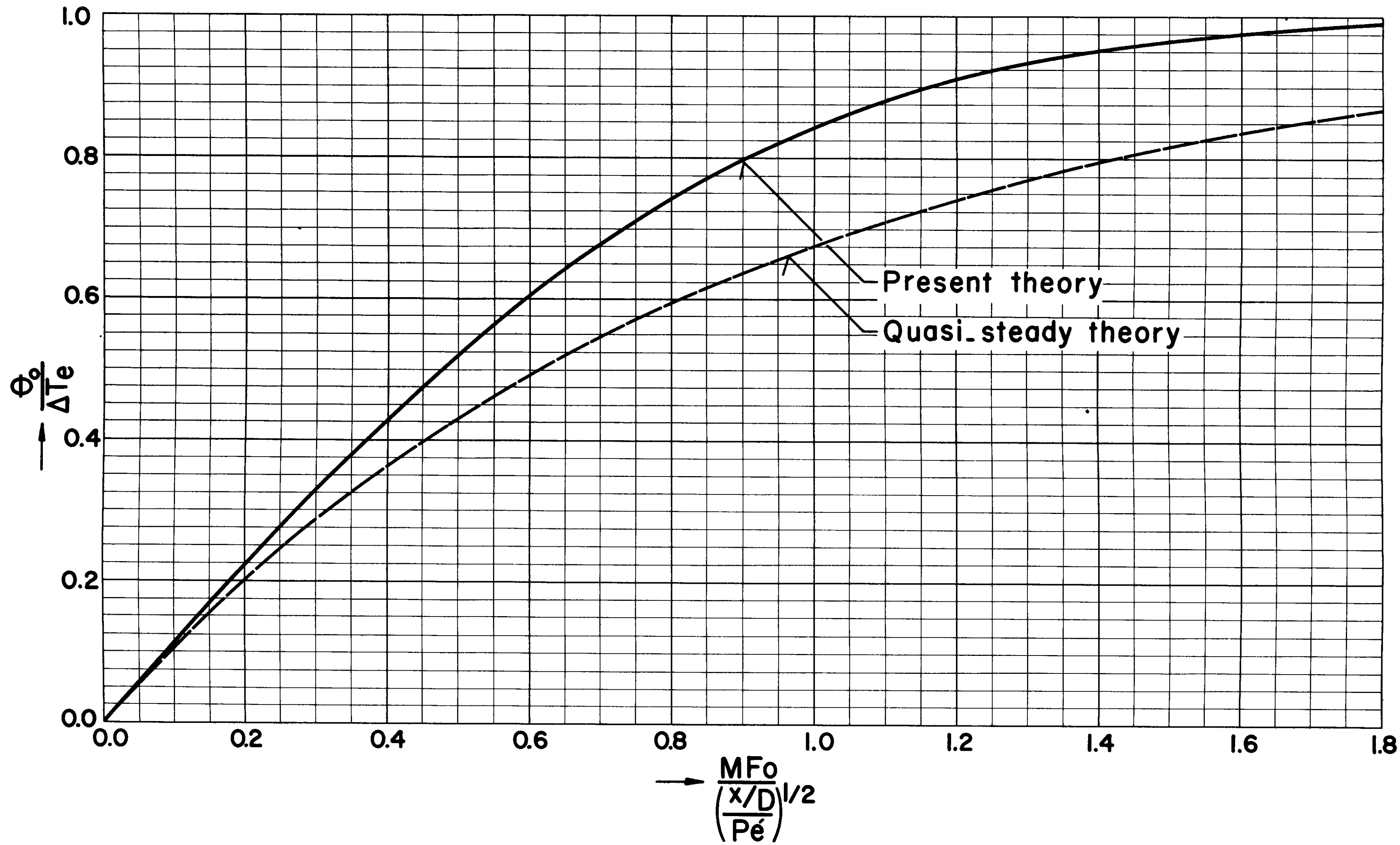
(a) Fluxes of enthalpy through control surfaces,
(b) Heat conduction through tube element and control surfaces,
(c) Time rate of change of internal energy within tube element and fluid control volume.

FIG. 4



10^6
10^5
10^4
10^3
10^2
10
1
10^-1
10^-2
10^-6
10^-5
10^-4
10^-3
10^-2
10^-1
1
5574 x 10^2
5576 x 10^2 (0.04%)
17632.7
17632.7 (0.00%)
558.166
557.595 (-0.10%)
Σ λ_k^2 e^-4λ_k^2 x/D/Pé
44.352
44.608 (0.58%)
Σ e^-4λ_k^2 x/D/Pé
13.8533
14.1062 (1.83%)
17.9054
17.6327 (-1.52%)
4.20558
4.46077 (6.07%)
1.14304
1.41062 (23.4%)
Liquid metals
Air
Water
Viscous oils
x/D / Pé

FIG. 5



1.0
0.8
0.6
0.4
0.2
0.0
$\frac{\Phi_o}{\Delta T_e}$
0.0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
1.6
1.8
$\frac{MFo}{\left(\frac{x/D}{Pé}\right)^{1/2}}$
Present theory
Quasi_steady theory

FIG. 6

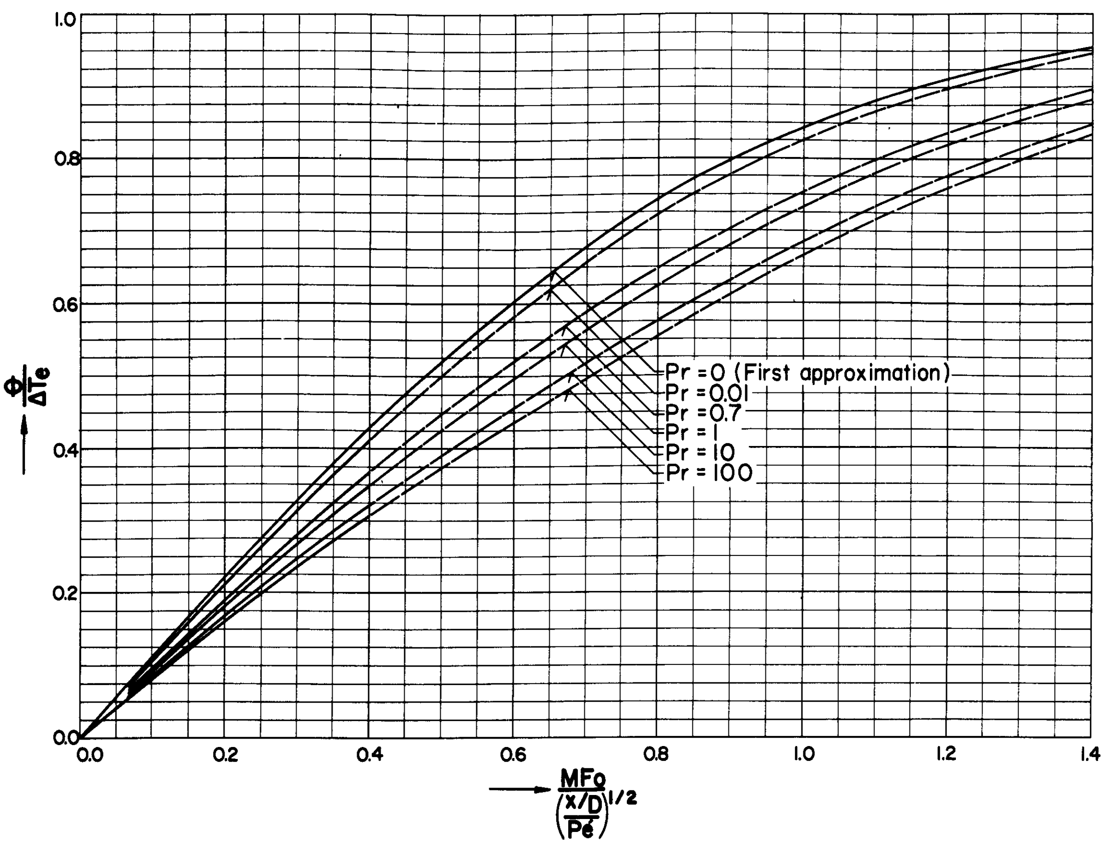

Φ/ΔTe
Pr = 0 (First approximation)
Pr = 0.01
Pr = 0.7
Pr = 1
Pr = 10
Pr = 100
MFo / (x/D / Pé)^1/2
0.0
0.2
0.4
0.6
0.8
1.0
1.2
1.4

Fig.7

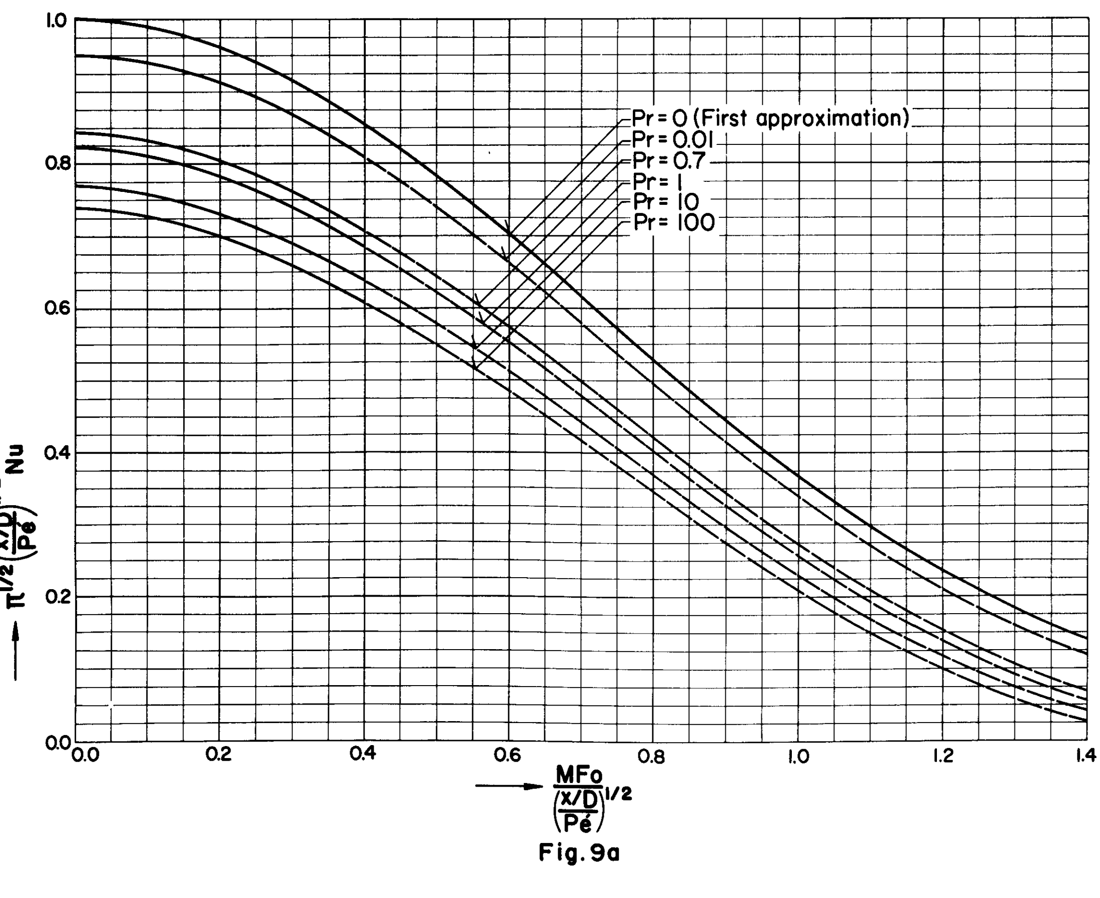

π^1/2 (x/D / Pé)^1/2 Nu
Pr = 0 (First approximation)
Pr = 0.01
Pr = 0.7
Pr = 1
Pr = 10
Pr = 100
MFo / (x/D / Pé)^1/2
0.0
0.2
0.4
0.6
0.8
1.0
1.2
1.4

Fig. 9a

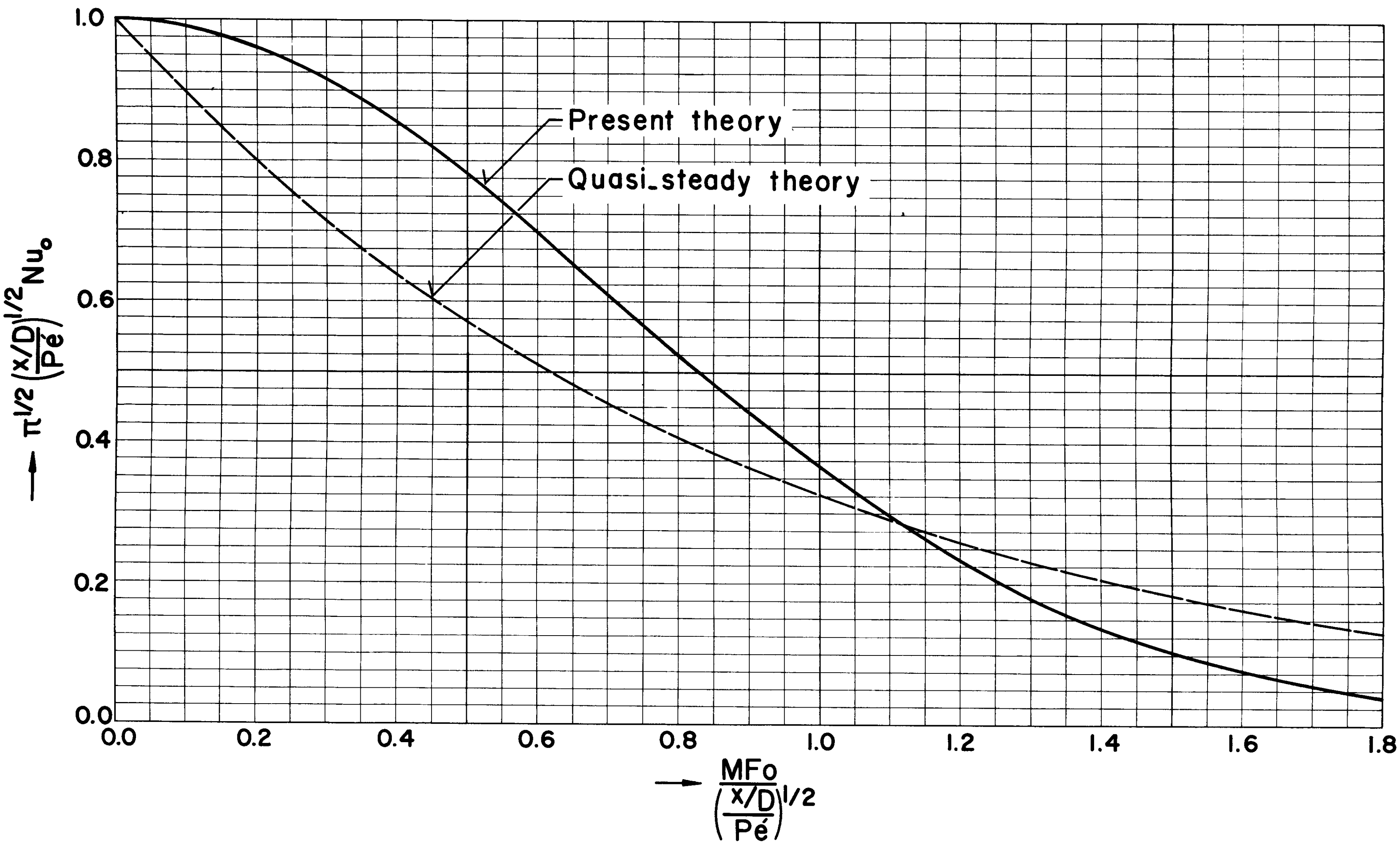
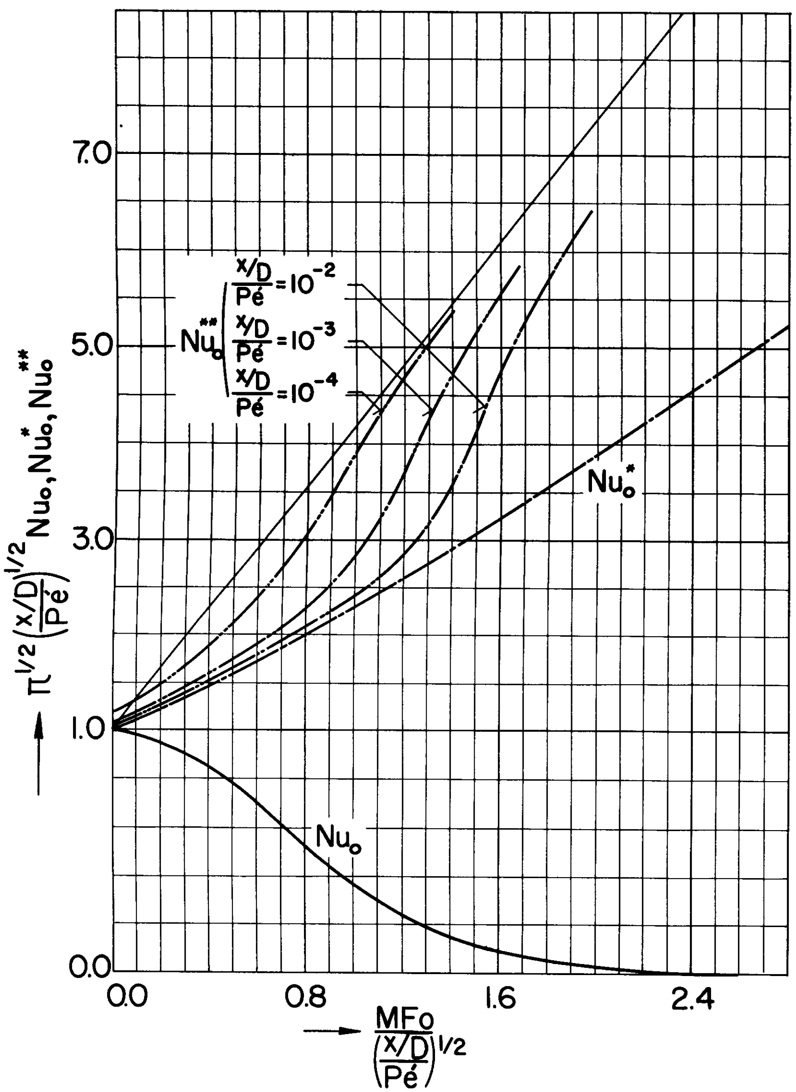
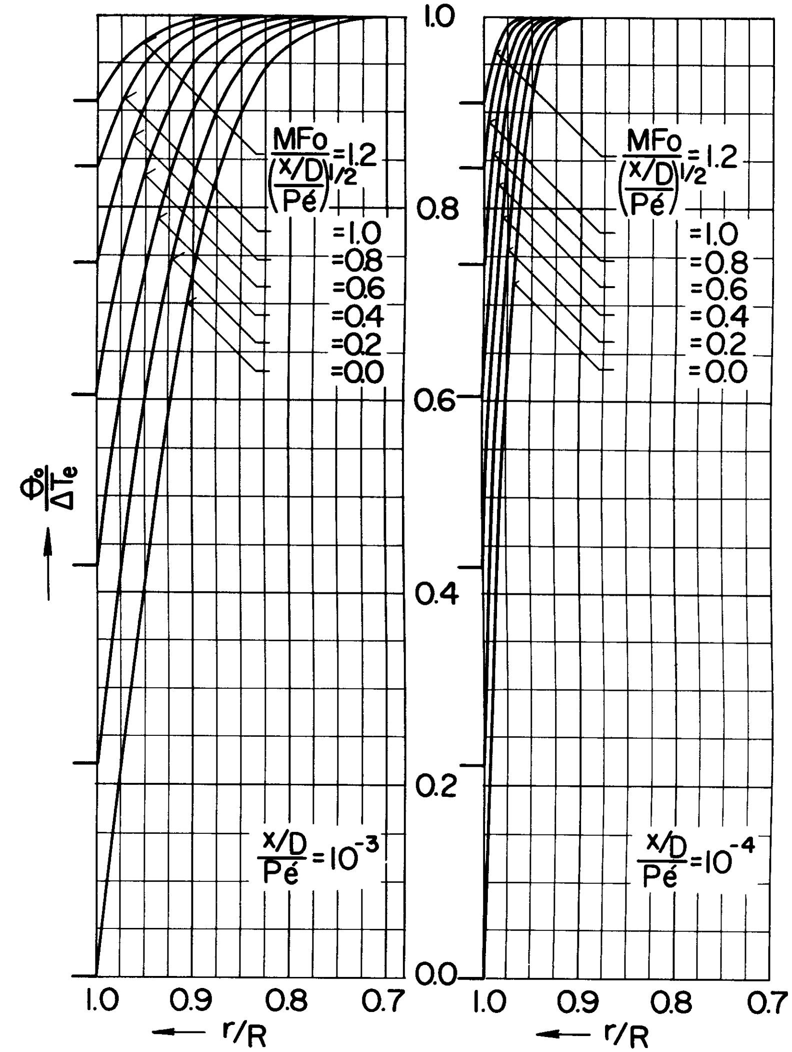

Present theory
Quasi-steady theory
1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
1.6
1.8
$\pi^{1/2}\left(\frac{x/D}{Pé}\right)^{1/2} Nu_o$
$\frac{MFo}{\left(\frac{x/D}{Pé}\right)^{1/2}}$

FIG. 8


7.0
5.0
3.0
1.0
0.0
0.0
0.8
1.6
2.4
$\pi^{1/2}\left(\frac{x/D}{Pé}\right)^{1/2} Nu_o, Nu_o^*, Nu_o^{**}$
$\frac{MFo}{\left(\frac{x/D}{Pé}\right)^{1/2}}$
Nu_o^{**}
$\frac{x/D}{Pé} = 10^{-2}$
$\frac{x/D}{Pé} = 10^{-3}$
$\frac{x/D}{Pé} = 10^{-4}$
Nu_o^*
Nu_o
Fig. 8a


1.0
0.8
0.6
0.4
0.2
0.0
$\frac{\Phi_o}{\Delta T_e}$
$\frac{MFo}{\left(\frac{x/D}{Pé}\right)^{1/2}} = 1.2$
=1.0
=0.8
=0.6
=0.4
=0.2
=0.0
$\frac{x/D}{Pé} = 10^{-3}$
$\frac{x/D}{Pé} = 10^{-4}$
1.0
0.9
0.8
0.7
r/R
Fig. 8c

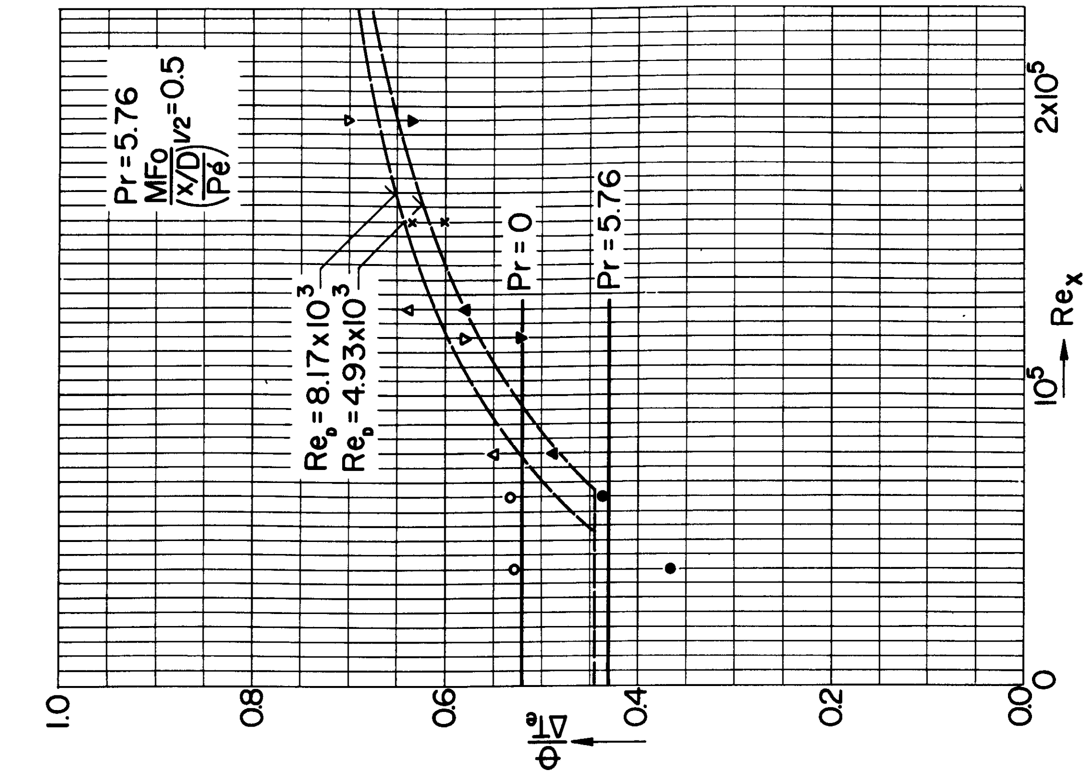

Pr = 5.76
$\frac{MFo}{\left(\frac{x/D}{Pé}\right)^{1/2}} = 0.5$
$Re_D = 8.17 \times 10^3$
$Re_D = 4.93 \times 10^3$
Pr = 0
Pr = 5.76
$\frac{\Phi}{\Delta T_e}$
Re_x
1.0
0.8
0.6
0.4
0.2
0.0
0
10^5
2×10^5

Fig.9d

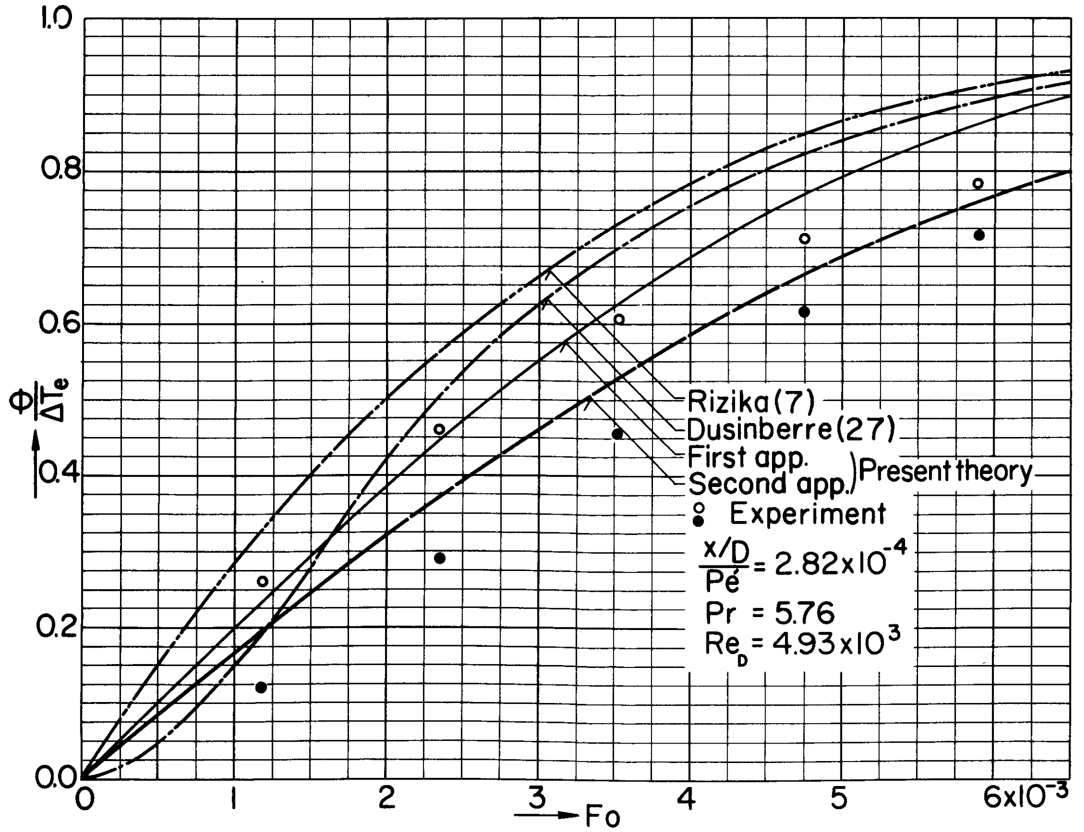

$\frac{\Phi}{\Delta T_e}$
Rizika(7)
Dusinberre(27)
First app.
Second app.
Present theory
Experiment
$\frac{x/D}{Pé} = 2.82 \times 10^{-4}$
Pr = 5.76
$Re_D = 4.93 \times 10^3$
Fo
0.0
0.2
0.4
0.6
0.8
1.0
0
1
2
3
4
5
6×10^{-3}

Fig.8b

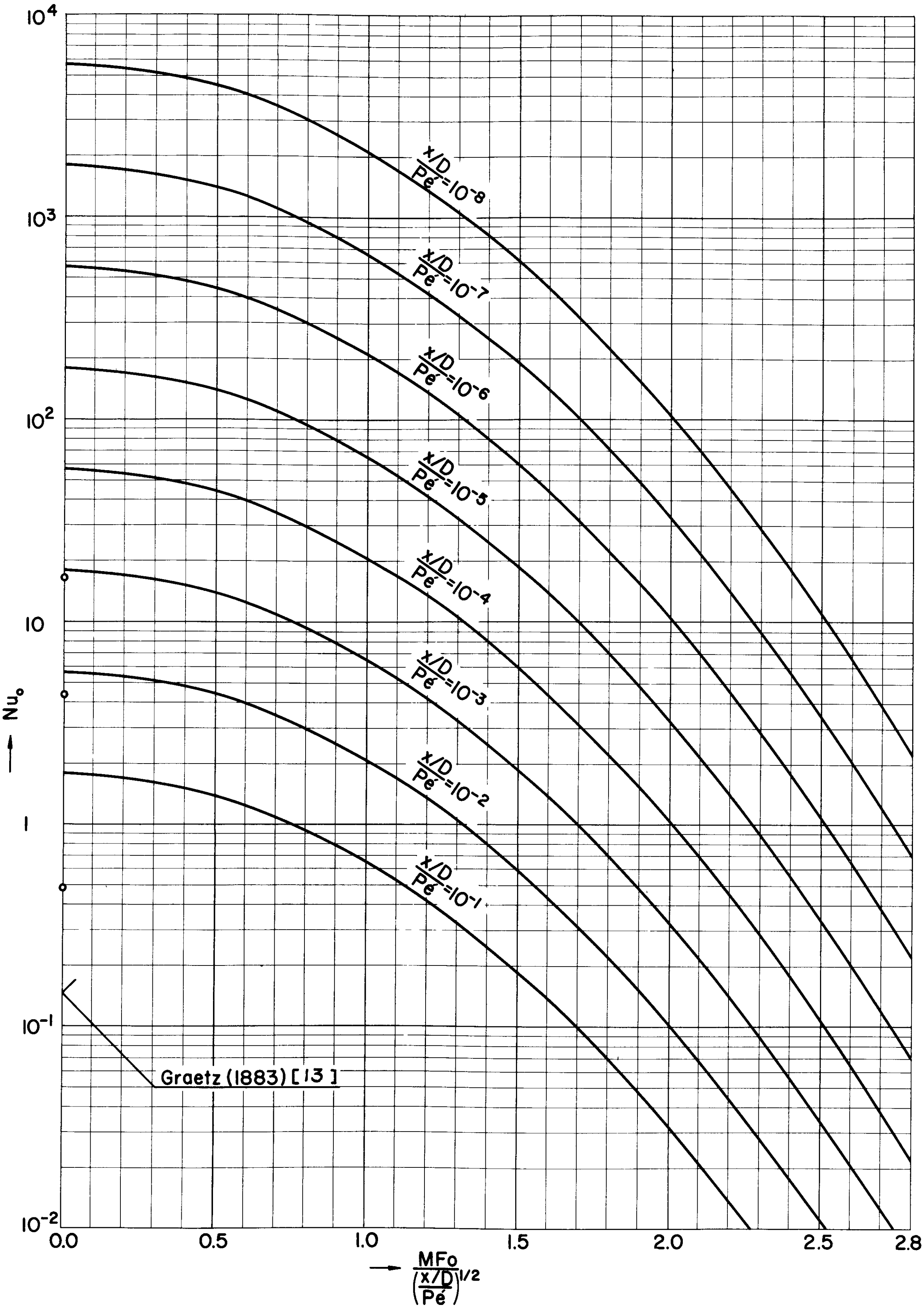
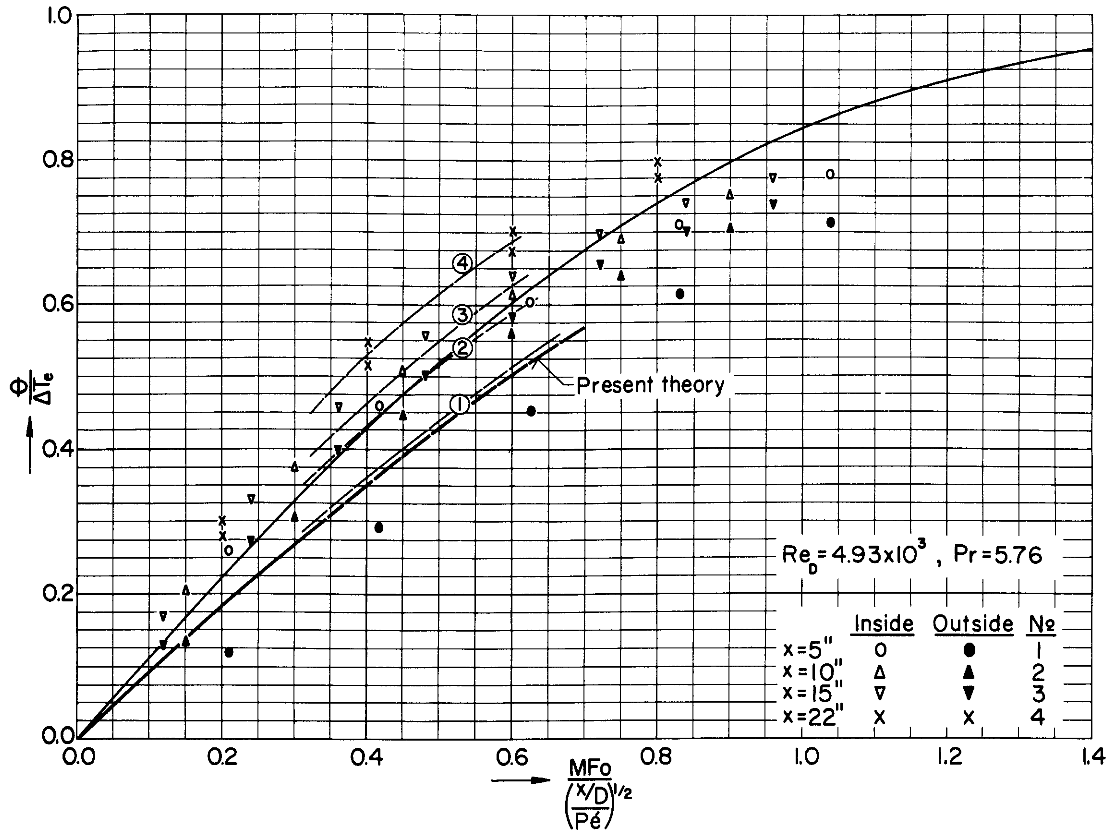
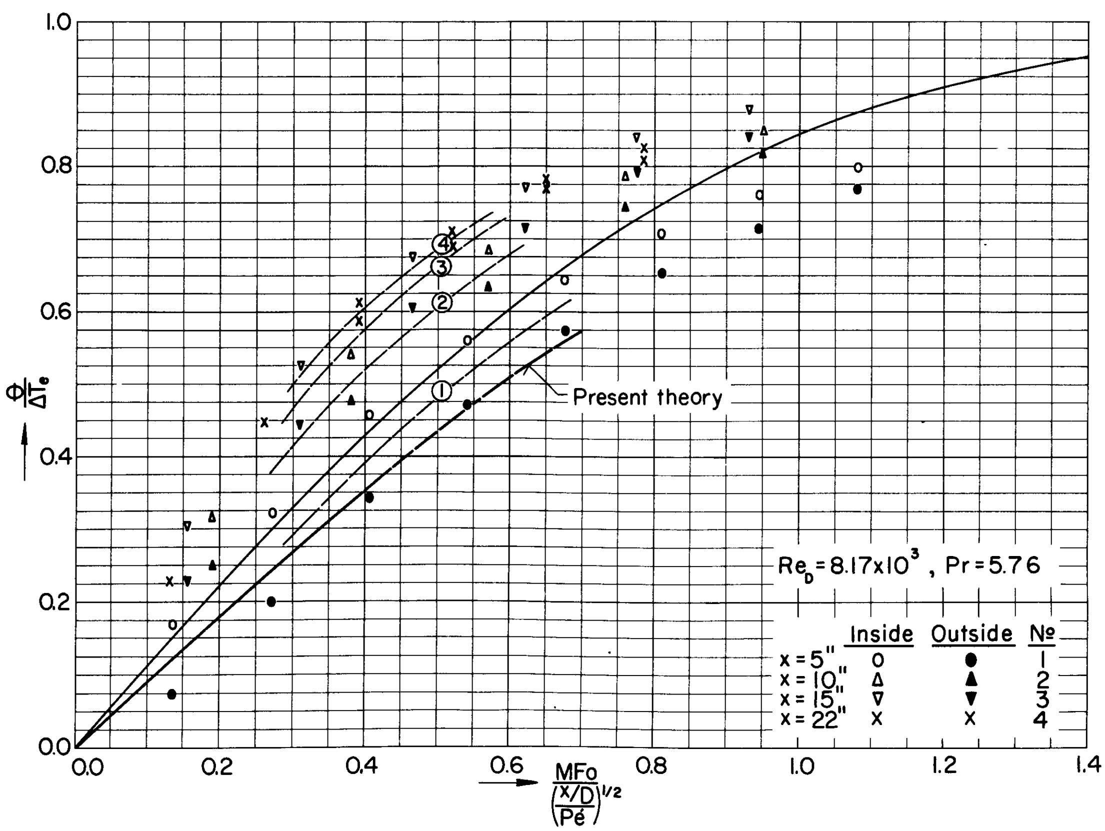

10^4
10^3
10^2
10
1
10^-1
10^-2
Nu_o
x/D / Pé = 10^-8
x/D / Pé = 10^-7
x/D / Pé = 10^-6
x/D / Pé = 10^-5
x/D / Pé = 10^-4
x/D / Pé = 10^-3
x/D / Pé = 10^-2
x/D / Pé = 10^-1
Graetz (1883) [13]
0.0
0.5
1.0
1.5
2.0
2.5
2.8
MFo / (x/D / Pé)^1/2

FIG. 9



1.0
0.8
0.6
0.4
0.2
0.0
Φ/ΔTₑ
0.0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
MFo / (x/D / Pé)^1/2
Present theory
④
③
②
①
Re_D = 4.93x10³ , Pr = 5.76
Inside Outside №
x=5" o ● 1
x=10" Δ ▲ 2
x=15" ∇ ▼ 3
x=22" x x 4

Fig. 9b



1.0
0.8
0.6
0.4
0.2
0.0
Φ/ΔTₑ
0.0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
MFo / (x/D / Pé)^1/2
Present theory
④
③
②
①
Re_D = 8.17x10³ , Pr = 5.76
Inside Outside №
x=5" o ● 1
x=10" Δ ▲ 2
x=15" ∇ ▼ 3
x=22" x x 4

Fig. 9c

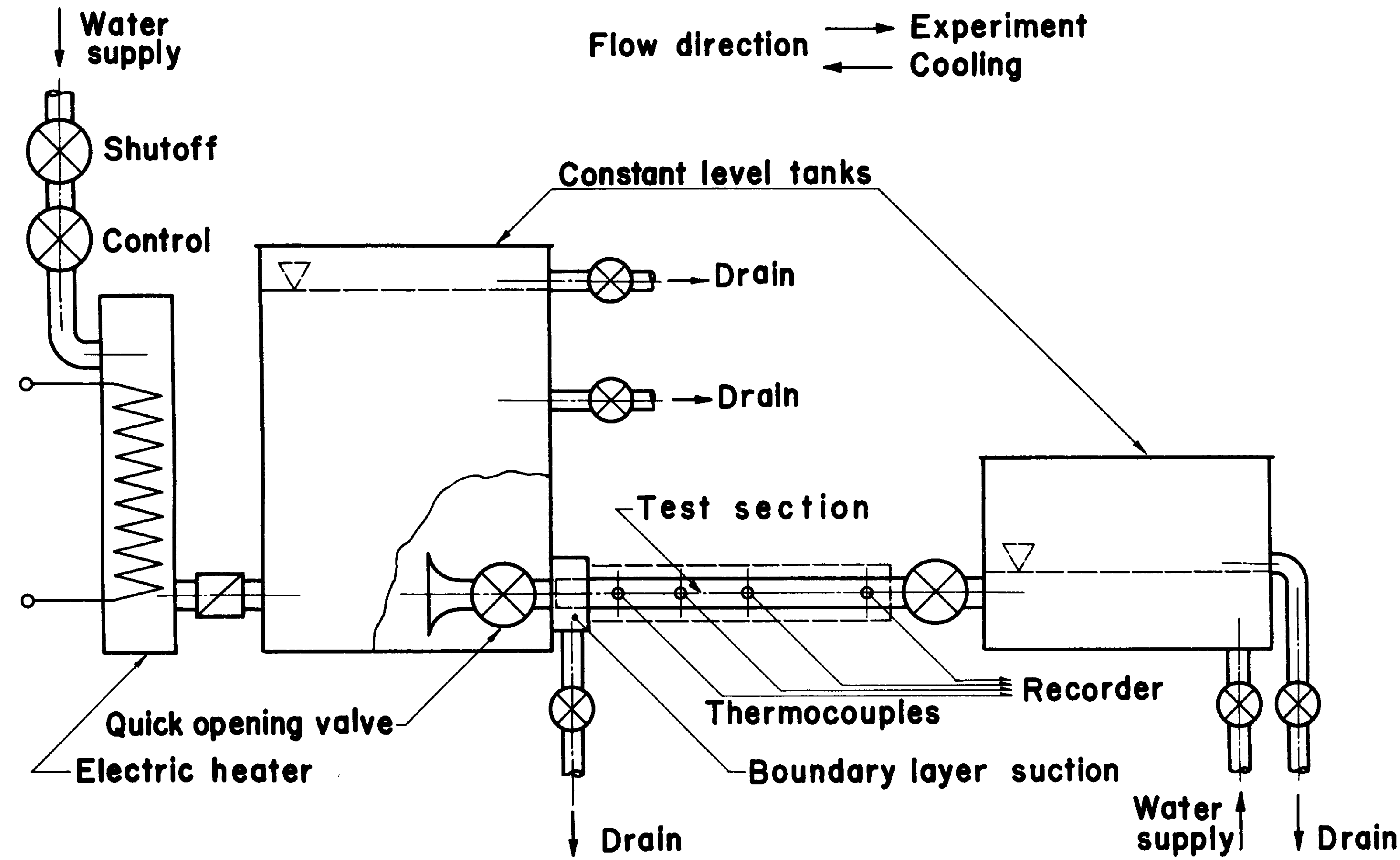
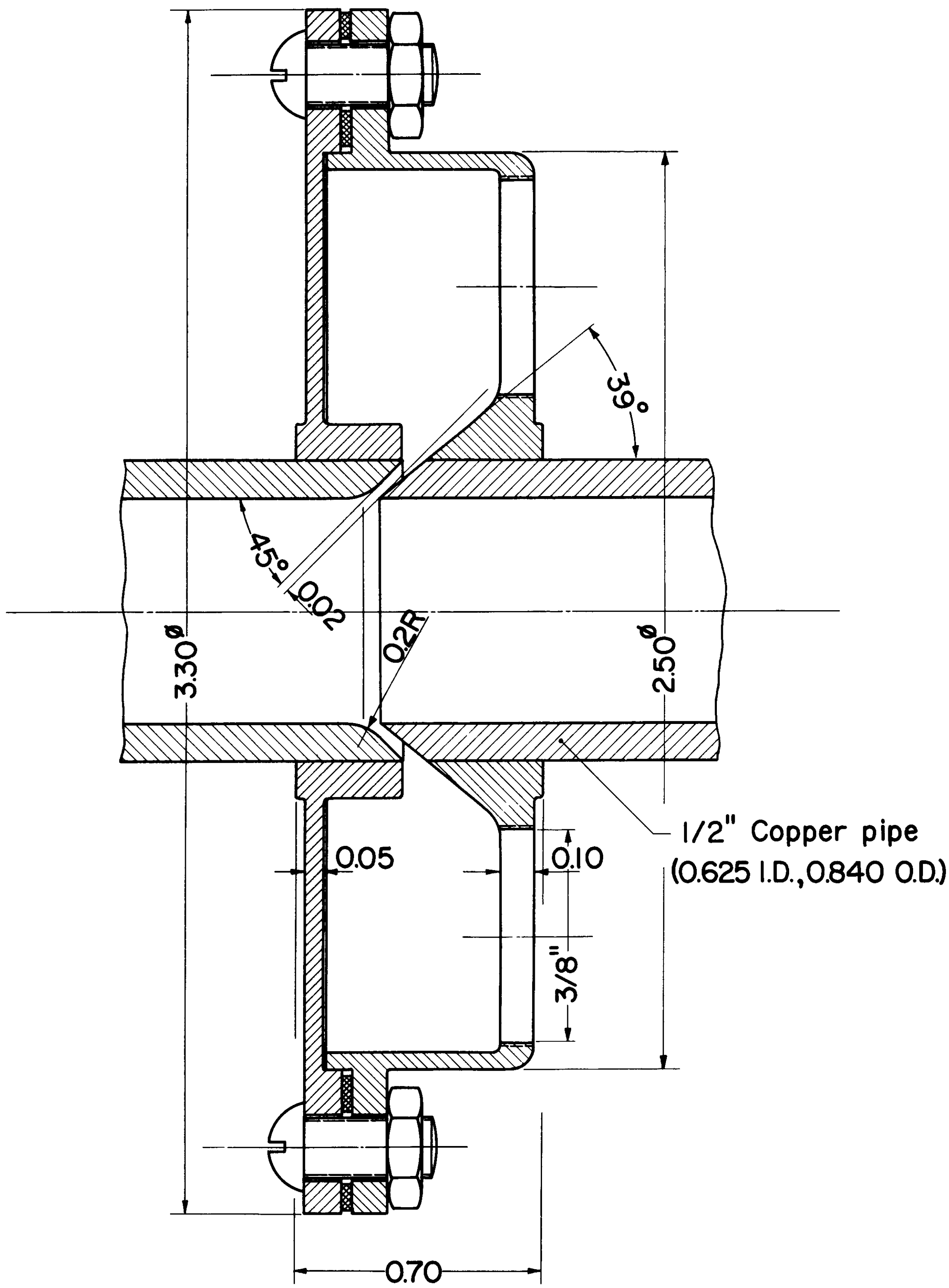


FIG. 10 SCHEMATIC DIAGRAM



Not to scale

FIG.11 BOUNDARY LAYER SUCTION SLOT

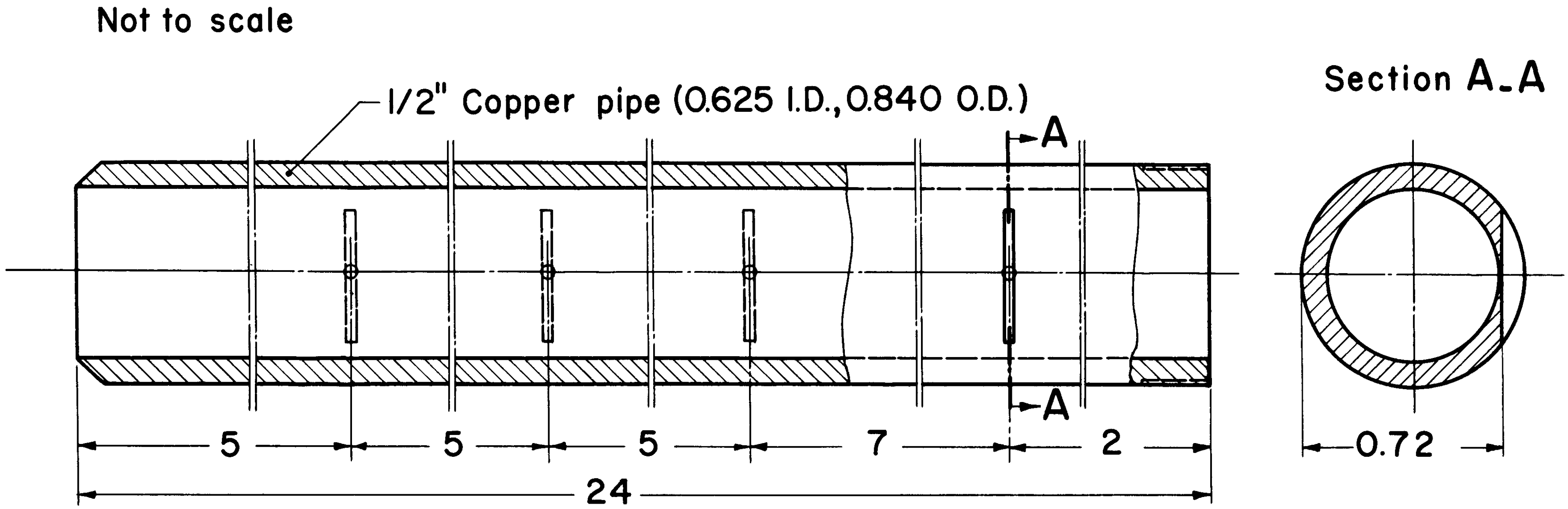
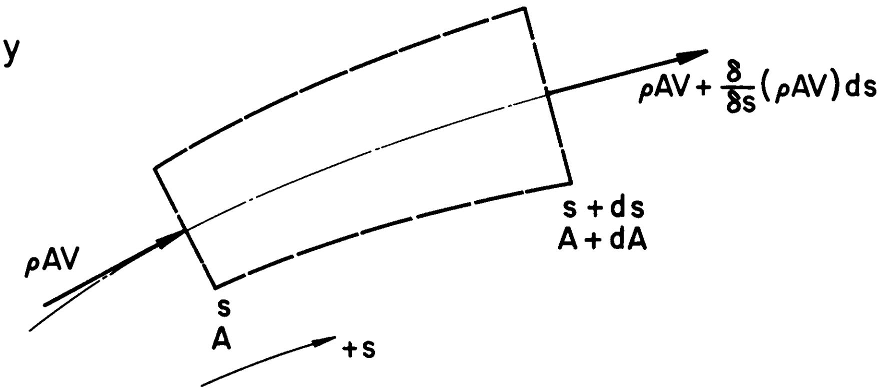


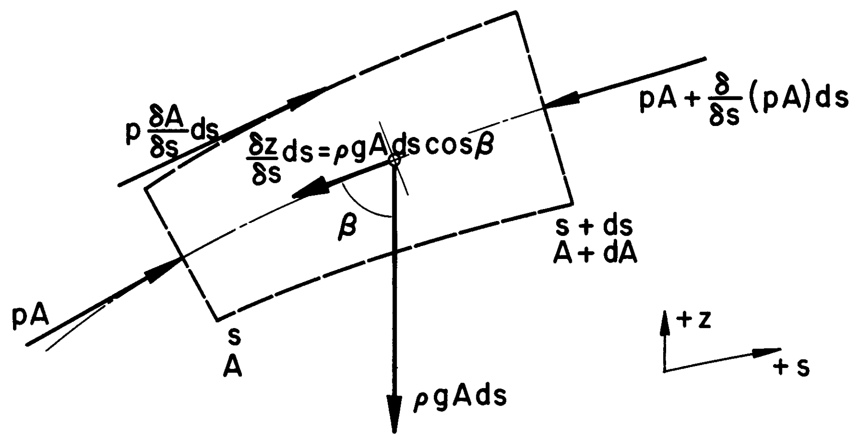
FIG.12 THERMOCOUPLE LOCATIONS

1. Continuity

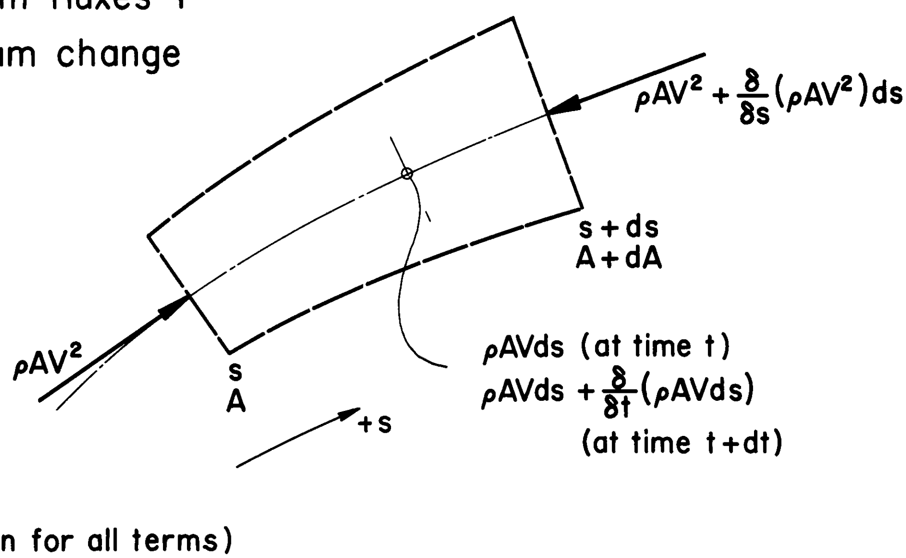


2. Momentum in streamline direction

a) Forces



b) Momentum fluxes +
Momentum change



(dt is common for all terms)

FIG. 13

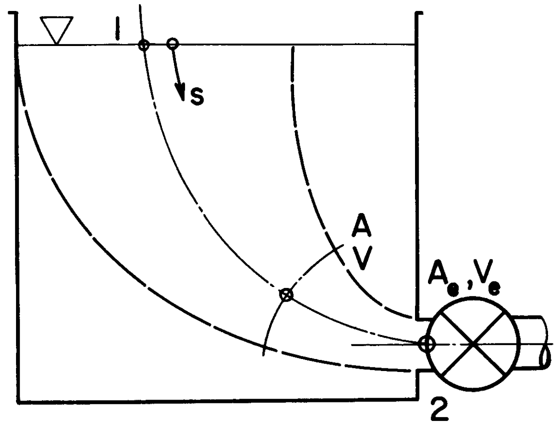


FIG. 14

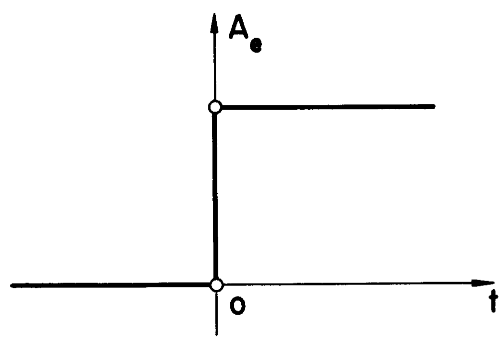


FIG. 15

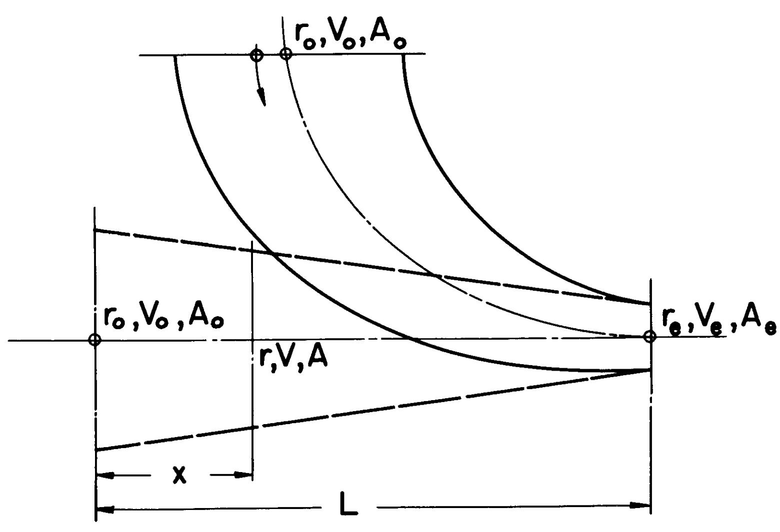


FIG. 16