

**MONOTONE CONTROL OF QUEUEING AND  
PRODUCTION/INVENTORY SYSTEMS**

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## Abstract

Weber and Stidham (1987) used submodularity to establish *transition monotonicity* (a service completion at one station cannot reduce the service rate at another station) for Markovian queueing networks that meet certain regularity conditions and are controlled to minimize service and queueing costs. We give an extension of monotonicity to other directions in the state space, such as arrival transitions, and to arrival routing problems. The conditions used to establish monotonicity, which deal with the boundary of the state space, are easily verified for many queueing systems. We also show that, without service costs, transition-monotone controls can be described by simple control regions and switching functions, extending earlier results. The theory is applied to production/inventory systems with holding costs at each stage and finished goods backorder costs.

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# 1 Introduction

The optimal control of arrival and service rates in networks of queues has been the topic of considerable research (see Stidham 1985, 1988 and the references therein). Most studies have dealt with specific network structures or systems with only two queues. Even in these special cases and with memoryless arrival and service processes, one usually must resort to numerical dynamic programming techniques to compute the optimal control. Alternatively, a certain form of control can be analyzed; examples include end-to-end controls for communication systems, and kanban and base stock controls for manufacturing systems.

Another fruitful line of research has been to study the *structure* of optimal policies. In many cases, the optimal policy has been shown to have intuitive monotonicity properties. Weber and Stidham (1987) use the preservation of submodularity under dynamic programming value-iteration to prove *transition monotonicity*: service completion at one station cannot reduce the optimal service rate at another station. Their result applies to a fairly broad class of queueing networks, which are assumed to be Markovian with all service rates controllable, controllable and uncontrollable arrival rates, and discounted or long-run average costs. In a series system with service rate control, service costs and convex holding costs, for example, they show that upon the addition of a customer to a queue, the optimal service rates at that queue and at downstream queues do not decrease and the optimal service rates upstream do not increase. Several other papers have used submodularity to establish some form of monotonicity (see, for example, Rosberg, Varaiya, and Walrand 1982 and Beutler and Teneketzis 1989). Hajek (1984) uses submodularity, which he calls multimodularity in Hajek (1985), to establish monotonicity for a system of two interacting queues with arrival routing, service rate control, and server allocation.

However, most proofs using submodularity are problem-specific. Weber and

Stidham's result, though it addresses general network structures, does not seem strong enough to produce monotonicity for Hajek's (1984) problem and the production/inventory model that motivated our research. We give an extension of monotonicity to other directions in the state space such as arrivals: an arrival cannot reduce the optimal service rates. Some arrival routing problems are also included in our model. The following assumptions are required:

1. Holding costs satisfy a submodularity condition (related to convexity),
2. The boundaries of the state space satisfy a geometric condition: only one control (or other vector used as a monotone direction) can cross each boundary in each direction, and
3. Service and arrival rates are either controllable to zero or are uncontrolled and state-independent.

Controls that choose between two directions, such as routing arrivals to one of two queues, are allowed but must meet an additional boundary crossing condition. The theory is powerful in the sense that it can be used to easily prove monotonicity for a variety of queueing control problems. We have used it to reproduce some of the two-station monotonicity results of Rosberg, Varaiya, and Walrand (1982) and Hajek (1984). Our theorem does not apply to routing decisions within a network or to server allocation (scheduling) problems. Since Hajek proved monotonicity for an allocation problem using submodularity, it seems likely that the assumptions could be weakened to include some of these problems.

Our monotonicity results apply to the following production/inventory problem. Items pass through a sequence of production stages before being placed in finished goods inventory, which is used to meet a Poisson demand. Each stage is modelled as an exponential single-server queue. Demand that cannot be met from inventory

is backordered and met by the next available finished item. Production and holding costs are incurred at each stage, as well as finished goods backorder costs. The problem is to control the production rates to minimize discounted or average costs. For a series system, we show that upon the addition of an item between stages, the production rates downstream do not decrease and the production rates upstream do not increase; also, a demand does not decrease production rates.

We also give some simplifying characterizations of the optimal policy under transition monotonicity. It is well-known that, in the absence of service costs, these problems have all-or-nothing (bang-bang) optimal policies. We show that they are characterized by switching functions with certain slope restrictions that bound a region in which all servers work at full capacity. When the system state reaches a boundary, one or more servers turn off. This simplification should ease the task of numerically computing optimal policies. Switching functions are also discussed in Hajek (1984), Stidham (1985, 1988), and Chen, Yang, and Yao (1991) for specific two-station networks. Developing numerical techniques that exploited these switching functions to compute optimal policies would be a useful area for further research.

We define a general queueing control problem in Section 2, then present monotonicity theorems in Section 3. Applications to production/inventory systems and to two interacting queues are discussed in Section 4. We will use the term increasing (decreasing) in the weaker sense of nondecreasing (nonincreasing) and denote a unit vector whose  $i$ th component is one by  $e_i$ .

## 2 Problem Description

We are primarily concerned with arrival rate, service rate, and routing control problems for queueing networks and related models of production/inventory systems.

However, the method being used to obtain monotonicity results applies to a broader class of Markov decision processes, so we will follow Weber and Stidham and use a more general notation. To accommodate certain routing problems, we define controls that choose between two transitions. If, instead, the rate of each transition is controlled independently, set  $d_i^0 = 0$  in what follows. Consider a continuous time Markov decision process with state  $x = (x_1, \dots, x_m) \in X \subseteq Z^m$ . The decision space is  $\{\mu = (\mu_1, \dots, \mu_q) : 0 \leq \mu_i \leq \bar{\mu}_i\}$  and, given a control  $\mu$  in state  $x$ , the transition  $x \rightarrow x + d_i^1$  occurs at rate  $\mu_i$  while the transition  $x \rightarrow x + d_i^0$  occurs at rate  $\bar{\mu}_i - \mu_i$ , for  $i \in \mathcal{C} = \{1, \dots, q\}$ . Let  $d_i = d_i^1 - d_i^0$ ; the control  $\mu_i$  pushes in the direction  $d_i$ . Because the system is memoryless, an optimal control need only depend on the current state  $x$ ; we denote this control  $\mu(x)$ . Also, if  $x + d_i^1 \notin X$ , then  $\mu_i(x) = 0$ .

Let  $d_{q+1}, \dots, d_{q+p}$  be the uncontrolled transitions with index set  $\mathcal{U} = \{q+1, \dots, q+p\}$ . To prevent transitions out of the state space, define the transition function  $D_i x = x + d_i$  if  $x + d_i \in X$  and  $D_i x = x$  otherwise;  $D_i$  occurs at the constant rate  $\lambda_i$ . The cost rate is a separable, nonnegative function of the state,

$$h(x) = \sum_{k=1}^m h_k(x_k),$$

with certain restrictions defined later, plus a continuous, separable function of the control,

$$c(\mu) = \sum_{i \in \mathcal{C}} c_i(\mu_i).$$

The functions  $c_i$  are assumed to be convex; if they are not, an equivalent optimization problem results when  $c_i$  is replaced by its convex lower envelope on  $[0, \bar{\mu}_i]$ . The objective is to minimize the expected discounted cost. An average cost criterion will be considered in Section 3.5.

In a queueing network control problem,  $x_i$  is the queue length (including customers in service) at queue  $i$ ;  $X = Z_+^m$ ;  $\mathcal{C}$  consists of a customer moving from queue  $j$  to

queue  $k$  ( $d_i^1 = e_k - e_j, d_i^0 = 0$ ), a customer departing the network from queue  $j$  ( $d_i^1 = -e_j, d_i^0 = 0$ ), controlled arrivals at queue  $j$  ( $d_i^1 = e_j, d_i^0 = 0$ ), and routing arrivals to queue  $j$  or  $k$  ( $d_i^1 = e_k, d_i^0 = e_j$ );  $\mu_i$  is a service, arrival, or routing rate control;  $\mathcal{U}$  may contain uncontrolled arrivals or departures;  $h_k(x_k)$  is the cost of holding  $x_k$  customers at queue  $k$ ; and  $c_i(\mu_i)$  is the cost of service (or arrivals) at rate  $\mu_i$ .

In production/inventory models, one of the state variables, say  $x_n$  measures finished goods inventory and is allowed to be negative, representing backorders. The uncontrolled transitions are demands which decrease the finished goods inventory. The controls  $\mu_i$  are production rates with costs  $c_i$ ; the costs  $h_k$  are holding costs at each station  $k \neq n$  and  $h_n$  is the finished goods holding and backorder cost. A complete description is given in Section 4.1.

For the infinite-horizon problem with discount rate  $\alpha > 0$ , the minimum expected cost is

$$V(x) = E_x \int_0^\infty e^{-\alpha t} [h(x(t)) + c(\mu(t))] dt,$$

where  $E_x$  denotes expectation given the initial state  $x(0) = x$ . We will uniformize the process as in Lippman (1975) by defining the potential event rate

$$\Lambda = \sum_{i \in \mathcal{C}} \bar{\mu}_i + \sum_{i \in \mathcal{U}} \lambda_i.$$

Let  $V_n(x)$  be the minimum  $n$ -stage expected discounted cost for the embedded discrete-time Markov decision process. Then  $V_n$  is well-defined and satisfies the dynamic programming equation

$$\begin{aligned} (\alpha + \Lambda)V_{n+1}(x) &= h(x) + \sum_{i \in \mathcal{U}} \lambda_i V_n(D_i x) \\ &+ \sum_{i \in \mathcal{C}} \min_{0 \leq \mu_i \leq \bar{\mu}_i} \left\{ c_i(\mu_i) + \mu_i V_n(x + d_i^1) + (\bar{\mu}_i - \mu_i) V_n(x + d_i^0) \right\}, \quad (2.1) \end{aligned}$$

where we define  $V_0(x) = 0$  and  $V_n(x) = \infty, x \notin X$ .

### 3 General Monotonicity Results

#### 3.1 Transition Monotonicity

Monotonicity results will be obtained by investigating the effect on the optimal control of moving in certain directions in the state space.

**Definition.** The control  $\mu(x)$  is **D-monotone** if

$$\mu_j(x) \leq \mu_j(x + d_i)$$

for all controls  $j \in \mathcal{C}$  and direction vectors  $d_i \in \mathbf{D}$  such that  $d_i \neq d_j$ .

We will only consider direction sets  $\mathbf{D}$  that contain all of the control directions  $d_i, i \in \mathcal{C}$ . If  $\mathbf{D} = \{d_i : i \in \mathcal{C}\}$  and  $d_i^0 = 0$  (i.e.,  $\mathbf{D}$  is the set of control transitions), we will say that  $\mu(x)$  is *control-monotone*; if  $\mathbf{D} = \{d_i : i \in \mathcal{C} \cup \mathcal{U}\}$  and  $d_i^0 = 0$  (i.e.,  $\mathbf{D}$  is the set of all transitions), we will say that  $\mu(x)$  is *transition-monotone*. For a queueing system with uncontrolled arrivals and controlled servers, control monotonicity means that after a service completion the service rate of other servers increases; transition monotonicity means that the service rate of all servers also increases after an arrival.

*Remark.* Under transition monotonicity,  $\mu_j$  is increasing in every other transition. If  $\{x(t), t \geq 0\}$  is an irreducible Markov chain under some control  $\mu(x)$ , then we can write  $-d_j = \sum_{i=i_1}^{i_n} d_i$  for some sequence of transitions not including  $d_j$ ; hence,  $\mu_j$  is decreasing in  $d_j$ . This monotonicity result might be called a law of diminishing



returns. Weber and Stidham use this reasoning for a cycle of queues.

Considering the minimization in (2.1), if

$$V_n(x + d_i^1) - V_n(x + d_i^0) \geq V_n(x + d_i^1 + d_j) - V_n(x + d_i^0 + d_j) \quad (3.1)$$

for all  $i \in \mathcal{C}$ ,  $d_i \neq d_j \in \mathbf{D}$ , and  $x$  such that all four points are in  $X$ , then the future cost of control  $\mu_i$  decreases when transition  $d_j$  occurs. Because  $c_i$  is convex, a  $\mathbf{D}$ -monotone optimal control must exist at stage  $n + 1$ . For simplicity, change  $x$  in (3.1) and write

$$V_n(x + d_i) - V_n(x) \geq V_n(x + d_i + d_j) - V_n(x + d_j). \quad (3.2)$$

Since  $V = \lim_{n \rightarrow \infty} V_n$ , if (3.2) holds with  $V_n$  replaced by  $V$  then  $\mathbf{D}$ -monotonicity also holds for the infinite-horizon problem. Condition (3.2) is an example of submodularity of a function on a lattice (Topkis 1978).

**Definition.** The function  $f : X \rightarrow R$  is *submodular* w.r.t.  $\mathcal{D}$  on  $X$  if

$$f(x + d_i) + f(x + d_j) - f(x) - f(x + d_i + d_j) \geq 0 \quad (3.3)$$

for all  $d_i \neq d_j \in \mathcal{D}$  and  $x$  such that all four points are in  $X$ .

The following condition will be used to guarantee that (3.2) holds sufficiently close to the boundary of  $X$ .

**Definition.**  $\mathbf{D}$  is *compatible* with  $X$  if for all  $d_i \neq d_j \in \mathbf{D}$ ,  $x + d_i$  and  $x + d_j \in X$  implies  $x$  and  $x + d_i + d_j \in X$ .

For most queueing control problems, we can think of  $X$  as the discrete equivalent of a polyhedral set and give a simple interpretation of compatibility: only one  $d_i$  can

cross each bounding hyperplane in each direction. Let  $\mathbf{D}(x) = \{d \in \mathbf{D} : x + d \in X\}$  be the set of feasible directions from state  $x$ . If  $\mathbf{D}$  is compatible with  $X$  and  $f$  is submodular w.r.t.  $\mathbf{D}$  on  $X$ , we will say that  $f$  is submodular w.r.t.  $\mathbf{D}(x)$ , since (3.3) holds for all  $x \in X$  and  $d_i \neq d_j \in \mathbf{D}(x)$ .

Weber and Stidham establish submodularity of  $V$  by showing that submodularity is preserved under value-iteration. The key step is the following lemma, which is proven in the Appendix (their proof is very similar but does not include choice controls).

**Lemma** *If  $\mathbf{D}$  contains all control directions  $d_i, i \in \mathcal{C}$ ,*

*(i)  $f(x)$  is submodular w.r.t.  $\mathbf{D}(x)$ , with  $f(x) = \infty, x \notin X$ ,*

*(ii)  $\mathbf{D}$  is compatible with  $X$ , and*

*(iii)  $d_i^0$  is always feasible, i.e.,  $x + d_i^0 \in X$  whenever  $x \in X$ ,*

*then for each  $d_k, k \in \mathcal{C}$ ,*

$$g(x) = \min_{0 \leq \mu \leq \bar{\mu}_k} \{c_k(\mu) + \mu f(x + d_k^1) + (\bar{\mu}_k - \mu) f(x + d_k^0)\}$$

*is also submodular w.r.t.  $\mathbf{D}(x)$ .*

Submodularity must also be preserved by uncontrolled transitions (the first sum in Equation 2.1). One type of transition that preserves submodularity is a state-independent transition,  $D_i x = x + d_i$  for all  $x \in X$ , i.e., Poisson arrivals. Some other transitions also work, depending on the geometry of  $\mathbf{D}$  and  $X$ .

**Definition.** The uncontrolled transition  $D_i, i \in \mathcal{U}$ , *preserves submodularity* w.r.t.  $\mathbf{D}(x)$  if  $f(D_i(x))$  is submodular whenever  $f(x)$  is submodular.

Monotonicity now follows easily from the lemma.

**Theorem (Monotonicity)** *If  $\mathbf{D}$  contains all control directions  $d_i, i \in \mathcal{C}$ ,*

*(i)  $h(x)$  is submodular w.r.t.  $\mathbf{D}(x)$ ,*

*(ii)  $\mathbf{D}$  is compatible with  $X$ ,*

*(iii)  $d_i^0$  is always feasible, and*

*(iv) all uncontrolled transitions preserve submodularity w.r.t.  $\mathbf{D}(x)$*

*then there exists a  $\mathbf{D}$ -monotone optimal control.*

*Proof.* First show that  $V_n(x)$  is submodular w.r.t.  $\mathcal{D}(x)$  using induction on  $n$ . Initially  $V_0(x) = 0$  is submodular. In (2.1),  $h(x)$  is submodular by (i), the first sum by (iv), and the second sum by the lemma and the inductive hypothesis. Since  $V_n \rightarrow V$ , it follows that  $V$  is submodular w.r.t.  $\mathbf{D}(x)$  and a  $\mathbf{D}$ -monotone optimal control exists.  $\square$

Note that the inductive proof establishes monotonicity for finite and infinite-horizon problems.

Weber and Stidham's monotonicity result is equivalent to the following specialization of our theorem.

**Corollary (Control Monotonicity)** *If  $\mathbf{D} = \{d_i : i \in \mathcal{C}\}, d_i^0 = 0$  for all  $i \in \mathcal{C}$ , and (i), (ii), and (iv) above, then there exists a control-monotone optimal control.*

The generalization of adding other directions to  $\mathbf{D}$  is very useful in some problems because it strengthens the monotonicity result; the strongest way to use this generalization is to consider one additional direction at a time. For systems with uncontrolled transitions, transition monotonicity (if it holds) tends to be a much stronger result than control monotonicity. The next two subsections describe some of the useful implications of transition monotonicity. Choice controls with  $d_i^0 \neq 0$  are useful in some arrival routing problems. We have not been able to establish monotonicity without condition (iii); with this condition, the theorem does not apply to problems with

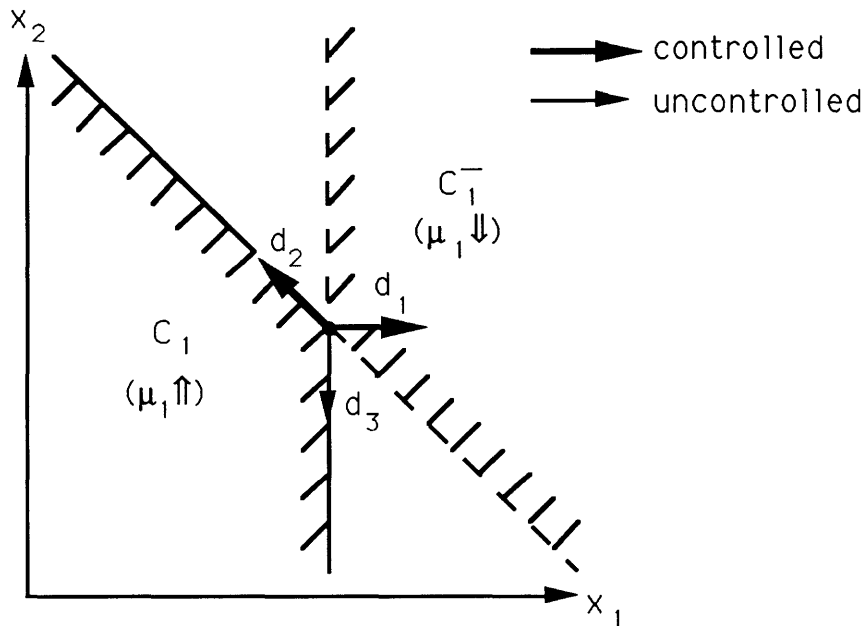


Figure 1: Increasing and Decreasing Directions for the Control  $\mu_1$

server allocation or routing between servers.

Not all networks of interest display monotonicity. In a tandem queue example of Weber and Stidham, where arrivals and the second server are controlled, but the first server is not controlled, the optimal arrival rate increases when a customer is added to the first queue. This behavior is not control-monotone; the theorem does not apply because condition (iv) is violated.

### 3.2 State Space Monotonicity

If  $\mathbf{D}$ -monotonicity holds with  $e_k$  or  $-e_k$  in  $\mathbf{D}$ , it can be used to establish monotonicity with respect to the  $x_k$  axis. Depending on the geometry of the problem, it may be possible to infer state space monotonicity from other direction sets.  $\mathbf{D}$ -monotonicity

establishes that the optimal control for a transition  $d_i$  increases after any other transition in  $\mathbf{D}$ . It will also increase after any sequence of transitions in  $\mathbf{D}$ ; hence, the convex cone  $C_i$  of  $\mathbf{D} \setminus \{d_i\}$  is a set of directions in which  $\mu_i$  is increasing and the negative convex cone  $C_i^-$  is a set of directions in which  $\mu_i$  is decreasing. Figure 1 shows these cones for the two-stage production/inventory system of Section 4.1 under transition monotonicity, where  $\mathbf{D} = \{d_1, d_2, d_3\}$ . If the vector  $e_k$ , giving the direction of the  $x_k$ -axis, lies within  $C_i$  or  $C_i^-$ , then monotonicity holds with respect to the  $x_k$  coordinate. For example, the control  $d_1$  (and  $d_2$ ) in Figure 1 is monotone with respect to both  $x_1$  and  $x_2$ . Control monotonicity corresponds to the convex cones generated by smaller sets  $\mathbf{D}$ . If only control monotonicity is established in Figure 1, the convex cone of  $\mu_1$  is the single direction  $d_2$ , which is not sufficient to establish state space monotonicity.

### 3.3 Control Regions and Switching Functions

It is well known that if the service cost  $c_i$  in (2.1) is linear, then an all-or-nothing (bang-bang) control is optimal, with  $\mu_i(x) = 0$  or  $\bar{\mu}_i$ . Bang-bang, transition-monotone controls can be characterized very simply as a control region. Let  $S_i = \{x \in X : \mu_i(x) = \bar{\mu}_i\}$ ,  $S'_i = \{x + d_i : x \in S_i\}$ , and  $B_i = S'_i \setminus S_i$  for  $i \in \mathcal{C}$ . In queueing terminology,  $S_i$  is the region where server  $i$  is on. As illustrated in Figure 2,  $B_i$  contains points on the boundary of  $S_i$  that are reached by one transition  $d_i$ . For any  $x \in S_i$ , consider the cone  $C_i$  for control  $\mu_i$  with vertex  $x$ . Since  $\mu_i$  is on at  $x$ , it must be on in all of  $C_i$  (similarly, if  $\mu_i$  were off at  $x$ , it would be off in all of  $C_i^-$ ); i.e.,  $C_i \cap X \subseteq S_i$ . But all transitions other than  $d_i$  lie in  $C_i$ , so they cannot cause the system state to leave  $S_i$ . By a translation argument, these transitions cannot leave  $S'_i$  either. Under such a control, the system never leaves the *control region*  $S' = \bigcap_{i \in \mathcal{C}} S'_i$ . This region consists of an interior  $S = \bigcap_{i \in \mathcal{C}} S_i$  in which all controls are

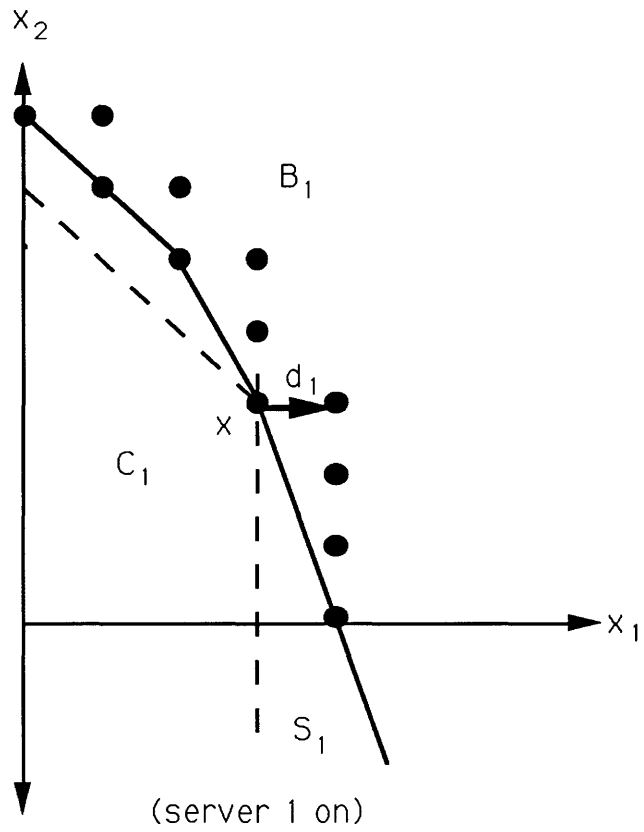


Figure 2: The cone  $C_1$  (to the left of the dashed line) is contained in the interior region  $S_1$  (to the left of the solid line);  $B_1$  contains the points outside of  $S_1$ .

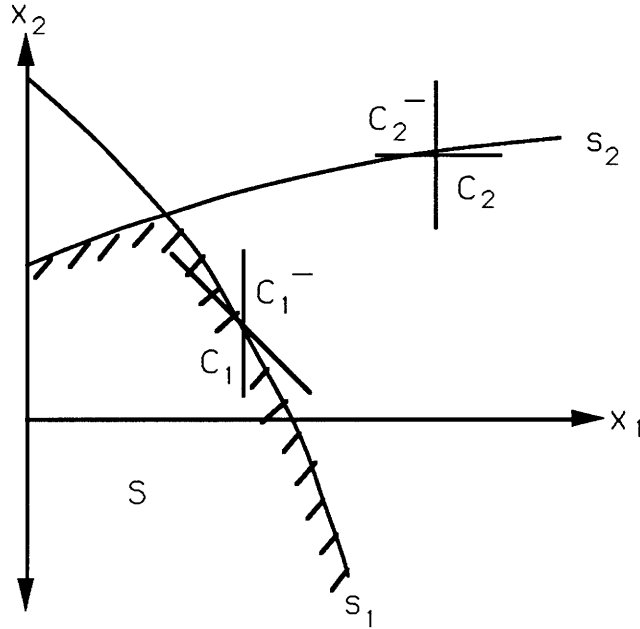


Figure 3: Switching Functions and the Interior Region

on and boundaries in which one or more controls are off. The system can only enter a boundary in which  $\mu_i$  is off through the transition  $d_i$ .

The control region can usually be defined using a *switching function* for each control  $\mu_i$ . Hajek proves that switching functions exist for a system of two interacting queues; we give sufficient conditions for the existence of switching functions in general  $\mathbf{D}$ -monotone systems. If the direction  $e_k$  lies in the cone  $C_i$  or  $C_i^-$  for control  $\mu_i$ , then  $S_i$  can be defined by a switching function  $s_i(x^{(k)})$ , where  $x^{(k)} = (x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_m)$ , such that  $\mu_i(x) = \bar{\mu}_i$  if and only if  $x_k \geq (\leq) s_i(x^{(k)})$ . In this case, the interior  $S$  is simply a subset of  $X$  bounded by the switching functions  $s_i$ . Moreover, the function  $s_i$  must lie between  $C_i$  and  $C_i^-$  for the control  $\mu_i$ . Figure 3 gives an idealization of these regions (the  $s_i$  are actually step functions). In queueing applications, these controls turn a server off when a boundary is reached; such controls have been analyzed on a continuous state space as reflected Brownian motion

(Harrison 1985).

### 3.4 Some Generalizations

Although compatibility is a useful condition because it is easily verified, other conditions can be used in the theorem. Let  $\delta X = \{x + d : x \in X, d \in \mathbf{D}, \text{ and } x + d \notin X\}$ . If it is possible to extend the cost function  $h$  onto the boundary  $\delta X$  in a way that preserves the submodularity of  $h$  and makes  $h$  arbitrarily large on  $\delta X$ , then  $h$  acts as a barrier and the optimal control for the unconstrained problem ( $V_0(x) = 0$  for all  $x$ ) will be feasible for the original problem. Since there are no boundaries, compatibility is not needed to prove the lemma for the unconstrained problem and establish transition monotonicity. Hajek uses this approach to prove monotonicity in a two-station network. However, extendibility of the cost function is at least as strong a condition; it implies compatibility, as the following argument shows.

Suppose that  $h$  is submodular w.r.t.  $\mathbf{D}$  on  $X$  and that for any  $M$  there exists an extension  $\tilde{h} : X \cup \delta X \rightarrow R$  of  $h$  that is submodular on  $X \cup \delta X$  with  $\tilde{h}(x) \geq M$ ,  $x \in \delta X$ . Given  $d_i, d_j \in \mathbf{D}$ ,  $i \neq j$ , and  $x$  such that  $x, x + d_i, x + d_j, x + d_i + d_j \in X \cup \delta X$ , the submodularity condition is

$$h(x + d_i) + h(x + d_j) - h(x) - h(x + d_i + d_j) \geq 0. \quad (3.4)$$

If  $x$  (or  $x + d_i + d_j$ ) is in  $\delta X$ , so that  $h(x) \geq M$ , then in order for (3.4) to hold for large  $M$ ,  $x + d_i$  or  $x + d_j$  must be in  $\delta X$ . The contrapositive of this statement is compatibility.

The compatibility condition can be weakened slightly if there are no service costs. Given four points in  $X$  on which  $f$  is submodular, as in (3.3),  $g(x)$  in the lemma is only submodular on these points if certain combinations of the four points obtained



Table 1: Comparison of Boundary Conditions

Case	Points in $X$	Compatible	$g$ Submodular
1	b,d,f,h	✓	✓
2	d,f,h		
3	b,f,h		✓
4	b,d,h		✓
5	b,d,f		
6	f,h	✓	✓
7	d,h	✓	✓
8	d,f		
9	b,h		✓
10	b,f		
11	b,d		
12	h	✓	✓
13	f		
14	d		
15	b		✓
16	-	✓	✓

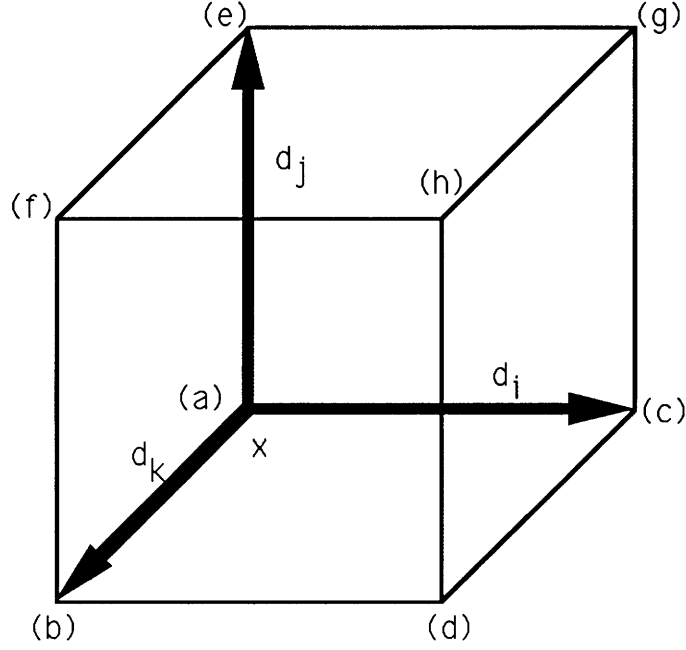


Figure 4: Points at which  $f$  is Evaluated

by adding  $d_k$ ,  $k \neq i$  or  $j$ , are in  $X$  (i.e., points (b), (d), (f), and (g) in Figure 4). Writing

$$g(x) = \bar{\mu}_k \min\{f(x), f(x + d_k)\},$$

a case-by-case analysis reveals that nine of the  $2^4$  possibilities guarantee submodularity. In contrast, compatibility allows only five of these possibilities (Table 1) and must hold for any four points  $x, x + d_i, x + d_j$ , and  $x + d_i + d_j$ , not just those defined as (b), (d), (f), and (g) are. If  $k = i$ , a similar analysis shows that compatibility is as weak as possible, allowing three out of  $2^2$  possible combinations of points in  $X$ . However, we have not found any meaningful examples where  $g(x)$  is submodular and conditions (i), (iii), and (iv) hold but compatibility does not hold.

We have assumed that the lower control limits are zero. Positive lower control limits  $\underline{\mu}_i \leq \mu_i \leq \bar{\mu}_i$  can be modelled by introducing a duplicate uncontrolled transition  $d_j = d_i, j \in \mathcal{U}, i \in \mathcal{C}$ , and resetting the control limits to  $\lambda_k = \underline{\mu}_i$  and  $0 \leq \mu_i \leq \bar{\mu}_i - \underline{\mu}_i$ .

However, the duplicate transition in  $\mathbf{D}$  adds the following conditions to the theorem:

1. By submodularity,  $2h(x + d_i) - h(x) - h(x + 2d_i) \geq 0$ ; i.e.,  $h$  is convex in the direction  $d_i$ .
2. By compatibility, if  $x + d_i \in X$  then  $x, x + 2d_i \in X$ ; i.e.,  $X$  has no boundaries in the directions  $\pm d_i$ .

### 3.5 Long-Run Average Costs

Monotonicity can be extended to the long-run average cost setting; the proof given by Weber and Stidham applies directly. If the state space  $X$  is finite (or  $h$  is bounded), well-known methods can be applied to let  $\alpha \rightarrow 0$ . In practice, this is not much of a restriction because it is usually clear how to truncate the state space. Note, however, that monotonicity holds for discounted optimal controls even when the system is unstable and cannot be truncated. For unbounded  $h$  we need the following assumptions:

1. There exists a policy with finite average cost.
2. For all  $x, y \in X$ , there exists a policy under which the undiscounted cost until first passage from  $x$  to  $y$  is finite.
3. The cost functions  $h_k$  are bounded below.
4. For some ordering of the states  $x \in X$ ,  $h(x) \rightarrow \infty$ .

The second assumption is a stronger form of irreducibility and the fourth guarantees that there are only finitely many good states. With these additional assumptions, the theorem holds for average cost optimal policies.

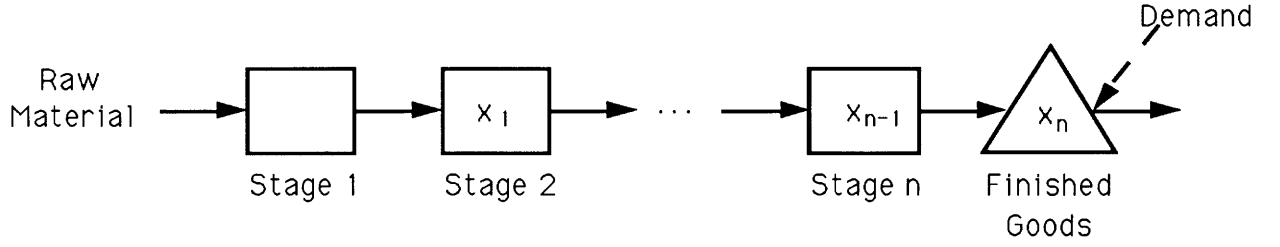


Figure 5: Series Production/Inventory System

## 4 Applications

### 4.1 Series Production/Inventory Systems

Consider an  $n$ -stage series production system in a make-to-stock environment with complete backordering (Figure 5). Each stage contains a single machine that operates as a  $M/1$  queue. Items are released into the system, processed at each stage, then placed in a finished goods inventory that services a Poisson demand. Demand that cannot be met from inventory is backordered and recorded as a negative inventory.

Denote the system state by  $x = (x_1, \dots, x_n)$ , where  $x_i$  is the number of items at stage  $i+1$ ,  $i = 1, \dots, n-1$  and  $x_n$  is the finished goods inventory. Because the supply of raw material is unlimited, there is no queueing and no state variable at stage 1. Stage  $i$  has production rate  $\mu_i$  controlled between 0 and  $\bar{\mu}_i$ , with transitions  $d_1 = e_1$  and  $d_i = e_i - e_{i-1}$ ,  $i = 2, \dots, n$ . Demand is an uncontrolled transition  $d_{n+1} = -e_n$  with rate  $\lambda$ . These transitions are illustrated in Figure 1 for a two-stage system. The state space is  $X = \{x \in Z^n : x_i \geq 0, i = 1, \dots, n-1\}$ . There is a work-in-process holding cost  $h_k$  after stage  $k$  (items available to stage  $k+1$ ),  $k = 1, \dots, n-1$ , a finished goods holding cost  $h_n$ , and a finished goods backorder cost  $b$ ; i.e.,

$$h(x) = \sum_{k=1}^{n-1} h_k x_k + h_n x_n^+ + b x_n^-. \quad (4.1)$$

No holding cost is assessed at stage 1 because it will contain at most one item in production and none queued. The production cost  $c(\mu)$  is approximately the same for all stable controls, and so is assumed to be zero. It is interesting to note that this production/inventory system differs from a series queue only in the removal of one boundary from  $X$ , the convex (as opposed to increasing) cost function, and which transition is uncontrolled. To establish transition monotonicity, we check the following conditions with  $\mathbf{D}$  containing all transitions.

1.  $h$  is submodular w.r.t.  $\mathbf{D}(x)$ . The submodularity condition (3.3) clearly holds when all four points lie on the same linear segment of  $h$ . The exception involves the transitions  $e_n - e_{n-1}$  and  $-e_n$  at  $x_n = 0$ ; here (3.3) is

$$h(x + e_n - e_{n-1}) + h(x - e_n) - h(x) - h(x - e_{n-1}) \geq 0.$$

Substituting (4.1) into the above gives

$$h(x) + h_n - h_{n-1} + h(x) + b - h(x) - h(x) + h_{n-1} = h_n + b \geq 0.$$

2.  $\mathbf{D}$  is compatible with  $X$ . The only transitions that cross the boundary  $x_k = 0, k < n$ , are  $e_k - e_{k-1}$  in the positive direction and  $e_{k+1} - e_k$  in the negative direction, so the geometric condition for compatibility is satisfied. A case-by-case check of compatibility can also be performed.

There are no choice controls ( $d_i^0 = 0$ ) and Poisson demand preserves submodularity, so we conclude that there exists a transition-monotone optimal control. To determine state space monotonicity, note that

$$\begin{aligned} e_k &= d_1 + \cdots + d_k \\ &= -(d_{k+1} + \cdots + d_n + d_{n+1}). \end{aligned}$$

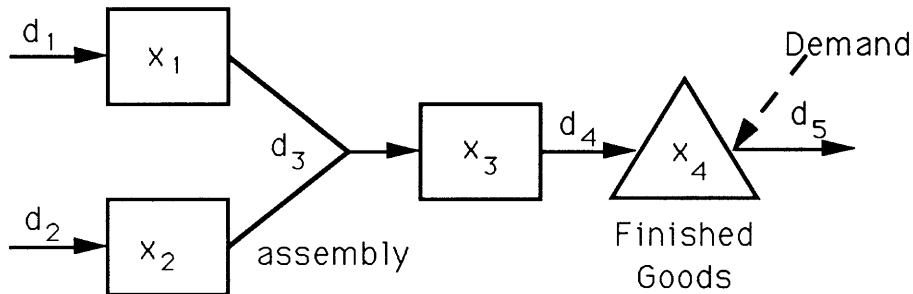


Figure 6: Assembly Production/Inventory System

For  $k < j$ , the first equality implies that  $\mu_j$  is increasing in  $x_k$  (upstream inventory); for  $k \geq j$ , the second equality implies that  $\mu_j$  is decreasing in  $x_k$  (downstream inventory). Furthermore, the optimal control is bang-bang with the following switching functions: stage  $i$  is on when  $x_i \leq s_i(x^{(i)})$  and off otherwise (recall that  $x_i$  is the inventory immediately downstream from stage  $i$  and that  $x^{(i)}$  is the vector of state variables other than  $x_i$ ). The rates of change of the switching functions are also constrained because the corresponding regions  $S_i$  must contain the cones  $C_i$ . In the two-stage system of Figure 3,  $s_1$  has a slope no greater than  $-1$  (more precisely, the step function  $s_i$  is decreasing and has no horizontal segment of length greater than one) and  $s_2$  is increasing. It is interesting to note that the base stock policy that has been proposed for this system (see Buzacott, Price, and Shanthikumar 1991) uses the limiting slopes of  $-1$  for  $s_1$  and zero for  $s_2$ . Similar characterizations of the optimal control of arrivals to two queues are given in Ghoneim and Stidham (1985).

## 4.2 Assembly and Disassembly Systems

For production/inventory systems with network topologies that include assembly, such as in Figure 6, arguments very similar to those in the last section can be used. Each stage except finished goods has a boundary  $x_k \geq 0$  which is crossed exactly once

in the increasing direction and once in the decreasing direction, so compatibility holds. The cost function  $h(x)$  is submodular as before. Transition monotonicity implies that  $\mu_j$  is increasing in upstream inventory and inventory in other branches, and decreasing in downstream inventory. For the example in Figure 6,  $\mu_1$  is increasing in  $x_1$  and  $x_2$  and decreasing in  $x_3$  and  $x_4$ ;  $\mu_3$  is increasing in  $x_1, x_2$ , and  $x_3$  and decreasing in  $x_4$ .

Reversing the above network topology gives a disassembly system, which might arise when a single production process yields multiple products. Similar arguments show that  $\mu_j$  is increasing in upstream inventory and decreasing in downstream inventory and inventory in other branches. Analogous results hold for queueing networks with assembly or disassembly topologies.

It should be noted that these results do not apply to systems with more than one transition in or out of a station. For example, if stations 1 and 2 in Figure 6 supply the same part to station 3,  $d_3 = e_3 - e_1 - e_2$  is replaced by two transitions,  $e_3 - e_1$  and  $e_3 - e_2$ , and compatibility is lost. A distribution network with more than one retail location supplied by a single supplier is also not compatible.

### 4.3 Two Interacting Queues

Most, but not all, of the monotonicity results of Hajek (1984) for two interacting queues follow from the general theory of Section 3. We discuss the application of the theorem to this problem to illustrate its limitations and a strategy that can sometimes be used when the conditions of the theorem are not met. In several simple networks, the uncontrolled transitions do not meet condition (iv) even for the minimal direction set  $\mathbf{D}$ . However, the submodularity conditions (iv) reduce to certain other conditions on  $V_n(x)$ , such as convexity. If these auxiliary conditions on  $V_n(x)$  can be shown to hold,  $\mathbf{D}$ -monotonicity can be established.

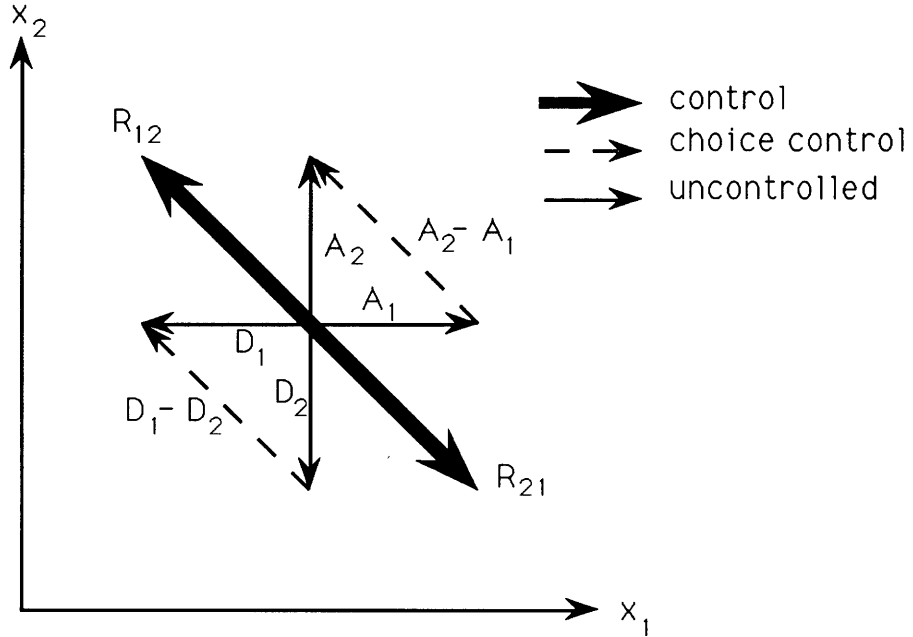


Figure 7: Transitions for Hajek's Two Interacting Queues

In Hajek's model (Figure 7) there are uncontrolled arrivals ( $A_1, A_2$ ) and departures ( $D_1, D_2$ ) from each queue. The controls route customers from queue 1 to queue 2 ( $R_{12}$ ) and vice-versa ( $R_{21}$ ), choose where to route an arrival ( $A_2 - A_1$ ), and choose which queue to service departures from ( $D_1 - D_2$ ). As usual, the state space is  $X = R_+^2$  and  $h(x)$  is a linear holding cost; there are no service costs. Several special cases are of interest. If  $R_{12}$  and  $D_2$  are considered controls and  $A_1$  the only uncontrolled transition, the system is a tandem queue with service rate control and the theorem establishes  $\{R_{12}, A_1, D_2\}$ -monotonicity as discussed in Weber and Stidham. Tedious but straightforward arguments show that the theorem also establishes  $\{R_{12}, A_1, D_2\}$ -monotonicity when the controls  $R_{12}$  and  $R_{21}$  and the uncontrolled transitions  $A_1, A_2$ , and  $D_2$  are considered. **D**-monotonicity has the same implications here as for the two-stage production/inventory system, i.e., the rate of  $R_{12}$  is increasing in  $x_1$  and decreasing in  $x_2$ .



Another special case is arrival routing: the control  $A_2 - A_1 = R_{12}$  chooses between  $d^0 = A_1$  and  $d^1 = A_2$ ; the uncontrolled transitions are  $D_1$  and  $D_2$ . Again let  $\mathbf{D} = \{R_{12}, A_1, D_2\}$ . Then  $D_1$  and  $D_2$  do not preserve submodularity for general functions  $f(x)$ ; the submodularity conditions for  $f(D_i x)$  require that  $f$  be convex and increasing in the  $x_1$  and  $x_2$  directions at points on the boundary ( $x_1 = 0$  or  $x_2 = 0$ ). Hajek notes that submodularity w.r.t.  $\mathbf{D}(x)$  implies convexity; a stochastic coupling argument shows that  $V_n(x)$  is increasing in  $x_k$ . With these auxiliary conditions,  $D_1$  and  $D_2$  preserve submodularity of  $V_n(x)$  and  $\mathbf{D}$ -monotonicity is established. This argument is very reminiscent of the approach used by Hajek (1984), Ghoneim and Stidham (1985), and Beutler and Teneketzis (1989) when they prove that submodularity plus auxiliary conditions are preserved under value-iteration. Condition (iv) identifies a candidate auxiliary condition.

When all of the transitions in Hajek's system are considered, our theorem does not seem to help in establishing monotonicity because the server allocation control ( $D_1 - D_2$ ) does not satisfy condition (iii).

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## Appendix - Proof of Lemma

We need to show that, for  $x \in X$  and  $d_i \neq d_j \in \mathbf{D}(x)$ ,

$$\Delta = g(x + d_i) + g(x + d_j) - g(x) - g(x + d_i + d_j) \geq 0. \quad (\text{A.1})$$

Let  $\mu(x)$  be the control that minimizes  $g(x)$ . Assume  $\mu_k(x + d_i) \geq \mu_k(x + d_j)$ .

*Case I:  $d_k \neq d_j$ .* If no boundaries are hit, submodularity of  $f$  and convexity of  $c_k$  give

$$\mu_k(x + d_i + d_k) \geq \mu_k(x + d_i) \geq \mu_k(x + d_j) \geq \mu_k(x),$$

or for brevity,  $\mu_{i+j} \geq \mu_i \geq \mu_j \geq \mu$ , motivating us to approximate  $\mu_{i+j}$  by  $\mu_i$  and  $\mu$  by  $\mu_j$ . Substituting into (A.1) and rearranging terms,

$$\begin{aligned} \Delta &\geq c_k(\mu_i) + \mu_i f(x + d_i + d_k^1) + (\bar{\mu} - \mu_i) f(x + d_i + d_k^0) \\ &\quad + c_k(\mu_j) + \mu_j f(x + d_j + d_k^1) + (\bar{\mu} - \mu_j) f(x + d_j + d_k^0) \\ &\quad - c_k(\mu_j) - \mu_j f(x + d_k^1) - (\bar{\mu} - \mu_j) f(x + d_k^0) \\ &\quad - c_k(\mu_i) - \mu_i f(x + d_i + d_j + d_k^1) - (\bar{\mu} - \mu_j) f(x + d_i + d_j + d_k^0) \\ &= (\bar{\mu} - \mu_j) [f(x + d_i + d_k^0) + f(x + d_j + d_k^0) - f(x + d_k^0) - f(x + d_i + d_j + d_k^0)] \\ &\quad + (\mu_i - \mu_j) [f(x + d_i + d_j + d_k^0) + f(x + d_i + d_k^1) - f(x + d_i + d_k^0) - f(x + d_i + d_j + d_k^1)] \\ &\quad + \mu_j [f(x + d_i + d_k^1) + f(x + d_j + d_k^1) - f(x + d_k^1) - f(x + d_i + d_j + d_k^1)]. \end{aligned}$$

By assumption (iii), all four points in the first bracketed expression are in  $X$ ; hence, it is nonnegative by the submodularity of  $f$  w.r.t.  $d_i$  and  $d_j$ . For the second expression,  $\mu_i \geq 0$  implies  $x + d_i + d_k^1 \in X$ . Also,  $x + d_i + d_j + d_k^0 \in X$  by assumption (iii). Then, by compatability of  $d_j$  and  $d_k$ , all four points are in  $X$  and the expression is nonnegative. Similarly,  $\mu_j > 0$  implies  $x + d_j + d_k^1 \in X$  and, since  $\mu_i \geq \mu_j > 0$ ,  $x + d_i + d_k^1 \in X$ . Compatability of  $d_i$  and  $d_j$  requires that all four points be in  $X$ ,

and the third expression is also nonnegative by the submodularity of  $f$  w.r.t.  $d_i$  and  $d_j$ .

*Case II:  $d_k = d_j$ .* Approximating  $\mu_{i+j}$  by  $\mu_j$  and  $\mu$  by  $\mu_i$  in (A.1) gives

$$\begin{aligned} \Delta \geq & (\bar{\mu} - \mu_i)[f(x + d_i + d_k^0) + f(x + d_j + d_k^0) - f(x + d_k^0) - f(x + d_i + d_j + d_k^0)] \\ & + (\mu_i - \mu_j)[f(x + d_i + d_k^1) + f(x + d_j + d_k^0) - f(x + d_k^1) - f(x + d_i + d_j + d_k^0)] \\ & + \mu_j[f(x + d_i + d_k^1) + f(x + d_j + d_k^1) - f(x + d_k^1) - f(x + d_i + d_j + d_k^1)]. \end{aligned}$$

The first and third terms are nonnegative as before; the second term is zero, since  $d_k = d_j$ .  $\square$