II. PLASMA DYNAMICS

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RESEARCH OBJECTIVES

This heading covers all of the work that is supported in part by the National Science Foundation and is under the over-all supervision of the Plasma Dynamics Committee of Massachusetts Institute of Technology. The general objective is to combine the technical knowledge of several departments, in a broad attempt to understand electrical plasmas, to control them, and to apply them to the needs of communication, propulsion, power conversion, and thermonuclear processes.

A. PLASMA PHYSICS[*]

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D. E. Baldwin		S. Yoshikawa

RESEARCH OBJECTIVES

The aim of this group continues to be the study of the fundamental properties of plasmas with more and more emphasis on high-density plasmas and plasmas in magnetic fields. To carry out this general objective, we have spent a great deal of effort on the production of plasmas of high-percentage ionization at low pressures under steady-state conditions, the achievement of which will allow us to carry on the fundamental studies in which we are most interested. At the present time, we have begun to achieve plasmas with high-percentage ionization by means of cesium plasmas, and we have several other schemes under way for producing them.

We are also studying ways of determining the characteristics of plasmas by means of microwaves, spectroscopic methods, and the diamagnetic effect of electrons. Along with these production and diagnostic studies, we are continuing measurements on the fundamental physics studies of loss and gain mechanisms of electrons in plasmas in magnetic fields. Considerable emphasis is being placed on studying the microwave radiation from plasmas, with and without magnetic fields, both as a tool for measuring the plasma temperature and thermal properties and as a means of understanding more about the motion of electrons and ions in magnetic fields.

Theoretical work has been concentrated on the study of waves in plasmas and of statistical theories of the nature of a plasma.

S. C. Brown

1. MAGNETO-AMBIPOLAR PLASMAS

An experimental program is being initiated for the study of dense plasmas in magnetic fields. The problems of interest are: (a) diffusion across the magnetic field, (b) instabilities in the diffusion process, (c) production of highly ionized plasmas, and (d) basic measurements on highly ionized plasmas. The initial experiments will center around a plasma column in a solenoid, 2 meters in length, so that both spatial and time resolution of plasma densities and electron temperatures can be obtained.

If diffusion is the governing loss mechanism of electrons and ions in a plasma column, the diffusion currents are given by

$$\text{Radial current density } \Gamma_r = - \frac{\partial}{\partial r} (D_{ab} n) \tag{1}$$

$$\text{Axial current density } \Gamma_z = - \frac{\partial}{\partial z} (D_a n) \tag{2}$$

where n is the electron or ion density, $D_a = (\mu_+ D_- + \mu_- D_+)/(\mu_+ + \mu_-)$ is the ambipolar diffusion coefficient, and $D_{ab} = D_a/(1 + \mu_+ \mu_- B^2)$ is the magneto-ambipolar diffusion coefficient. For low pressures and low magnetic fields, density decay is faster than electron energy decay, and hence D_a can be assumed constant. The decay of plasma density along a cylindrical tube can now be obtained by setting $\nabla \cdot \Gamma = 0$. If the column is assumed infinitely long, then the density in the lowest mode is given by

$$n = n_0 J_0(r/\Lambda_r) \exp\left[-z/\Lambda_r \left(1 + \mu_+ \mu_- B^2\right)^{1/2} \right] \tag{3}$$

where $n_0 J_0(r/\Lambda_r)$ is the density maintained at $z = 0$ by some active discharge, and Λ_r is the radial diffusion length, $R/2.405$. Other assumptions are: (a) no attachment
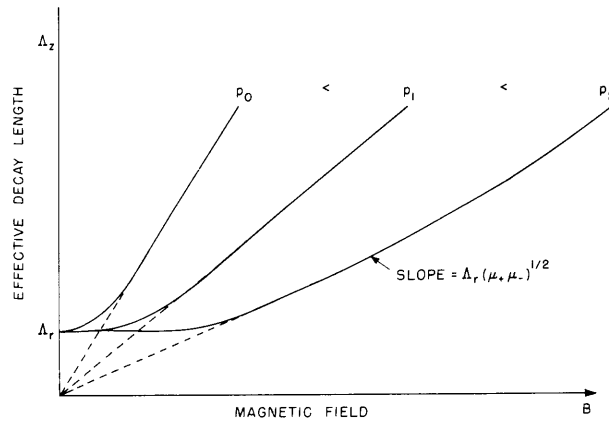

Fig. II-1. Effective decay length of plasma column in an axial magnetic field.
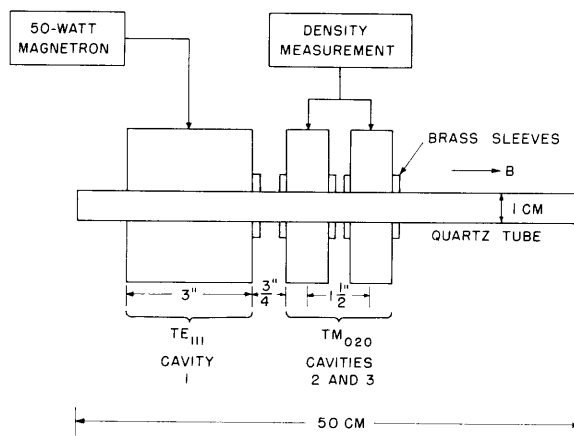
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Fig. II-2. Experimental apparatus.

or recombination, and (b) μ_+ and μ_- are independent of temperature. From Eq. 3, the decay length of the plasma column in the axial direction is

$$\Lambda_z = \Lambda_r \left(1 + \mu_+ \mu_- B^2\right)^{1/2}$$

and is shown plotted in Fig. II-1. Thus we can measure radial current losses by measuring the axial decay of density.

A preliminary microwave discharge experiment has been completed for checking the theoretical diffusion loss of a plasma across a magnetic field. A quartz tube, 50 cm long and 1 cm in diameter, was positioned with its axis along the magnetic field. The tube was thoroughly baked and pumped at 400°C, and hydrogen was admitted to the tube from a uranium hydride source at a pressure of from 1 to 10 microns. Three microwave cavities were placed along the tube, as shown in Fig. II-2. An S-band 50-watt magnetron was used to feed power into the TE_{111} mode of cavity 1 to produce a discharge at one end of the quartz tube. Two C-band cavities operating in the TM_{020} mode, and spaced as shown in Fig. II-2, were used to measure the density at two positions along the tube.

The experiment consisted of measuring the ratio of the densities in the two C-band cavities as a function of pressure, magnetic field, and electron density. As the measurements progressed, it became evident that consistent data were not obtainable, even with considerable effort. There was large scatter in the density ratio as a function of magnetic field, and at no time did the ratio reach unity — which the theory predicted for magnetic fields larger than approximately 10 gauss. Also, for a fixed magnetic field, there was no consistent variation of the ratio with either pressure or electron density. Some constrictions were observed as in previous microwave discharges. However, in these cases there was no direct correlation with the electron density ratio
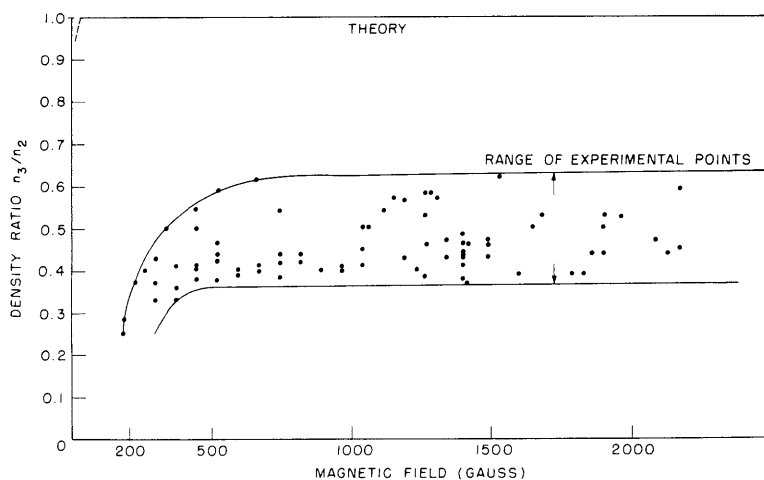
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Fig. II-3. Plot of density ratio versus magnetic field.

variations. Because of these facts, all of the measured ratios are plotted together in Fig. II-3 as a function of magnetic field and limits to the experimental points are indicated.

The results of this preliminary experiment clearly show that diffusion is not the controlling loss mechanism of the electrons. Examination of the quartz tube subsequent to the experiment shows internal depositions of a brown substance, which is most likely to be silicon reduced from SiO_2 by atomic hydrogen. During such a reaction water vapor is made, and it is probable that attachment to these impurities in the discharge is the dominant loss mechanism. These low-pressure discharges in hydrogen are plagued by this difficulty, and if further experiments are performed in this region, spectral analysis of the impurities will be necessary.

D. R. Whitehouse, Judith S. Vaughen

2. SYNCHROTRON RADIATION LOSS FROM A HOT PLASMA

In Quarterly Progress Report No. 55, pages 11-16, we reported computations on the contribution of synchrotron emission to the net energy loss by radiation from a plasma in a magnetic field. These calculations have been improved, in order to remove the plasma-temperature limitation of the method.

As a matter of convenience, we consider a plasma of electrons whose velocities, $v_{\parallel} = \beta_{\parallel} c$, along the magnetic field are negligible compared with their transverse components. We thus make use of Schwinger's formula (1) for I_n, the power radiated in the nth harmonic by a single electron of velocity $v_{\perp} = \beta c$ whose orbit is stationary.

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$$I_n(\beta) = \frac{2n\omega_b^2 e^2}{4\pi\epsilon_o c}(1-\beta^2)\left[J_{2n}(2n\beta) - \frac{1}{\beta}J_{2n+1}(2n\beta) - \frac{1-\beta^2}{\beta}\sum_{\nu=1}^{\infty}J_{2n+2\nu+1}(2n\beta)\right] \quad (1)$$

As in the previous report, we take the spectral distribution as collision broadening with collision frequency ν

$$I_n(\omega)\,d\omega = \int I_n(\beta)\left\{\frac{(\nu/\pi)\,d\omega}{[\omega-\omega_o(\beta)]^2 + \nu^2}\right\}df(\beta) \quad (2)$$

where

$$\omega_o(\beta) = n\omega_b(1-\beta^2)^{1/2} \quad (3)$$

Were we considering a plasma with significant orbital drift along B-lines, Eq. 1 would have to be in its angular-dependent form, and to include terms in β_{11}. In such a case, Doppler-shift effects would be included in Eq. 3. Since we only include two degrees of freedom, the electron-velocity distribution function is the two-dimensional relativistic Maxwellian in an approximate form that is valid for $\frac{kT}{mc^2} \ll \frac{8}{15}$. Thus

$$df(\beta) = \left(\frac{mc^2}{kT}\right)\gamma^4 \exp\left\{-\left(\frac{mc^2}{kT}\right)(\gamma-1)\right\}\beta d\beta \quad (4)$$

in which

$$\gamma = (1-\beta^2)^{-1/2}$$

The previous calculations approximated the bracketed expression in Eq. 1 by the leading term in the series expansion of the Bessel functions. Investigation of the next higher term places the following limit on that calculation: $n > \frac{\omega}{\omega_b} \gg (n^2-n-1)^{1/2}$. Thus the previous conclusions drawn for 10-kev and 50-kev plasmas are invalid.

A better approximation makes the following substitution:

$$J_p(x) = \frac{1}{p!}\left(\frac{x}{2}\right)^p \exp\left(-a_p x^2\right)$$

where the exponential factor (2) approximates $\Lambda_p(x)$, and a_p has been found empirically to be $\frac{0.258}{1.03 + p}$. This approximation is good within 2 per cent for $x < p/2$; this inequality implies $n > \frac{\omega}{\omega_b} > 0.85\,n$.
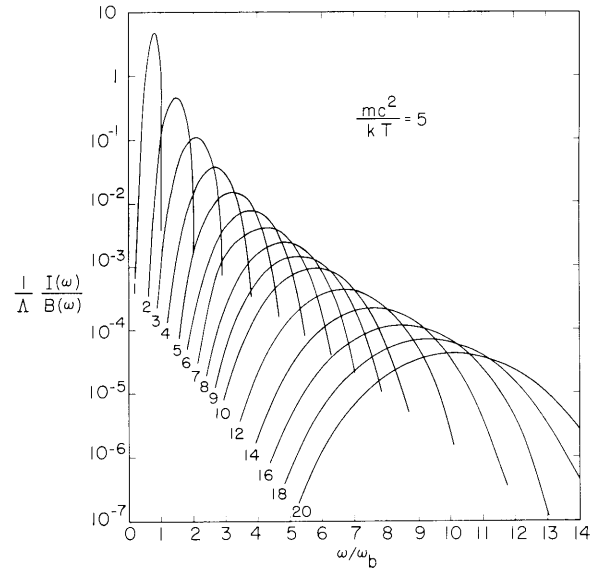
In view of these limitations, we have made no approximations to Eq. 1. In evaluating Eq. 2, we have gone to the limit $\nu/\omega_o \to 0$, so that the factor in braces takes on the character of a delta function.
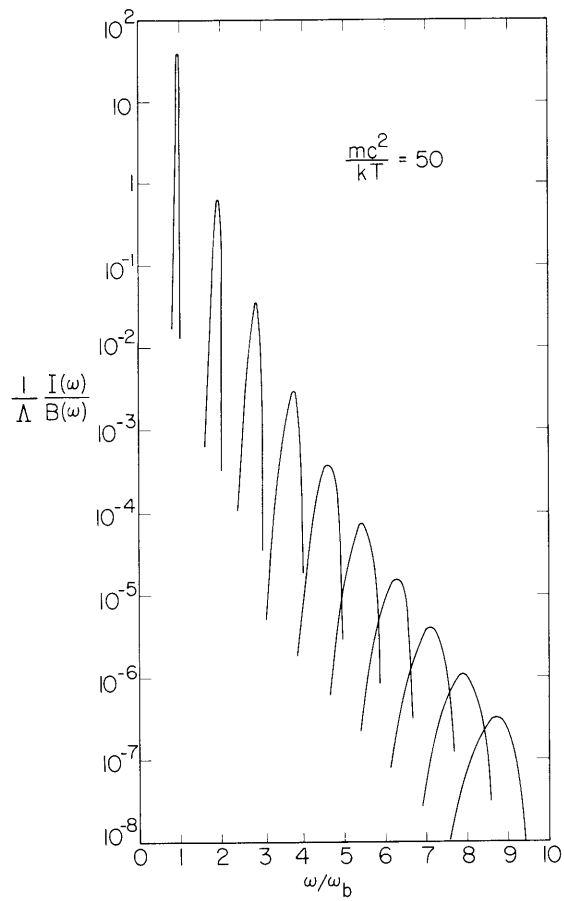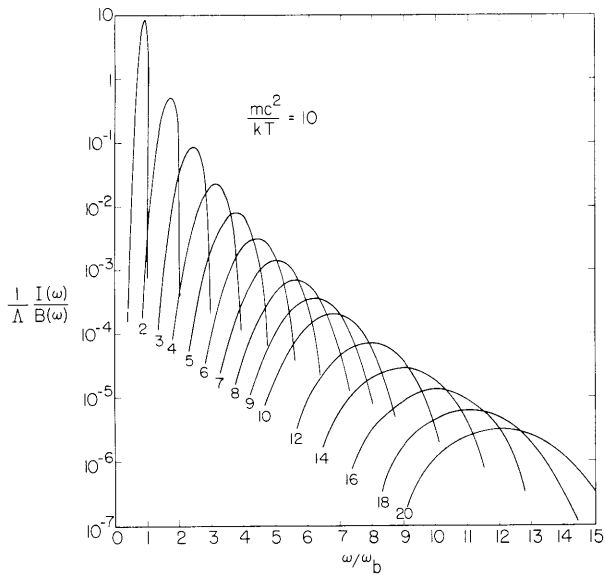
Fig. II-4. Synchrotron radiation from a plasma: (a) $mc^2/kT = 50$;
(b) $mc^2/kT = 10$; and (c) $mc^2/kT = 5$.

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Thus for a plasma slab of electron density $N = m\epsilon_o \omega_p^2 / e^2$ and thickness L, the ratio of the power emitted to that of a black body at the same temperature, and within the same frequency interval, is

$$\frac{I(\omega)}{B(\omega)} = \frac{\pi\Lambda}{n^2}\left(\frac{mc^2}{kT}\right)^2 a^3 \frac{\exp\left[-\frac{mc^2}{kT}\left(\frac{1}{a}-1\right)\right]}{(1-a^2)^{1/2}} \left\{ J_{2n}[2n(1-a^2)^{1/2}] - \frac{1}{(1-a^2)^{1/2}} J_{2n} \right.$$

$$\left. + [2n(1-a^2)^{1/2}] - \frac{a}{(1-a^2)^{1/2}} \sum_{\nu=1}^{\infty} J_{2n+2\nu+1}[2n(1-a^2)^{1/2}] \right\}$$

where $\Lambda = \left(\omega_p^2 L\right)/(\omega_+ c)$, and $a = \omega/n\omega_b$, and the typical range of values for Λ is 10^3-10^5.

The ratio $\frac{I(\omega)}{\Lambda B(\omega)}$ has been computed for $mc^2/kT = 50$, 10, and 5. The results are shown in Fig. II-4.

For $mc^2/kT = 50$, $\Lambda = 10^4$, it is seen that the black-body cutoff frequency is approximately $5\omega_b$. This is to be compared with the results of Trubnikov and Kudryavtsev (3), who find the cutoff at approximately $3\omega_b$.

For $mc^2/kT = 10$ and $\Lambda = 120$, Beard (4) finds the intensity at $\omega = 10\omega_b$ to be 10^{-4} of the black-body emission. Our result is larger than this by a factor of 25.

Finally, for $mc^2/kT = 5$ and $\Lambda = 10^4$, our results give the cutoff frequency to be at approximately $10\omega_b$. For a magnetic field of 10 kilogauss, a black-body emission up to this frequency is approximately 20 kw/m^2. To obtain the net radiation loss, the long tail seen in Fig. II-4c, which extends to higher frequencies, must be included, as well as the bremsstrahlung.

In order to demonstrate the effect of this synchrotron radiation on the energy balance of a practical thermonuclear reactor, the magnitude of the radiation will now be compared with the generated thermonuclear power and with the bremsstrahlung loss. We now consider the case in which $\beta = N_\pm k(T_+ + T_-)/\left(B^2/2\mu_o\right) = 1$. (Note that this β is not to be confused with β in Eqs. 1-4.) If other values of β arise through changes in B, but not in N or T, the values of the thermonuclear power density and the bremsstrahlung remain unchanged, but the magnetic radiation is multiplied by $\beta^{-3/2}$, provided that corresponding values of L are multiplied by $\beta^{-1/2}$. The magnetic radiation is computed by assuming the plasma to radiate as a black body up to the frequency ω_M, where $\sum_n I_n(\omega)/B(\omega) = 1$, and not at all thereafter. Since it is a surface effect, the magnetic radiation has been divided by L for comparison with the thermonuclear yield and the bremsstrahlung loss. The neglect of the contribution of the long tail extending above ω_M to the magnetic radiation could easily result in underrating the loss several times. The bremsstrahlung is computed (5) from $P_{brems} = 5.4 \times 10^{-31} T^{1/2} N^2$ watts/cm^3.
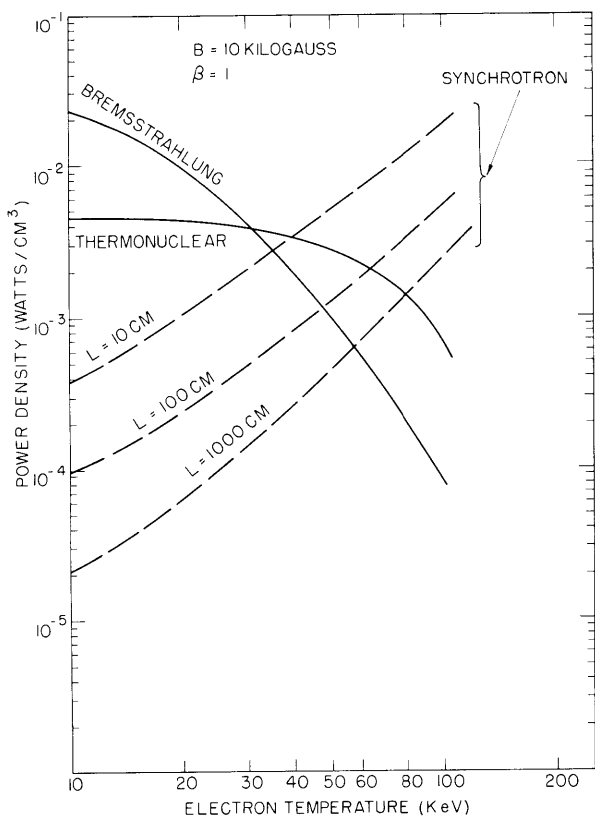
Fig. II-5. Comparison of synchrotron radiation with bremsstrahlung and thermonuclear yield. B = 10 kilogauss.
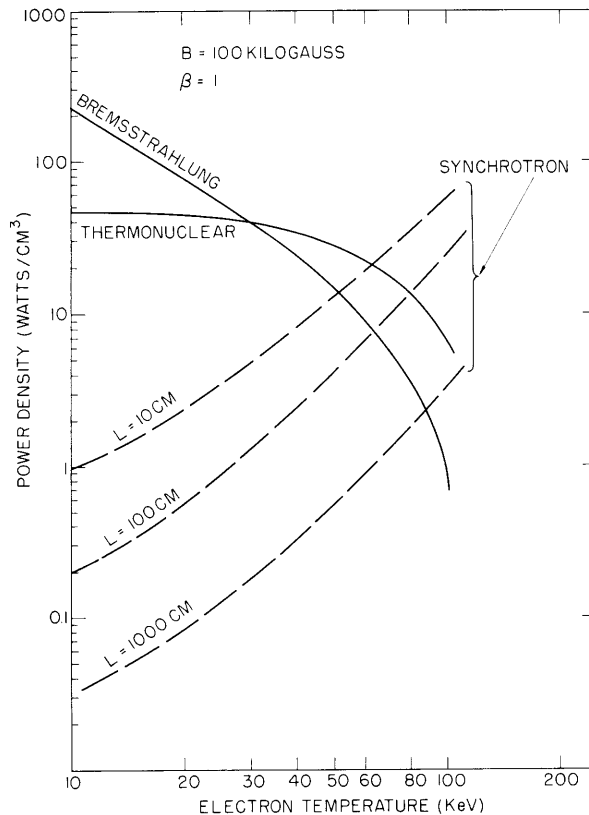
Fig. II-6. Comparison of synchrotron radiation with bremsstrahlung and thermonuclear yield. B = 100 kilogauss.

The thermonuclear power is computed under the assumption of a D-D reaction taking place, without subsequent burning of the tritium that is formed. Thus, from Post (5), we have $P_{TN} = 2.9 \times 10^{-13} N_D^2 \langle \sigma v \rangle_{DD}$ watts/cm^3. The values of $\langle \sigma v \rangle_{DD}$ are also obtained from data of Post (5). Values are computed for B = 10 kilogauss and B = 100 kilogauss, and the results are summarized in Figs. II-5 and II-6.

The authors wish to acknowledge the assistance of Elizabeth J. Campbell, Maida Karakashian, and Margaret E. Wirt, of the Joint Computing Group, M.I.T., in the computations leading to Fig. II-4.

<div align="right">J. L. Hirshfield, D. E. Baldwin</div>

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3. CESIUM PLASMA

Efforts to produce a highly ionized plasma with cesium (1) continue. An absolute measurement of the electron density in the plasma has been carried out with the use of a microwave cavity. The electric field used was the TM_{020} mode, aligned with the electric field parallel to the dc magnetic field. This cavity is shown in Fig. II-7, and the results of the density measurements are plotted there for various temperatures of the hot filament that ionizes the cesium beam. These microwave measurements are in substantial agreement with the densities calculated from the ion current (1).

In order to know the percentage of ionization in such a cesium plasma, it is necessary to measure the very low pressure of the neutral cesium. For this purpose, a cesium pressure gauge, which is shown schematically in Fig. II-8, is being constructed.

The collimating apertures (marked A in Fig. II-8) produce a well-defined beam of
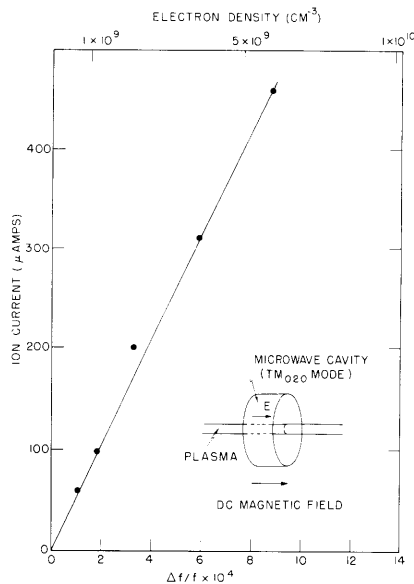

Fig. II-7. Measurements of charged-particle density with the use of a microwave cavity.
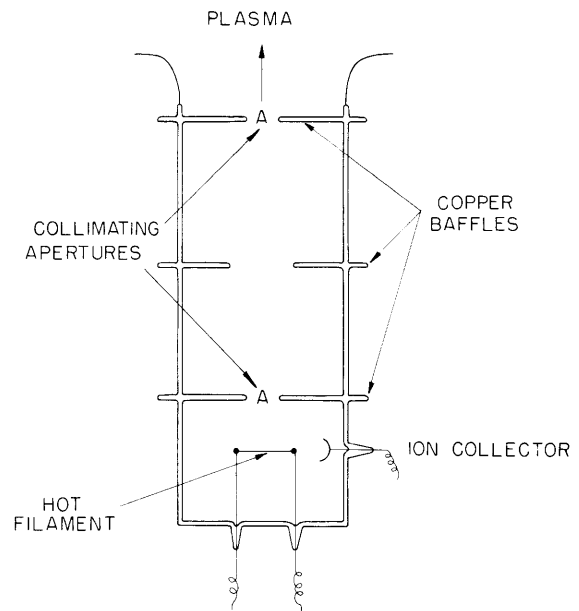
Fig. II-8. Cesium pressure gauge.

neutral cesium moving with thermal velocity. The tungsten filament is kept at a suf-
ficiently high temperature that all the arriving cesium is ionized. The ion collector,
which is held at a negative potential with respect to the tungsten filament, draws off all
of the ions produced. The resultant ion current, which is measured, is then a measure
of the pressure of the neutral cesium. The conversion factor from ion current to cesium
pressure is calculable directly from kinetic theory and the geometry of the pressure
gauge.

R. B. Hall, G. Bekefi

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Research Laboratory of Electronics, M.I.T., Oct. 15, 1959, pp. 16-19.

4. HEAT TRANSPORT IN PLASMAS

In a plasma, energy is carried mainly by electrons that drift through the medium
as a result of externally applied fields, electron temperature gradients, and density
gradients. Let the plasma be subjected to a uniform magnetic field $\vec{B}$ and an arbitrary
electric field $\vec{E}$. Assuming for simplicity, a Maxwellian distribution of electron veloc-
ities, and an electron-atom collision frequency ν_c that is independent of electron veloc-
ity, we obtain (1, 2) for the heat flux $\vec{H}$ (watts-meter^{-2})

$$\vec{H} = -\frac{5}{3}\frac{nU}{m}\overleftrightarrow{T} \cdot \left[\vec{\nabla}eV + \frac{4}{3}\vec{\nabla}U + \frac{2}{3}\frac{U}{n}\vec{\nabla}n\right] \tag{1}$$

Here e, m, and n are the charge, mass, and concentration of the electrons, respec-
tively; $U = 3/2\, kT$ is the average electron energy; V is the dc electric potential applied
across the plasma; and $\overleftrightarrow{T}$ is a tensor given by

$$\overleftrightarrow{T} = \begin{Vmatrix} \dfrac{\nu_c}{\nu_c^2 + \omega_b^2} & \dfrac{\omega_b}{\nu_c^2 + \omega_b^2} & 0 \\[4mm] -\dfrac{\omega_b}{\nu_c^2 + \omega_b^2} & \dfrac{\nu_c}{\nu_c^2 + \omega_b^2} & 0 \\[4mm] 0 & 0 & \dfrac{1}{\nu_c} \end{Vmatrix}$$

where ω_b is the cyclotron frequency, eB/m. The conservation of energy principle
requires that in a given elementary volume of plasma,

$$\frac{\partial nU}{\partial t} = S - \vec{\nabla} \cdot \vec{H} - nG\nu_c(U - U_e) \tag{2}$$

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The parameter S denotes the rate of heating of electrons by external sources; the last term of Eq. 2 represents the rate of loss of energy of an electron upon collision with an atom; G is the fractional excess energy loss per collision; and U_e is the equilibrium energy to which the electron decays when S is removed. Loss of energy resulting from excitation and ionization is neglected in the calculations.

Consider, now, a long cylinder of plasma of radius R with a magnetic field B applied along the cylinder axis, z. A section of the plasma column is heated by dc or rf fields, and thus the electrons are raised to a higher mean energy. We wish to investigate the time-independent decay of electron energy with distance z, outside the heated region, where S and V are taken to be zero. In the absence of a magnetic field, or in weak magnetic fields, the variation of U with z is effected by diffusion cooling to the radial walls (last term on the right-hand side of Eq. 1) and by axial gradients of electron density. To take account of density gradients, Eqs. 1 and 2 must be solved in conjunction with the following equations for the electron flux $\vec{\Gamma}$, and for the conservation of particles:

$$\vec{\Gamma} = -\frac{2}{3m} \overset{\leftrightarrow}{\tau} \cdot \vec{\nabla} Un \tag{3}$$

$$\frac{\partial n}{\partial t} = -\vec{\nabla} \cdot \vec{\Gamma} \tag{4}$$

Diffusion is the only loss mechanism considered. If we assume that radial temperature variations can be neglected, the time-independent decay of energy with z reduces to the solution of the nonlinear equation,

$$\frac{d^2 \mathcal{U}}{dz^2} - 2\beta \frac{d \mathcal{U}}{dz} = a^2 \left[1 - \frac{1}{\mathcal{U}} \right] \tag{5}$$

where $\mathcal{U} = U/U_e$, and $a^2 = \left(9mGv_c^2 / 10U_e \right)$. The second term on the left-hand side of Eq. 5 represents the contribution from gradients of electron density; the parameter β is the axial decay constant of the product (nU), which varies as

$$nU = \text{constant} \times J_0(2.405r/R) \exp(-\beta z) \tag{6}$$

with J_0, the zero-order Bessel function, and r, the radial distance in the plasma. In the limit of ambipolar diffusion, β is given in terms of the electron and ion mobilities, μ_-, μ_+, by

$$\beta = \frac{2.405}{R} \left[1 + \mu_+ \mu_- B^2 \right]^{-1/2} \tag{7}$$

Equation 5 can be solved approximately in the limit of very low and very high temperatures:

a. When the electrons are heated slightly above their equilibrium temperature, so that $(U-U_e)/U_e \ll 1$, then

$$\mathscr{U} - 1 = \text{constant} \times \exp(-\gamma z) \qquad (8)$$

where $\gamma = [(\alpha^2+\beta^2)^{1/2}-\beta]$. We note that as the magnetic field becomes very large ($\beta \rightarrow 0$), the energy decay becomes independent of B and R. If, on the other hand, the magnetic field is weak (or zero), β can greatly exceed α, and $\gamma \rightarrow \alpha^2/2\beta$.

b. In the range of high electron temperatures, $U \gg U_e$, and large magnetic fields, $\alpha > \beta$, we find that

$$\alpha z/\sqrt{2} = \mathscr{U}_M^{1/2} - \mathscr{U}^{1/2} \qquad (9)$$

where $\mathscr{U} = \mathscr{U}_M$ when $z = 0$.

The thermal effects discussed above can influence considerably the spatial decay of the electron concentration along the z-axis of the plasma column. After we eliminate the electron energy U from Eq. 6, by making a substitution from Eq. 8 or Eq. 9, we make the following observations: For moderate magnetic fields, less than a certain critical magnetic field, the electron density n decays monotonically with z (for all values of z), but it decays at a greater rate than it does when thermal effects are omitted. The rate of decay becomes smaller, the larger the magnetic field. For a magnetic field that is greater than the critical field, the electron density first increases with z (for small values of z) and then, once again, decreases monotonically. It is unlikely that the initial increase of n above the value that exists at z = 0 is physically realizable. At best, n could remain constant over a certain distance z and then fall off. Nevertheless, the tendency for n to increase with z may give rise to instabilities in that section of the plasma column.

The critical behavior of the plasma occurs when the energy decay constant γ of Eq. 8 is approximately equal to the decay constant β of Eqs. 6 and 7. Equating these two decay constants, we obtain the following relation for the critical magnetic field, B_c:

$$B_c \approx 5(fT)^{1/2} \, (M/mR) \text{ gauss} \qquad (10)$$

Equation 10 was derived for a monatomic gas; G of Eq. 10 is 2m/M, with m and M, the electron and ion masses, respectively. Temperature T of Eq. 10 is given in
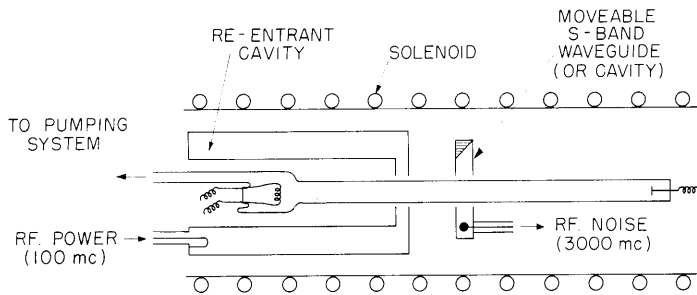

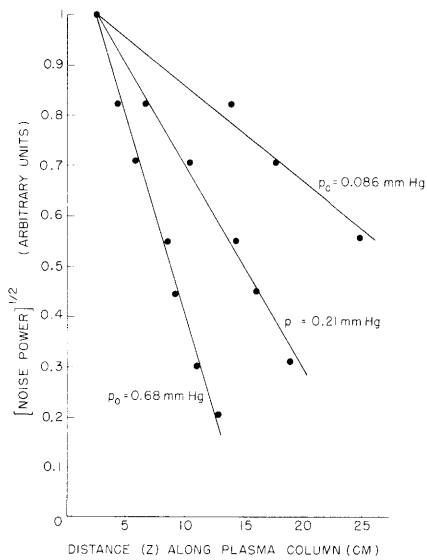Fig. II-9. Schematic diagram of equipment.

Fig. II-10. Square root of noise power as a function of density along a plasma column.

electron volts, and the plasma radius R in centimeters. The parameter $f = \nu_{c+}/\nu_{c-}$ represents the ratio of the ion to the electron collision frequency and its magnitude is dependent on the gas used and on the electron and ion temperatures. For helium, at an electron temperature of 1 ev, and an ion temperature of 0.04 ev, $f \approx 0.014$; for a tube of radius 1 cm, $B_c \approx 4200$ gauss.

Recent observations (3, 4) of the behavior of the positive column of dc discharges, subjected to an axial magnetic field, disclosed an onset of instabilities at certain critical magnetic fields; the instabilities were accompanied by an increase of diffusion to the radial walls. The magnitudes of the observed critical magnetic fields and their dependence upon the discharge parameters show a resemblance to those given by Eq. 10.

Measurements of heat conduction in a helium plasma of low degree of ionization are being performed. The plasma cylinder is 76 cm long and 2.5 cm in diameter. Magnetic fields up to 2500 gauss are applied axially. A section of plasma, 3 cm long, is heated within the gap of a re-entrant cavity that resonates at a frequency of 100 mc; 130 watts of rf power are available for heating (see Fig. II-9). The electron energy U is deduced from the microwave noise emitted (5) by a section of the plasma column that is 1 cm long. The radiation is sampled by a waveguide or cavity at a frequency of 3000 mc (bandwidth 2 mc). The noise is fed into a radiometer (5) that is capable of detecting temperature changes of 2°K. The noise output is proportional to the electron energy U. Figure II-10 shows preliminary measurements of [noise power]$^{1/2}$ as a function of z for three helium gas pressures, p_o. The ordinate is in arbitrary units because the noise output has not yet been calibrated in absolute electron temperatures. For convenience of presentation, the three curves were normalized to the same noise power at a value $z \doteq 2$ cm. Since U is known only within a constant of proportionality, the decay constant γ cannot be found. However, measurements support the relationship between U and z given by Eq. 9.

G. Bekefi

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B. PLASMA ELECTRONICS[*]

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Prof. A. Bers   W. D. Getty   S. D. Rothleder
Prof. D. J. Rose   W. Larrabee   W. C. Schwab
K. Arons   L. M. Lidsky   D. V. Turnquist
    A. Peskoff

RESEARCH OBJECTIVES

During the coming year, our research program will be concerned with the following problems:
1. Plasma waveguides.
2. Electron-beam stimulated plasma oscillations.
3. High-power microwave gaseous discharges.
4. Low-pressure gas-arc plasmas.

1. Plasma Waveguides

This is primarily a theoretical program, and its objective is to find the modes of propagation in a closed waveguide partly filled with a plasma, in the presence of a magnetic field. In many ways this is analogous to the ferrite loaded waveguide problem; but there are important theoretical differences that arise from the nonrigid nature of the plasma.

First, an "ideal plasma" with no random velocities (T=0° K) will be assumed. Later, an attempt will be made to include the effect of finite temperature.

2. Electron-Beam Stimulated Plasma Oscillations

This program is centered around a projected experiment in which it is planned to send a high-power, pulsed, electron beam down the axis of a low-pressure gas arc. A small-signal analysis indicates that there will be a very rapid amplification of a narrow frequency band around ω_{pa} of the arc. Starting from shot-noise fluctuations on the beam, the oscillation amplitudes should build up to saturation level within a few centimeters. Our present theoretical studies are aimed at predicting the saturated level of these noise-like oscillations. Preliminary estimates indicate that the plasma electrons may acquire approximately 10 per cent of the incident beam energy. If this oscillation energy is subsequently randomized by collisions, then a very high plasma temperature might be achieved with an incident electron beam of 1-2 amp at 10-20 kv.

Experiments will be made on a gas-arc facility that is now under construction.

3. High-Power Microwave Gaseous Discharges

Extensive measurements have been carried out on cw microwave discharges with available power of approximately 100 watts. There is evidence that the plasma density and electron temperature increase with power, but cw measurements have not been made much beyond 1 kw. In the experiments that are now set up, a tunable, 1-Mw, 10 μsec, pulsed magnetron will be used. Electron density during the pulse will be measured in much the same way as in the cw experiments. There is some doubt as to whether or not the discharge will reach equilibrium within the pulse duration of 10 μsec. Consequently, we shall attempt to measure electron density and conductivity as a function of time, during the pulse duration.

L. D. Smullin, A. Bers

4. Low-Pressure Gas-Arc Plasmas

An experimental program for the production of highly ionized steady-state plasmas, by using fast pumping techniques has been initiated.   This work is based on methods originally developed at Oak Ridge National Laboratory.   The plasma column is formed in a solenoidal magnetic field, with gas feed through the cathode region of the arc. With the use of graphite or of other refractory electrodes, a plasma is to be produced with density, 10^{11}-10^{13}/cm^3, particle energy 1-100 ev, and volume 10-10^3 cm^3, the values depending on the power level.   These plasmas will be used for studying plasma oscil-lations, wave propagation, ion cyclotron heating, and for other purposes.

D. J. Rose, L. D. Smullin

1. ELECTRON-BEAM STIMULATED PLASMA OSCILLATION

a. Introduction

An experiment (1) is being designed to study the possibility of heating a plasma by means of an intense electron beam. The ordered kinetic energy of the drifting beam will be used to excite and sustain oscillations of the electrons in the plasma through which it drifts. In turn, the organized kinetic energy of oscillation will be randomized by two main processes: (a) collisions between the oscil-lating plasma electrons and ions or neutral gas molecules; and (b) by the large-signal effects that will cause the trajectories of neighboring electrons to intersect, and thus effect randomization of their velocities in one dimension. The second mech-anism has been discussed by Buneman (2).
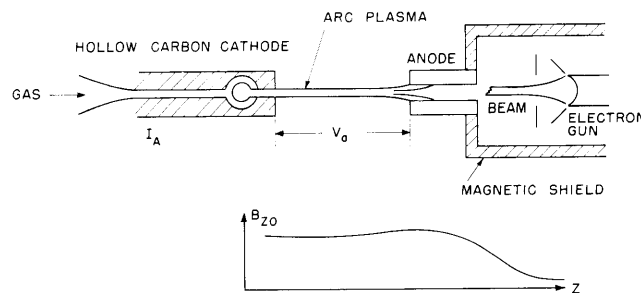

Fig. II-11.   Schematic diagram of apparatus for observation of interactions between an electron beam and a plasma.

The experimental apparatus that is being constructed is illustrated schematically in Fig. II-11. The plasma for the experiment is provided by a low-pressure gas arc emanating from a hollow carbon cathode. Typical ranges for the parameters of such a discharge are:

Anode-cathode distance, 25-40 cm

V_a, 100-200 volts

I_a, 0.5-10 amps

B_{zo}, 100-1000 gauss

ω_{pa}, 10^{11} rad/sec

Highly ionized (>95 per cent).

The electron gun will be pulsed at approximately 10-20 kv, in order to produce a beam of several amperes. The beam plasma frequency is

$$\omega_{pb} = \frac{3.3 \times 10^8}{b} (K'V_o')^{1/2}$$

where b is the beam radius in centimeters, V_o' is the voltage in kv, and K' is the "microperveance" $\left(I_o = 10^{-6} K' V_o^{3/2} \right)$. Reasonable values of ω_{pb} are in the range 5-10 × 10^9 rad/sec.

The object of the experiment is to determine how much of the 10-50 kw of injected beam power can be transformed into random energy of the plasma.

An attempt is being made to gain a theoretical understanding of the phenomena involved. One of the theoretical difficulties stems from the extremely strong inter-action that appears to take place between beam and plasma. In the neighborhood of the arc plasma frequency, ω_{pa}, disturbances grow as $\exp(\alpha - j\beta_e) z$, where $\beta_e = \omega_{pa}/v_o = 2\pi/\lambda p$, v_o is the beam drift velocity, and α may range from 1/4 to $2\beta_e$. Thus, in one plasma wavelength (a few millimeters), a disturbance may grow by a factor e up to e^{12}. Exactly how rapid the growth will be depends on plasma temperature, beam diameter, etc. Thus, even initial noise fluctuations are sufficient to produce very large oscillation levels at a few centimeters beyond the point where the beam enters the plasma. The nonlinear phenomena that cause the oscillation amplitudes to reach finite limits are still not understood.

b. Some Small-Signal Relations in a Plasma Reactive-Medium Amplifier

The basic small-signal equations for the interaction between a beam and a plasma are:

$$\frac{j\omega v_a}{\eta} = -\frac{1}{j\omega \epsilon_o} (J_a + J_b) \tag{1}$$

$$j\omega J_a = j\omega \epsilon_o \frac{\omega_{pa}^2}{\eta} v_a \tag{2}$$

$$\left(\frac{j\omega}{v_{ob}} - j\beta\right) \frac{v_{ob}v_b}{\eta} = -\frac{1}{j\omega\epsilon_o}(J_a + J_b) \tag{3}$$

$$\left(\frac{j\omega}{v_{ob}} - j\beta\right) J_b = j\omega\epsilon_o \frac{\omega_{pb}^2}{v_{ob}\eta} v_b \tag{4}$$

where v_a, J_a, ω_{pa}, are the ac velocity, convection current, and plasma frequency of the fixed plasma (temperature assumed to be zero); v_{ob}, J_{ob} are the dc velocity and current of the electron beam; similarly, for the ac terms. Equations 1-4 can be solved for β. Thus

$$\beta = \frac{\omega}{v_{ob}} \left(1 \pm \frac{\frac{\omega_{pb}}{\omega}}{\left(1 - \frac{\omega_{pa}^2}{\omega^2}\right)^{1/2}}\right) \tag{5}$$

From Eqs. 1, 3, and 5 we can write

$$\frac{v_a}{v_{ob}} = \delta \frac{\omega_{pb}}{\omega\left(1 - \frac{\omega_{pa}^2}{\omega^2}\right)^{1/2}} \tag{6}$$

where $v_b = \delta v_{ob}$.

Similarly, from Eqs. 1, 2, and 5, letting $J_b = aJ_{ob}$ and $\omega_{pb}^2 = \eta J_{ob}/\epsilon v_{ob}$, we obtain

$$\frac{v_a}{v_{ob}} = a\left(\frac{\omega_{pb}}{\omega}\right)^2 \frac{1}{\left(1 - \frac{\omega_{pa}^2}{\omega^2}\right)} \tag{7}$$

If the plasma were a lossless resonator, which is implied from Eqs. 6 and 7, the ac velocity modulation would be infinite at $\omega = \omega_{pa}$. If we replace the denominator $\left(1 - \omega_{pa}^2/\omega^2\right)^{1/2}$ with a function that includes collisions and temperature, $F(\nu_c, T, \omega)$, with a finite maximum value at $\omega \approx \omega_{pa}$, then

$$\frac{V_a}{V_{ob}} = \left(\frac{v_a}{v_{ob}}\right)^2 \approx \delta^2 \left(\frac{\omega_{pb}}{\omega_{pa}} F\right)^2 \tag{8}$$

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$$\frac{V_a}{V_{ob}} = \left(\frac{v_a}{v_{ob}}\right)^2 \approx a^2 \left(\frac{\omega_{pb}}{\omega_{pa}} F\right)^4 \tag{9}$$

The function F can be found from Figs. II-13 and II-14.

We now have two expressions for the ac velocity of the plasma electrons, in terms of the beam drift velocity. If we assume that the system saturates because of the non-linear behavior of the beam, then Eqs. 8 and 9 allow us to estimate the maximum value of v_a. If we assume that $\omega_{pb}/\omega_{pa} = 0.1$, $V_{ob} = 10$ kv, $V_T = 100$ volts (effective temperature of plasma), then $(\omega_{pb}/\omega_{pa})$ F = 0.6, and $V_a = 0.36 \, \delta^2 V_{ob}$.

Equation 4 can be rewritten as

$$\frac{J_b}{J_{ob}} \left(\frac{\omega_{pb}}{\omega_{pa}} F\right) = \frac{v_b}{v_{ob}} \tag{10a}$$

or

$$a \left(\frac{\omega_{pb}}{\omega_{pa}} F\right) = \delta \tag{10b}$$

The absolute upper bound on a is 2 (perfect bunching of the beam into impulses). More practically, it may be approximately from 0.5 to 0.7, and this would make δ approximately 1/4. From this assortment of numbers, we estimate $V_a \approx 100$ volts.

c. Effect of Temperature on the Growth of Plasma Oscillations

The zero-temperature dispersion equation for electron-beam-plasma interaction is not valid near plasma resonance, partly on account of the neglect of the random energy of the electrons. If the plasma electrons are assumed to have random energy, a new dispersion equation that is valid at plasma resonance and contains an upper bound on the space-charge wave amplification constant can be derived.

The mathematical model used for the plasma and beam is exactly the same as that used for the zero-temperature case, except that the plasma electrons have a random velocity spread that is symmetric about the origin. Theoretically, the random fluctuations of the electron velocities require the introduction of a distribution function, in order for the electronic current and charge density to be evaluated. The distribution function f(v, z, t) must satisfy the Boltzmann equation, which is solved by using the perturbation method and assuming that first-order quantities vary with z and t as exp(jωt-Γz). We obtain the basic equations

$$f_1 = -\eta \frac{\frac{df_0(v)}{dv}}{j\omega - \Gamma v} E_1 \tag{11}$$

$$J_1 = -j\omega e \eta E_1 \int_v \frac{f_0(v)\, dv}{(j\omega - \Gamma v)^2} \tag{12}$$

($\eta = -e/m$), where f_1, J_1, and E_1 are the complex coefficients of the first-order perturbations of the distribution function, longitudinal current, and longitudinal electric field, respectively. If we combine Eqs. 11 and 12 with the sinusoidal steady-state wave equation relating J_1 and E_1, we obtain the integral form of the dispersion equation

$$\frac{\eta e}{\epsilon_0} \int_v \frac{f_0(v)\, dv}{(j\omega - \Gamma v)^2} = 1 \tag{13}$$

This equation, or its equivalent, has been obtained by several authors (3, 4). We shall now evaluate Eq. 13 for three equilibrium distribution functions $f_0(v)$.

i. Zero-temperature dispersion equation

As an example of the utility of Eq. 13, we derive the zero-temperature dispersion as a special case. The equilibrium distribution for beam and plasma is

$$f_0(v) = n_a \delta(v) + n_b \delta(v - v_{bo})$$

where $\delta(v)$ is a delta-function at $v = 0$. The densities n_a and n_b are assumed to be constant. From Eq. 13 we obtain directly

$$\Gamma = j\beta_e \pm j \frac{\beta_{pb}}{\left(1 - \frac{\omega_{pa}^2}{\omega^2}\right)^{1/2}}; \quad \beta_e = \frac{\omega}{v_{bo}}, \quad \beta_{pb} = \frac{\omega_{pb}}{v_{bo}}$$

The beam and plasma electron resonant frequencies are ω_{pb} and ω_{pa}, respectively.

ii. Triple-stream dispersion equation

We now split the plasma electrons into two beams with the same speeds but opposite directions. The equilibrium distribution is

$$f_0(v) = \frac{n_a}{2} \delta(v + v_a) + \frac{n_a}{2} \delta(v - v_a) + n_b \delta(v - v_{bo})$$

From Eq. 13 we obtain

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$$\left(\frac{\omega_{pa}}{\omega}\right)^2 \frac{1 + \frac{u^2}{3R}}{\left(1 - \frac{u^2}{3R}\right)^2} + \left(\frac{\omega_{pb}}{\omega}\right)^2 \frac{1}{(u-1)^2} = 1 \tag{14}$$

where $u = \Gamma/j\beta_e$ and $R = v_{bo}^2/3v_a^2$. The parameter R is the ratio of the time average kinetic energy of a beam electron to the total average random energy of a "plasma" electron, if we assume similar "random" motion in the x and y directions. In the experiment that is being considered, this ratio is equal to at least several hundred, therefore an expansion of the first term of Eq. 14 in powers of u^2/R seems justified. We have

$$\left(\frac{\omega_{pa}}{\omega}\right)^2 \left(1 + \frac{u^2}{R}\right) + \left(\frac{\omega_{pb}}{\omega}\right)^2 \frac{1}{(u-1)^2} = 1 \tag{15}$$

This dispersion equation is identical to that obtained by Boyd (3) for the case of a Maxwell-Boltzmann distribution function.

iii. Maxwell-Boltzmann dispersion equation

We assume that at equilibrium the plasma electrons have a uniform distribution in configuration space and a Maxwellian velocity distribution. Therefore

$$f_o(v) = n_b \delta(v - v_{bo}) + n_a \left(\frac{1}{2\pi v_p^2}\right)^{1/2} \exp\left(-\frac{v^2}{2v_p^2}\right)$$

where $v_p^2 = kT/m$. We obtain from Eq. 13

$$\left(\frac{\omega_{pb}}{\omega}\right)^2 \frac{1}{(u-1)^2} + \left(\frac{\omega_{pa}}{\omega}\right)^2 \left(\frac{1}{2\pi v_p^2}\right)^{1/2} \int_{-\infty}^{\infty} \frac{\exp\left(-\frac{v^2}{2v_p^2}\right)}{\left(1 - u\frac{v}{v_{bo}}\right)^2} dv = 1$$

The integral can be evaluated by contour integration, as demonstrated by Sumi (5). An alternative method, mentioned by Boyd, is to expand the denominator of the integrand in a power series in uv/v_{bo}, and integrate term by term. The resulting series, in powers of u^2/R, is

$$\left(\frac{\omega_{pb}}{\omega}\right)^2 \frac{1}{(u-1)^2} + \left(\frac{\omega_{pa}}{\omega}\right)^2 \left[1 + \frac{u^2}{R} + \frac{5}{3}\frac{u^4}{R^2} + \dots\right] = 1$$

and, to first-order, is identical with Eq. 15 if we equate v_a and v_p. Taking only the first two terms of the series, we obtain a normalized form for which graphical solutions
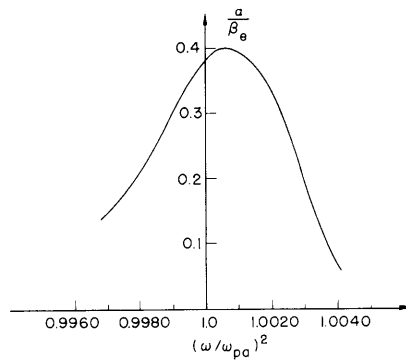
Fig. II-12. Amplification constant of reactive-medium amplifier versus $(\omega/\omega_{pa})^2$. $(R = 2500$, and $(\omega_{pb}/\omega_{pa})^2 = 10^{-4}$.)
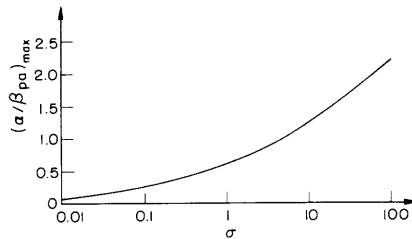
Fig. II-13. Maximum value of amplification constant (a/β_{pa}) versus σ, where $\sigma = R(\omega_{pb}/\omega_{pa})^2$. The constant R is the ratio of the beam voltage to the plasma electron temperature in volts. The function F, equivalent to $\left(1 - \omega_{pa}^2 / \omega^2\right)^{-1/2}$, is given by $F = (\omega_{pa}/\omega_{pb})(a/\beta_{pa})$.

Fig. II-14. Maximum value of amplification constant in cm^{-1} versus temperature of plasma electrons (V_T). (Beam velocity of 6×10^7 meters per second and plasma frequency $\omega_{pa} = 10^{11}$ rad/sec are assumed.)

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have been given by Boyd, Field, and Gould (4) in terms of the parameters σ and Λ:

$$(u-1)^2 (u^2+\Lambda) + \sigma = 0 \tag{16}$$

with

$$\sigma = R \left(\frac{\omega_{pb}}{\omega_{pa}} \right)^2$$

$$\Lambda = R \left(1 - \frac{\omega^2}{\omega_{pa}^2} \right)$$

$$R = \frac{1/2 \, mv_{bo}^2}{3/2 \, kT}$$

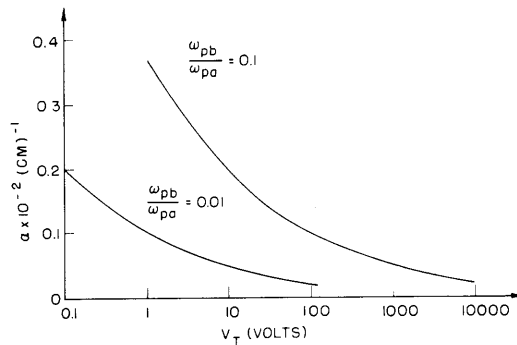The root of Eq. 16 that is of primary interest is in the first quadrant of the u-plane. Using Boyd's (3) data, we have plotted a graph of the imaginary part of this root $[-\alpha/\beta_e$, where $\alpha = Re(\Gamma)]$ as a function of frequency, for a particular value of R and σ (Fig. II-12). If the energy ratio is large, the peak of the graph occurs near $\omega = \omega_{pa}$. For R = 100, and σ in the range $0.001 < \sigma < 4.0$, the peak occurs within 2 per cent of ω_{pa}, and as R increases, the frequency for maximum growth approaches ω_{pa}. At this frequency, $\Lambda = 0$, and we can solve Eq. 16 directly. A graph of $(\alpha/\beta_{pa})_{max}$ as a function of σ is given in Fig. II-13.

The same data are plotted in Fig. II-14 for a specific beam velocity and plasma frequency, ω_{pa}. The large values of the growth constant (α) indicate that we might expect an appreciable growth of the initial velocity fluctuations within fractions of a centimeter.

L. D. Smullin, W. D. Getty

References

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4. G. D. Boyd, L. M. Field, and R. W. Gould, Interactions between an electron stream and an arc discharge plasma, Microwave Research Institute Symposium Series, Vol. VIII, Polytechnic Institute of Brooklyn, New York, 1958, p. 367.

5. M. Sumi, Theory of spatially growing plasma waves, J. Phys. Soc. (Japan) 14, 653 (May 1959).

2. PROPAGATION BETWEEN PARALLEL PLATES FILLED WITH A GYRO-ELECTRIC MEDIUM TRANSVERSE TO THE STATIC MAGNETIC FIELD

We shall consider the case of electromagnetic propagation between infinite parallel plates filled with a gyroelectric medium, in which the direction of propagation is transverse to the applied magnetic field. If we define a right-hand rectangular coordinate system x, y, z, and assume that the applied magnetic field is in the z-direction, the tensor permitivity of the medium is

$$\bar{\bar{\epsilon}} = \epsilon_o \begin{bmatrix} k_1 & jk_2 & 0 \\ -jk_2 & k_1 & 0 \\ 0 & 0 & k_3 \end{bmatrix}$$

where

$$k_1 = 1 - \frac{\omega_p^2}{\omega^2 - \omega_o^2}, \quad k_2 = \frac{\omega_p^2 \frac{\omega_c}{\omega}}{\omega^2 - \omega_c^2}, \quad k_3 = 1 - \frac{\omega_p^2}{\omega^2}$$

and ω is the operating frequency; ω_p, the plasma frequency; and ω_c, the cyclotron frequency; all are in radians per second.

Assuming time dependence of $\exp(j\omega t)$, and space dependence of the form $\exp(j\bar{\beta}\cdot\bar{r})$, we obtain from Maxwell's equations

$$-\beta^2 \bar{E} + \bar{\beta}(\beta_x E_x + \beta_y E_y + \beta_z E_z) + k^2 \bar{\bar{\kappa}} \cdot \bar{E} = 0 \tag{1}$$

where

$$\bar{\beta} = \bar{i}_x \beta_x + \bar{i}_y \beta_y + \bar{i}_z \beta_z, \quad \beta^2 = \bar{\beta} \cdot \bar{\beta}$$

$$\bar{r} = \bar{i}_x x + \bar{i}_y y + \bar{i}_z z$$

$$k^2 = \omega^2 \mu_o \epsilon_o$$

$$\bar{\bar{\kappa}} = \frac{1}{\epsilon_o} \bar{\bar{\epsilon}}$$

Equation 1 may be conveniently expressed in matrix form:

$$
\begin{bmatrix}
-\beta^2 + \beta_x^2 + k^2 k_1 & \beta_x \beta_y + jk_2 k^2 & \beta_x \beta_z \\
\beta_x \beta_y - jk_2 k^2 & -\beta^2 + \beta_y^2 + k^2 k_1 & \beta_y \beta_z \\
\beta_x \beta_z & \beta_y \beta_z & -\beta^2 + \beta_z^2 + k^2 k_3
\end{bmatrix}
\begin{bmatrix}
E_x \\
E_y \\
E_z
\end{bmatrix}
=
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\tag{2}
$$

We now consider several special cases. Figure II-15 shows plates parallel to the z-axis and separated by distance d. We consider propagation in the x-direction with no variation in the z-direction $(\beta_z = 0)$.

In this case, matrix 2 reduces to

$$
\begin{bmatrix}
k^2 k_1 - \beta_y^2 & \beta_x \beta_y + jk_2 k^2 & 0 \\
\beta_x \beta_y - jk_2 k^2 & k^2 k_1 - \beta_x^2 & 0 \\
0 & 0 & k^2 k_3 - \beta_x^3 - \beta_y^2
\end{bmatrix}
\tag{3}
$$

We observe from matrix 3 that E_z is uncoupled from E_x and E_y.

(a) TE Modes

If $E_x = E_y = 0$, we can obtain a nonzero E_z if

$$
\beta_x^2 + \beta_y^2 = k_3 k^2
\tag{4}
$$

E_z must vanish on the conducting plates at $y = 0$, $y = d$. Thus E_z must have a y dependence of the form $\sin\left(\frac{n\pi}{d} y\right)$, where n is a positive integer. This implies that $\beta_y = \pm \frac{n\pi}{d}$. Thus, for this set of modes to exist, we have

$$
\beta_x^2 = k_3 k^2 - \left(\frac{n\pi}{d}\right)^2
\tag{5}
$$

For unattenuated propagation, β_x is real. This implies that $k_3 k^2 > \left(\frac{n\pi}{d}\right)^2$, and that $k_3 > 0$, or $\omega > \omega_p$. Here, the cutoff frequency for the nth mode is $\sqrt{k_3}\, k = \frac{n\pi}{d}$. This case is completely analogous to the TE_{no} modes between air-filled parallel plates.

(b) TM Modes

With reference to matrix 3, we may have a nontrivial solution for E_x and E_y if the 2×2 submatrix relating them has its determinant equal to zero. This implies that

$$
\beta_x^2 + \beta_y^2 = k_1 k^2 \left[1 - \left(\frac{k_2}{k_1}\right)^2 \right]
\tag{6}
$$
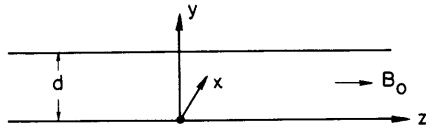
Fig. II-15. Plates parallel to z-axis; propagation in x-direction.
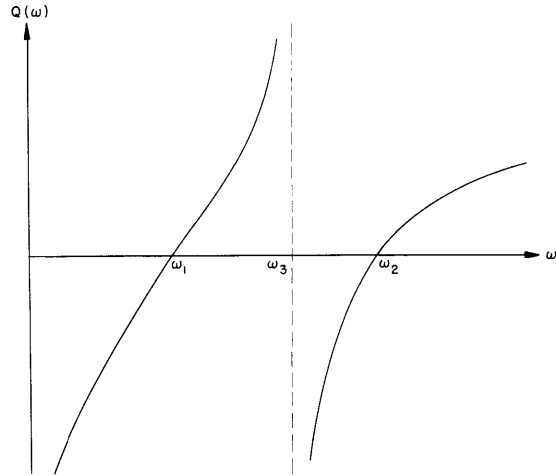
$$\omega_1 = \left(\omega_p^2 + \left(\frac{\omega_c}{2}\right)^2\right)^{1/2} - \frac{\omega_c}{2}$$

$$\omega_3 = \left(\omega_p^2 + \omega_c^2\right)^{1/2}$$

$$\omega_2 = \left(\omega_p^2 + \left(\frac{\omega_c}{2}\right)^2\right)^{1/2} + \frac{\omega_c}{2}$$

Fig. II-16. Plot of parameter Q against frequency ω.

Fig. II-17. Plates perpendicular to z-axis; propagation in y-direction.

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Since E_x must vanish at $y = 0$ and $y = d$, β_y must equal $\pm \frac{n\pi}{d}$. This implies that

$$\beta_x^2 = k_1 k^2 \left[1 - \left(\frac{k_2}{k_1} \right)^2 \right] - \left(\frac{n\pi}{d} \right)^2 \tag{7}$$
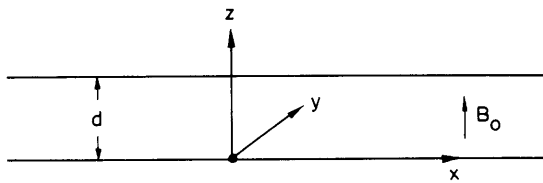
A propagating mode can only occur if β_x is real. For this to be so, the quantity Q must be positive, where

$$Q = k_1 \left[1 - \left(\frac{k_2}{k_1} \right)^2 \right] \tag{8}$$

Let us determine when Q changes sign. It will do so when $Q = 0$, and when Q has a singularity. $Q = 0$ when $k_1 = \pm k_2$, and this occurs at frequencies

$$\omega_1 = \left(\omega_p^2 + \left(\frac{\omega_c}{2} \right)^2 \right)^{1/2} - \left(\frac{\omega_c}{2} \right), \quad \omega_2 = \left(\omega_p^2 + \left(\frac{\omega_c}{2} \right)^2 \right)^{1/2} + \left(\frac{\omega_c}{2} \right)$$

Q has a singularity when $k_1 = 0$. This occurs at $\omega_3 = \left(\omega_p^2 + \omega_c^2 \right)^{1/2}$. Clearly, $\omega_1 < \omega_3 < \omega_2$. Knowing that $Q(0) \rightarrow -\infty$ and $Q(\infty) = 1$, we may sketch the function $Q(\omega)$ (Fig. II-16).

Referring to Eq. 7, we observe that only for $\omega_1 < \omega < \omega_3$ and $\omega > \omega_2$ is a propagating mode (β_x real) possible. In this frequency range, the lowest mode cutoff condition is $\sqrt{Q} k = \pi/d$. The cutoff condition for the nth mode is $\sqrt{Q} k = \pi n/d$.

(c) B_o Normal to Plates

We now consider the case shown in Fig. II-17, in which the magnetic field is normal to the plates. We consider propagation in the y-direction with no variation in the x-direction ($\beta_x = 0$). In this case matrix 2 reduces to

$$\begin{bmatrix} -\beta_y^2 - \beta_z^2 + k_1 k^2 & jk_2 k^2 & 0 \\ -jk_2 k^2 & -\beta_z^2 + k_1 k^2 & \beta_y \beta_z \\ 0 & \beta_y \beta_z & -\beta_y^2 + k_3 k^2 \end{bmatrix} \tag{9}$$

We observe from matrix 9 that E_x, E_y, and E_z are coupled. For a mode to exist, the determinant of matrix 9 must equal zero. This condition, after simplification, yields

$$k_1 Y^2 + (k_1 + k_3) YZ + k_3 Z^2 - \left(k_1^2 + k_1 k_3 - k_2^2 \right) Y - 2k_1 k_3 Z + k_3 \left(k_1^2 - k_2^2 \right) = 0 \tag{10}$$

where

$$Y = \frac{\beta_y^2}{k^2}, \quad Z = \frac{\beta_z^2}{k^2}$$

For any modes to exist,

$$\beta_z = \pm \frac{n\pi}{d}, \quad Z = \frac{n^2\pi^2}{d^2 k^2} > 0$$

If we restrict our attention to a particular mode (n fixed), an assumed positive value of Z specifies the dimensionless parameter $kd = \omega d(\mu_o \epsilon_o)^{1/2}$.

Equation 10 is quadratic in Y with coefficients that are functions of the frequency-dependent quantities k_1, k_2, and k_3, and of the parameter Z. To investigate the existence of propagating modes at a particular frequency, we calculate k_1, k_2, k_3 at that frequency (we assume that we know ω_p and ω_c) and solve for Y as a function of Z. If there is a range of $Z > 0$ for which one or both roots of Eq. 10 is positive, then there are propagating modes.

A somewhat more restricted, but interesting, result is found from the cutoffs for $\beta_y = 0$. In this case, $Y = 0$, which implies, from Eq. 10, that

$$Z^2 - 2k_1 Z + k_1^2 - k_2^2 = 0 \tag{11}$$

The roots of this equation are $Z = k_1 \pm k_2$, which corresponds to

$$Z = \left(\frac{\omega d}{c}\right)^2 = \frac{(n\pi)^2}{k_1 \pm k_2}$$

However, we know already that $k_1 = -k_2$ at $\omega = \omega_1$, and $k_1 = k_2$ at $\omega = \omega_2$. An examination of the characteristics of k_1 and k_2 versus ω shows that for $\omega_1 < \omega < \omega_2$, $k_1 + k_2 > 0$. For $\omega < \omega_1$ and $\omega > \omega_2$, $k_1 - k_2 > 0$. This set of conditions suggests that a propagating mode can exist at all frequencies for a properly chosen d. This is different from the other cases considered, in which it was found that in some frequency ranges no propagating mode was possible.

B. Reiffen

3. PROPAGATION IN PLASMA WAVEGUIDES

The study of propagation in plasma-loaded waveguides continues. In the past, this problem has been studied with the assumption that the motion of the plasma ions was negligible compared with that of the plasma electrons (1, 2, 3). We now include the effect of the plasma ions.

For the cylindrical system of Fig. II-18, we assume quasi-static modes that are radially symmetric, and we arrive at the determinantal and boundary-matching equations

$$\beta_z^2 \left[\frac{\omega_{pi}^2}{\omega^2} + \frac{\omega_{pe}^2}{\omega^2} - 1 \right] = \beta_r^2 \left[1 + \frac{\omega_{pi}^2}{\omega_{ci}^2 - \omega^2} + \frac{\omega_{pe}^2}{\omega_{ce}^2 - \omega^2} \right] \tag{1}$$

$$\beta_r b \frac{J_1(\beta_r b)}{J_0(\beta_r b)} \left[1 + \frac{\omega_{pi}^2}{\omega_{ci}^2 - \omega^2} + \frac{\omega_{pe}^2}{\omega_{ce}^2 - \omega^2} \right] = -\beta_z b \frac{I_1(\beta_z b) + \dfrac{I_0(\beta_z a)}{K_0(\beta_z a)} K_1(\beta_z b)}{I_0(\beta_z b) - \dfrac{I_0(\beta_z a)}{K_0(\beta_z a)} K_0(\beta_z b)} \tag{2}$$

The quantities ω_{pi} and ω_{ci} are, respectively, the ion plasma and cyclotron frequency, and ω_{pe} and ω_{ce} are, respectively, the electron plasma and cyclotron frequency. The simultaneous solution of Eqs. 1 and 2 gives the dispersion relation, β_z, as a function of ω. The propagation constant β_z may be pure real or pure imaginary. In the second case, we have attenuation rather than propagation. The radial wave number β_r may be pure real (body waves) or pure imaginary (surface waves).
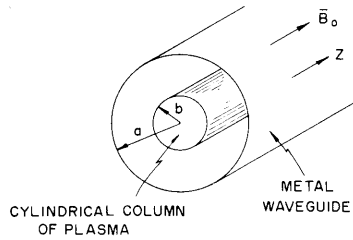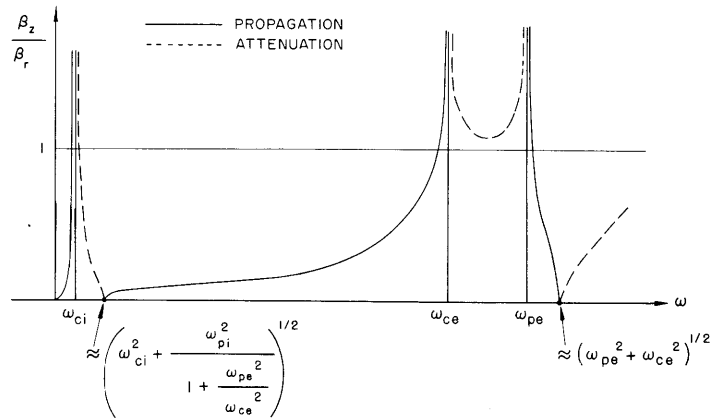

Fig. II-18.   Geometry of system.

Fig. II-19.   Dispersion in plasma-filled waveguide.

For the very special case of the completely filled waveguide, that is, with a = b, $\beta_r b$ is constant with frequency, and assumes the discrete values of the roots of $J_0(\beta_r b) = 0$. A typical dispersion characteristic for this case is sketched qualitatively in Fig. II-19. For frequencies much higher than ω_{pi} and ω_{ci}, the dispersion relation becomes independent of the ions.

The case of the partially filled waveguide is being explored and will be reported on later.

P. Chorney

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4. COVARIANT FORMULATION FOR WAVE MOTIONS IN A PLASMA

Small-amplitude wave motions of the electrons in an unbounded, homogeneous plasma in a constant external magnetic field have been studied for situations in which relativistic electrons may be present in the plasma. The effect of close encounters between particles has not been included in this study. The problem is formulated by use of a relativistic generalization of the collision-free Boltzmann equation, which is coupled to the electromagnetic field equations. These coupled equations are reduced to a single integrodifferential equation for the perturbation in the distribution function by expressing the fields in terms of this function. The equation is linearized by neglecting quantities of second order in the perturbation. From this linearized equation the dispersion relations for plane waves in the plasma are deduced. These are then applied to a plasma in thermal equilibrium at a high temperature at which the random electron velocities are relativistic, and to a low-temperature plasma penetrated by a relativistic plasma beam.

a. Covariant Form of the Boltzmann Equation

The usual form of the Boltzmann equation is applicable only when particles with non-relativistic velocities are considered. This is the result of using Newtonian mechanics to describe the dynamics of the particles when the Boltzmann equation is derived. It is a simple matter to extend it to the relativistic domain by using relativistic

mechanics (1, 2, 3). A little care must be taken, however, in defining densities, volumes, and so forth, to conform to the relativistic space-time geometry. The best way to do this is to use a completely covariant formulation, that is, to write the equations in such a manner that their form is independent of the particular choice of coordinate system in which they are written. To do this, it is convenient to introduce the following variables.

The spatial coordinates are denoted by

$$x_i = (x_1, x_2, x_3) \tag{1}$$

where i and subsequent Roman indices run from 1 to 3. The space-time coordinates are represented by

$$x_\alpha = (x_1, x_2, x_3, ict) \tag{2}$$

where α and all Greek indices run from 1 to 4. The proper time, or arc length in four-dimensional space-time is found from

$$c^2 d\tau^2 = -dx_\alpha dx_\alpha = c^2 dt^2 - dx_1^2 - dx_2^2 - dx_3^2 \tag{3}$$

The four-velocity of a particle is defined by

$$u_\alpha = \frac{dx_\alpha}{d\tau} \tag{4}$$

The sum of the squares of its spatial components is denoted by

$$u_i u_i \equiv u^2 \tag{5}$$

In terms of u^2, $d\tau$ may be related to dt by

$$dt = d\tau \left(1 + \frac{dx_1^2 + dx_2^2 + dx_3^2}{c^2 d\tau^2}\right)^{1/2} = d\tau \left(1 + \frac{u^2}{c^2}\right)^{1/2} \tag{6}$$

The four-velocity may therefore be written in terms of its first three components as

$$u_\alpha = \left(u_1, u_2, u_3, ic[1+u^2/c^2]^{1/2}\right) \tag{7}$$

The invariant volume element in u_i-space is

$$dU = \frac{du_1 du_2 du_3}{(1+u^2/c^2)^{1/2}} \tag{8}$$

The distribution function $f(x_\alpha, u_\alpha)$ is defined to be a function of the eight

variables x_α, u_α. For a physical particle only three of the four velocity components, say u_i, are independent. Nevertheless, as we shall see, it is convenient to define f as a function of the four components, u_α. The value of the distribution function on the hypersurface $u_4 = ic(1+u^2/c^2)^{1/2}$ in four-dimensional velocity space is denoted by

$$\widetilde{f}(x_\alpha, u_i) = f\left(x_\alpha, u_i, ic[1+u^2/c^2]^{1/2}\right) \tag{9}$$

and $\widetilde{f}$ is required to be an invariant, equal to the density of particles in the six-dimensional space x_i, u_i at time t. The proper density of particles in configuration space at time t may be found in terms of the quantities in Eqs. 8 and 9. It is

$$n(x_\alpha) = \iiint \widetilde{f}(x_\alpha, u_i) \, dU \tag{10}$$

The average of any function $h(x_\alpha)$ over velocity space is

$$\left\langle h(x_\alpha) \right\rangle = \frac{\iiint \widetilde{f}(x_\alpha, u_i) \, \widetilde{h}(u_i) \, dU}{\iiint \widetilde{f}(x_\alpha, u_i) \, dU} \tag{11}$$

where a tilde over any function of u_α denotes its value in the subspace $u_4 = ic(1+u^2/c^2)^{1/2}$. For convenience, the tilde will be deleted hereafter when it appears in three-dimensional velocity integrals since, in this case, the integrand must always be taken at $u_4 = ic(1+u^2/c^2)^{1/2}$.

It is also possible to write these averages as integrals over the whole of the four-dimensional velocity space, by observing that for any function $\psi(u_\alpha)$,

$$\widetilde{\psi}(u_i) = \int_0^{i\infty} \psi(u_\alpha) \, \delta\left(u_4 - ic[1+u^2/c^2]^{1/2}\right) du_4 \tag{12}$$

Multiplying by dU and integrating over u_i, we obtain

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \widetilde{\psi}(u_i) \, \frac{du_1 du_2 du_3}{(1+u^2/c^2)^{1/2}}$$

$$= \int_0^{i\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \psi(u_\alpha) \, \frac{\delta\left(u_4 - ic[1+u^2/c^2]^{1/2}\right)}{(1+u^2/c^2)^{1/2}} \, du_1 du_2 du_3 du_4$$

$$= \int_0^{i\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \psi(u_\alpha) \, 2c \, \delta\left(u_\alpha u_\alpha + c^2\right) du_1 du_2 du_3 du_4 \tag{13}$$

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so that $\left\langle h(x_\alpha) \right\rangle$ may be expressed as

$$\left\langle h(x_\alpha) \right\rangle = \frac{\int_0^{i\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(x_\alpha, u_\alpha) \, f(x_\alpha, u_\alpha) \, 2c\delta\left(u_\alpha u_\alpha + c^2\right) du_1 du_2 du_3 du_4}{\int_0^{i\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x_\alpha, u_\alpha) \, 2c\delta\left(u_\alpha u_\alpha + c^2\right) du_1 du_2 du_3 du_4} \tag{14}$$

and $f(x_\alpha, u_\alpha) \cdot 2c\delta\left(u_\alpha u_\alpha + c^2\right)$ is the density of particles at t in x_i, u_α-space. That is, $f(x_\alpha, u_\alpha) \, 2c\delta\left(u_\alpha u_\alpha + c^2\right) dx_1 dx_2 dx_3 du_1 du_2 du_3 du_4$ is the number of particles at time t with spatial coordinates in the interval $(x_i, x_i + dx_i)$ and with four-velocity components in the interval $(u_\alpha, u_\alpha + du_\alpha)$.

After a short interval of time, δt, these particles will be located with spatial coordinates in the range $(x_i + \delta x_i, x_i + \delta x_i + d\{x_i + \delta x_i\})$, and velocity in the range $(u_\alpha + \delta u_\alpha, u_\alpha + \delta u_\alpha + d\{u_\alpha + \delta u_\alpha\})$. The change in position is given by

$$\delta x_i = u_i \delta\tau \tag{15}$$

The force equation for a relativistic particle is

$$F_\alpha = m \frac{du_\alpha}{d\tau} \tag{16}$$

where F_α is the four-vector force, and m the rest mass of the particle. Therefore,

$$\delta u_\alpha = \frac{F_\alpha}{m} \delta\tau \tag{17}$$

Since the number of particles under consideration is conserved during the interval $\delta\tau$,

$$f(x_\alpha, u_\alpha) \, dx_1 dx_2 dx_3 du_1 du_2 du_3 du_4 = f(x_\alpha + \delta x_\alpha, u_\alpha + \delta u_\alpha) \, d(x_1 + \delta x_1) \ldots d(u_4 + \delta u_4) \tag{18}$$

where the factor $2c\delta\left(u_\alpha u_\alpha + c^2\right)$ appearing in the density of particles in x_i, u_α-space is an invariant, and has been divided out of Eq. 18. The two seven-dimensional volume elements are related by

$$d(x_1 + \delta x_1) \ldots d(u_4 + \delta u_4) = J \, dx_1 \ldots du_4 \tag{19}$$

in which J is the Jacobian of the transformation from t to t + δt, and hence the distributions are related by

$$f(x_\alpha + \delta x_\alpha, u_\alpha + \delta u_\alpha) = \frac{1}{J} f(x_\alpha, u_\alpha) \tag{20}$$

The Jacobian is

$$J = \frac{\partial(x_1 + \delta x_1, \ldots u_4 + \delta u_4)}{\partial(x_1, \ldots u_4)}$$

$$= \begin{vmatrix} \dfrac{\partial(x_i + \delta x_i)}{\partial x_j} & \dfrac{\partial(x_i + \delta x_i)}{\partial u_\beta} \\[2em] \dfrac{\partial(u_\alpha + \delta u_\alpha)}{\partial x_j} & \dfrac{\partial(u_\alpha + \delta u_\alpha)}{\partial u_\beta} \end{vmatrix}$$

$$= \begin{vmatrix} \dfrac{\partial x_i}{\partial x_j} + \dfrac{\partial}{\partial x_j}(u_i \delta\tau) & \dfrac{\partial x_i}{\partial u_\beta} + \dfrac{\partial}{\partial u_\beta}(u_i \delta\tau) \\[2em] \dfrac{\partial u_\alpha}{\partial x_j} + \dfrac{\partial}{\partial x_j}\left(\dfrac{F_\alpha}{m}\delta\tau\right) & \dfrac{\partial u_\alpha}{\partial u_\beta} + \dfrac{\partial}{\partial u_\beta}\left(\dfrac{F_\alpha}{m}\delta\tau\right) \end{vmatrix}$$

$$= \begin{vmatrix} \delta_{ij} & \delta_{i\beta}\,\delta\tau \\[2em] \dfrac{1}{m}\dfrac{\partial F_\alpha}{\partial x_j}\,\delta\tau & \delta_{\alpha\beta} + \dfrac{1}{m}\dfrac{\partial F_\alpha}{\partial u_\beta}\,\delta\tau \end{vmatrix}$$

$$= 1 + \frac{1}{m}\frac{\partial F_\alpha}{\partial u_\alpha}\,\delta\tau + 0(\delta\tau^2) \tag{21}$$

Therefore, to first order in $\delta\tau$, we have

$$f(x_\alpha + \delta x_\alpha, u_\alpha + \delta u_\alpha) = f(x_\alpha, u_\alpha) - \frac{1}{m}\frac{\partial F_\alpha}{\partial u_\alpha}f(x_\alpha, u_\alpha)\,\delta\tau \tag{22}$$

Making an expansion of $f(x_\alpha + \delta x_\alpha, u_\alpha + \delta u_\alpha)$ to first order in $\delta\tau$ about (x_α, u_α) yields

$$f(x_\alpha + \delta x_\alpha, u_\alpha + \delta u_\alpha) = f(x_\alpha, u_\alpha) + \frac{\partial f}{\partial x_\alpha}\delta x_\alpha + \frac{\partial f}{\partial u_\alpha}\delta u_\alpha = f(x_\alpha, u_\alpha) + u_\alpha \frac{\partial f}{\partial x_\alpha}\delta\tau + \frac{F_\alpha}{m}\frac{\partial f}{\partial u_\alpha}\delta\tau \tag{23}$$

Subtracting Eq. 23 from Eq. 22 yields

$$0 = u_\alpha \frac{\partial f}{\partial x_\alpha}\,\delta\tau + \frac{F_\alpha}{m}\frac{\partial f}{\partial u_\alpha}\,\delta\tau + \frac{f}{m}\frac{\partial F_\alpha}{\partial u_\alpha}\,\delta\tau \tag{24}$$

or, dividing out $\delta\tau$ and rearranging results in

$$u_\alpha \frac{\partial f}{\partial x_\alpha} + \frac{1}{m} \frac{\partial}{\partial u_\alpha} (F_\alpha f) = 0 \tag{25}$$

This is the covariant generalization of the Boltzmann equation. For forces F_α of electromagnetic origin, the force is related to the electromagnetic field by

$$F_\alpha = q\, u_\beta\, F_{\alpha\beta} \tag{26}$$

where q is the charge on the particle suffering the force, and $F_{\alpha\beta}$ is the antisymmetric field tensor:

$$F_{\alpha\beta} = \begin{bmatrix} 0 & -B_3 & B_2 & \dfrac{E_1}{c} \\[2mm] B_3 & 0 & -B_1 & \dfrac{E_2}{c} \\[2mm] -B_2 & B_1 & 0 & \dfrac{E_3}{c} \\[2mm] -\dfrac{E_1}{c} & -\dfrac{E_2}{c} & -\dfrac{E_3}{c} & 0 \end{bmatrix} \tag{27}$$

Since $F_{\alpha\beta}$ is a function only of x_α,

$$\frac{\partial F_\alpha}{\partial u_\alpha} = q\, F_{\alpha\beta} \frac{\partial u_\beta}{\partial u_\alpha} = q\, F_{\alpha\beta}\, \delta_{\alpha\beta} = 0 \tag{28}$$

because of the antisymmetry of the field tensor. For electromagnetic forces, the Boltzmann equation may therefore be written

$$u_\alpha \frac{\partial f}{\partial x_\alpha} + \frac{F_\alpha}{m} \frac{\partial f}{\partial u_\alpha} = 0 \tag{29}$$

b.   Small Perturbations of a Plasma in a Magnetic Field

The characteristics of small-amplitude perturbations of the electrons in a homogeneous, unbounded plasma in a constant external magnetic field will now be studied. The perturbations are assumed to be plane waves, $\exp[i(k \cdot x - \omega t)]$. The dispersion relations, that is, the relations between the wave number k and the frequency ω for plane waves, will be found for all such waves that may propagate in the plasma. From these dispersion relations, all characteristics of arbitrary disturbances of the plasma can be determined.

The four-force in the Boltzmann equation is expressible in terms of the electromagnetic field present in the plasma. This, in turn, is related to the perturbation in

the motion of the plasma electrons, which is actually the source of the field. Hence it is possible to eliminate the field from the Boltzmann equation by writing them in terms of the distribution function. The resulting equation for the distribution function is then linearized by assuming that the perturbations are small, and hence terms in the equation that are smaller than first order in the perturbation are dropped. The equation is an integrodifferential equation. It is written in cylindrical coordinates, and through use of an integrating factor is converted to an integral equation. By taking the three velocity moments of this equation, a set of three homogeneous algebraic equations in the perturbations of the three velocity moments is obtained. The coefficients in these equations are known. The vanishing of their determinant leads to the dispersion relations.

The distribution function is written

$$f(x_\alpha, u_\alpha) = f^O(u_\alpha) + f^1(x_\alpha, u_\alpha) = f^O(u_\alpha) + g(u_\alpha) \exp(ik_\beta x_\beta) \tag{30}$$

where f^O is independent of x_α because the undisturbed state of the plasma is homogeneous and time-independent. The perturbation distribution function f^1 is factored into a function g of velocity and a function of space-time $\exp(ik_\alpha x_\alpha)$, with

$$k_\alpha = (k_1, k_2, k_3, i\omega/c) \tag{31}$$

which represents a plane wave.

Similarly, the field tensor is broken into the tensor $F^O_{\alpha\beta}$ in the undisturbed plasma, and $F^1_{\alpha\beta}$ caused by the perturbation.

$$F_{\alpha\beta} = F^O_{\alpha\beta} + F^1_{\alpha\beta} \tag{32}$$

The field in the undisturbed plasma is a magnetic field in the 3-direction, so that

$$F^O_{\alpha\beta} = \begin{bmatrix} 0 & -B & 0 & 0 \\ B & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{33}$$

The perturbation field tensor may be written in terms of the four-vector potential, A^1_α as

$$F^1_{\alpha\beta} = \frac{\partial A^1_\beta}{\partial x_\alpha} - \frac{\partial A^1_\alpha}{\partial x_\beta} \tag{34}$$

where the potential is expressible in terms of the perturbation current, J^1_α, as

$$A_\alpha^1(x_\nu) = \frac{\mu_0}{4\pi} \int d\vec{x'} \frac{J_\alpha^1\left(\vec{x'}, t - \frac{|\vec{x} - \vec{x'}|}{c}\right)}{|\vec{x} - \vec{x'}|} \tag{35}$$

where the integral is taken over all configuration space, and J_α^1 is related to f^1 by

$$J_\alpha^1(x_\nu) = -e \int u_\alpha f^1(x_\nu, u_\nu) dU = -e \exp(ik_\nu x_\nu) \int u_\alpha g \, dU \tag{36}$$

Substituting this into the expression for A_α^1 yields

$$A_\alpha^1(x_\nu) = -\frac{e\mu_0}{4\pi} \cdot \int u_\alpha g \, dU \cdot \int d\vec{x'} \frac{e^{i\left[k \cdot \vec{x'} - \omega\left(t - \frac{|\vec{x} - \vec{x'}|}{c}\right)\right]}}{|\vec{x} - \vec{x'}|} \tag{37}$$

Carrying out the spatial integration yields

$$A_\alpha^1(x_\nu) = -\frac{e\mu_0}{k^2 - \frac{\omega^2}{c^2}} \exp(ik_\nu x_\nu) \int u_\alpha g \, dU \tag{38}$$

and the field tensor becomes

$$F_{\alpha\beta}^1 = -\frac{ie\mu_0}{k^2 - \frac{\omega^2}{c^2}} \exp(ik_\nu x_\nu) \int (k_\alpha u_\beta - k_\beta u_\alpha) g \, dU \tag{39}$$

With the use of the definitions given in Eqs. 30 and 32 the Boltzmann equation for the unperturbed plasma becomes

$$-\frac{e}{m} u_\beta F_{\alpha\beta}^0 \frac{\partial f^0}{\partial u_\alpha} = 0 \tag{40}$$

and the equation for the perturbed quantities becomes

$$u_\alpha \frac{\partial f^1}{\partial x_\alpha} - \frac{e}{m} u_\beta F_{\alpha\beta}^0 \frac{\partial f^1}{\partial u_\alpha} - \frac{e}{m} u_\beta F_{\alpha\beta}^1 \frac{\partial f^0}{\partial u_\alpha} = 0 \tag{41}$$

in which the term $u_\beta F_{\alpha\beta}^1 \left(\partial f^1 / \partial u_\alpha\right)$ has been dropped because it is of second order in the small perturbation. With the use of Eqs. 30 and 39, the first-order Boltzmann equation finally becomes

$$k_\alpha u_\alpha g + \frac{ie}{m} u_\beta F_{\alpha\beta}^0 \frac{\partial g}{\partial u_\alpha} + \frac{e^2 \mu_0/m}{k^2 - \frac{\omega^2}{c^2}} u_\beta \frac{\partial f^0}{\partial u_\alpha} \int (k_\alpha u_\beta' - k_\beta u_\alpha') g \, dU = 0 \tag{42}$$

It is now convenient to introduce cylindrical coordinates (ρ, ϕ, u_3) in velocity space with the axis of symmetry along the magnetic field.

In these coordinates

$$u_\beta F^o_{\alpha\beta} \frac{\partial f}{\partial u_\alpha} = B\left(u_1 \frac{\partial f}{\partial u_2} - u_2 \frac{\partial f}{\partial u_1}\right) = B \frac{\partial f}{\partial \phi} \tag{43}$$

Substituting Eq. 43 in Eq. 40 yields

$$\frac{\partial f^o}{\partial \phi} = 0 \tag{44}$$

or, equivalently, the zero-order distribution function must be independent of ϕ. Substituting Eq. 43 in Eq. 42 leads to

$$\Omega_c \frac{\partial g}{\partial \phi} - ik_\alpha u_\alpha g - \frac{ie^2 \mu_o/m}{k^2 - \frac{\omega^2}{c^2}} u_\beta \frac{\partial f^o}{\partial u_\alpha} \int (k_\alpha u'_\beta - k_\beta u'_\alpha)\, g dU = 0 \tag{45}$$

where $\Omega_c = eB/m$ is the cyclotron frequency. This equation is a first-order differential equation with respect to the variable ϕ. By using the integrating factor $\exp\left(\frac{i}{\Omega_c} \int k_\alpha u_\alpha d\phi\right)$ it may be converted to

$$g(\rho, \phi, u_3, u_4) = \frac{ie^2 \mu_o/m\Omega_c}{k^2 - \frac{\omega^2}{c^2}} \exp\left(\frac{i}{\Omega_c} \int k_\alpha u_\alpha d\phi\right)\left[\Phi(\rho, u_3, u_4)\right.$$

$$\left. + \int e^{-\frac{i}{\Omega_c} \int k_\alpha u_\alpha d\phi} u_\beta \frac{\partial f^o}{\partial u_\alpha} \int (k_\alpha u'_\beta - k_\beta u'_\alpha)\, g(u'_\nu)\, dU' \cdot d\phi\right] \tag{46}$$

where Φ is an arbitrary function independent of ϕ. The wave is assumed to propagate in the 1-3 plane with no loss in generality, since f^o is independent of ϕ, so that we may take

$$k_\alpha = (k_t, 0, k_3, i\omega/c) \tag{47}$$

By using Eq. 47, the integrating factor becomes $\exp\left\{\frac{i}{\Omega_c}\left[k_t\rho \sin\phi + k_3 u_3 - \omega(1+u^2/c^2)^{1/2}\right]\right\}$. With the aid of the Bessel-function expansion

$$\exp\left(\frac{ik_t\rho}{\Omega_c} \sin\phi\right) = \sum_{n=-\infty}^{\infty} \exp(in\phi)\, J_n\left(\frac{\rho k_t}{\Omega_c}\right) \tag{48}$$

the integral in Eq. 46 can be evaluated. The right-hand side of Eq. 46, after some

manipulation, can be written as a linear combination of the three quantities $\int u_i g dU$, in which the coefficients are known functions. Multiplying this equation successively by the three components u_j, and integrating over velocity, yields the three equations

$$\int u_j g dU = \frac{e^2/\epsilon_o m}{\omega^2 - k^2 c^2} M_{ij} \int u_i g dU \tag{49}$$

In terms of the quantities

$$F \equiv \rho k_3 \frac{\partial \tilde{f}^o}{\partial u_3} - \left(k_3 u_3 - \omega[1+u^2/c^2]^{1/2}\right) \frac{\partial \tilde{f}^o}{\partial \rho}$$

$$D \equiv k_3 u_3 - \omega(1+u^2/c^2)^{1/2}$$

$$E \equiv u_3 - \frac{k_3}{\omega}(1+u^2/c^2)^{1/2}$$

$$H \equiv k_3 u_3 + \frac{k_t^2 c^2 - \omega^2}{\omega}(1+u^2/c^2)^{1/2}$$

the elements M_{ij} can be expressed as

$$M_{ij} = \int_{-\infty}^{\infty} \int_{0}^{\infty} m_{ij} \frac{\rho d\rho du_3}{(1+u^2/c^2)^{1/2}} \tag{50}$$

where the nine quantities m_{ij} are

$$m_{11} = -\frac{4\pi\Omega_c^2}{k_t^2} \frac{HF}{\rho} \Sigma$$

$$m_{21} = -\frac{4\pi i \Omega_c^2}{k_t} \frac{HF}{D} \Sigma'$$

$$m_{31} = \frac{4\pi\Omega_c^2}{k_t} \frac{u_3 HF}{\rho D} \Sigma - 2\pi \frac{k_3 k_t c^2}{\omega} \frac{\partial \tilde{f}^o}{\partial u_3} \frac{u_3(1+u^2/c^2)^{1/2}}{D}$$

$$m_{12} = \frac{4\pi i \Omega_c^2}{k_t} F \Sigma'$$

$$m_{22} = -\pi\rho \frac{F}{D} \left[4\Omega_c^2 \Sigma'' - 1\right]$$

$$m_{32} = -4\pi i \Omega_c^2 \frac{u_3 F}{D} \Sigma'$$

$$m_{13} = \frac{4\pi \Omega_c^2}{k_t} \frac{EF}{\rho} \Sigma$$

$$m_{23} = 4\pi i \Omega_c^2 \frac{EF}{D} \Sigma'$$

$$m_{33} = -4\pi \Omega_c^2 \frac{u_3 EF}{\rho D} \Sigma + 2\pi \frac{\omega^2 - k_3^2 c^2}{\omega} \frac{\partial \tilde{f}^o}{\partial u_3} \frac{u_3 (1+u^2/c^2)^{1/2}}{D}$$

in which

$$\Sigma = \sum_{n=1}^{\infty} \frac{n^2 J_n^2 \left(\frac{\rho k_t}{\Omega_c}\right)}{n^2 \Omega_c^2 - D^2}$$

$$\Sigma' = \sum_{n=1}^{\infty} \frac{n^2 J_n \left(\frac{\rho k_t}{\Omega_c}\right) J_n' \left(\frac{\rho k_t}{\Omega_c}\right)}{n^2 \Omega_c^2 - D^2}$$

$$\Sigma'' = \sum_{n=1}^{\infty} \frac{n^2 J_n'^2 \left(\frac{\rho k_t}{\Omega_c}\right)}{n^2 \Omega_c^2 - D^2}$$

The condition for the existence of a solution to Eq. 49 is that the determinant of the coefficients vanish; that is,

$$\det \left| M_{ij} - \frac{\epsilon_o m}{c^2} (\omega^2 - k^2 c^2) \delta_{ij} \right| = 0 \tag{51}$$

c. A Stationary Plasma Penetrated by a Moving Plasma

Equation 51 will now be applied to a zero-temperature stationary plasma through which a second zero-temperature plasma passes with an arbitrary velocity, less than that of light, in the 3-direction. For zero magnetic field, the dispersion relation, for a wave polarized along the 2-axis, is

$$\omega^2 - k^2 c^2 - \Omega_a^2 - \Omega_b^2 = 0 \tag{52}$$

and, for a wave polarized in the 3-1 plane, it is

51

$$\left(\omega^2-k^2c^2-\Omega_a^2-\Omega_b^2\right)\left[\left(\omega^2-\Omega_a^2\right)\left(kc\cos\theta-\frac{\omega(1+\Lambda^2)^{1/2}}{\Lambda}\right)^2-\frac{\omega^2\Omega_b^2}{\Lambda^2}\right]$$

$$+ k^2c^2\Omega_a^2\Omega_b^2 \sin^2\theta = 0 \tag{53}$$

where

$$\Omega_a = \left(N_a e^2 / \epsilon_o m\right)^{1/2} \qquad \text{(stationary plasma)}$$

$$\Omega_b = \left(N_b e^2 / \epsilon_o m\right)^{1/2} \qquad \text{(moving plasma)}$$

are the plasma frequencies of the two plasmas, θ is the angle between the velocity of the moving plasma and the direction of propagation of the wave, and $u = c\Lambda$ is the velocity of the moving plasma.

Thus the propagation of the transverse wave polarized perpendicular to the plasma velocity is independent of the velocity of the moving plasma. It behaves exactly like a transverse wave in a single plasma with density $N_a + N_b$.

For waves polarized in the 1-3 plane, no simple longitudinal or transverse waves are permissible (except for propagation along the plasma velocity or in the nonrelativistic limit, $\Lambda \ll 1$). In general, all allowable waves have both longitudinal and transverse components. The two special cases, $\theta = 0$ and $\theta = \pi/2$ will now be considered in detail.

For $\theta = \pi/2$, the dispersion relation becomes

$$\left(\omega^2-k^2c^2-\Omega_a^2-\Omega_b^2\right)\left[\left(\omega^2-\Omega_b^2\right)\frac{1+\Lambda^2}{\Lambda^2}-\frac{\Omega_b^2}{\Lambda^2}\right]+\frac{k^2c^2}{\omega^2}\Omega_a^2\Omega_b^2 = 0 \tag{54}$$

or, solving for k^2c^2 gives

$$k^2c^2 = \frac{\omega^2\left(\omega^2-\omega_2^2\right)\left(\omega^2-\omega_4^2\right)}{\left(\omega^2-\omega_1^2\right)\left(\omega^2-\omega_3^2\right)} \tag{55}$$

where

$$\omega_{1,3}^2 = \frac{1}{2}\left(\Omega_a^2+\frac{\Omega_b^2}{1+\Lambda^2}\right)\mp\frac{1}{2}\left[\left(\Omega_a^2+\frac{\Omega_b^2}{1+\Lambda^2}\right)^2+\frac{4\Omega_a^2\Omega_b^2\Lambda^2}{1+\Lambda^2}\right]^{1/2}$$

$$\omega_2^2 = \Omega_a^2+\frac{\Omega_b^2}{1+\Lambda^2}$$

$$\omega_4^2 = \Omega_a^2 + \Omega_b^2$$

The mapping of real k into the ω-plane is given in Fig. II-20. There are gaps for $0 < |\omega| < \omega_2$ and $\omega_3 < |\omega| < \omega_4$ in which there are evanescent (4) waves. In these ranges of frequency k is pure imaginary. In the nonrelativistic limit $\Lambda \ll 1$, $\omega_2 \to \omega_3 \to \omega_4 = \left(\Omega_a^2 + \Omega_b^2\right)^{1/2}$, and there results only a single gap below the plasma frequency $\left(\Omega_a^2 + \Omega_b^2\right)^{1/2}$. For frequencies outside the gaps simple unattenuated waves propagate.

For $\theta = 0$ Eq. 53 gives a transverse wave polarized in the 1-direction which satisfies Eq. 52, and a longitudinal wave that satisfies the dispersion relation

$$\left(\omega^2 - \Omega_a^2\right)\left(kc - \frac{\omega\left(1+\Lambda^2\right)^{1/2}}{\Lambda}\right)^2 - \frac{\omega^2 \Omega_b^2}{\Lambda^2} = 0 \tag{56}$$

or, if we solve for k, we have

$$k = \frac{\omega}{c\Lambda}\left[\left(1+\Lambda^2\right)^{1/2} \pm \frac{\Omega_b}{\left(\omega^2 - \Omega_a^2\right)^{1/2}}\right] \tag{57}$$

The mappings of real ω into the k-plane are given in Figs. II-21 and II-22 for the positive and negative signs, respectively. For the positive sign, simple unattenuated waves with $|k| > k_0$ can propagate. On the other hand, for $|k| < k_0$ the frequency must be complex. These waves represent instabilities of the plasma. For nonzero temperature it is found that k no longer approaches infinity at $\omega = \Omega_a$, but the mapping of real ω into complex k forms a continuous contour as ω goes from $-\infty$ to $+\infty$. The instability is therefore convective (4), that is, a disturbance localized in space at a given instant of time will grow in time and move away.

For the negative sign (Fig. II-22) simple waves can propagate at all wavelengths. However, for frequencies in the range $|\omega| < \Omega_a$ only evanescent waves are permissible.

4. Plasma in Thermal Equilibrium

The propagation of waves in a plasma in thermal equilibrium will now be studied. The distribution function (5) for such a plasma is

$$\tilde{f}^0 = \frac{N}{4\pi c^3} \frac{\sigma}{K_1(\sigma)} \exp\left[-\sigma\left(1+u^2/c^2\right)^{1/2}\right] \tag{58}$$

where $\sigma = mc^2/\kappa T$, and K_1 is the modified Bessel function. The results for several special cases will be given.

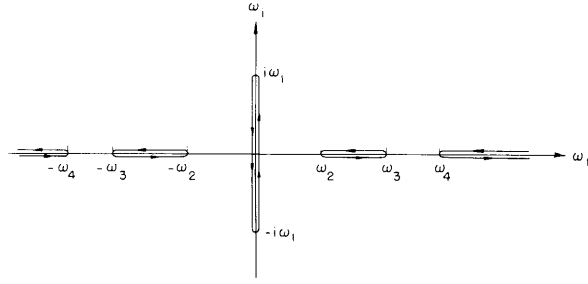For a longitudinal wave propagating along the magnetic field the dispersion

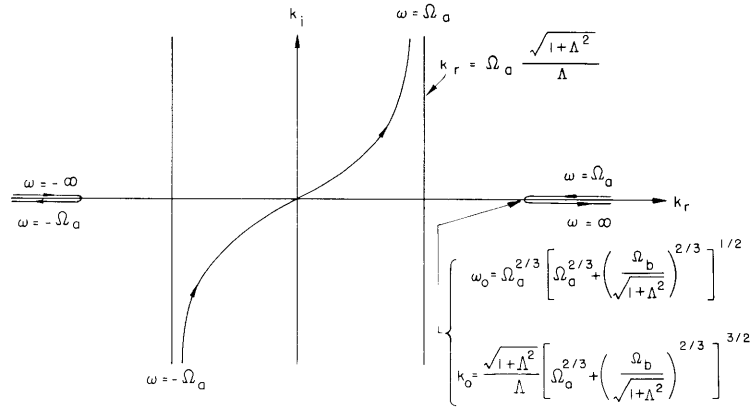Fig. II-20. Mapping of real k into the complex ω-plane for a wave propagating across the moving plasma.

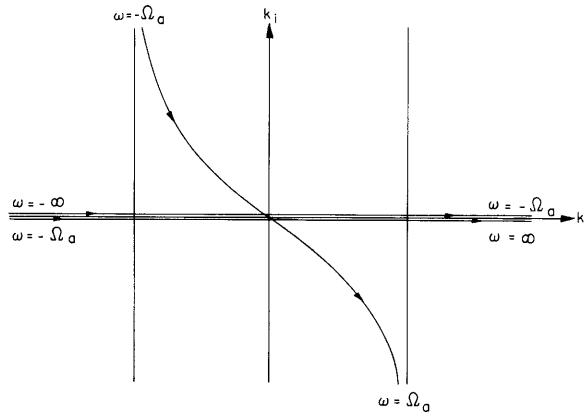Fig. II-21. Mapping of real ω into the k-plane for a wave propagating along the moving plasma (+).

Fig. II-22. Mapping of real ω into the k-plane for a wave propagating along the moving plasma (-).

relation is

$$1 = -\frac{\Omega_p^2}{\omega} \frac{\sigma^2}{4\pi c^5 K_1(\sigma)} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp[-\sigma(1+u^2/c^2)^{1/2}]$$

$$\times \frac{u_3^2 du_1 du_2 du_3}{\left(ku_3 - \omega\left(1 + \frac{u^2}{c^2}\right)^{1/2}\right)\left(1 + \frac{u^2}{c^2}\right)^{1/2}} \tag{59}$$

The Landau damping of the wave can be calculated for low temperature by finding the imaginary part of this integral. The integral over u_3 is taken along the real axis, except for a small semicircle above the pole of the integrand. First, it may be noted that for phase velocities greater than light, the integrand has no singularity, the integral is real, and there is no damping. This would be expected because there are no trapped electrons moving with the wave if its phase velocity is greater than that of light. For $(\omega/k) < c$ the imaginary part of ω is

$$-\sqrt{\frac{\pi}{2}} \sigma^{3/2} \frac{\Omega_p^2 \omega^2}{kc(k^2 c^2 - \omega^2)} \exp\{\sigma[1-(1-\omega^2/k^2 c^2)^{-1/2}]\} \tag{60}$$

For $(\omega/k) \ll c$ this reduces to the usual result for Landau damping (6). For low temperature, $\kappa T/mc^2 \ll 1$, Eq. 59 may be expanded in powers of $\kappa T/mc^2$.

To first order, we have

$$\omega^2 = \Omega_p^2 + \Omega_p^2\left(\frac{3k^2 c^2}{\omega^2} - 1\right)\frac{\kappa T}{mc^2} \tag{61}$$

Except for the $-\Omega_p^2 \dfrac{\kappa T}{mc^2}$ term this is the usual result. This extra term is caused by relativistic effects. It indicates that waves may propagate in the plasma with frequencies below the plasma frequency for a finite temperature. As the temperature gets very large, the cutoff frequency approaches the limiting value

$$\omega_{cutoff}^2 \to \frac{2}{3} \Omega_p^2 \tag{62}$$

This value is obtained by solving Eq. 59 for k = 0, in the limit $\sigma \to 0$.

For transverse waves propagating along the magnetic field, we have

$$\omega^2 - k^2 c^2 = -\frac{\Omega_p^2 \omega}{4\pi c^5} \frac{\sigma^2}{K_1(\sigma)} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{-\sigma\left(1+\frac{u^2}{c^2}\right)^{1/2}} u_1^2 du_1 du_2 du_3}{\left(1+\frac{u^2}{c^2}\right)^{1/2}\left(ku_3 - \omega\left(1+\frac{u^2}{c^2}\right)^{1/2} \pm \Omega_c\right)} \tag{63}$$

For $\Omega_c = 0$, cutoff again occurs at $\omega^2 = \frac{2}{3}\Omega_p^2$ for $\sigma \to 0$.

For low temperature, Eq. 63 can be expanded as follows:

$$\omega^2 = k^2 c^2 + \frac{\Omega_p^2}{1 \pm \frac{\Omega_c}{\omega}} + \frac{\Omega_p^2}{\left(1 \pm \frac{\Omega_c}{\omega}\right)^3}\left[\frac{k^2 c^2}{\omega^2} - \left(1 \pm \frac{\Omega_c}{\omega}\right)\left(1 \mp \frac{3}{2}\frac{\Omega_c}{\omega}\right)\right]\frac{\kappa T}{mc^2} \tag{64}$$

Here, the relativistic correction to the usual dispersion relation (the second term in brackets) is more complicated than for the longitudinal wave; it is a function of the magnetic field.

For a transverse wave propagating across the magnetic field and polarized parallel to the field the dispersion relation is

$$\omega^2 - k^2 c^2 = \frac{\Omega_p^2 \sigma^2}{c^5 K_1(\sigma)} \int_0^{\infty} \frac{u^4 \exp[-\sigma(1+u^2/c^2)^{1/2}]}{1 + \frac{u^2}{c^2}}$$

$$\cdot \left[\frac{1}{3} - 2\Omega_c^2 \sum_1^{\infty} \frac{n^2 \int_0^1 \lambda(1-\lambda^2)^{1/2} J_n^2\left(\frac{ku\lambda}{\Omega_c}\right) d\lambda}{n^2 \Omega_c^2 - \omega^2\left(1 + \frac{u^2}{c^2}\right)}\right] du \tag{65}$$

We see from Eq. 65 that the resonances are not sharply peaked at multiples of Ω_c — the usual result of nonrelativistic calculations — but are spread out somewhat by the factor $1 + \frac{u^2}{c^2}$ that occurs in the denominator of the summand. This arises because the cyclotron frequency of a relativistic electron depends upon its velocity. Expanding Eq. 65 to first order in $\kappa T/mc^2$ yields

$$\omega^2 = k^2 c^2 + \Omega_p^2 + \Omega_p^2\left(\frac{k^2 c^2}{\omega^2 - \Omega_c^2} - 1\right)\frac{\kappa T}{mc^2} \tag{66}$$

Aside from the -1 in the parentheses, this is the usual result.

For waves propagating across the magnetic field polarized perpendicular to the magnetic field, the longitudinal and transverse waves are coupled so that the dispersion relation is considerably more complicated. It is

$$
\left[\frac{2\sigma^2\Omega_p^2\Omega_c^2}{c^5 K_1(\sigma)k^2c^2} \int_0^1 \int_0^\infty \exp[-\sigma(1+u^2/c^2)^{1/2}] \sum_1^\infty \frac{n^2 J_n^2\left(\frac{ku\lambda}{\Omega_c}\right)}{n^2\Omega_c^2 - \omega^2\left(1+\frac{u^2}{c^2}\right)} \frac{u^2\lambda du d\lambda}{(1-\lambda^2)^{1/2}} + 1\right]
$$

$$
\cdot\left[\frac{-\sigma^2\Omega_p^2}{2c^5 K_1(\sigma)} \int_0^1 \int_0^\infty \exp[-\sigma(1+u^2/c^2)^{1/2}] \left\{1 - 4\Omega_c^2 \sum_1^\infty \frac{n^2 J_n'^2\left(\frac{ku\lambda}{\Omega_c}\right)}{n^2\Omega_c^2 - \omega^2\left(1+\frac{u^2}{c^2}\right)}\right\}\right.
$$

$$
\left.\cdot\frac{u^4\lambda^3 du d\lambda}{(1-\lambda^2)^{1/2}\left(1+\frac{u^2}{c^2}\right)} + \omega^2 - k^2c^2\right] = \left[\frac{2\sigma^2\Omega_p^2\Omega_c^2}{c^5 K_1(\sigma)k} \int_0^1 \int_0^\infty \exp[-\sigma(1+u^2/c^2)^{1/2}]\right.
$$

$$
\left.\cdot\sum_1^\infty \frac{n^2 J_n\left(\frac{ku\lambda}{\Omega_c}\right) J_n'\left(\frac{ku\lambda}{\Omega_c}\right)}{n^2\Omega_c^2 - \omega^2\left(1+\frac{u^2}{c^2}\right)} \frac{u^3\lambda^2 du d\lambda}{(1-\lambda^2)^{1/2}\left(1+\frac{u^2}{c^2}\right)^{1/2}}\right]^2 \tag{67}
$$

For low temperatures this relation becomes

$$
\left[\omega^2 - \frac{\Omega_p^2}{1-\frac{\Omega_c^2}{\omega^2}}\left\{1+\left(\frac{3k^2c^2}{\omega^2-4\Omega_c^2} - \frac{\omega^2+4\Omega_c^2}{\omega^2-\Omega_c^2}\right)\frac{\kappa T}{mc^2}\right\}\right]\cdot\left[\omega^2 - k^2c^2 - \frac{\Omega_p^2}{1-\frac{\Omega_c^2}{\omega^2}}\right.
$$

$$
\left.\cdot\left\{1+\left(\frac{\omega^2+8\Omega_c^2}{\omega^2-4\Omega_c^2}\frac{k^2c^2}{\omega^2} - \frac{\omega^2+4\Omega_c^2}{\omega^2-\Omega_c^2}\right)\frac{\kappa T}{mc^2}\right\}\right] = \frac{\Omega_p^4\Omega_c^2\omega^2}{\left(\omega^2-\Omega_c^2\right)^2}
$$

$$
\cdot\left[1+\left(\frac{6k^2c^2}{\omega^2-4\Omega_c^2} - \frac{\frac{7}{2}\omega^2+\frac{3}{2}\Omega_c^2}{\omega^2-\Omega_c^2}\right)\frac{\kappa T}{mc^2}\right]^2 \tag{68}
$$

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C. PLASMA MAGNETOHYDRODYNAMICS AND ENERGY CONVERSION[*]

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RESEARCH OBJECTIVES

1. The aim of the magnetohydrodynamics group is still directed at understanding the collective behavior of plasmas with special emphasis on the character of hydromagnetic waves in a bounded medium, instabilities in plasmas, and strong shocks.

The study of waves in a finite domain is of importance to the general problem of hydromagnetic resonators. Since these devices are used in several schemes for the heating of a plasma, the need for a thorough understanding of the properties of resonators is evident.

The hydromagnetic stabilities that we propose to investigate will be those associated with a mean flow of the plasma. Since it is not permissible at times to neglect the effect of the electric forces, several cases proposed for consideration are not pure hydromagnetic instabilities, but belong instead to the class of electrostatic instabilities.

Our program for the study of shocks continues to be the one indicated in Quarterly Progress Report No. 52, page 20, and deals with investigations of the structure of shocks, the radiation from shocks, and the dynamics of plane shocks in plasmas.

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2. The objectives of the energy conversion group are twofold:

(a) To study problems of magnetohydrodynamic flow, in order to obtain a better understanding of the phenomena involved. For instance, the problem of interaction between the ionized gas behind a shock wave and a magnetic field will be studied in detail for both weak and strong interactions, and for steady and time-variant magnetic fields.

(b) To study systems in which energy conversion occurs between flow energy in a conduction liquid or gas and an electrical system. These studies will include steady and nonsteady flow, and dc and ac electrical systems.

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