XII. STATISTICAL COMMUNICATION THEORY

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RESEARCH OBJECTIVES

This group is interested in a variety of problems in statistical communication theory. Our current research is primarily concerned with: nonlinear systems with gaussian inputs, general nonlinear operators, characterization of random signals, optimum detection of noisy frequency-modulated signals, pulse code modulation systems, a level selector tube, removal of noise in phonograph records, and extension of the dynamic range in tape recording.

1. The study of nonlinear systems with gaussian inputs involves the generalization of Dr. Norbert Wiener's orthogonal expansion for a nonlinear system with a single input and a single output to an expansion for a nonlinear system with multiple gaussian inputs and a single output. Optimum filtering and prediction are two possible applications. The subclass of nonlinear systems that are composed of no-memory squaring devices and linear devices is receiving special attention because of the comparative simplicity in analysis.

2. A primary objective in the study of general nonlinear operators is a rigorous treatment of the nonlinear feedback problem. The mathematical technique of iteration has been found to be of substantial value in this study.

3. In the study of the characterization of random signals emphasis is given to the characterization in terms of a finite set of numbers. Such a characterization is necessary when the random signals in a problem are required to be applied to a computer in the form of numbers. The characterization is also important in the study of the memory of a nonlinear system.

4. A statistical approach to the problem of finding the best methods of demodulating and filtering frequency-modulated signals is being undertaken. A large part of the research consists of finding mathematical signal representation methods that are manageable.

5. The research on the transmission of messages by pulse code modulation in a noisy channel is primarily the optimization of performance according to different criteria and to the parameters and components that are under the control of the designer. It includes an investigation of the implications of the different criteria under study.

6. Work is continuing on the development of a level selector tube for measuring first-order and second-order probability distributions and determining optimum nonlinear systems.

7. An application of nonlinear filtering techniques to the removal of noise in phonograph records is planned. This work includes a detailed study of the properties of the noise and should permit further improvement of the filter reported in a thesis by D. A. Shnidman ("Nonlinear Filter for the Reduction of Record Noise," S. M. Thesis, Department of Electrical Engineering, M. I. T., June 1959).

8. It is desirable for laboratory work as well as for the commercial recording of music to extend the dynamic range available in tape recording. The dynamic range is limited above by the nonlinear effect of magnetic saturation and below by the random tape noise. As an application of statistical methods to nonlinear problems this problem will be studied with the goal of increasing the useful dynamic range.

Y. W. Lee

A. CHARACTERIZATION OF RANDOM SIGNALS WITH AN APPLICATION TO NONSTATIONARY LINEAR SYSTEMS

1. The general problem

Let $x(t)$ be a random process with zero mean in the interval $[a, b]$. We want to characterize the member functions of this random process in terms of a set of N random variables, $\{a_1, \ldots, a_N\}$. We shall call these random variables a set of "observations" on the random process. In general, these observations would be derived from the process by a set of functionals $\{T_i\}$:

$$\begin{aligned} a_1 &= T_1[x(t)] \\ a_2 &= T_2[x(t)] \\ &\vdots \\ a_N &= T_N[x(t)] \end{aligned} \tag{1}$$

The way in which the set of observations characterizes the member functions is described by defining an approximation function which is a function of the set $\{a_i\}$:

$$z(t) = F(t, a_1, \ldots, a_N)$$

Then $z(t)$ is a random process which in some way approximates $x(t)$.

We now define a distortion measure, D, which is a measure of the degree of approximation attained. One analytically useful distortion measure is the mean-square error between the random processes $x(t)$ and $z(t)$.

$$D[\{T_i\}, F] = E\left[\int_a^b \{x(t)-z(t)\}^2\, dt\right] \tag{2}$$

The ultimate problem in the representation of a random process is to find the set of functionals $\{T_i^*\}$ and the approximation function F^* which minimize the distortion D. Any number of other problems in which the $\{T_i\}$ and F are restricted to certain classes may be defined. Tufts (1, 2) considered the problem in which the set $\{a_n\}$ is the set of periodic time samples $\{x(nT_o)\}$, and F is restricted to be of the form

$$F[t_1\{a_n\}] = \sum_n a_n\, \phi(t-nT_o) \tag{3}$$

that is, a linear sum of interpolation functions.

2. Restriction of F to be linear

The class of problem that is considered in this report is the minimization of D with respect to $\{T_i\}$ and F when F is restricted to be linear in the variables $a_1, \ldots, a_N$. This minimization is achieved for every N by the set of linear observations

$$\begin{aligned} a_1 &= T_1^*[x(t)] = \int_a^b x(t)\, \phi_1(t)\, dt \\ &\;\;\vdots \\ a_N &= T_N^*[x(t)] = \int_a^b x(t)\, \phi_N(t)\, dt \end{aligned} \tag{4}$$

and the approximation function

$$z(t) = F^*[t, a_1, \ldots, a_N] = \sum_{n=1}^{N} a_n\, \phi_n(t) \tag{5}$$

where $\{\phi_i(t)\}$ is the orthonormal set of eigenfunctions which are solutions of the integral equation

$$\int_a^b R(s,t)\, \phi_n(t)\, dt = \beta_n\, \phi_n(s) \qquad a \leqslant s \leqslant b \tag{6}$$

and which are ordered in such a way that

$$\beta_1 \geqslant \beta_2 \geqslant \beta_3 \geqslant \ldots \tag{7}$$

In the rest of this report, all eigenfunctions will be assumed to be ordered in this way. In Eq. 6, the kernel $R(s,t)$ is the autocorrelation function of the random process $x(t)$ and is defined as

$$R(s,t) = E[x(s)x(t)] \tag{8}$$

The minimized distortion is

$$D = E[x^2(t)] - \sum_{n=1}^{N} \beta_n \tag{9}$$

The proof was outlined in earlier reports (3, 4). This result was also proved by A. Koschmann (5). It follows immediately from the following lemma.

LEMMA. If the kernel $K(s,t)$ is nonnegative definite and of integrable square, then the maximum of the sum

$$\sum_{n=1}^{N} \int_a^b \int_a^b K(s,t)\, \psi_n(s)\, \psi_n(t)\, ds\, dt \tag{10}$$

when the set $\{\psi_n(s)\}$ varies under the conditions

$$\int_a^b \psi_m(t)\, \psi_n(t)\, dt = \begin{cases} 0 \text{ for } m \neq n \\ 1 \text{ for } m = n \end{cases} \qquad m, n = 1, \ldots, N \tag{11}$$

is attained for the set $\{\phi_n\}$ of the first N eigenfunctions of $K(s,t)$. This maximum is $\sum_{n=1}^{N} \lambda_n$, where the λ_n are the first N eigenvalues of $K(s,t)$. (See ref. 9, p. 459, for a similar theorem for differential equations.)

If the random process $[x(t)]$ is gaussian, it can be shown that Eqs. 4 and 5 are optimum with no restriction on F.

3. A weighted mean-square error

An extension of this same problem can be made by considering a new distortion defined as the weighted mean-square error:

$$D = E\left[\int_a^b G^2(t)\{x(t)-z(t)\}^2\, dt\right] \qquad G(t) \text{ is a nonnegative function} \tag{12}$$

It is easily shown that the minimum of D is attained if we use the following set of linear observations:

$$\begin{aligned} a_1 &= \int_a^b G(t)\, x(t)\, \gamma_1(t)\, dt \\ &\vdots \\ a_N &= \int_a^b G(t)\, x(t)\, \gamma_N(t)\, dt \end{aligned} \tag{13}$$

and the approximation function

$$F(t, a_1, \ldots, a_N) = \sum_{n=1}^{N} a_n\, \gamma_n(t)$$

where the set $\{\gamma_n(t)\}$ is the set of normalized eigenfunctions of

$$\int_a^b G(s)\, G(t)\, R(s,t)\, \gamma_n(t)\, dt = \beta_n\, \gamma_n(s) \qquad a \leqslant s \leqslant b \tag{14}$$

In this problem one or both of the limits may be infinite because our only condition is that the kernel, $G(s)\,G(t)\,R(s,t)$, be of integrable square over the interval. Thus, we may consider the characterization of the past of a signal by using the interval $[-\infty, 0]$. This problem is pertinent to a characterization of nonlinear systems described by Bose (6).

4. Characterization of one process from another process linearly correlated with it

Let $x(t)$ and $y(t)$ be two zero-mean random processes with autocorrelation and cross-correlation, $R_{xx}(s,t)$, $R_{yy}(s,t)$, and $R_{yx}(s,t)$. We want to approximate $x(t)$ by $z(t) = F[t, a_1, \ldots, a_N]$, in which the set of random variables $\{a_i\}$ is derived from $y(t)$. We again define the distortion D as in Eq. 2, and place the linear constraint on F. It is found that D is minimized with respect to $\{T_i\}$ and F by the set of linear observations

$$\begin{aligned} a_1 &= T_1^*[x(t)] = \int_a^b x(t)\, g_1(t)\, dt \\ &\;\;\vdots \\ a_N &= T_N^*[x(t)] = \int_a^b x(t)\, g_N(t)\, dt \end{aligned} \tag{15}$$

and the approximation function

$$z(t) = F^*(t, a_1, \ldots, a_N) = \sum_{n=1}^{N} a_n\, \phi_n(t) \tag{16}$$

The set $\{g_i(s)\}$ and the orthonormal set $\{\phi_i(s)\}$ are solutions to the following integral equations:

$$\begin{aligned} \beta_n \int_a^b R_{yy}(u,s)\, g_n(s)\, ds &= \int_a^b g_n(t)\, dt \int_a^b R_{yx}(u,s)\, R_{yx}(t,s)\, ds \\ \int_a^b R_{yy}(s,t)\, g_n(t)\, dt &= \int_a^b R_{yx}(s,t)\, \phi_n(t)\, dt \end{aligned} \tag{17}$$

and the minimized mean-square error is given by

$$D = E[x^2(t)] - \sum_{n=1}^{N} \beta_n \tag{18}$$

It can be shown that this result is equivalent to first passing the random process $y(t)$ through an optimum nonstationary linear filter, $h(s,t)$, the output of which is

$$z(s) = \int_a^b h(s,t)\, y(t)\, dt \tag{19}$$

and then characterizing the resulting process in the optimum manner described by Eqs. 4 and 5.

The optimum $h(s,t)$ is the solution to

$$R_{yx}(s,t) = \int_a^b R_{yy}(s,u)\, h(t,u)\, du \tag{20}$$

which is similar to an equation of Booton (7).

5. Sample problem

As an example let us consider the random process $x(t)$ to be a random signal, $s(t)$, which consists of N orthonormal waveforms, $s_1(t), \ldots, s_N(t)$, occurring with probabilities $P_1, \ldots, P_N$, respectively, where $P_1 \geq P_2 \geq \ldots \geq P_N$ and $\sum_{n=1}^{N} P_n = 1$. Let $y(t)$ be this signal plus independent white noise with autocorrelation function $R_{nn}(s,t) = N_o \delta(s-t)$. It can be shown that

$$\begin{aligned} R_{yx}(s,t) &= R_{ss}(s,t) = \sum_{n=1}^{N} P_n\, s_n(s)\, s_n(t) \\ R_{yy}(s,t) &= R_{ss}(s,t) + R_{nn}(s,t) = \sum_{n=1}^{N} P_n\, s_n(s)\, s_n(t) + N_o\, \delta(s-t) \end{aligned} \tag{21}$$

Equations 17 now become

$$\begin{aligned} &\beta_n \sum_{m=1}^{N} P_m\, s_m(u) \int_a^b s_m(s)\, g_n(s)\, ds + \beta_n N_o\, g_n(u) \\ &\quad = \sum_{m=1}^{N} P_m^2\, s_m(u) \int_a^b s_m(s)\, g_n(s)\, ds \\ &\sum_{m=1}^{N} P_m\, s_m(s) \int_a^b s_m(t)\, g_n(t)\, dt + N_o\, g_n(s) \\ &\quad = \sum_{m=1}^{N} P_m\, s_m(s) \int_a^b s_m(t)\, \phi_n(t)\, dt \end{aligned} \tag{22}$$

The set of solutions is given by

$$\begin{aligned} g_i(s) &= \frac{P_i}{P_i + N_o}\, s_i(s) \qquad i = 1, \ldots, N \\ \phi_i(s) &= s_i(s) \qquad\qquad\quad i = 1, \ldots, N \end{aligned} \tag{23}$$

and the mean-square error is

$$D = 1 - \sum_{i=1}^{N} \frac{P_i^2}{P_i + N_o} \tag{24}$$

Here the observations $\{a_i\}$ are given by

$$a_i = \frac{P_i}{P_i + N_o} \int_a^b x(t)\, s_i(t)\, dt \tag{25}$$

or if

$$h_i(t) = s_i(b-t) \frac{P_i}{P_i + N_o} \tag{26}$$

we have

$$a_i = \int_a^b x(t)\, h_i(b-t)\, dt \tag{27}$$

which is the output at time b of a matched filter of impulse response $h_i(t)$.

6. An application to the theory of optimum nonstationary systems

The optimum nonstationary linear filter $h(s,t)$, the output of which is given in Eq. 19, is, in general, difficult to realize, but in a certain special case it can be approximated by a combination of stationary filters, as we shall now show.

If we assume that t is a parameter, the function $h(s,t)$ can be expanded in the series

$$h(s,t) = h_t(s) = \sum_{n=1}^{\infty} g_n(t)\, \phi_n(s) \tag{28}$$

where $\{\phi_n\}$ is an orthonormal set and

$$g_n(t) = \int_0^T h(s,t)\, \phi_n(s)\, ds \tag{29}$$

On interchanging the order of summation and integration in Eq. 19 we obtain for $z(s)$

$$z(s) = \sum_{n=1}^{\infty} \phi_n(s) \int_0^T g_n(t)\, y(t)\, dt \tag{30}$$

which is in the form of Eq. 16. We can then conclude that on the basis of the results of the preceding sections the finite series $\sum_{n=1}^{N} \phi_n(s)\, g_n(t)$, where $\{g_n(t)\}$ and $\{\phi_n(t)\}$ are sets of solutions to Eqs. 17, approximates the filter $h(s,t)$ in a most rapidly convergent manner, in the sense that the mean-square error of the outputs is minimized for

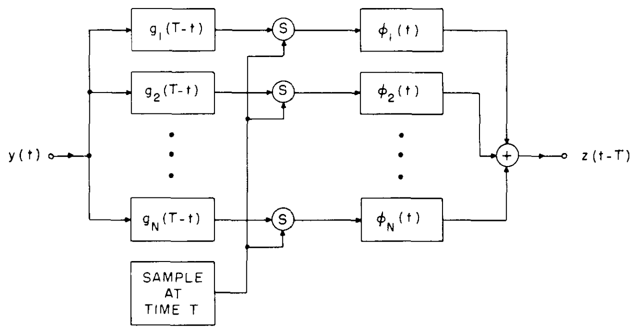

Fig. XII-1. An approximation to a nonstationary linear system.

each N. The resulting system, which has a delay of T sec, is shown in Fig. XII-1. This type of system would be the most useful in cyclostationary processes, that is, when

$$\begin{aligned} R_{yy}(s,t) &= R_{yy}(s+T,t+T) \\ R_{yx}(s,t) &= R_{yx}(s+T,t+T) \end{aligned} \tag{31}$$

because in this case the $\{g_i(t)\}$ and $\{\phi_i(t)\}$ are optimum for every interval of the form $[nT,(n+1)T]$.

Koschmann (8) considered a similar problem, that of optimizing a set of coefficients $\{b_i\}$ in such a way that

$$E[(z(T_1)-x(T_1))^2] \tag{32}$$

is minimized (where $z(T_1) = \sum_{n=1}^{\infty} b_n \int_0^T g_n(t)\, y(t)\, dt$ and $0 \leq T_1 \leq T$). He showed that the optimum set $\{b_i\}$ must be a solution to the set of equations

$$\begin{aligned} &\sum_{n=1}^{\infty} b_n \int_0^T \int_0^T R_{yy}(u,v)\, g_n(u)\, g_m(v)\, du\, dv \\ &\qquad = \int_0^T R_{xy}(t,s)\, g_m(s)\, ds \qquad m = 0, 1, 2, \ldots \end{aligned} \tag{33}$$

If we set $b_i = \phi_i(t)$ and use the second of Eqs. 17, we obtain

$$\begin{aligned} &\sum_{n=1}^{\infty} \phi_n(t) \int_0^T \phi_n(u)\, du \left\{\int_0^T R_{xy}(u,v)\, g_m(v)\, dv\right\} \\ &\qquad = \int_0^T R_{xy}(t,s)\, g_m(s)\, ds \end{aligned} \tag{34}$$

The series on the left is an orthonormal series with Fourier coefficients and it converges in the mean (9) to the function on the right. Under certain conditions it can be shown (10) that the series converges at every point and that therefore $\phi_i(T_1)$ is a solution of Eqs. 33 for every T_1.

The sets of functions $\{g_i(t)\}$ and $\{\phi_i(t)\}$, which are solutions to Eqs. 17, then minimize Eq. 32 for every T_1. With these sets the series does not necessarily converge in a most rapid manner for each T_1 but only on the average over the interval.

7. Optimality in the case of additive noise

It may be of interest to know whether or not Eqs. 4 and 5 are still optimum when the random variables $\{a_i\}$ are each subjected to additive noise. For example, the set $\{a_i\}$ might be used as a coding of the random signal x(t), in which case the random variables a_i would be subject to noise while they are being transmitted through a communications link. Then we would have for the approximation function $F[t, a_1+\epsilon_1, \dots, a_N+\epsilon_N]$, where the set $\{\epsilon_i\}$ is an arbitrary set of random variables. It is found that when the ϵ_i are independent of the signal and of one another and when their means are zero, then Eqs. 4 and 5 are still optimum.

K. L. Jordan, Jr.

References

1. D. W. Tufts, A sampling theorem for stationary random processes, Quarterly Progress Report, Research Laboratory of Electronics, M. I. T., July 15, 1958, pp. 111-116.

2. D. W. Tufts, Quarterly Progress Report No. 53, Research Laboratory of Electronics, M.I.T., April 15, 1959, pp. 87-93.

3. K. L. Jordan, Jr., Minimization of truncation error in series expansions of random processes, Quarterly Progress Report No. 52, Research Laboratory of Electronics, M.I.T., Jan. 15, 1959, pp. 70-78.

4. A proof was given by the author in Quarterly Progress Report No. 53, Research Laboratory of Electronics, M. I. T., April 15, 1959, pp. 84-86. This is incorrect because $K_1(t, s)$ cannot be considered positive definite as was assumed in Eq. 17 of that report.

5. A. Koschmann, On the filtering of non-stationary time series, Ph. D. Thesis, Purdue University, August 1954, pp. 66-70; Proceedings of the National Electronics Conference, 1954, p. 126, cf. Theorem 2.

6. A. G. Bose, A theory of nonlinear systems, Technical Report 309, Research Laboratory of Electronics, M. I. T., May 15, 1956.

7. R. C. Booton, An optimization theory for time-varying linear systems with non-stationary statistical inputs, Proc. IRE 40, 980 (1952), cf. Eq. (24).

8. A. Koschmann, op. cit., p. 23.

(References continued on following page)

9. For a definition of "convergence in the mean," see R. Courant and D. Hilbert, Methods of Mathematical Physics (Interscience Publishers, Inc., New York, 1953), p. 110.

10. F. Riesz and B. Sze.-Nagy, Functional Analysis (F. Ungar Publishing Company, New York, 1955), p. 244 (cf. theorem).

B. GRAVITATIONAL FIELDS

A preliminary study has been made of the practicability of using gravitational fields for the detection of accelerating masses and for communication purposes. In this report, static and time-variant gravitational fields are treated under the simplifying assumption of small masses and velocities. This treatment is not intended to be the most elegant one. Rather, our aim is to establish a physical understanding of gravitational fields by using well-known concepts; for this purpose, the analogy between gravitational and electromagnetic fields has been stressed. The resulting equations predict transverse gravity waves that propagate at the velocity of light. However, the efficiency of radiation of these waves is small because of the low wave impedance, which is approximately $9\pi \times 10^{-19}$ g-ohms.

In order to establish the basic definitions, let us derive the equations of static gravitational fields. Throughout this report the symbols for the gravitational field quantities were chosen to indicate their counterparts in electromagnetic fields. The mks system of units is used.

1. Static gravitational fields

Consider two masses, m_1 and m_2. From Newton's law of gravitation, the force of attraction between the two masses is

$$\vec{F} = \frac{m_1 m_2}{4\pi k R^2} \vec{i}_R \tag{1}$$

Now let

$$\vec{F} = m_2 \vec{G} \tag{2}$$

where

$$\vec{G} = \frac{m_1}{4\pi k R^2} \vec{i}_R \tag{3}$$

Note that k, as defined, is negative because "like" masses attract each other. Also, define a vector $\vec{D}$:

$$\vec{D} = k\vec{G} = \frac{m_1}{4\pi R^2} \vec{i}_R \tag{4}$$

Now consider an isolated mass and define the total flux emanating from the mass:

$$\psi = \oint_A \vec{D} \cdot d\vec{a}$$

If a sphere of radius R is constructed with the isolated mass at its center, then, from symmetry, we have

$$\oint_A \vec{D} \cdot d\vec{a} = 4\pi R^2 D$$

Thus

$$\vec{D} = \frac{\psi}{4\pi R^2} \vec{i}_R$$

From Eq. 4 we note that

$$\psi = m_1$$

We can generalize this result to

$$\oint_A \vec{D} \cdot d\vec{a} = M \tag{5}$$

where M is the total mass within the closed area. By applying the divergence theorem to Eq. 5, we obtain

$$\oint_A \vec{D} \cdot d\vec{a} = \int_V \nabla \cdot \vec{D}\, dv = M = \int_V \rho\, dv$$

where ρ is the mass density. Thus we have

$$\int_V [\nabla \cdot \vec{D} - \rho]\, dv = 0$$

Since this must be true for any volume, we require that

$$\nabla \cdot \vec{D} = \rho \tag{6}$$

Further, since

$$\frac{1}{R^2} \vec{i}_R = -\nabla \frac{1}{R}$$

We can write Eq. 4 in the form

$$\vec{D} = -\frac{m}{4\pi} \nabla \frac{1}{R}$$

Thus

$$\nabla \times \vec{D} = 0 \tag{7}$$

since, if ϕ is a scalar, then

$$\nabla \times \nabla\phi = 0$$

2. The gravitational field of a moving mass

We shall now derive equations that relate the gravitational field of a mass as seen by an observer stationary with respect to the mass to the gravitational field as seen by an observer moving with respect to the mass. For this derivation, only special relativity will be used.

We begin by defining a normalized velocity, β:

$$\beta = \frac{v}{c} \qquad \text{(c is the velocity of light)}$$

and a quantity, γ:

$$\gamma = [1-\beta^2]^{1/2}$$

Let a mass, M, be at rest in a primed coordinate system (x', y', z', t') which is moving with a constant velocity, v, along the x-axis of an unprimed coordinate system

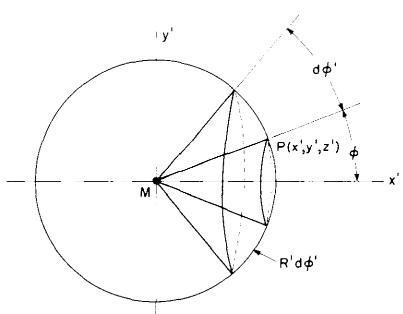

Fig. XII-2. The geometry for calculating the gravitational field of a moving mass.

(x, y, z, t). Consider a sphere of radius R' relative to M with M as its center. Parallel to the y'-z' plane construct a zone that intercepts an angle $d\phi'$ and passes through the point $P(x', y', z')$, as shown in Fig. XII-2.

By symmetry, the flux density over the zone will be constant and of magnitude D'. The total flux passing through the zone is thus

$$d\psi' = D'[2\pi y' R' d\phi'] \tag{8}$$

Now consider the sphere relative to the unprimed coordinate system. From special relativity, x is shortened by a factor γ.

$$x = \gamma x' \tag{9}$$

Since there is no motion in the y or z directions, we have

$$\begin{aligned} y &= y' \\ z &= z' \end{aligned} \tag{10}$$

Thus the sphere in the primed coordinate system is an oblate spheroid in the unprimed coordinate system.

We now make the assumption that $\vec{D}$ is radial (1). Then the flux through the zone in the unprimed coordinate system is

$$d\psi = D[2\pi y R d\phi] \tag{11}$$

Also, under a Lorentz transformation, mass (and thus flux) is not conserved. Rather, $m_o = \gamma m$, where m_o is the rest mass. Hence our assumption is

$$d\psi' = \gamma d\psi \tag{12}$$

By substituting Eqs. 8 and 11 in Eq. 12 we obtain

$$D'[2\pi y' R' d\phi'] = \gamma D[2\pi y R d\phi]$$

or

$$R'G'd\phi' = \gamma RGd\phi \tag{13}$$

Now, since $y' = y$, and $\gamma x' = x$, we have

$$R' \sin \phi' = R \sin \phi$$

and

$$\gamma R' \cos \phi' = R \cos \phi \tag{14}$$

By division we obtain

$$\tan \phi' = \gamma \tan \phi$$

or

$$\phi' = \tan^{-1}[\gamma \tan \phi] \tag{15}$$

Equation 15 implies that

$$\cos \phi' = \frac{1}{[1 + \gamma^2 \tan^2 \phi]^{1/2}} \tag{16}$$

$$\sin \phi' = \frac{\gamma \tan \phi}{[1 + \gamma^2 \tan^2 \phi]^{1/2}} \tag{17}$$

and by differentiating Eq. 15 we obtain

$$d\phi' = \frac{\gamma \sec^2 \phi}{1 + \gamma^2 \tan^2 \phi} \, d\phi \tag{18}$$

By substituting Eqs. 14, 16, 17, and 18 in Eq. 13, we obtain

$$G = \frac{1}{\gamma} G' \sec \phi \cos \phi' \tag{19}$$

In determining the transformation of the x-component of G we note that our assumption that $\vec{D}$ is radial implies that $G_x = G \cos \phi$. Thus, from Eq. 19 we have

$$G_x = G \cos \phi = \frac{1}{\gamma} G' \cos \phi'$$

or

$$G_x = \frac{1}{\gamma} G'_x \tag{20}$$

In determining the transformation of the y (or z) component of G we also note that $G_y = G \sin \phi$. Thus from Eq. 19 we have

$$G_y = G \sin \phi = \frac{1}{\gamma} G' \tan \phi \cos \phi' = \frac{1}{\gamma} G' \frac{\tan \phi}{\tan \phi'} \sin \phi'$$

and by using Eq. 15 we obtain

$$G_y = \frac{1}{\gamma^2} G' \sin \phi'$$

or

$$G_y = \frac{1}{\gamma^2} G'_y \tag{21}$$

We have thus shown that if a mass is moving in the x-direction with a velocity βc, and if the field of the moving mass relative to the mass is $\vec{G'}$, then the field, $\vec{G}$, that a stationary observer sees is

$$G_x = \frac{1}{\gamma} G'_x \tag{20}$$

$$G_y = \frac{1}{\gamma^2} G'_y \tag{21}$$

$$G_z = \frac{1}{\gamma^2} G'_z \tag{22}$$

Note that, except for a scale factor of γ, $\vec{G}$ transforms in the same manner as the electric field, $\vec{E}$, of a charge (2).

3. The gravitational field between two moving masses

Our next step in the derivation is the calculation of the force that an observer sees acting between two masses when both are moving in relation to him.

Consider (see Fig. XII-3) a mass, m_1, moving with a velocity v_1 along the x-axis of an unprimed coordinate system and a mass, m_2, moving with a velocity v_2 parallel to v_1 through a point P(x, y, 0). We want to compute the field acting on m_2 as seen by an observer who is stationary in the unprimed coordinate system. Let us allow the observer to introduce a mass, m_o, which is also at rest in the unprimed system so that the total field acting on m_2, as seen by an observer moving with m_2, will be zero. The observer in the frame of reference of m_2 will then observe no accelerating force acting on m_2. Since the frame of reference of m_2 and that of m_o are inertial frames, it follows that the observer who is stationary in the unprimed coordinate system will also observe m_2 to be unaccelerated. The field that the stationary observer must produce with m_o is, then, the negative of the field that he observes.

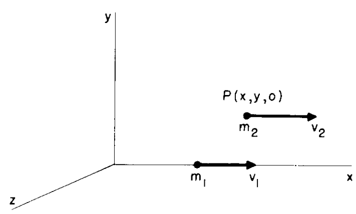

Fig. XII-3. The geometry for calculating the gravitational field between two moving masses.

We define three normalized velocities:

β_1 = the velocity of m_1 in relation to the unprimed coordinate system

β_2 = the velocity of m_2 in relation to the unprimed coordinate system

β_{12} = the velocity of m_1 in relation to m_2

By the velocity addition theorem (3), these velocities are related by

$$\beta_{12} = \frac{\beta_1 - \beta_2}{1 - \beta_1\beta_2}$$

or, equivalently,

$$\gamma_{12} = \frac{\gamma_1\gamma_2}{1 - \beta_1\beta_2} \tag{23}$$

We also define three gravitational fields at the point P:

$\vec{G}_1$ = the gravitational field caused by m_1 in relation to m_1

$\vec{G}_o$ = the gravitational field caused by m_o in relation to m_o

$\vec{G}'$ = the gravitational field caused by m_o and m_1 in relation to m_2

Then, from Eqs. 20 and 21, we have

$$G'_x = \frac{1}{\gamma_{12}} G_{1x} + \frac{1}{\gamma_2} G_{ox}$$
$$G'_y = \frac{1}{\gamma_{12}^2} G_{1y} + \frac{1}{\gamma_2^2} G_{oy} \tag{24}$$

In order for G' to be zero we thus require that

$$G_{ox} = -\frac{\gamma_2}{\gamma_{12}} G_{1x}$$
$$G_{oy} = -\left(\frac{\gamma_2}{\gamma_{12}}\right)^2 G_{1y} \tag{25}$$

Now G_o is the field that the stationary observer must produce. According to our discussion above, the field G that he observes is $-G_o$. Thus

$$G = -G_o \tag{26}$$

or, from Eq. 25, we have

$$G_x = \left(\frac{\gamma_2}{\gamma_{12}}\right) G_{1x} \tag{27}$$

$$G_y = \left(\frac{\gamma_2}{\gamma_{12}}\right)^2 G_{1y} \tag{28}$$

where G_x and G_y are the x and y components of the gravitational field that a stationary observer sees acting on m_2.

By use of Eq. 23 we can write Eq. 27 as

$$G_x = [1-\beta_1\beta_2] \frac{G_{1x}}{\gamma_1} \tag{29}$$

Now define

$$G_{sx} = \frac{1}{\gamma_1} G_{1x}$$

and

$$G_{dx} = -\frac{\beta_1\beta_2}{\gamma_1} G_{1x} \tag{30}$$

In terms of these field quantities, Eq. 29 can be written as

$$G_x = G_{sx} + G_{dx} \tag{31}$$

Note that G_{sx} is the x-component of the gravitational field that would be measured if m_2 were stationary. Hence G_{dx} is the additional x-component of the field resulting from the motion of m_2 through the field of m_1.

By a similar use of Eq. 23, we can also expand Eq. 28:

$$G_y = [1-\beta_1\beta_2]^2 \frac{G_{1y}}{\gamma_1^2} \tag{32}$$

Now define

$$\begin{aligned} G_{sy} &= \frac{1}{\gamma_1^2} G_{1y} \\ G_{dy} &= -\frac{\beta_1\beta_2[2-\beta_1\beta_2]}{\gamma_1^2} G_{1y} \end{aligned} \tag{33}$$

In terms of these field quantities, Eq. 32 can be written as

$$G_y = G_{sy} + G_{dy} \tag{34}$$

Note again that G_{sy} is the y-component of the gravitational field that would be measured if m_2 were stationary. Hence G_{dy} is the additional y-component of the field resulting from the motion of m_2 through the field of m_1.

That additional component of the field acting on m_2 as a result of the motion of m_2 through the field of m_1 will be called the dynamic field, G_d:

$$\begin{aligned} \vec{G}_d &= \vec{i}_x G_{dx} + \vec{i}_y G_{dy} \\ \vec{G}_d &= -\frac{\beta_1\beta_2}{\gamma_1}\left[\vec{i}_x G_{1x} + \vec{i}_y \frac{2-\beta_1\beta_2}{\gamma_1} G_{1y}\right] \end{aligned} \tag{35}$$

For clarity in our equations, we shall now restrict ourselves to the case of small velocities for which $\beta_1 \ll 1$. With this approximation

$$\begin{aligned} \vec{G}_d &= -\beta_1\beta_2[\vec{i}_x G_{1x} + 2\vec{i}_y G_{1y}] \\ &= -\beta_1\beta_2 \vec{G}_1 - \vec{i}_y \beta_1\beta_2 G_{1y} \end{aligned} \tag{36}$$

and since $\vec{D} = k\vec{G}$, we have

$$\vec{G}_d = -\frac{1}{kc^2}[v_1v_2\vec{D}_1 + \vec{i}_y v_1 v_2 D_{1y}] \tag{37}$$

Now define a constant, μ:

$$\mu = \frac{1}{kc^2} \tag{38}$$

Equation 37 then becomes

$$\vec{G}_d = -\mu[v_1 v_2 \vec{D}_1 + \vec{i}_y v_1 v_2 D_{1y}] \tag{39}$$

In vector form, Eq. 39 may be written as

$$\vec{G}_d = -\mu[\vec{V}_1 \cdot \vec{V}_2]\, \vec{D}_1 + \mu \vec{V}_2 \times [\vec{V}_1 \times \vec{D}_1] \tag{40}$$

Now note that if v_2 were in the z-direction, the orientation would be such that the Lorentz contraction does not occur. If v_2 were in the y-direction, we could break v_1 and v_2 into two components: one parallel to and one perpendicular to $\vec{i}_R$ and apply Eq. 40. And so we can say that Eq. 40 is general.

We now define three vector field quantities:

$$\vec{H} = \vec{V}_1 \times \vec{D}_1 \tag{41}$$

$$\vec{B} = \mu \vec{H} \tag{42}$$

$$\vec{G}_{ds} = -\mu[\vec{V}_1 \cdot \vec{V}_2]\, \vec{D}_1 \tag{43}$$

In terms of these fields we can write Eq. 40 as

$$\vec{G}_d = \vec{G}_{ds} + \vec{V}_2 \times \vec{B} \tag{44}$$

We shall now derive certain properties of $\vec{G}_{ds}$ and $\vec{H}$ which will be needed in the next section. By taking the curl of Eq. 43 we obtain

$$-\nabla \times \vec{G}_{ds} = [\mu \vec{V}_1 \cdot \vec{V}_2]\, \nabla \times \vec{D}_1 + [\nabla(\mu \vec{V}_1 \cdot \vec{V}_2)] \times \vec{D}_1$$

However, from Eq. 7, we have

$$\nabla \times \vec{D}_1 = 0$$

Also, since $\vec{V}_1$ and $\vec{V}_2$ are constant vectors, we have

$$\nabla(\mu \vec{V}_1 \cdot \vec{V}_2) = 0$$

Thus we conclude that

$$\nabla \times \vec{G}_{ds} = 0 \tag{45}$$

and

$$\vec{G}_{ds} = \nabla \phi_{ds} \tag{46}$$

To determine ϕ_{ds}, we have

$$\vec{D}_1 = -\frac{m_1}{4\pi} \nabla \frac{1}{R} \tag{47}$$

Hence from Eq. 43

$$\vec{G}_{ds} = \frac{\mu m_1 \vec{V}_1 \cdot \vec{V}_2}{4\pi} \nabla \frac{1}{R}$$

and

$$\phi_{ds} = \frac{\mu m_1 \vec{V}_1 \cdot \vec{V}_2}{4\pi R} \tag{48}$$

From Eqs. 41 and 47 we can write

$$\vec{H} = \frac{m_1}{4\pi} \left(\nabla \frac{1}{R} \right) \times \vec{V}_1 \tag{49}$$

But since $\nabla \times V_1 = 0$, Eq. 49 can be written as

$$\vec{H} = \frac{m_1}{4\pi} \nabla \times \left(\frac{\vec{V}_1}{R} \right) \tag{50}$$

Therefore, we conclude that

$$\nabla \cdot \vec{H} = 0 \tag{51}$$

since, if $\vec{A}$ is any vector,

$$\nabla \cdot \nabla \times \vec{A} = 0$$

4. Time-variant gravitational fields

The field of a mass moving with a uniform velocity must be carried along with the mass. This implies that the time and space derivatives are not independent. Their relation may be expressed as

$$\frac{\partial}{\partial t} = -\vec{V} \cdot \nabla \tag{52}$$

This means that, at a given point, the amount of change of any given field parameter in the time dt is equal to that of the same field parameter, at a fixed time, over the distance ds = -vdt along the direction of the motion of the mass.

Consider the flux density vector, $\vec{D}$. Then by applying Eq. 52 we have

$$\frac{\partial \vec{D}}{\partial t} = -(V \cdot \nabla)\, \vec{D} \tag{53}$$

To evaluate the right-hand side of this equation, we shall let $\vec{B} = \vec{V}$ and $\vec{A} = \vec{D}$ in the vector identity:

$$\nabla \times (\vec{A} \times \vec{B}) = \vec{A}\, \nabla \cdot \vec{B} - \vec{B}\, \nabla \cdot \vec{A} + (\vec{B} \cdot \nabla)\, \vec{A} - (\vec{A} \cdot \nabla)\, \vec{B} \tag{54}$$

Since $\vec{V}$ is a constant vector, we have

$$\nabla \cdot \vec{V} = 0$$

and

$$(D \cdot \nabla)\, \vec{V} = 0$$

Thus, from Eq. 54, we obtain

$$(\vec{V} \cdot \nabla)\, \vec{D} = \nabla \times (\vec{D} \times \vec{V}) + \vec{V}\, \nabla \cdot \vec{D} \tag{55}$$

But, from Eq. 6, we have

$$\vec{V}\, \nabla \cdot \vec{D} = \rho \vec{V} = \vec{J} \tag{56}$$

where J is the momentum density.

From Eq. 41 we note that

$$\nabla \times (\vec{D} \times \vec{V}) = -\nabla \times \vec{H} \tag{57}$$

By substituting Eqs. 55, 56, and 57 in Eq. 53, we obtain the first of our gravitational field equations:

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \tag{58}$$

The second field equation is obtained by applying Eq. 52 to the H-field:

$$-\frac{\partial \vec{H}}{\partial t} = (V \cdot \nabla)\, \vec{H} \tag{59}$$

To evaluate the right-hand side of this equation we again apply Eq. 52, and, since $\nabla \cdot \vec{H} = 0$ (Eq. 51), we obtain

$$(\vec{V} \cdot \nabla)\, \vec{H} = \nabla \times (\vec{H} \times \vec{V}) \tag{60}$$

We now substitute Eq. 60 in Eq. 59 and multiply by μ:

$$-\frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{B}\times\vec{V}) \tag{61}$$

Here, as defined in Eq. 44, $\vec{V} = -\vec{V}_2$. Thus, from Eqs. 44 and 45,

$$\nabla \times (\vec{B}\times\vec{V}) = \nabla \times (\vec{G}_d) \tag{62}$$

However, the total field, G, is the sum of the static and dynamic fields:

$$\vec{G} = \vec{G}_s + \vec{G}_d$$

and since, from Eq. 7, the static field is irrotational, we can write Eq. 62 as

$$\nabla \times (\vec{B}\times\vec{V}) = \nabla \times \vec{G} \tag{63}$$

Substituting Eq. 63 in Eq. 61, we obtain the second gravitational field equation:

$$\nabla \times G = -\frac{\partial B}{\partial t} \tag{64}$$

To summarize: We have derived the following gravitational field equations:

$$\nabla \times \vec{G} = -\frac{\partial \vec{B}}{\partial t} \tag{64}$$

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \tag{58}$$

$$\nabla \cdot \vec{H} = 0 \tag{51}$$

$$\vec{B} = \mu \vec{H} \tag{42}$$

$$\nabla \cdot \vec{D} = \rho \tag{6}$$

$$\vec{D} = k \vec{G} \tag{4}$$

$$\vec{J} = \rho \vec{V} \tag{56}$$

$$k\mu c^2 = 1 \tag{38}$$

where c is the velocity of light, and k is defined by Eq. 1 and has the value (see ref. 4)

$$k = -\frac{10^{11}}{26.6\pi} \qquad \frac{\text{kg sec}^2}{\text{m}^3} \tag{65}$$

We now postulate that the equations above are the general gravitational field equations under the following approximations:

a. The gravitational fields are sufficiently weak that the deformation of the metric tensor can be neglected.

b. The velocity of any masses considered is small as compared with the velocity of light.

We now note that solutions of the Maxwell equations are also solutions of these gravitational equations because they are identical in form. Thus, a gravity wave is predicted which propagates at a velocity of $[\mu k]^{-1} = c$ with a wave impedance of

$$z_o = \sqrt{\frac{\mu}{k}} = \frac{1}{-kc}$$

$$= 8.9\pi \times 10^{-19} \text{ g-ohms} \tag{66}$$

The very low wave impedance implies that the efficiency of radiation of these waves is very low. Further, they will be difficult to detect.

M. Schetzen

References

1. The assumption that D' and D are radial implies that the gravitational fields are small, so that the deformation of the metric tensor may be neglected.

2. See, for example, F. K. Richtmyer and E. H. Kennard, Introduction to Modern Physics (McGraw-Hill Publishing Company, New York, 4th edition, 1947), p. 134.

3. See, for example, A. P. French, Principles of Modern Physics (John Wiley and Sons, Inc., New York, 1958), p. 159.

4. International Critical Tables of Numerical Data, Vol. 1, Physics, Chemistry, and Technology (National Research Council, Washington, D.C., 1926), p. 395.

C. REALIZABILITY OF NONLINEAR FILTERS AND FEEDBACK SYSTEMS

This report attempts to answer the question: What is a good enough model of a physical filter — a model that will not lead to impossible results when it is used in feedback problems?

A feedback system is the embodiment of the solution of a pair of simultaneous equations. For example, the feedback system $\underline{G}$, which is shown in Fig. XII-4 is described by the equations

$$\begin{aligned} y &= \underline{H}(e) \\ e &= x + y \end{aligned} \tag{1}$$

(in which $\underline{H}$ is an operator) relating the three functions of time x(t), y(t), and e(t). A solution eliminates one of the three from consideration, and leaves an explicit relation, for example,

$$y = \underline{G}(x) \tag{2}$$

in which G(x) is the solution of the equation

$$\underline{G}(x) = \underline{H}(x+\underline{G}(x)) \tag{3}$$

In a physical system — we are not restricting ourselves to stable systems — every input produces a well-defined and unique output. Hence, if the mathematical model is

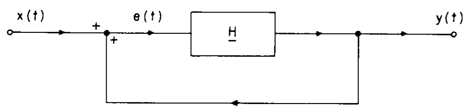

Fig. XII-4. A feedback system.

to correspond to reality, it must at least have a unique solution. The solution must also be one that can be approximated by a physical system. This limits the class of useful models, for many idealizations which it might be convenient to use yield impossible solutions or have no solution, even if they satisfy the usual linear criterion of "no response before excitation."

1. Examples of unrealizable systems

Consider the linear system obtained by letting $\underline{H}$ be a pure gain of magnitude 2. Applying the conventional feedback equation, we get

$$\begin{aligned} Y(\omega) &= \frac{H(\omega)}{1 - H(\omega)} X(\omega) \\ &= -2X(\omega) \end{aligned} \tag{4}$$

in which $X(\omega)$, $Y(\omega)$, and $H(\omega)$ are the frequency spectra of the input, output, and the operator $\underline{H}$, respectively. The response to a unit step is an inverted step of amplitude 2. It is easy enough to verify by substituting the value of y that we have found in Eq. 1 that this is indeed a solution. However, it is one that the physical system never exhibits because the inevitable delay at the highest frequencies results in instability; the output becomes infinite instantaneously if the delay is zero. Hence this model is useless, although it can be made useful by including an arbitrarily small delay.

Consider, next, the servomechanism illustrated by Fig. XII-5, consisting of a relay-controlled motor. The relay is idealized as a no-memory device with two states, and the motor is assumed to have a linear transfer characteristic. The response to a small pulse does not exist, even though we might be tempted to describe its behavior as hunting of zero amplitude and infinite frequency. If we attempt to find the output by

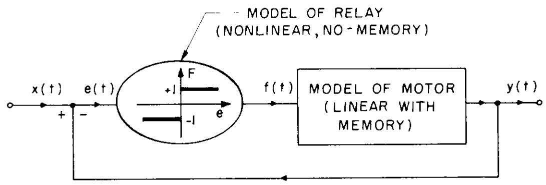

Fig. XII-5. Unrealizable model of a relay-controlled motor servomechanism.

iteration, the sequence does not converge. This system, then, is not realizable unless the slope in the transition region is made finite, or a time delay is included.

In both of these examples the open-loop model is useful because it can be realized in a limiting sense, while the closed-loop model cannot be realized at all. It is essential to take the limit after the loop is closed, not before.

2. Properties that determine a realizability class of filters

A class of filters that has sufficient conditions for realizability must be capable of giving valid results in problems involving addition, cascading, feedback, or any combination of these; it must never be subject to any of the difficulties that we have described. The following properties determine such a class, which we shall refer to as a realizability class:

a. Every filter belonging to the class can be approximated by a physical device. The meaning of this will be explored more carefully below.

b. When a filter belonging to the class is placed in a feedback loop, the feedback equations have a unique solution, so that the derived system is itself a filter. It is a trivial assertion that the same holds true for the system derived from the sum or cascade of two filters.

This property eliminates the possibility of the difficulty in the second example.

c. When a filter is derived by summing or cascading two filters belonging to the class, or by placing a filter belonging to the class in a feedback loop, an arbitrarily good physical approximation to the derived filter can be obtained by making close enough physical approximations to the original filter or pair, and placing the approximations in a feedback loop, sum or cascade.

For example, consider a filter $\underline{H}$ that belongs to the class, and is placed in a feedback loop. The resulting closed-loop system $\underline{G}$ is related to $\underline{H}$ by the equation

$$\underline{G} = \underline{H} * (\underline{I}+\underline{G}) \tag{5}$$

in which the star denotes the cascading operation, and I is the identity operator. This equation is the operator form of Eq. 3, and property (b) ensures that it has a solution for $\underline{G}$. If now $\underline{H}_p$ is any physical device, then it gives rise to a physical feedback system, $\underline{G}_p$, which satisfies the corresponding relation

$$\underline{G}_p = \underline{H}_p * (\underline{I}+\underline{G}_p) \tag{6}$$

The error in approximating $\underline{G}$ by $\underline{G}_p$ is given by

$$\underline{G} - \underline{G}_p = \underline{H} * (\underline{I}+\underline{G}) - \underline{H}_p * (\underline{I}+\underline{G}_p) \tag{7}$$

Property (c) asserts that $\underline{G} - \underline{G}_p$ can be made as small as desired by making $\underline{H} - \underline{H}_p$ small enough. (Property (a) ensures that it is, in fact, possible to build $\underline{H}_p$ so that $\underline{H} - \underline{H}_p$ is arbitrarily small.)

This property disposes of the difficulty which the first example exhibits; for this class of filters it is immaterial whether limits are taken before or after closing the loop.

d. The filter derived by summing or cascading a pair of filters that belong to the class, or by placing a filter that belongs to the class in a feedback loop, itself belongs to the class. This property ensures the usefulness of the model in problems involving arbitrary combinations of summation, cascading, and feedback.

3. Definition of the class of sufficient filters

We shall define a class of filters to which we shall refer as the class of sufficient filters; this class possesses all of the properties described in section 2, and is therefore a realizability class.

A filter is defined to be an operator $\underline{H}$, which maps inputs, e, into outputs, y. Thus

$$y = \underline{H}(e) \tag{8}$$

Here, $\underline{H}$ is defined for all e that are measurable functions of time, t. Each e is defined for all t, and is bounded on all finite time intervals, $0 \leq t \leq T$, and has the property that $e(t) = 0$ for $t < 0$. The corresponding outputs, y, are similar time functions.

A filter, $\underline{H}$, belongs to the class of sufficient filters if for every pair of inputs x_1, x_2 that belong to a collection of time functions, uniformly bounded on every finite time interval $0 \leq t \leq T$, the following inequality is valid for all t in that interval.

$$\left|\underline{H}(x_1, t)-H(x_2, t)\right| \leq k \left\{ \max_{0 \leq \tau \leq t} \left| \int_0^{\tau} [x_1(\sigma)-x_2(\sigma)\ d\sigma \right| \right\} \tag{9}$$

in which the left-hand side denotes the absolute value at time t of the difference between the responses of the filter $\underline{H}$ to the inputs x_1 and x_2.

This condition can be interpreted as a condition of inertia. If the difference between any two inputs at time σ, $[x_1(\sigma)-x_2(\sigma)]$, is regarded as a force, then the integral on the right-hand side of Eq. 9 signifies an impulse or change of momentum. If, next, the corresponding difference between the outputs at time t, $H(x_1, t) - H(x_2, t)$, is regarded as a velocity, then Eq. 9 states that the absolute velocity is always less than the maximum absolute preceding impulse, multiplied by some constant, k. Hence there is an effective mass that is always greater than 1/k.

The class of sufficient filters has, in addition to all the properties of a realizability class, the very useful property that every feedback problem involving a sufficient filter can be solved by iteration; that is, if $\underline{H}$ is a sufficient filter and $\underline{G}$ is a feedback system related to $\underline{H}$ by the expression

$$\underline{G} = \underline{H} * (\underline{I}+\underline{G}) \tag{10}$$

then $\underline{G}$ can be found by means of the iteration,

$$\begin{aligned} &\underline{G}_o = \underline{H} \\ &\underline{G}_n = \underline{H} * (I+G_{n-1}), \quad n = 1, 2, \dots \\ &\underline{G} = \lim_{n\to\infty} \underline{G}_n \end{aligned} \tag{11}$$

which converges as a consequence of Eq. 9. In fact, the iteration represented by Eq. 11 is the means of proving properties (b), (c), and (d) stated in section 2. (These proofs are omitted because they are long and tedious.)

4. Nature of approximation

When we say that a filter can be approximated by a physical device, we mean that: Given any finite time interval, and any set of inputs that are uniformily bounded on it, it is possible to construct a device whose output in response to any of these inputs does not differ from the filter output by more than some arbitrarily small error over that time interval.

We define the following norm as a measure of the size of any function (or error) over an interval in time, $a \leqslant t \leqslant b$:

$$\|x\|_a^b = \max_{a\leqslant t\leqslant b} \left| \int_a^t x(\sigma)\, d\sigma \right| \tag{12}$$

The distance between two functions, x and y, is $\|x-y\|_a^b$.

5. Method of approximation

To prove that a filter can be approximated by a physical device, we show how to synthesize the device out of elements that are assumed to be realizable, not ideally, but with tolerances on accuracy and restrictions on range of operation.

To do so, we use the fact that any filter that is uniformly continuous and restricted to a compact collection of inputs can be approximated by a finite-state device. Compactness is a property that requires that the inputs be restricted in a certain way; for example, they might be bounded in size and their spectrum limited in its behavior at high frequencies. The difficulties associated with high-frequency limitations have been avoided here by choosing a (rather unusual) norm, suggested by Brilliant (1), which de-emphasizes high frequencies – their effects are disregarded.

The compactness of the inputs ensures that the collection can be divided into a finite number of parts or "neighborhoods" with the property that the distance between any two inputs in a neighborhood does not exceed some maximum value, which can be specified before the division (1).

The division is accomplished (see Fig. XII-6) by sampling the function at regular time intervals, quantizing the samples into a finite number of levels, and delaying the

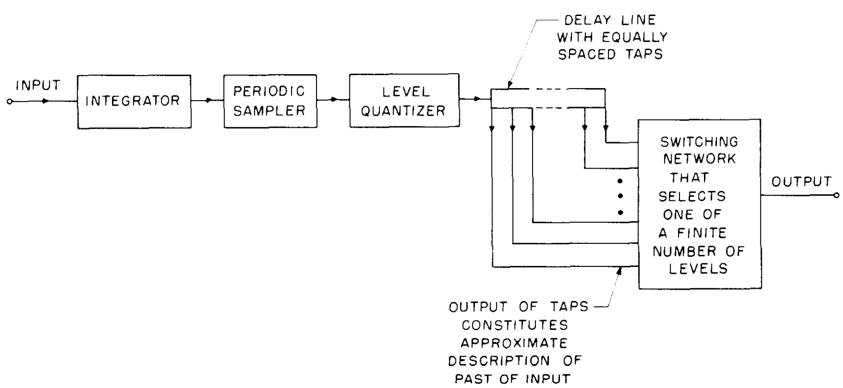

Fig. XII-6. Synthesis of a physical approximation to a filter.

samples by means of a series of delay lines, as suggested by Singleton (2). Thus, at any time, the entire past of the input is approximately determined by the outputs of the delay lines. These outputs operate a level selector through a switching device, approximately reproducing the output, one increment at a time. The same output corresponds to every input in a neighborhood. However, this procedure merely results in an error that is small enough because the filter model is uniformly continuous, as a consequence of Eq. 9.

The integrator that precedes the system has the purpose of changing inputs of bounded height into inputs of bounded slope, in order to make the subsequent quantization a valid means of determining the past.

This clearly impracticable scheme is very useful as a means for proving that our model can be related to a physical device.

6. Application of sufficient filters

To determine whether or not a filter is a good model of a physical device, we check its sufficiency by applying Eq. 9. A linear filter is sufficient if the variation, v(t), of its impulse response, $h(\gamma)$, given by the expression,

$$v(t) = |h(0)| + \int_{0^+}^{t} \left|\frac{d}{dt} h(\gamma)\right| d\gamma \tag{13}$$

exists (is not infinite) for all finite t. If it exists, then the constant, k, in Eq. 9 equals v(t).

A no-memory device is never sufficient by itself. If it is monotonic, then it is sufficient when it is cascaded with any sufficient filter, provided that the absolute value of its slope is bounded. If it has bounded slope but is not monotonic, then it is essential that the sufficient filter precede it in cascade.

It is probably true that the class of sufficient filters is general enough to describe most physical systems, provided that effects such as those of stray inertia and saturation are taken into consideration whenever they are essential to the operation of a system.

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References

1. M. B. Brilliant, Theory of the analysis of nonlinear systems, Technical Report 345, Research Laboratory of Electronics, M. I. T., March 3, 1958.

2. H. E. Singleton, Theory of nonlinear transducers, Technical Report 160, Research Laboratory of Electronics, M. I. T., Aug. 12, 1950.