Summary of Time-Dependent Perturbation Theory

\[ \langle \Omega | T [\phi(x_1) \cdots \phi(x_n)] | \Omega \rangle = \left( \text{sum of all connected diagrams with } n \text{ external points} \right) \]

Example: The four-point function:

\[
\langle \Omega | T \phi_1 \phi_2 \phi_3 \phi_4 | \Omega \rangle \\
= \quad \text{term 1} + \quad \text{term 2} + \quad \text{term 3} + \quad \text{term 4} + \ldots \\
+ \quad \text{term 5} + \quad \text{term 6} + \quad \text{term 7} + \ldots \\
+ \quad \text{term 8} + \ldots + \quad \text{term 9} + \ldots.
\]

---

Cross Sections and Decay Rates

Definition of Cross Section:

\[
\begin{array}{c}
\rho_B \\
\ell_B
\end{array} \xrightarrow{v} \begin{array}{c}
\rho_A \\
\ell_A
\end{array}
\]

where

- \( \rho_A \) and \( \rho_B \) = number density of particles
- \( A \) = cross sectional area of beams
- \( \ell_A \) and \( \ell_B \) = lengths of particle packets

Then

\[
\sigma = \frac{\text{Number of scattering events of specified type}}{\rho_A \ell_A \rho_B \ell_B A}.
\]

Note that \( \sigma \) depends on frame.

Special case: 1 particle in each beam, so \( \rho_A \ell_A A = 1 \) and \( \rho_B \ell_B A = 1 \). Then

\[
\text{Number of events} = \frac{\sigma}{A}.
\]
**Definition of Decay Rate:**

\[ \Gamma \equiv \frac{\text{Number of decays per unit time}}{\text{Number of particles present}} . \]

Number of surviving particles at time \( t \):

\[ N(t) = N_0 e^{-\Gamma t} . \]

Mean lifetime:

\[ \tau = \frac{1}{N_0} \int_0^\infty dt \left( -\frac{dN}{dt} \right) t = 1/\Gamma . \]

Half-life:

\[ e^{-\Gamma t} = \frac{1}{2} \quad \Rightarrow \quad t_{1/2} = \tau \ln 2 . \]

---

**Unstable Particles as Resonances (Preview):**

Unstable particles are not eigenstates of \( H \); they are resonances in scattering experiments.

In nonrelativistic quantum mechanics, the Breit-Wigner formula

\[ f(E) \propto \frac{1}{E - E_0 + i\Gamma/2} \quad \Rightarrow \quad \sigma \propto \frac{1}{(E - E_0)^2 + \Gamma^2/4} . \]

The “full width at half max” of the resonance = \( \Gamma \).

In the relativistic theory, the Breit-Wigner formula is replaced by a modified (Lorentz-invariant) propagator:

\[ \frac{1}{p^2 - m^2 + i\Gamma} \approx \frac{1}{2E_{\bar{p}}(p^0 - E_{\bar{p}} + i(m/E_{\bar{p}})\Gamma/2) , \]

which can be seen using

\[ (p^0)^2 - |\vec{p}|^2 - m^2 = (p^0)^2 - E_{\bar{p}}^2 = (p^0 + E_{\bar{p}})(p^0 - E_{\bar{p}}) \approx 2E_{\bar{p}}(p^0 - E_{\bar{p}}) . \]
**Cross Sections and the S-Matrix**

**Initial and Final States—In- and Out-States:**

Recall our discussion of particle creation by an external source,

\[(\Box + m^2)\phi(x) = j(x),\]

where \(j\) was assumed to be nonzero only during a finite interval \(t_1 < t < t_2\).

- In that case, the Fock space of the free theory for \(t < t_1\) defined the in-states, the Fock space of the free theory for \(t > t_2\) defined the out-states, and we could calculate exactly the relationship between the two.
- We started in the in-vacuum and stayed there. The amplitude

\[\langle \vec{p}_1 \, \vec{p}_2 \ldots \vec{p}_N, \text{out} | 0, \text{in} \rangle\]

was then interpreted as the amplitude for producing a set of final particles with momenta \(\vec{p}_1 \ldots \vec{p}_N\).

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For interacting QFT’s, it is more complicated. The interactions do not turn off, and affect even the 1-particle states. It is still possible to define in- and out-states \(|\vec{p}_1 \ldots \vec{p}_N, \text{in}\rangle\) and \(|\vec{p}_1 \ldots \vec{p}_N, \text{out}\rangle\) with the following properties:

- They are exact eigenstates of the full Hamiltonian.
- At asymptotically early times, wavepackets constructed from \(|\vec{p}_1 \ldots \vec{p}_N, \text{in}\rangle\) evolve as free wavepackets. (The pieces of this ket that describe the scattering vanish in stationary phase approximation at early times.) These states are used to describe the initial state of the scattering.
- At asymptotically late times, wavepackets constructed from \(|\vec{p}_1 \ldots \vec{p}_N, \text{out}\rangle\) evolve as free wavepackets. These states are used to describe the final state.
**Wavepacket States:**

One-particle incoming wave packet:

\[ |\phi\rangle = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E_k}} \phi(\vec{k}) \begin{pmatrix} \vec{k}, \text{in} \end{pmatrix}, \]

where

\[ \langle \phi | \phi \rangle = 1 \implies \int \frac{d^3k}{(2\pi)^3} |\phi(\vec{k})|^2 = 1. \]

Two-particle initial state:

\[ |\phi_A \phi_B, \vec{b}, \text{in} \rangle = \int \frac{d^3k_A}{(2\pi)^3} \frac{d^3k_B}{(2\pi)^3} \frac{\phi_A(\vec{k}_A) \phi_B(\vec{k}_B)e^{-i\vec{b} \cdot \vec{k}_B}}{\sqrt{(2E_A)(2E_B)}} \begin{pmatrix} \vec{k}_A \vec{k}_B, \text{in} \end{pmatrix}, \]

where \( \vec{b} \) is a vector which translates particle \( B \) orthogonal to the beam, so that we can construct collisions with different impact parameters.

Multiparticle final state:

\[ \langle \phi_1 \ldots \phi_n, \text{out} | = \left( \prod_{f=1}^n \int \frac{d^3p_f}{(2\pi)^3} \frac{\phi_f(\vec{p}_f)}{\sqrt{2E_f}} \right) \langle \vec{p}_1 \ldots \vec{p}_n, \text{out} |. \]

---

**The S-Matrix:**

Definition:

\[ S |\Psi, \text{out} \rangle = |\Psi, \text{in} \rangle. \]

Therefore

\[ \langle \Psi, \text{out} | \Psi, \text{in} \rangle = \langle \Psi, \text{out} | S |\Psi, \text{out} \rangle. \]

But \( S \) maps a complete set of orthonormal states onto a complete set of orthonormal states, so \( S \) is unitary. Therefore

\[ \langle \Psi, \text{out} | S |\Psi, \text{out} \rangle = \langle \Psi, \text{out} | S^\dagger S |\Psi, \text{out} \rangle = \langle \Psi, \text{in} | S |\Psi, \text{in} \rangle, \]

so P&S often do not label the states as in or out.

---
\textbf{S, T, and M:}

No scattering \implies \text{final} = \text{initial}, so separate this part of S:

\[ \mathcal{S} \equiv 1 + iT. \]

But T must contain a momentum-conserving $\delta$-function, so define

\[ \langle \vec{p}_1 \ldots \vec{p}_n | iT | k_A k_B \rangle \equiv (2\pi)^4 \delta^{(4)} \left( k_A + k_B - \sum p_f \right) \cdot i\mathcal{M}(k_A k_B \to \{p_f\}). \]

\textbf{Evaluation of the Cross Section:}

The probability of scattering into the specified final states is just the square of the
S-matrix element, summed over the final states:

\[ \mathcal{P} \left( \{AB, \vec{b}\} \to \{p_1 \ldots p_n\} \right) = \left( \prod_{j=1}^{n} \int \frac{d^3 p_f}{(2\pi)^3} \frac{1}{2E_f} \right) \left| \langle \vec{p}_1 \ldots \vec{p}_n | S | \phi_A \phi_B, \vec{b} \rangle \right|^2. \]

To relate to the cross section, think of a single particle $B$ scattering off of a particle $A$, with impact parameter vector $\vec{b}$:

\[ \begin{figure}[h]
\centering
\begin{tikzpicture}
\node (A) at (0,0) {$A$};
\node (B) at (2,1) {$B$};
\draw[dashed] (A) -- (B);
\draw (B) -- (B -| A) node [midway, above=1cm] {$\vec{b}$};
\end{tikzpicture}
\end{figure} \]

Remembering that the cross section can be viewed as the cross sectional area
blocked off by the target particle,

\[ d\sigma = \int d^2 b \mathcal{P} \left( \{AB, \vec{b}\} \to \{p_1 \ldots p_n\} \right). \]
Substituting the expression for $\mathcal{P}$ and writing out the wavepacket integrals describing the initial state,

$$
\begin{align*}
\mathcal{P} & = \int \mathcal{D}^2 b \left( \prod_{f=1}^{n} \frac{\mathcal{D}^3 p_f}{(2\pi)^3 2E_f} \right) \int \mathcal{D}^3 k_A \frac{\phi_A(k_A)}{(2\pi)^3} \int \mathcal{D}^3 k_B \frac{\phi_B(k_B)}{(2\pi)^3} \sqrt{(2E_A)(2E_B)} \times \int \frac{\mathcal{D}^3 \vec{k}_A}{(2\pi)^3} \int \frac{\mathcal{D}^3 \vec{k}_B}{(2\pi)^3} \frac{\phi_A^*(\vec{k}_A) \phi_B^*(\vec{k}_B)}{\sqrt{(2E_A)(2E_B)}} e^{i\vec{k}_A \cdot \vec{p}} \times \left\langle \vec{p}_1 \ldots \vec{p}_n | S | \vec{k}_A \vec{k}_B \right\rangle \left\langle \vec{p}_1 \ldots \vec{p}_n | S | \vec{k}_A \vec{k}_B \right\rangle^* .
\end{align*}
$$

This can be simplified by using

$$
\begin{align*}
\int \mathcal{D}^2 b e^{i\vec{k} \cdot (\vec{p}_B - \vec{p}_A)} & = (2\pi)^2 \delta^{(2)}(\vec{k}_B - \vec{k}_A) , \\
\left\langle \vec{p}_1 \ldots \vec{p}_n | S | \vec{k}_A \vec{k}_B \right\rangle & = i\mathcal{M} \left( \vec{k}_A \vec{k}_B \rightarrow \{\vec{p}_f\} \right) (2\pi)^4 \delta^{(4)} (k_A + k_B - \sum p_f) , \\
\left\langle \vec{p}_1 \ldots \vec{p}_n | S | \vec{k}_A \vec{k}_B \right\rangle^* & = -i\mathcal{M}^* \left( \vec{k}_A \vec{k}_B \rightarrow \{\vec{p}_f\} \right) (2\pi)^4 \delta^{(4)} (k_A + k_B - \sum p_f) .
\end{align*}
$$

We first integrate over $\vec{k}_A$ and $\vec{k}_B$ using $\delta^{(2)}(\vec{k}_B - \vec{k}_A)$ and

$$
\begin{align*}
\delta^{(4)} (k_A + k_B - \sum p_f) & = \delta^{(2)} (k_A^+ + k_B^+ - \sum p_f^+) \delta (k_A^z + k_B^z - \sum p_f^z) \\
& \times \delta \left( \vec{E}_A + \vec{E}_B - \sum E_f \right) ,
\end{align*}
$$

where the beam is taken along the $z$-axis. After integrating over $\vec{k}_A$ and $\vec{k}_B$, we are left with

$$
\begin{align*}
\int \mathcal{D}^2 \vec{k}_B \delta \left( k_A^z + k_B^z - \sum p_f^z \right) \delta \left( \vec{E}_A + \vec{E}_B - \sum E_f \right) & = \int \mathcal{D}^2 \vec{k}_A \delta \left( F(k_A^z) \right) ,
\end{align*}
$$

where the first $\delta$-function was used to integrate $k_B^z$, and

$$
F(k_A^z) = \sqrt{|k_A^z|^2 + (k_A^z)^2 + m^2} + \sqrt{|k_B^z|^2 + (\sum p_f^z - k_A^z)^2 + m^2} - \sum E_f .
$$

Then

$$
\int \mathcal{D}^2 \vec{k}_A \delta \left( F(k_A^z) \right) = \frac{1}{|df|} ,
$$

evaluated where $F(k_A^z) = 0$ .
Rewriting
\[ F(\vec{k}_A) = \sqrt{|\vec{k}_A|^2 + (\vec{k}_A^z)^2 + m^2 + \sqrt{|\vec{k}_B|^2 + \left(\sum p_f^z - \vec{k}_A^z\right)^2 + m^2 - \sum E_f}, \]

one finds
\[ \left| \frac{dF}{d\vec{k}_A^z} \right| = \frac{\vec{k}_A^z E_A - \sum p_f^z - \vec{k}_A^z}{E_A} \]

Remembering the \( \delta \)-function constraint \( \delta \left( \vec{k}_A^z + \vec{k}_B^z - \sum p_f^z \right) \) from the previous slide, one has
\[ \left| \frac{dF}{d\vec{k}_A^z} \right| = \left| \frac{\vec{k}_A^z}{E_A} - \frac{\vec{k}_B^z}{E_B} \right| = \left| v_A^z - v_B^z \right|. \]

What values of \( \vec{k}_A \) satisfy the constraint \( F(\vec{k}_A) = 0 \)? There are two solutions, since \( F(\vec{k}_A) = 0 \) can be manipulated into a simple quadratic equation. (To see this, move one square root to the RHS of the equation and square both sides. The \( (\vec{k}_A^z)^2 \) term on each side cancels, leaving only linear terms and a square root on the LHS. Isolate the square root and square both sides again, obtaining a quadratic equation.) One solution gives \( \vec{k}_A = \vec{k}_A \), and the other corresponds to \( A \) and \( B \) approaching each other from opposite directions. Assume that the initial wavepacket is too narrow to overlap the 2nd solution.

---

Then
\[ d\sigma = \left( \prod_{f=1}^{n} \frac{d^3 p_f}{(2\pi)^3 2 E_f} \right) \int \frac{d^3 k_A}{(2\pi)^3} \int \frac{d^3 k_B}{(2\pi)^3} \frac{|\phi_A(\vec{k}_A)|^2 |\phi_B(\vec{k}_B)|^2}{(2 E_A)(2 E_B) |v_A^z - v_B^z|} \times \left| \mathcal{M} (\vec{k}_A \vec{k}_B \rightarrow \{\vec{p}_f\}) \right|^2 (2\pi)^4 \delta^{(4)} \left( k_A + k_B - \sum p_f \right). \]

Define the relativistically invariant n-body phase space measure
\[ d\Pi_n(P) = \left( \prod_{f=1}^{n} \frac{d^3 p_f}{(2\pi)^3 2 E_f} \right) (2\pi)^4 \delta^{(4)} \left( P - \sum p_f \right), \]

and assume that \( E_A(\vec{k}_A), E_B(\vec{k}_B), |v_A^z - v_B^z|, \left| \mathcal{M} (\vec{k}_A \vec{k}_B \rightarrow \{\vec{p}_f\}) \right|^2 \), and \( d\Pi_n(k_A + k_B) \) are all sufficiently slowly varying that they can be evaluated at the central momenta of the two initial wavepackets, \( \vec{k}_A = \vec{p}_A \) and \( \vec{k}_B = \vec{p}_B \). Then the normalization of the wavepackets implies that
\[ \int \frac{d^3 k_A}{(2\pi)^3} \int \frac{d^3 k_B}{(2\pi)^3} |\phi_A(\vec{k}_A)|^2 |\phi_B(\vec{k}_B)|^2 = 1. \]
so finally

\[ d\sigma = \frac{|\mathcal{M}(\vec{p}_A \vec{p}_B \rightarrow \{\vec{p}_f\})|^2}{(2E_A)(2E_B)|v_A^z - v_B^z|} \ d\Pi_n(p_A + p_B). \]

This formula holds whether the final state particles are distinguishable or not. In calculating a total cross section, however, one must not double-count final states. If the final state contains \( n \) identical particles, one must either restrict the integration or divide the answer by \( n! \).

---

**Special Case: Two-Particle Final State:**

In the center of mass (CM) frame, \( \vec{p}_A = -\vec{p}_B \) and \( E_{cm} = E_A + E_B \), so

\[
d\Pi_2(p_A + p_B) = \left( \prod_{j=1}^{2} \frac{d^3 p_f}{(2\pi)^3} \frac{1}{2E_f} \right) (2\pi)^4 \delta^{(4)}(p_A + p_B - \sum p_f) \\
= \frac{d^3 p_1}{(2\pi)^3} \frac{d^3 p_2}{(2\pi)^3} \frac{1}{(2E_1)(2E_2)} (2\pi)^4 \delta^{(4)}(E_{cm} - E_1 - E_2) \\
= \frac{d\Omega}{(2\pi)^3} \frac{p_1^2}{(2\pi)^3} \frac{1}{(2E_1)(2E_2)} (2\pi)^4 \delta\left(E_{cm} - \sqrt{p_1^2 + m_1^2} - \sqrt{p_2^2 + m_2^2}\right) \\
= \frac{d\Omega}{16\pi^2 E_{cm}} \frac{p_1}{E_1 + E_2} \left| \frac{p_1}{E_1} + \frac{p_1}{E_2} \right|^{-1} \\
= \frac{d\Omega}{16\pi^2 E_{cm}}.
\]
The two-particle final state, center-of-mass cross section is then

\[
\left( \frac{d\sigma}{d\Omega} \right)_{\text{cm}} = \frac{|\vec{p}_A| |\mathcal{M}(\vec{p}_A \vec{p}_B \rightarrow \vec{p}_1 \vec{p}_2)|^2}{64\pi^2 E_A E_B (E_A + E_B) |v_A^z - v_B^z|}.
\]

If all four masses are equal, then

\[E_A = E_B = \frac{1}{2} E_{\text{cm}}\]

and

\[|v_A^z - v_B^z| = \frac{2 |\vec{p}_A|}{E_A} = \frac{4 |\vec{p}_A|}{E_{\text{cm}}},\]

so

\[
\left( \frac{d\sigma}{d\Omega} \right)_{\text{cm}} = \frac{\mathcal{M}^2}{64\pi^2 E_{\text{cm}}^2} \quad (\text{all masses equal}).
\]

---

**Evaluation of the Decay Rate:**

The formula for decay rates is more difficult to justify, since decaying particles have to be viewed as resonances in a scattering experiment. For now we just state the result. By analogy with the formula for cross sections,

\[
d\sigma = \frac{|\mathcal{M}(\vec{p}_A \vec{p}_B \rightarrow \{\vec{p}_f\})|^2}{(2E_A)(2E_B)|v_A^z - v_B^z|} d\Pi_n(p_A + p_B),
\]

we write

\[
d\Gamma = \frac{|\mathcal{M}(\vec{p}_A \rightarrow \{\vec{p}_f\})|^2}{2E_A} d\Pi_n(p_A).
\]

Here \(\mathcal{M}\) cannot be defined in terms of an \(S\)-matrix, since decaying particles cannot be described by wavepackets constructed in the asymptotic past. \(\mathcal{M}\) can be calculated, however, by the Feynman rules that Peskin & Schroeder describe in Section 4.6. If some or all of the final state particles are identical, then the same comments that were made about cross sections apply here.