

## VIII. MICROWAVE ELECTRONICS\*

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### A. ELECTROMAGNETIC WAVES IN DENSE ELECTRON-BEAM WAVEGUIDES AND THEIR INTERACTION WITH ELECTROMAGNETIC FIELDS OF GAPS

We have previously considered (1, 2, 3) this problem under the assumption that the space-charge densities in the electron stream are weak ( $\omega_p/\omega \ll 1$ ). We shall now attempt to remove this restriction and solve the problem for arbitrary space-charge densities, that is, for arbitrary  $\omega_p/\omega$ . In the present discussion we shall consider electron beams that are confined by infinite magnetic fields. The extension of the following results to electron beams with zero curl of the generalized momentum, and focused by finite magnetic fields, is straightforward and will be given in a later report.

#### 1. Natural Waves

Consider an electron beam inside a perfectly conducting metallic waveguide, as shown in Fig. VIII-1. The system is assumed to be uniform in the z-direction, and the magnetic focusing field is assumed to be of large amplitude so that the electron motion is constrained to be entirely in the z-direction. The linearized Maxwell equations are

$$\nabla \times \bar{\mathbf{E}} = -j\omega\mu_0 \bar{\mathbf{H}} \quad (1)$$

$$\nabla \times \bar{\mathbf{H}} = j\omega\epsilon_0 \bar{\mathbf{E}} + \bar{\mathbf{i}}_z J \quad (2)$$

$$\nabla \cdot \bar{\mathbf{E}} = \frac{\rho}{\epsilon_0} \quad (3)$$

$$\nabla \cdot \bar{\mathbf{H}} = 0 \quad (4)$$

and the linearized force and continuity equations are

$$\left( j\beta_e + \frac{\partial}{\partial z} \right) V = E_z \quad (5)$$

$$\left( j\beta_e + \frac{\partial}{\partial z} \right) J = j\omega\epsilon_0 \beta_p^2 V \quad (6)$$

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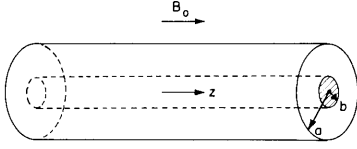


Fig. VIII-1. Electron-beam waveguide.

We have assumed a time-dependence  $e^{j\omega t}$  and introduced the kinetic voltage  $V = (m/e) v_0 v$ . It can be seen from these equations that the only solutions that interact with the electron beam are those for TM waves, that is, those having a longitudinal electric field,  $E_z$ . For TM waves

with a longitudinal dependence of the form  $e^{-j\beta z}$ , we obtain the following wave equations. (Hereafter, superscript  $p$  will denote field quantities applying to the region inside the electron beam, and superscript  $a$  will denote field quantities applying to the region outside the electron beam.)

Inside the beam:

$$\left(\nabla_T^2 + p^2\right) E_z^p = 0 \quad (7)$$

$$p^2 = (-\beta^2 + k^2) \left[ 1 - \frac{\beta_p^2}{(\beta - \beta_e)^2} \right] \quad (8)$$

Outside the beam:

$$\left(\nabla_T^2 + q^2\right) E_z^a = 0 \quad (9)$$

$$q^2 = -\beta^2 + k^2 \quad (10)$$

The boundary conditions provide another relationship between the propagation constants  $p$  and  $q$  and the geometry of the particular system. In general, this relationship is of the form

$$D(p, q, a, b) = 0 \quad (11)$$

The transverse electromagnetic fields are given in terms of the longitudinal electric field by

$$\bar{E}_T = -\frac{j\beta}{q} \nabla_T E_z \quad (12)$$

and

$$\bar{H}_T = -\frac{j\omega\epsilon_0}{q} \bar{i}_z \times \nabla_T E_z \quad (13)$$

Thus the solution of Eqs. 7-11, together with Eqs. 5, 6, 12, and 13, determines all of the electromagnetic fields and the motion of the electrons. (In general, the propagation

constants of this system may be complex; we shall then write  $\Gamma$  for  $j\beta$ , where  $\Gamma$  is complex.) The kinetic power theorem provides a set of power orthogonality relations (4) for the waves of such a lossless system. Consider two waves with dependence  $e^{-\Gamma n z}$  and  $e^{-\Gamma_m z}$ . We then have

$$\int_A \left[ \hat{E}_n \times \hat{H}_m^* + \hat{E}_m^* \times \hat{H}_n + \hat{V}_n \hat{J}_m^* + \hat{V}_m^* \hat{J}_n \right] \cdot d\bar{a} = 0; \quad \Gamma_n \neq \Gamma_m^* \quad (14)$$

in which the circumflex signs on the field quantities indicate that their  $z$ -dependence is omitted. The integration is carried out over the entire cross section of the system. The orthogonality relationship of Eq. 14 allows us to expand arbitrary electromagnetic and kinetic fields at a particular cross section of the system. We thus have

$$\bar{E} = \sum_m a_n \hat{E}_n e^{-\Gamma_n z} \quad (15)$$

$$\bar{H} = \sum_n a_n \hat{H}_n e^{-\Gamma_n z} \quad (16)$$

$$\bar{V} = \sum_n a_n \hat{V}_n e^{-\Gamma_n z} \quad (17)$$

$$\bar{J} = \sum_n a_n \hat{J}_n e^{-\Gamma_n z} \quad (18)$$

$$a_m = \frac{\int_A \left[ \bar{E}_T \times \hat{H}_{mT}^* + \hat{E}_{mT}^* \times \bar{H}_T + \bar{V} \hat{J}_m^* + \hat{V}_m^* \bar{J} \right] \cdot da}{2 \operatorname{Re} \int_A \left[ \hat{E}_{mT} \times \hat{H}_{mT}^* + \hat{V}_m \hat{J}_m^* \right] \cdot d\bar{a}} \quad (19)$$

Equation 19 gives the excitation of the beam modes in terms of a specified set of fields  $\bar{E}_T$ ,  $\bar{H}_T$ ,  $\bar{V}$ , and  $\bar{J}$  at the reference plane  $z = 0$ .

## 2. Interaction of Beam Waves with Fields of a Gap

Consider the system of Fig. VIII-2, in which a gap has been introduced into the waveguide wall. We shall assume some structure external to the gap (for example, a resonant cavity) which will support electromagnetic fields, and, for the present, we shall let this structure be arbitrary. The problem of the excitation of the electron-beam waveguide by the external fields is similar to the problem of the homogeneous, empty, rectangular waveguide with a T junction (5) or to the problem of a homogeneous, empty, circular waveguide with a gap (6). Because of the presence of the gap in the wall the

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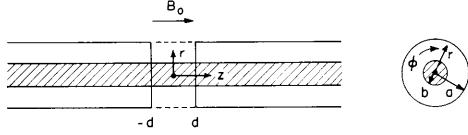


Fig. VIII-2. Electron-beam waveguide with a gap.

electromagnetic fields of the present system must satisfy nonuniform boundary conditions at radius  $a$ . The exact solution for the electromagnetic fields in the gap can only be obtained through a simultaneous solution of the field problem inside the electron-beam waveguide and outside the

gap, in the electromagnetic circuit structure, and by subsequently matching these fields at the gap boundary. For gaps whose extent is short compared with one-half of a free-space wavelength a simpler approach can be used (7). The electric field in the gap at radius  $a$  can be considered quasi-static, and hence a reasonable guess as to its nature can be made. These gap fields can now be considered to arise from, and to give rise to, excitations in the electron-beam waveguide and in the external electromagnetic circuit. In this report we shall consider the excitation of the electron-beam waveguide and leave the circuit-excitation problem for another time.

The nonuniform nature of the boundary conditions at  $r = a$  requires us to write the electromagnetic fields in terms of Fourier integrals. We write

$$E_z^{a,p}(r, z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{E}^{a,p}(r, \beta) e^{-j\beta z} d\beta \quad (20)$$

Here, we have assumed that there is no  $\phi$  variation and that the superscript  $a$  and  $p$  refer, respectively, to the regions outside and inside the electron beam. This field must satisfy the wave equations (Eqs. 7 and 9), and, consequently, can be written as

$$E_z^{a,p}(r, z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi^{a,p}(p, q, r) \mathcal{E}(\beta) e^{-j\beta z} d\beta \quad (21)$$

The function  $\Phi(p, q, r)$  is determined by the wave equations and the boundary conditions for particular systems. The function  $\mathcal{E}(\beta)$  is determined by the boundary conditions at radius  $r = a$ . Two simple situations for the gap field at radius  $r = a$  are illustrated in Fig. VIII-3a and VIII-3b. For the case of gridless gaps with thin-walled edges we have

$$E_z^a(a, z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} J_0(\beta d) e^{-j\beta z} d\beta \quad (22)$$

for which

$$\mathcal{E}(\beta) = \frac{J_0(\beta d)}{\Phi^a(p, q, r=a)} \quad (23)$$

and for the case of gridded gaps with thick-walled edges, we have

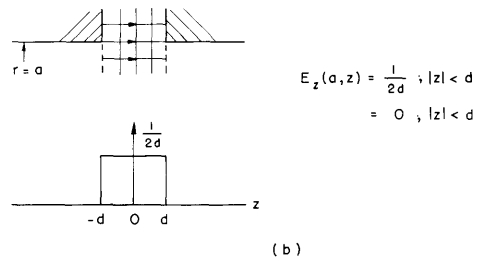
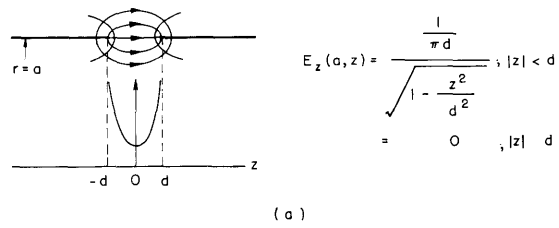


Fig. VIII-3. Static-field solutions at the gap edge: (a) gridless, thin-walled gap; (b) gridded, thick-walled gap.

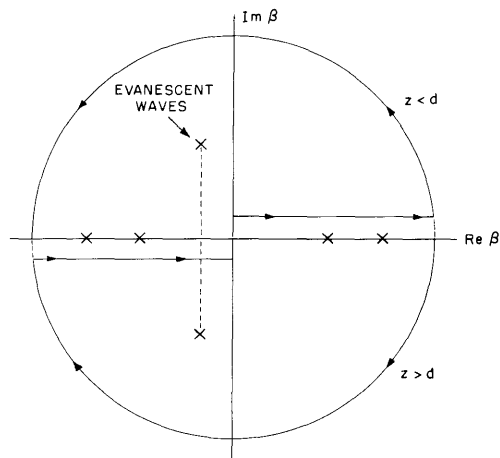


Fig. VIII-4. Integration path for Eq. 27.

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$$E_z^a(a, z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\sin \beta d}{\beta d} e^{-j\beta z} d\beta \quad (24)$$

for which

$$\mathcal{E}(\beta) = \frac{\frac{\sin \beta d}{\beta d}}{\Phi^a(p, q, r=a)} \quad (25)$$

Here, we are interested in gridless gaps; consequently we make use of Eq. 23. Equation 21 can then be written as

$$E_z^{a,p}(r, z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\Phi^{a,p}(p, q, r)}{\Phi^a(p, q, r=a)} J_0(\beta d) e^{-j\beta z} d\beta \quad (26)$$

As an illustration, consider a solid electron beam of radius  $b$  and a drift tube of radius  $a$ . The solution of the wave equations (Eqs. 7 and 9), subject to the boundary conditions at the radius of the beam  $r = b$ , gives

For  $0 \leq r \leq b$ :

$$E_z^p(r, z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{J_0(pr)}{D(p, q, b, a)} J_0(\beta d) e^{-j\beta z} d\beta \quad (27)$$

For  $b \leq r \leq a$ :

$$E_z^a(r, z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{D(p, q, b, r)}{D(p, q, b, a)} J_0(\beta d) e^{-j\beta z} d\beta \quad (28)$$

where

$$D(p, q, b, r) = pb J_1(pb) H_{0,0}(qr, qb) - qb J_0(pb) H_{0,1}(qr, qb) \quad (29)$$

and the biradial functions are

$$H_{m,n}(x, y) = \frac{\pi}{2} [J_m(x)N_n(y) - N_m(x)J_n(y)]$$

Equations 27 and 28 can be evaluated by methods of contour integration in the complex  $\beta$  plane. For large values of  $z$ , we shall only encounter propagating waves in the electron-beam waveguide. These propagating waves will have propagation constants that are pure real. For these waves, the integration of Eqs. 27 and 28 can be carried out as illustrated in Fig. VIII-4. The results of the integration are as follows.

For  $z \gg d$ :

$$E_z^p(r, z) = \sum_n -j \left[ \frac{J_0(pr) J_0(\beta d)}{D'(p, q, b, a)} e^{-j\beta z} \right]_{\beta_n, q_n, p_n} \quad (30)$$

$$E_z^a(r, z) = \sum_n -j \left[ \frac{D(p, q, b, r) J_0(\beta d)}{D'(p, q, b, a)} e^{-j\beta z} \right]_{\beta_n, q_n, p_n} \quad (31)$$

where  $D'$  is the derivative of  $D$  with respect to  $\beta$ , and the summation is over the real  $\beta$  roots, determined by

$$D(p, q, b, a) = 0 \quad (32)$$

and Eqs. 8 and 10.

A. Bers

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7. The approach outlined here was suggested by Professor L. J. Chu, of Massachusetts Institute of Technology.

