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A. ON THE REALIZATION OF AN N$^{\text{th}}$-ORDER G MATRIX

As pointed out in a previous report (1), a given $n \times n$ G matrix may still be realizable even though the realization technique leading to an (n+1)-node network fails. It was also shown that its realization in a 2n-node network, in which the terminal pairs appropriate to the given G matrix pertain to alternate branches in a linear tree, is the most general realization possibility for this matrix. In order to apply the (n+1)-node technique to this situation, however, one must first augment the given n^{th}-order G matrix to one of order (2n−1); and this augmentation must fulfill the obvious requirement that subsequent abridgment will re-establish the original matrix. Our first topic, therefore, is concerned with this augmentation process.

1. Augmentation of the Given G Matrix

Suppose we consider a matrix B of order m with its rows and columns partitioned into groups of r and s. The sum r + s, of course, equals m. In this partitioned form we can write

$$B = \left[\begin{array}{c|c} b_{rr} & b_{rs} \\ \hline b_{sr} & b_{ss} \end{array}\right] \tag{1}$$

In a set of equilibrium equations having the matrix B, the first r variables pertain to accessible terminal pairs. The remaining variables we wish to suppress or eliminate. As is well known (2), the abridged matrix pertinent to the desired r points of access is given by

$$B_{abr} = b_{rr} - b_{rs} \cdot b_{ss}^{-1} \cdot b_{sr} \tag{2}$$

This same result applies if the r retained variables are not necessarily those designated by the suffixes 1 . . . r but are <u>any</u> r of the m original variables. Instead of forming b_{rr} by deleting all rows and columns designated by suffixes larger than r, we delete all rows and columns pertinent to the variables we wish to suppress. The submatrix b_{ss} then consists of the elements located upon the intersections of these deleted rows and columns; the elements in b_{rs} are the remaining ones in the deleted columns, and those in b_{sr} are the remaining ones in the deleted rows – the relative positions of all elements remaining unaltered.

The important point to recognize now is that if the matrix B has rank s, then B_{abr}

given by Eq. 2 is identically zero. This fact may readily be seen. Suppose we construct B by writing down s independent rows, each containing (r+s) elements, and regarding these rows as represented by the partitioned matrix

$$\left[b_{sr} \;\vdots\; b_{ss}\right] \tag{3}$$

Here we assume that the square submatrix b_{ss} is nonsingular. If we now construct r additional rows that are linear combinations of the rows in matrix 3, these can be written

$$\ell_{rs} \times \left[b_{sr} \;\vdots\; b_{ss}\right] \tag{4}$$

in which ℓ_{rs} is a matrix (with r rows and s columns) effecting these row combinations. The matrix

$$B = \left[\begin{array}{c:c} \ell_{rs}b_{sr} & \ell_{rs}b_{ss} \\ \hdashline b_{sr} & b_{ss} \end{array}\right] \tag{5}$$

clearly has rank s and its abridgment, according to the process indicated in Eq. 2, yields a null result.

The desired augmentation of a given G matrix, as mentioned above, may therefore be accomplished in the following manner. To be specific, let us consider a matrix of order 3:

$$G = \begin{bmatrix} G_{11} & G_{12} & G_{13} \\ G_{21} & G_{22} & G_{23} \\ G_{31} & G_{32} & G_{33} \end{bmatrix} \tag{6}$$

If we wish to augment to order 4, we first insert a null row and column so as to have

$$G_{exp} = \begin{bmatrix} G_{11} & 0 & G_{12} & G_{13} \\ 0 & 0 & 0 & 0 \\ G_{21} & 0 & G_{22} & G_{23} \\ G_{31} & 0 & G_{32} & G_{33} \end{bmatrix} \tag{7}$$

to which we shall refer as the <u>expanded</u> form of G. The expanded form can alternately have zeros in any row and column other than the second, but in the present discussion we shall always choose for this purpose the even-numbered rows and columns, since these pertain to the branches in a linear tree that do not represent accessible terminal pairs, and hence are associated with the variables to be eliminated.

The desired augmented G matrix is now formed by adding to G_{exp} any 4^{th}-order matrix B or rank 1:

$$G_{aug} = G_{exp} + B \tag{8}$$

The abridgment of G_{aug} according to the process indicated in Eq. 2, in which the deleted row and column is the second, clearly yields the original matrix in Eq. 6.

If we wish to augment G in Eq. 6 to order 5, we form

$$G_{exp} = \begin{bmatrix} G_{11} & 0 & G_{12} & 0 & G_{13} \\ 0 & 0 & 0 & 0 & 0 \\ G_{21} & 0 & G_{22} & 0 & G_{23} \\ 0 & 0 & 0 & 0 & 0 \\ G_{31} & 0 & G_{32} & 0 & G_{33} \end{bmatrix} \tag{9}$$

and then get the desired G_{aug} from Eq. 8, in which B is any 5^{th}-order matrix of rank 2.

If G_{aug} for the expanded form, Eq. 7, is realizable as a 5-node network with linear tree, then this network yields matrix 6 with its implied terminal pairs 1, 2, 3 identified, respectively, with the tree branches 1, 3, and 4. If G_{aug} for the expanded form, Eq. 9, is realizable as a 6-node network with linear tree, then the given matrix 6 is realized with its implied terminal pairs 1, 2, 3 identified with tree branches 1, 3, 5, respectively. For G of order 3, the second network is the most general that need be considered; and the freedom inherent in the construction of the 5^{th}-order matrix B of rank 2 provides all the leeway that exists for the realization of G, beyond the rather restrictive conditions imposed by its realization in a 4-node network (the so-called (n+1)-node realization conditions). The question to be answered next is concerned with how we can make the most of this leeway.

2. Construction of the Auxiliary Matrix B

By the process just described, the given $n \times n$ G matrix can be augmented to an order m, which may be n + 1, m + 2, . . . up to 2n - 1. (It can be augmented still further, of course, but, for reasons already mentioned, this would not yield a greater realizability potential.) Realization of the given G matrix is successful if G_{aug} yields an (m+1)-node network with linear tree and all positive branch conductances.

As shown in the previous report (3), the branch conductances pertinent to a G matrix of order m are given by

$$
T\,G\,T = \begin{bmatrix} g_{11} & g_{12} & g_{13} & \cdots & g_{1m} \\ & g_{22} & g_{23} & \cdots & g_{2m} \\ & & g_{33} & \cdots & g_{3m} \\ & & & \cdots\cdots & \\ & & & & g_{mm} \end{bmatrix} = g \tag{10}
$$

with

$$
T = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ -1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 1 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & \cdots & -1 & 1 \end{bmatrix} \tag{11}
$$

If we identify G in Eq. 10 with G_{aug} in Eq. 8, then it is clear that we can write

$$
g_{aug} = T\,G_{aug}\,T, \quad g_{exp} = T\,G_{exp}\,T, \quad g = T\,B\,T \tag{12}
$$

and have

$$
g_{aug} = g_{exp} + g \tag{13}
$$

In other words, the additive property applies here. We can separately compute a set of branch conductances pertinent to the expanded G matrix as elements of a matrix g_{exp} and a set of branch conductances pertinent to the auxiliary matrix B as elements of a matrix g, and then form the resulting branch conductances simply by adding respective members of these two sets together.

Elements of the matrix g_{exp} pertinent to G_{exp} are not all positive; neither are those of the matrix g pertinent to B; and the crux of the whole problem is to construct B in such a way that the positive elements in g swamp, or at least cancel, the corresponding negative ones in g_{exp}, and vice versa, so that the branch conductance matrix g_{aug} has no negative elements.

Although a trial-and-error procedure is a possible approach to this problem (assume a matrix B of proper order and rank, compute g, add it to g_{exp}, and see if all resulting elements are positive or zero; if not, revise the structure of B, and so on), it is better to devise a method for the construction of g matrices directly and thus eliminate the B matrix from the procedure altogether. This scheme can be accomplished as follows.

Consider representation for the elements of a branch conductance matrix g, Eq. 10, given by

$$g_{ik} = -h_{i-1} \cdot h_k \qquad \text{for } i \leq k \leq m, \text{ and } i = 1, 2, \ldots m \tag{14}$$

in which $h_o, h_1, h_2, \ldots h_m$ is the vector set (forming a closed polygon in m-dimensional space) of a matrix

$$H = \begin{bmatrix} h_{01} & 0 & 0 & \cdot & 0 \\ h_{11} & h_{12} & 0 & \cdot & 0 \\ h_{21} & h_{22} & h_{23} & \cdot & \vdots \\ \vdots & \vdots & \vdots & \vdots & h_{m-1,m} \\ h_{m1} & h_{m2} & h_{m3} & \cdot & h_{mm} \end{bmatrix} \tag{15}$$

so that we have

$$g_{ik} = -\sum_{\nu=1}^{i} h_{i-1,\nu} h_{k\nu} \qquad \text{for } k \geq i, \text{ and } i = 1, 2, \ldots m \tag{16}$$

as given in the previous report (5). The rank of H is the rank of the matrix B in the representation g = T B T. Hence if we want g to be representative of a matrix B of rank 1, we choose an H matrix having a single column. Let us simplify the notation in this case by denoting the elements in this column $h_o, h_1, \ldots h_m$. Then we have

$$g_{11} = -h_o h_1, g_{12} = -h_o h_2, \cdots g_{1m} = -h_o h_m \tag{17}$$

$$g_{22} = -h_1 h_2, g_{23} = -h_1 h_3, \cdots g_{2m} = -h_1 h_m \tag{18}$$

$$g_{33} = -h_2 h_3, g_{34} = -h_2 h_4, \cdots g_{3m} = -h_2 h_m \tag{19}$$

and so forth.

From Eqs. 17 and 18 we see that

$$g_{22} = \frac{g_{11} g_{12}}{-h_o^2}, g_{23} = \frac{g_{11} g_{13}}{-h_o^2}, \ldots g_{2m} = \frac{g_{11} g_{1m}}{-h_o^2} \tag{20}$$

and from Eqs. 17 and 19 that

$$g_{33} = \frac{g_{12} g_{13}}{-h_o^2}, g_{34} = \frac{g_{12} g_{14}}{-h_o^2}, \ldots g_{3m} = \frac{g_{12} g_{1m}}{-h_o^2} \tag{21}$$

and so forth, while from Eq. 17 alone we have

$$(g_{11}+g_{12}+\ldots+g_{1m}) = -h_o(h_1+h_2+\ldots+h_m) \tag{22}$$

But since columns in the H matrix must sum to zero,

$$h_1 + h_2 + \ldots + h_m = -h_o \tag{23}$$

and so Eq. 22 yields

$$h_o^2 = g_{11} + g_{12} + \ldots + g_{1m} \tag{24}$$

This result, together with Eqs. 20, 21, and their continuation, shows that if we want to construct a g matrix pertinent to a B matrix of rank 1, we may do so by writing down any values we please for the elements $g_{11}, g_{12}, \ldots g_{1m}$ in the first row of g, and then calculate the elements in the remaining rows by the simple relation

$$g_{ik} = \frac{-g_{1,i-1}g_{1k}}{\sum\limits_{k=1}^{m} g_{1k}} \qquad \text{for } k \geqslant i, \text{ and } i = 2, 3, \ldots m \tag{25}$$

A network having these branch-conductance values has the peculiar property that if we pick <u>any</u> linear tree, and assign terminal pairs to all but one of its branches (this one can be <u>any</u> tree branch), then the resulting conductance matrix is identically zero!

The elements $g_{11}, g_{12}, \ldots g_{1m}$, which we choose arbitrarily (except that their sum be nonzero), are the conductances of branches forming a starlike tree. There are many different starlike trees in a full graph; correspondingly there are many sets of <u>m</u> elements in the matrix g that can be chosen arbitrarily, and the rest can be computed from these by formula 25 (suitably modified as to indices). The elements in the first row of g are just a possible set. For the present, we shall not elaborate on this point.

More significant is the fact that here we have a method for the construction of a branch conductance matrix g appropriate to a B matrix of any desired rank. If B_1 and B_2 are like-order matrices, each of rank 1, then (except in degenerate cases) their sum $B_1 + B_2$ has rank 2; and if we add three such matrices we get one of rank 3, and so forth. Since, as we have pointed out, the additive property applies to the relation between a matrix B and the corresponding branch conductance matrix g, we can form two matrices g_1 and g_2 according to this method by using formula 25, and add them to get a g matrix appropriate to a matrix B of rank 2. In the network having the branch conductances of this g matrix, we can pick <u>any</u> linear tree, assign terminal pairs to all but two of its branches (<u>any</u> two), and get a resulting conductance matrix $G \equiv 0$.

The extension of this procedure to the construction of a g matrix appropriate to a matrix B of any desired order and rank is thus clear. A few simple numerical examples might be interesting at this point. Consider the branch conductance matrix

$$g_1 = \begin{bmatrix} 1 & -2 & 2 & -3 \\ & -1 & 1 & -\frac{3}{2} \\ & & -2 & 3 \\ & & & -3 \end{bmatrix} \tag{26}$$

which is constructed by writing down a first row at random (its sum in this case is -2) and then rapidly computing the remaining elements by formula 25. Thus the (22)-element is $-[1\cdot(-2)]/(-2) = -1$; the (23)-element is $-[1\cdot(2)]/(-2) = 1$; the (24)-element is $-[1\cdot(-3)]/(-2) = -3/2$, and so on. The corresponding B matrix may readily be calculated by the method discussed in the previous report (1). We have, in this case,

$$T^{-1}\, g = \begin{bmatrix} 1 & -2 & 2 & -3 \\ & -3 & 3 & -\frac{9}{2} \\ & & 1 & -\frac{3}{2} \\ & & & -\frac{9}{2} \end{bmatrix} \tag{27}$$

and so

$$B_1 = T^{-1}\, g\, T^{-1} = \begin{bmatrix} -2 & -3 & -1 & -3 \\ -3 & -\frac{9}{2} & -\frac{3}{2} & -\frac{9}{2} \\ -1 & -\frac{3}{2} & -\frac{1}{2} & -\frac{3}{2} \\ -3 & -\frac{9}{2} & -\frac{3}{2} & -\frac{9}{2} \end{bmatrix} \tag{28}$$

which clearly has rank 1.

In like manner we construct

$$g_2 = \begin{bmatrix} 1 & -2 & 1 & 2 \\ & 1 & -\frac{1}{2} & -1 \\ & & 1 & 2 \\ & & & -1 \end{bmatrix} \tag{29}$$

for which the sum of elements in the first row is 2. Correspondingly, we get, by the same process illustrated in Eqs. 27 and 28,

$$B_2 = \begin{bmatrix} 2 & 1 & 3 & 2 \\ 1 & \frac{1}{2} & \frac{3}{2} & 1 \\ 3 & \frac{3}{2} & \frac{9}{2} & 3 \\ 2 & 1 & 3 & 2 \end{bmatrix} \tag{30}$$

which has rank 1. Adding B_1 and B_2, we have

$$B = B_1 + B_2 = \begin{bmatrix} 0 & -2 & 2 & -1 \\ -2 & -4 & 0 & -\frac{7}{2} \\ 2 & 0 & 4 & \frac{3}{2} \\ -1 & -\frac{7}{2} & \frac{3}{2} & -\frac{5}{2} \end{bmatrix} \tag{31}$$

Now let us abridge this matrix so as to eliminate variables 3 and 4 (rows and columns 3 and 4 are deleted). For formula 2, we have

$$\begin{aligned} b_{rs} \times b_{ss}^{-1} \times b_{sr} &= \begin{bmatrix} 2 & -1 \\ 0 & -\frac{7}{2} \end{bmatrix} \times \begin{bmatrix} 4 & \frac{3}{2} \\ \frac{3}{2} & -\frac{5}{2} \end{bmatrix}^{-1} \times \begin{bmatrix} 2 & 0 \\ -1 & -\frac{7}{2} \end{bmatrix} \\ &= \frac{2}{49} \times \begin{bmatrix} 2 & -1 \\ 0 & -\frac{7}{2} \end{bmatrix} \times \begin{bmatrix} 5 & 3 \\ 3 & -8 \end{bmatrix} \times \begin{bmatrix} 2 & 0 \\ -1 & -\frac{7}{2} \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ -2 & -4 \end{bmatrix} \end{aligned} \tag{32}$$

and since

$$b_{rr} = \begin{bmatrix} 0 & -2 \\ -2 & -4 \end{bmatrix} \tag{33}$$

we see that the abridgment yields a null matrix, as it should.

Alternately, let us abridge matrix 31 so as to eliminate variables 2 and 4. We then have

$$\begin{aligned} b_{rs} \times b_{ss}^{-1} \times b_{sr} &= \begin{bmatrix} -2 & -1 \\ 0 & \frac{3}{2} \end{bmatrix} \times \begin{bmatrix} -4 & -\frac{7}{2} \\ -\frac{7}{2} & -\frac{5}{2} \end{bmatrix}^{-1} \times \begin{bmatrix} -2 & 0 \\ -1 & \frac{3}{2} \end{bmatrix} \\ &= \frac{4}{9} \times \begin{bmatrix} -2 & -1 \\ 0 & \frac{3}{2} \end{bmatrix} \times \begin{bmatrix} \frac{5}{2} & -\frac{7}{2} \\ -\frac{7}{2} & 4 \end{bmatrix} \times \begin{bmatrix} -2 & 0 \\ -1 & \frac{3}{2} \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 2 & 4 \end{bmatrix} \end{aligned} \tag{34}$$

which is b_{rr} in this case. Hence the abridgment again yields an identically zero result, as it should.

3. The Procedure

In the given G matrix of order n, which is not (n+1)-node realizable, insert a null row and column to form a tentative G_{exp}, and from this calculate a branch conductance matrix g_{exp} having some positive and some negative elements. Construct a branch conductance matrix g by the method of formula 25, choosing positive elements in the first row to cancel negative ones in the first row of g_{exp}, as well as any other negative ones in this matrix, and at the same time allowing positive ones in g_{exp} to absorb resulting negative elements in g. Some trial-and-error manipulation will soon reveal whether or not all resultant elements in the sum $g_{exp} + g$ can be made non-negative. If they can, we have a solution; if not, we can revise G_{exp} by placing the null rows and columns in different positions, or we can next form a G_{exp} by inserting two nonadjacent null rows and columns in the given G. We now construct two matrices g_1 and g_2 by the method of formula 25, and hence a matrix $g = g_1 + g_2$, so that all elements in the sum $g_{exp} + g$ are non-negative. Since we now have more free choices of elements, our chances for obtaining a solution are better than before.

Ultimately, we can form a matrix G_{exp} by inserting n - 1 null rows and columns in G so that all even-numbered rows and columns in the expanded matrix are null. A branch conductance matrix $g = g_1 + g_2 + \dots g_{n-1}$, in which the g_k are constructed by the method of formula 25, affords the maximum number of arbitrary choices available in the process of obtaining a resultant matrix $g_{exp} + g$ having no negative elements. The fact that g is a linear combination of the component g_k matrices is a distinct advantage in utilizing the available free choices most effectively.

Thus we can first construct g_1 so that $g_{exp} + g_1$ has elements as nearly non-negative as may be had by the free choices available in this construction process. Next, we construct g_2, utilizing the additional free choices (the same in number as before) as in the previous step so that the sum $g_{exp} + g_1 + g_2$ has elements as nearly non-negative as may be. Continuing in this way, each step follows essentially the same pattern with the same objective and carries the result closer to the desired goal. Additional study, however, needs to be directed toward developing a systematic procedure that will clearly indicate whether or not a solution exists in a given situation.

Meanwhile, the present method easily yields numerous solutions in situations that ordinarily are regarded as being rather difficult to solve. As an example, consider the matrix, taken from Slepian and Weinberg (4),

$$G = \begin{bmatrix} 9 & 5 & -1 \\ 5 & 9 & 5 \\ -1 & 5 & 9 \end{bmatrix} \tag{35}$$

and its expanded form

$$G_{exp} = \begin{bmatrix} 9 & 0 & 5 & -1 \\ 0 & 0 & 0 & 0 \\ 5 & 0 & 9 & 5 \\ -1 & 0 & 5 & 9 \end{bmatrix} \tag{36}$$

which yields the branch conductance matrix

$$g_{exp} = \begin{bmatrix} 9 & -5 & 6 & -1 \\ & 5 & -6 & 1 \\ & & 4 & 5 \\ & & & 4 \end{bmatrix} \tag{37}$$

For the matrix g, suppose we write

$$g = \begin{bmatrix} x_1 & 5 & x_3 & x_4 \\ & \dfrac{-5x_1}{a} & \dfrac{-x_1x_3}{a} & \dfrac{-x_1x_4}{a} \\ & & \dfrac{-5x_3}{a} & \dfrac{-5x_4}{a} \\ & & & \dfrac{-x_3x_4}{a} \\ & & & \end{bmatrix} \tag{38}$$

in which we have left three of the free choices arbitrary, and have fixed the fourth so as to cancel one of the negative elements in the first row of g_{exp}. The sum of the first-row elements is denoted

$$a = x_1 + x_3 + x_4 + 5 \tag{39}$$

If we choose a negative value for x_1 and positive values for x_3 and x_4, as well as for the sum $\underline{a}$, then all second-row elements in the matrix 38 are positive, and if we specify that

$$\frac{-x_1x_3}{a} = 6 \tag{40}$$

then our only remaining concern is with the (33)-, (34)-, and (44)-elements for which the conditions read

$$\frac{5x_3}{a} \leqslant 4, \quad \frac{5x_4}{a} \leqslant 5, \quad \frac{x_3 x_4}{a} \leqslant 4$$

or

$$\frac{x_3}{a} \leqslant \frac{4}{5}, \quad \frac{x_4}{a} \leqslant 1, \quad \frac{x_3}{a} \leqslant \frac{4}{x_4} \tag{41}$$

Use of Eq. 40 yields

$$(-x_1) \geqslant \frac{15}{2}, \quad a \geqslant x_4, \quad (-x_1) \geqslant \frac{3x_4}{2} \tag{42}$$

If we choose $(-x_1) = 15/2$, then Eq. 39 yields

$$a = -\frac{15}{2} + x_3 + x_4 + 5 \geqslant x_4 \tag{43}$$

from which

$$x_3 \geqslant \frac{5}{2} \tag{44}$$

Several possible solutions are now readily obtained. If we let $x_3 = 5/2$ and $a = x_4$, we get

$$g = \begin{bmatrix} -\frac{15}{2} & 5 & \frac{5}{2} & \frac{25}{8} \\ & 12 & 6 & \frac{15}{2} \\ & & -4 & -5 \\ & & & -\frac{5}{2} \end{bmatrix} \tag{45}$$

Alternately, the choices $x_3 = 6$, $x_4 = 4$ lead to $a = 15/2$, and we have

$$g = \begin{bmatrix} -\frac{15}{2} & 5 & 6 & 4 \\ & 5 & 6 & 4 \\ & & -4 & -\frac{8}{3} \\ & & & -\frac{16}{5} \end{bmatrix} \tag{46}$$

which evidently is also acceptable.

If we let $(-x_1) = 8$, Eq. 39 yields

$$a = -3 + x_3 + x_4 \geqslant x_4 \tag{47}$$

or

$$x_3 \geqslant 3 \tag{48}$$

Here the choice $x_3 = 3$ and $a = x_4$ yields still another solution:

$$g = \begin{bmatrix} -8 & 5 & 3 & 4 \\ & 10 & 6 & 8 \\ & & -\frac{15}{4} & -5 \\ & & & -3 \end{bmatrix} \tag{49}$$

4. Normalization

If we let

$$a = \sum_{k=1}^{m} g_{1k} \tag{50}$$

as in the preceding example, and introduce a normalized g matrix, $\bar{g} = g/a$, with elements

$$\bar{g}_{ik} = g_{ik}/a \tag{51}$$

then the first-row elements of this normalized matrix sum to unity:

$$\sum_{k=1}^{m} \bar{g}_{1k} = 1 \tag{52}$$

and the remaining elements are given by the simpler formula

$$\bar{g}_{ik} = -\bar{g}_{1,i-1}\bar{g}_{1k} \qquad \text{for } k \geqslant i, \text{ and } i = 2, 3, \ldots m \tag{53}$$

If this normalization is applied to each of the matrices $g_1, g_2, \ldots$ in the process described above, then we must distinguish corresponding normalization factors $a_1, a_2, \ldots$. In the first step, $\bar{g}_1$ is added to g_{exp}/a_1 with the object of obtaining no negative elements. In the second step, $\bar{g}_2$ is added to $(g_{exp}+g_1)/a_2$ with the object of obtaining no negative elements, and so forth.

In the situation for which only one step is needed (as in the example just considered) we can regard $g_{exp}/a = \bar{g}_{exp}$ as a correspondingly normalized matrix derived from normalized matrices G_{exp}/a and G/a. This result then amounts simply to recognizing that if we inject into the process of constructing a branch conductance matrix g (according to formula 25) the simplifying constraint that the sum of its first-row elements be unity, we can still obtain all the results that we have in the absence of this constraint by considering all finite nonzero admittance-level factors as multipliers for the given matrix G.

As far as positive multipliers are concerned, this result would mean that normalization (or constraining the sum of first-row elements in g to be unity) is, in effect, no constraint at all, for a realization of the given G matrix multiplied by a positive constant is as good as a realization without this multiplier. However, the sum given by Eq. 50 can be negative as well as positive, and this tells us that we can try to multiply the given G matrix by negative constants, as well as by positive ones, when constructing g matrices on a normalized basis. This point of view may be helpful in certain situations.

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References

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2. For example, see E. A. Guillemin, Introductory Circuit Theory (John Wiley and Sons, Inc., New York, 1953), pp. 525-526.

3. E. A. Guillemin, Quarterly Progress Report No. 56, op. cit.; see Eq. 13 in which T is given by Eq. 7.

4. P. Slepian and L. Weinberg, Synthesis applications of paramount and dominant matrices, Proc. National Electronics Conference, Vol. 14 (1958), pp. 611-630.

5. E. A. Guillemin, Quarterly Progress Report No. 56, op. cit.; see Eqs. 26, 27, 28.