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SOME ABSTRACT PIVOT ALGORITHMS

by

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Some Abstract Pivot Algorithms

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Abstract: Several problems in the theory of combinatorial geometries (or matroids) are solved by means of algorithms which involve the notion of "abstract pivots". The main example is the Edmonds-Fulkerson partition theorem, which is applied to prove a number of generalized exchange properties for bases.

1. Introduction

The theory of combinatorial geometries (or matroids, as they were first called [18]) concerns properties of a matrix which depend only on a knowledge of which sets of columns are independent. This paper concerns a number of problems and related algorithms in combinatorial geometry which derive from the abstract analog of "pivoting" in matrices. In matrix theory, a pivot is a single application of the Gauss-Jordan elimination process, which eliminates one variable from a set of equations. In abstract combinatorial geometries, the existence of pivots is assumed as an axiom, in the form of a replacement property for bases:

if S and T are maximal independent sets (bases) and $x \in S$, there exists an element $y \in T$ such that $(S-x) \cup y$ is a basis.

If we think of S and T as sets of columns in a matrix M, with S an identity submatrix, then replacing x by y corresponds to a "pivot about position x in column y."

This replacement property allows one to recover some of the algebraic structure of matrices in combinatorial form. As one example of this, Rota has observed [15] that many determinant identities have "analogs" which are valid in any combinatorial geometry. Such results are obtained by ignoring the values of determinants and considering only whether or not they are zero (i.e. whether the underlying sets of vectors constitute a bases). For example, one can use determinants to prove the following exchange property* for bases (which is stronger than the replacement axiom but follows from it):

if S and T are bases, and $x \in S$, there exists an element $y \in T$ such that both $(S-x) \cup y$ and $(T-y) \cup x$ are bases.

The argument for matrices is as follows: If S is represented by an identity matrix, and T is an arbitrary nonsingular square matrix, then

$$\det T = \sum_{y \in T} \pm \det((S-x) \cup y) \cdot \det((T-y) \cup x)$$

as can be seen by expanding det T by cofactors along "row x". Since $\det T \neq 0$, some term on the right must be nonzero and the result follows.

It is not hard to give a "determinant-free" proof of the exchange property (see [2],[3]) and this proof shows that the property holds in any combinatorial geometry.

In [8] one of the authors obtained a "multiple exchange property" for bases, which corresponds to the Laplace expansion theorem for determinants in the same way that the ordinary exchange property corresponds to expansion by cofactors: if S and T are bases, and $X \subseteq S$, then there exists a subset $Y \subseteq T$ such that $(S-X) \cup Y$ and $(T-Y) \cup X)$ are both bases.

The proof by determinants is virtually identical to the one just described when X is a singleton. However a proof valid in any geo-

*The distinction drawn here between "replacement" and "exchange" does not correspond to standard terminology.

metry is much more difficult. In this paper we give a new constructive proof, by describing an elementary pivot algorithm for carrying out the exchange.

We will also show how a number of results related to the multiple exchange property can be expressed as "abstract pivot theorems", and describe the pivot algorithms associated with them. Among other things, we will show how Greene's exchange theorem follows immediately from the powerful "matroid partition theorem" of Edmonds and Fulkerson [7]. We describe this theorem in section 2, including an algorithm which, although not essentially new, takes on a particularly simple form in the present context. In section 3, we describe a number of "multiple exchange theorems", all of which can be reduced to the Edmonds-Fulkerson theorem, and hence can be proved by elementary pivot techniques. In section 4, we raise a new question: can a multiple exchange of k vectors be carried out by a sequence of k single exchanges? We conjecture that some permutation of the vectors can be exchanged sequentially, and prove that this is the case for k=2.

2. Pivot Operations and the Edmonds-Fulkerson Theorem

Recall that a combinatorial geometry G(X) consists of a finite set X together with a collection of subsets of X called bases, such that (i) all bases have the same size and (ii) if S and T are bases, and $x \in S$, then there exists an element $y \in T$ such that $(S-x) \cup y$ is a basis. A set A is called independent if it is contained in some basis.

If it is possible to associate the elements of X with columns of a matrix M in such a way that bases correspond to maximal independent sets of columns, we say that G(X) is coordinatized by M. Examples show that not every geometry can be coordinatized by a matrix; nevertheless most arguments involving the elementary tools of linear algebra - independence, dependence, linear closure, dimension, etc. - carry over to combinatorial geometries with no difficulty. The reader can safely assume that any such argument appearing in this paper can be derived solely from the axioms for bases.

We mention two important properties: first the rank of a subset A, denoted r(A), is defined as the maximum size of an independent subset of A and obeys the submodular law:

$$r(A \cup B) + r(A \cap B) \leq r(A) + r(B).$$

Second, if S is a basis, and $y \notin S$, we say that y depends on the set C(y,S) of elements $x \in S$ such that $(S-x) \cup y$ is a basis. More generally, we say that y depends on a set A if there exists a basis S such that $C(y,S) \subseteq A$. The set $y \cup C(y,S)$ is called the circuit determined by y and S, and is a minimal (in the sense of set-inclusion) dependent set. Most important for our purposes is the fact that "dependence" is transitive: if y depends on A, and every element of A depends on B, then y depends on B. We will make free use of these ideas without attempting to justify our reasoning - the reader can refer to [14] or [18] for a detailed development.

Suppose that G(X) is coordinatized by matrix M, and S is a basis whose columns in M are coordinate vectors. (This means that M is in reduced echelon form with respect to the columns corresponding to S.)

For any $y \notin S$, the elements of $C(y,S)$ can be identified immediately by looking at the nonzero entries in column y. Each element $x \in C(y,S)$ can be replaced by y to form a new basis $T = (S-x) \cup y$. We call the operation of transforming S into T a <u>pivot about x in y (with respect to S)</u>. Whenever such a pivot is possible, that is, whenever $x \in C(y,S)$, we write

$$y \xrightarrow[S]{} x.$$

These symbols define a directed graph with vertex set X and a multi-labelled set of directed edges, with one label type for each basis S.[†] In concrete terms, each pivot represents a single application of the Gauss-Jordan elimination process (applied to the column y). Much of this paper concerns the interpretation of these symbols in special situations.

It will be convenient to know when a chain of pivots

$$x \xrightarrow[S]{} y \xrightarrow[T]{} z \ldots \xrightarrow[U]{} w$$

can be carried out simultaneously. That is, if a basis appears several times in the chain, we need conditions which guarantee that all of the replacements involving it can be made at once. The following lemma provides a very useful condition of this type, which applies even when the bases $S, T, \ldots, U$ come from different geometries.

[†]A related structure, called a <u>basis graph</u>, has been studied by several authors (Bondy [1], Holzmann and Harary [10], Maurer [12],[13]). The structures are formally distinct, however, since the vertices in a basis graph are bases, with edges defined by pivots. Here, the vertices are elements of X and each basis determines a class of edges.

Lemma 2.1 Suppose that $y_0, y_1, \ldots, y_k$ are elements of X and $B_1, B_2, \ldots, B_k$ are bases of geometries $G_1(X), G_2(X), \ldots, G_k(X)$ respectively. (Neither the B_i's nor the G_i's are required to be distinct.) Suppose that

$$y_0 \xrightarrow[B_1]{} y_1 \xrightarrow[B_2]{} \cdots y_{k-1} \xrightarrow[B_k]{} y_k$$

is a chain of pivots. Assume further that this chain is minimal, in the sense that no shorter path from y_0 to y_k exists using the labels $B_1, B_2, \ldots, B_k$. Then each of the sets

$$B_i' = (B_i - y_a - y_b - \cdots - y_c) \cup y_{a-1} \cup y_{b-1} \cup \cdots \cup y_{c-1}$$

(where $B_i = B_a = B_b = \ldots = B_c$) is a basis in $G_i(X)$, $i = 1, 2, \ldots, k$.

Proof: We observe that, for each B_i, the pivots on elements of B_i can be carried out sequentially, provided that the last ones are made first. If B_i appears only once in the list, then B_i' is trivially a basis (by definition of $y_{i-1} \xrightarrow[B_i]{} y_i$). If B_i appears more than once, then $y_{i-1} \xrightarrow[B_i]{} y_i$ can still be performed unless some member of the circuit $C(y_{i-1}, B_i)$, say y_j, has been removed from B_i in an earlier pivot. But then there exists an arc $y_{i-1} \xrightarrow[B_i]{} y_j$ with $j > i$, which violates the assumption of minimal length.

Next we describe the <u>matroid partition theorem</u> of Edmonds and Fulkerson. The question is this: suppose that $G_1(X), G_2(X), \ldots, G_k(X)$ are geometries defined on the same set X. <u>Under what conditions is it possible to partition X into blocks B_i such that, for each i, B_i is independent in G_i?</u> Moreover, how can one find such a partition if it exists?

In terms of matrices, the problem can be described as follows: suppose that $M_1, M_2, \ldots, M_k$ are matrices, each having $|X|$ columns, which are stacked on top of each other to form a large matrix M^*. Under what conditions is it possible to partition the columns of M^* into sets B_i so that for each i, the submatrix of M_i determined by B_i has independent columns. The answer is contained in the following:

<u>Theorem 2.2 (Edmonds, Fulkerson) A partition of X into sets B_i, independent in G_i, exists if and only if for each</u> $A \subseteq X$, $|A| \leq r_1(A) + r_2(A) + \ldots + r_k(A)$, <u>where $r_i(A)$ denotes the rank of A in G_i.</u>

Necessity of this condition is trivial, so it suffices to prove that a partition exists whenever the conditions are satisfied. We now give an algorithm, based on pivot operations, which shows this:

Suppose that $B_1, B_2, \ldots, B_k$ are subsets of X with the property that B_i is a basis of G_i, for each i. If $\bigcup B_i = X$, we are done, since we can form a partition into independent sets by removing duplicated elements. If $\bigcup B_i \neq X$, let $y \in X - \bigcup B_i$. We must show how to rearrange the elements of $\bigcup B_i$ into new sets B_i' with the same property, and add y to one of them. If this is always possible, we can continue until X is exhausted, and a partition is obtained.

The algorithm is based on a labelling procedure:

<u>Step (0)</u> Label the element y.

<u>Step (1)</u> For each labelled element y', label every unlabelled element z such that $y' \underset{B_i}{\rightarrow} z$ for some B_i.

<u>Step (2)</u> If an element common to two bases, say B_i and B_j, has been labelled, stop. Otherwise go back to step 1.

When the labelling procedure stops, there is a chain

$$y = y_0 \underset{B^{(1)}}{\longrightarrow} y_1 \underset{B^{(2)}}{\longrightarrow} y_2 \rightarrow \cdots \rightarrow y_{j-1} \underset{B^{(j)}}{\longrightarrow} y_j$$

where y_j is common to two bases, say $B^{(j)}$ and B_k. (It is understood that bases can appear several times in the list.) Now define, for each $i = 1, \ldots, k$,

$$B_i' = \begin{cases} B_i \text{ if } B_i \text{ does not appear in the list} \\ (B_i - y_a - y_b - \cdots y_c) \cup y_{a-1} \cup y_{b-1} \cup \cdots \cup y_{c-1} \\ \text{if } B_i = B^{(a)} = B^{(b)} = \cdots = B^{(c)} \end{cases}$$

From the nature of the labelling algorithm, it is clear that the chain from y to y_i is minimal. Hence the previous lemma applies, and it follows that each B_i' is a basis in G_i. Clearly $\bigcup B_i' = y \cup \bigcup B_i$, and we have added y as desired.

It remains to show that the labelling process terminates — that is, some element common to two bases is eventually labelled. Suppose to the contrary, that the algorithm proceeds until Step (1) no longer labels anything new. If we denote the set of labelled elements by L, then L depends on $L \cap B_i$ in each geometry G_i, and the sets $L \cap B_i$ are disjoint. Hence

$$\sum r_i(L) = \sum |L \cap B_i| \leq |L| - 1$$

since $y \in L$ but $y \notin \bigcup B_i$. This contradicts our hypothesis, and the proof is complete.

In the concrete matrix version of the problem, it should be noted that no matrix operations are necessary until the end of each cycle (adding an element y). The labelling is done entirely by scanning the nonzero elements of each column. After the new bases $B_1', B_2', \ldots, B_k'$ have been found, one performs row operations on each M_i to put it in canonical form with respect to B_i', but it is not necessary to do this sooner. For example, in the picture below, if $B_1 = \{x_3, x_4, x_5\}$ and $B_2 = \{x_2, x_3\}$, and $y = x_1$, the circles and arrows illustrate the relations

$$x_1 \xrightarrow[B_2]{} x_2 \xrightarrow[B_1]{} x_4 \xrightarrow[B_2]{} x_3$$

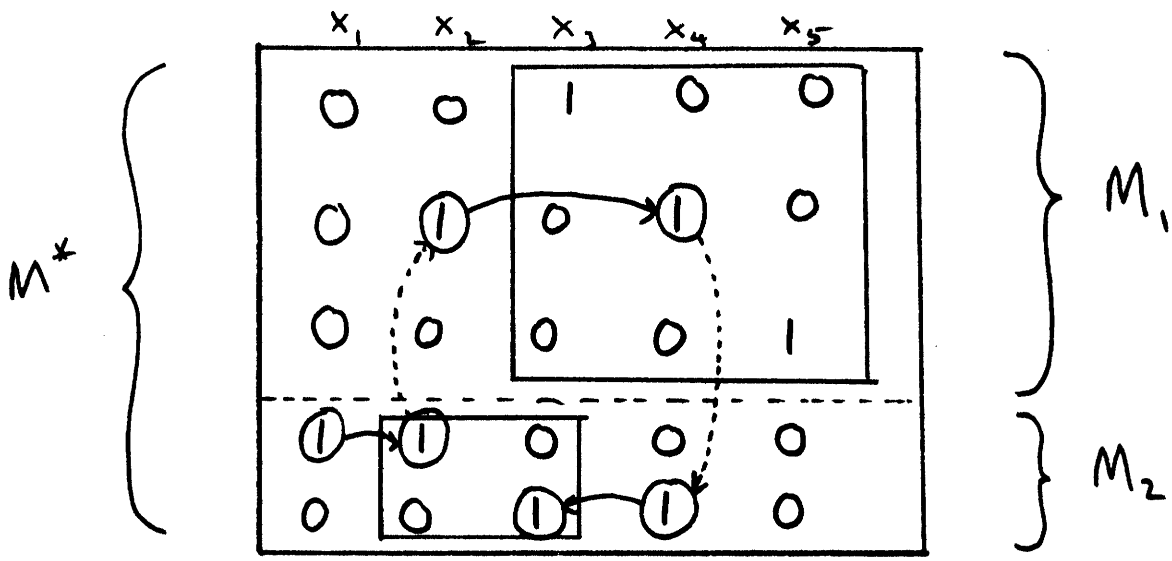

Fig. 1

(In fact, this is all the labelling which takes place). According to the algorithm, we construct new bases

$$B_1' = (B_1 - x_4) \cup x_2 = \{x_2, x_3, x_5\}$$

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HERE IS SUMMARY OF INPUTS

VARIABLE                  DIST TYPE     MEAN          STD DEV
INVESTMENT COST           N             3             0.666667
RESIDUAL VALUE            %             0             0
LIFETIME IN YEARS         A             5.73          1.84854
ANNUAL BENEFITS           A             0.888         0.472711

ARE THESE INPUTS ALL OK, Y OR N ?Y


CRITERIA OPTIONS ARE:
 INPUT YOUR CRITERION OPTION:1,2,3 OR 4 ?1

INPUT COST OF CAPITAL FOR ANNUAL DISCOUNT RATE AS % ?10

RANGE OF-3.8831  8.57964 SUGGESTS 31 CELLS OF 0.402024
INPUT PREFERRED INTERVAL ?.4
THIS YIELDS 32 INTERVALS OF 0.4


MEANS OF INV,RESID,LIFE,BEN ARE:  3  0  5.73  0.888
THE CRITERION NPV AT THESE MEANS IS  0.873177

THESE 200 VALID SIMULATIONS OF NPV HAVE
MEAN OF 0.761805 AND STD DEV OF 2.53541


 EACH X IN HISTOGRAM IS 0.5 SIMULATED NPV

  MIN     CUM%     Q(K)

-4        0.5       1   XX
-3.6      2         3   XXXXXX
-3.2      4         4   XXXXXXXX
-2.8      8         8   XXXXXXXXXXXXXXXX
-2.4      11.5      7   XXXXXXXXXXXXXX
-2        17        11  XXXXXXXXXXXXXXXXXXXXXX
-1.6      21.5      9   XXXXXXXXXXXXXXXXXX
-1.2      31.5      20  XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
-0.8      36        9   XXXXXXXXXXXXXXXXXX
-0.4      44.5      17  XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
 0        49.5      10  XXXXXXXXXXXXXXXXXXXX
 0.4      58        17  XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
 0.8      61.5      7   XXXXXXXXXXXXXX
 1.2      67        11  XXXXXXXXXXXXXXXXXXXXXX
 1.6      72        10  XXXXXXXXXXXXXXXXXXXX
 2        79        14  XXXXXXXXXXXXXXXXXXXXXXXXXXXX
 2.4      82        6   XXXXXXXXXXXX
 2.8      85.5      7   XXXXXXXXXXXXXX
 3.2      87.5      4   XXXXXXXX
 3.6      89        3   XXXXXX
 4        90.5      3   XXXXXX
 4.4      92.5      4   XXXXXXXX
 4.8      93        1   XX
 5.2      94        2   XXXX
 5.6      95.5      3   XXXXXX
 6        96        1   XX
 6.4      96        0
 6.8      97        2   XXXX
 7.2      98        2   XXXX
 7.6      99        2   XXXX
 8        99.5      1   XX
 8.4      100       1   XX
```

and $B_2' = (B_2 - x_3 - x_2) \cup x_4 \cup x_1 = \{x_1, x_4\}$

which provide a complete partition of X.

A variation on the Edmonds-Fulkerson theorem which can be proved by similar methods is the <u>matroid intersection theorem</u> (due to Edmonds): If $G_1(X)$ and $G_2(X)$ are two geometries defined on the same set X, then there exists a subset $S \subseteq X$ of size k which is independent in both G_1 and G_2 if and only if $k \leq r_1(A) + r_2(X - A)$ for all $A \subseteq X$. The connection between matroid intersection and matroid partition is well known, and a labelling algorithm similar to the one given above can be constructed. Such an algorithm has been described by Lawler [11]. (See also Edmonds [5],[6])

3. Multiple Exchange Theorems

The following theorem was proved by Greene [8] (and independently by Brylawski [2]).

<u>Theorem 3.1: Let S and T be bases of a combinatorial geometry $G(X)$, and let $A \subseteq S$. Then there exists a subset $B \subseteq T$ such that $(S-A) \cup B$ and $(T-B) \cup A$ are both bases.</u>

If S is a singleton, it is not difficult (see [2],[3]) to show this. For matrices it can be proved immediately by assuming that S is a coordinate basis. The columns of T are represented by a nonsingular matrix and the result is equivalent to the following:

Theorem 3.2 Let M be a nonsingular matrix, whose rows have been partitioned into two parts A and A'. Then it is always possible to permute the columns of M in such a way that the principal minors corresponding to A and A' are nonzero.

This follows easily from the Laplace expansion theorem for determinants, but the question of how to carry out the exchange is much less obvious. Greene's original proof provided an efficient but unattractive algorithm. However, it is much more convenient to observe that the multiple exchange property is a trivial consequence of the Edmonds-Fulkerson theorem. Hence an elementary algorithm is easily obtained.

To see this, consider the geometries $G_1(T) = G/A$ and $G_2(T) = G/S-A$ defined on T by "factoring out" A and S-A. That is, we define rank functions

$$r_1(U) = r(U \cup A) - r(A)$$

$$r_2(U) = r(U \cup (S-A)) - r(S-A).$$

It is easy to see that exchanging A for a subset of T is equivalent to partitioning T into sets B_1 and B_2 which are bases in G_1 and G_2, respectively. According to the theorem, this can be done provided that

$$|U| \leq r_1(U) + r_2(U)$$

for every subset $U \subseteq T$. But

$$r_1(U) + r_2(U) = r(U \cup A) + r(U \cup (S-A)) - |S|$$

$$= r(U \cup A) + r(U \cup (S-A)) - r(U \cup A \cup (S-A))$$
$$\geq r((U \cup A) \cap (U \cup (S-A)))$$

by the submodular law. But

$$r((U \cup A) \cap (U \cup (S-A))) = r(U) = |U|$$

and this completes the proof.

Remark: In order to apply the Edmonds-Fulkerson algorithm, it is not necessary to compute the factor geometries G/A and $G/S-A$. The algorithm can be applied directly, provided that we start with bases $B_1 \cup A$ and $B_2 \cup (S-A)$, $B_1 \subseteq T$, $B_2 \subseteq T$, and modify step (1) by requiring that elements of S are never labelled.

The multi-part partition theorem in fact proves a stronger result:

Theorem 3.3 Let S and T be bases of $G(X)$ and let $\Pi = \{S_1, S_2, \ldots, S_k\}$ be a partition of S. Then there exists a partition $\Pi' = \{T_1, \ldots, T_k\}$ of T with the property that, for each $i = 1,2,\ldots,k$, the set $(S-S_i) \cup T_i$ is a basis of $G(X)$.

Proof: To extend the argument used to prove the multiple exchange theorem we need the following extended submodular inequality (easily proved by induction, using the ordinary submodular law): if $P_1, P_2, \ldots, P_k$ are subsets of any geometry, then

$$\sum_{i=1}^{k} r(P_i) \geq r\left(\bigcap_1^k P_i\right) + r\left(P_1 \cup \bigcap_2^k P_i\right) + r\left(P_2 \cup \bigcap_3^k P_i\right) + \ldots + r(P_{k-1} \cup P_k).$$

To prove the theorem, let $G_i = T/S-S_i$, $i = 1,\ldots,k$. If $A \subseteq T$, then $r_i(A) = r(A \cup (S-S_i)) - |S-S_i|$, so that

$$\sum_{i=1}^{k} r_i(A) = \sum_{i=1}^{k} r(A \cup (S-S_i)) - (k-1)|S|.$$

Let $P_i = A \cup (S-S_i)$ in the above inequality. Then $r(P_i \cup \bigcap_{i+1}^{k} P_j) = |S|$ for each $i = 1,\ldots,k-1$, and $r(\bigcap_1^k P_i) = |A|$. Hence $\sum r_i(A) \geq |A| + (k-1)|S| - (k-1)|S| = |A|$, for every subset $A \subseteq T$. By the Edmonds-Fulkerson theorem, T can be partitioned into sets T_i such that T_i is independent in G_i for each i. It is easy to show that this implies $T_i \cup (S-S_i)$ is a basis in G for each i.†

If Π is taken to be the trivial partition of S into $|S|$ parts, we obtain the following result of Brualdi [3]:

<u>Theorem 3.4</u> <u>If S and T are bases of $G(X)$, there exists a one-to-one correspondence ϕ: $S \to T$ such that $(S-x) \cup \phi(x)$ is a basis for all $x \in S$.</u>

There are elementary examples which show that the last two results are <u>replacement</u> theorems rather than <u>exchange</u> theorems. That is, for example, it is not always possible to have $(S-x) \cup \phi(x)$ <u>and</u> $(T-\phi(x)) \cup x$ <u>simultaneously</u> bases for all $x \in S$. (See [3]. Dilworth [4] obtained similar results in a related but somewhat more special case.)

It is interesting to note that the Edmonds-Fulkerson partition theorem proves a result which is apparently stronger

† The referee has pointed out that Theorem 3.3 can be derived directly from Theorem 3.1 by an induction argument.

than the multiple exchange theorem. This is most clearly seen by examining the analog of Brualdi's theorem when one of the sets is not required to be a basis. We ask: under what conditions, if S is a basis and T is an arbitrary set of size $|S|$, does there exist an injective map σ: $S \to T$ such that $(S-x) \cup \sigma(x)$ is a basis for each $x \in S$. If T is represented by an arbitrary square matrix, the Edmonds-Fulkerson theorem in this case gives necessary and sufficient conditions for some term in the determinant expansion of T to be nonzero. (These conditions are equivalent to the well-known "matching conditions" of P. Hall [9], as can be easily verified.) Brualdi's theorem, on the other hand, gives only a sufficient condition: that the columns of T be independent. In an analogous way, the 2-part case of the Edmonds-Fulkerson theorem gives a result which is apparently stronger than Green's multiple exchange property.

We remark that, when applied to Brualdi's Theorem, the algorithm which we describe in section 2 is essentially equivalent to the so-called "Hungarian method" - or "alternating chain" method - for finding a matching in a bipartite graph.

4. Sequential Exchange Properties

In this section, we consider the question: can a multiple exchange be carried out by a series of single exchanges? Here we mean exchange rather than replacement:

If $x \in S$ and $y \in T$, a <u>single exchange</u> of x for y is a pair of pivots $x \underset{T}{\longrightarrow} y$, $y \underset{S}{\longrightarrow} x$. A <u>replacement</u> is a single pivot $x \underset{T}{\longrightarrow} y$ or $y \underset{S}{\longrightarrow} x$. There are five questions which one might reasonably ask:

<u>Question 1: If $A \subseteq S$ can be exchanged for $B \subseteq T$, is it always possible to do this with $|A|$ single exchanges?</u>

<u>Question 2: If $A = \{a_1,\ldots,a_k\}$ is it always possible to exchange A for some $B \subseteq T$ by exchanging $a_1,a_2,\ldots,a_k$ in order?</u>

<u>Question 3: If $A \subseteq S$ can be exchanged for $B \subseteq T$, is there always some set of single exchanges which carries this out?</u>

<u>Question 4: If $A = \{a_1,\ldots,a_k\}$, is there always a permutation σ such that A can be exchanged for some B by exchanging $a_{\sigma(1)}, a_{\sigma(2)},\ldots,a_{\sigma(k)}$ in order?</u>

<u>Question 5: Is it possible to exchange A for some B by some sequence of exchanges?</u>

In this paper, we will partially answer these questions as follows:

(i) The answer to questions 1 and 2 is <u>no</u>.

(ii) The answer to question 4 is <u>yes</u> if $k = 2$.

<u>Conjecture</u>: <u>Questions 3 and 4 (and hence 5) can be answered affirmatively for all k</u>.

First, the counterexamples: let M be the matrix

$$\begin{matrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{matrix}$$
$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

<u>Counterexample 1</u>: If $S = \{x_1,x_2,x_3\}$ and $T = \{x_4,x_5,x_6\}$, then $\{x_1,x_2\}$ can be exchanged for $\{x_4,x_5\}$ but it is not possible to achieve this by two single exchanges.

<u>Counterexample 2</u>: Let S and T be as above. Then $\{x_1,x_3\}$ can be exchanged for $\{x_4,x_5\}$ via $x_3 \leftrightarrow x_5$, $x_1 \leftrightarrow x_4$. However, it is not possible to exchange $\{x_1,x_3\}$ for <u>anything</u> by switching x_1 first and then x_3.

We now prove two lemmas in order to affirm question 4 when $k = 2$;

Lemma 4.1: Suppose that S and T are bases of a combinatorial geometry, and suppose that there exists a closed alternating chain of pivots

$$x_1 \underset{S}{\rightarrow} y_1 \underset{T}{\rightarrow} x_2 \underset{S}{\rightarrow} y_2 \rightarrow \cdots y_n \underset{T}{\rightarrow} x_{n+1} = x_1$$

(Here we assume that the x's are in T and the y's are in S). If this cycle is minimal, in the sense that it contains no chords $x_i \underset{S}{\longrightarrow} y_j$, $i \neq j$, or $y_i \underset{T}{\longrightarrow} x_j$, $i \neq j-1$, then $\{x_1,\ldots,x_n\}$ can be exchanged for $\{y_1,y_2,\ldots,y_n\}$.

Proof: This is a special case of the lemma on sequential pivots described in section 2.

Next, we have the following lemma, which should not be confused with the (false) assertion in Question 1:

Lemma 4.2: Suppose that S and T are bases and $A \subseteq S$, $B \subseteq T$, with $|A| = |B| = k$. If A can be exchanged for B, it is possible to carry out this exchange by means of 2k replacements (or pivots).

Proof: Consider the directed graph whose vertices are the elements of $A \cup B$, and whose edges are given by the symbols $a \underset{T}{\rightarrow} b$, $b' \underset{S}{\rightarrow} a'$. First observe that every $a \in A$ is connected to some $b \in B$ by an edge $a \underset{T}{\rightarrow} b$, since otherwise a depends on $T - B$, which is impossible since A can be exchanged for B. Similarly, each $b \in B$ is connected to some $a \in A$. Hence there exist directed cycles, and we choose one which is minimal. By the previous lemma, this permits us to exchange some subset $A_0 \subseteq B$ for some subset $B_0 \subseteq B$, using $2k_0$ replacements, where $k_0 = |A_0| = |B_0|$. Now repeat the process for $A - A_0$, $B - B_0$, and so forth until the exchange is complete.

Remark: It is possible to use the previous two lemmas to construct a labelling algorithm for multiple exchange directly. However, it is entirely equivalent to the one previously described so we omit the details.

If our conjecture is true, the $2k$ pivots described in the previous lemma can be arranged so that each successive pair $x \rightarrow y$, $y \rightarrow x$ is an exchange. Next we show that this is always the case if $k = 2$.

Theorem 4.3: Let S and T be bases, and let $\{x_1, x_2\} \subseteq S$. Then, after relabelling x_1 and x_2 if necessary, it is possible to find a sequence of exchanges

$$x_1 \underset{T}{\rightarrow} y_1 \underset{S}{\rightarrow} x_1$$

$$x_2 \underset{T'}{\rightarrow} y_2 \underset{S'}{\rightarrow} x_2$$

for some $y_1, y_2 \in T$. (Here $S' = (S - x_1) \cup y_1$, $T' = (T - y_1) \cup x_1$.)

Proof: Suppose that x_1 has been exchanged for y_1 (as is always possible). If x_2 can now be exchanged for some y_2, we are done, so assume that x_2 can be exchanged only for x_1.

This implies that $S'' = (S-x_2) \cup y_1$ and $T'' = (T-y_1) \cup x_2$ are both bases. On the other hand, we know that $\{x_1,x_2\}$ can be exchanged for something, say $\{y_2,y_3\}$. Hence, in S' and T', $\{y_1,x_2\}$ can be exchanged for $\{y_2,y_3\}$. Similarly, $\{y_1,x_1\}$ can be exchanged for $\{y_2,y_3\}$ in S'' and T''. By the previous lemma, each of these exchanges can be carried out by four pivots, which we represent by the following diagrams:

$$y_1 \xrightarrow[T']{} y_2 \xrightarrow[S']{} x_2 \xrightarrow[T']{} y_3 \xrightarrow[S']{} y_1$$

$$y_1 \xrightarrow[T'']{} y_2 \xrightarrow[S'']{} x_1 \xrightarrow[T'']{} y_3 \xrightarrow[S'']{} y_1$$

We can assume that the diagrams have this form, since any chords would permit a sequential exchange immediately, and the possibility

$$y_1 \xrightarrow[T'']{} y_3 \xrightarrow[S'']{} x_1 \xrightarrow[T'']{} y_2 \xrightarrow[S'']{} y_1$$

for the second diagram is excluded by the fact that the arc $x_1 \xrightarrow[T'']{} y_3$ must be present. (This follows from the existence of arcs $x_1 \xrightarrow[T'']{} x_2$ and $x_2 \xrightarrow[T']{} y_3$, since T' is the result of replacing x_2 by x_1 in T''.) From the fact that both chains are chordless, we infer that neither $y_2 \xrightarrow[S']{} y_1$ nor $y_2 \xrightarrow[S'']{} y_1$ occurs. Hence y_2 depends on both $S'-y_1 = S-x_2$ and $S''-y_1 = S-x_1$. But then y_2 depends on $S-x_1-x_2$, which contradicts the fact that $\{y_2,y_3\}$ can be exchanged for $\{x_1,x_2\}$. This completes the proof.

<u>Note</u>:

After submitting this manuscript, the authors learned that D. E. Knuth had independently discovered the same proof of the Edmonds-Fulkerson partition theorem ("Matroid Partitioning") Stanford Technical Report Stan CS-73-342, March 1973). Knuth also employs an "arrow" notation which is similar to ours.

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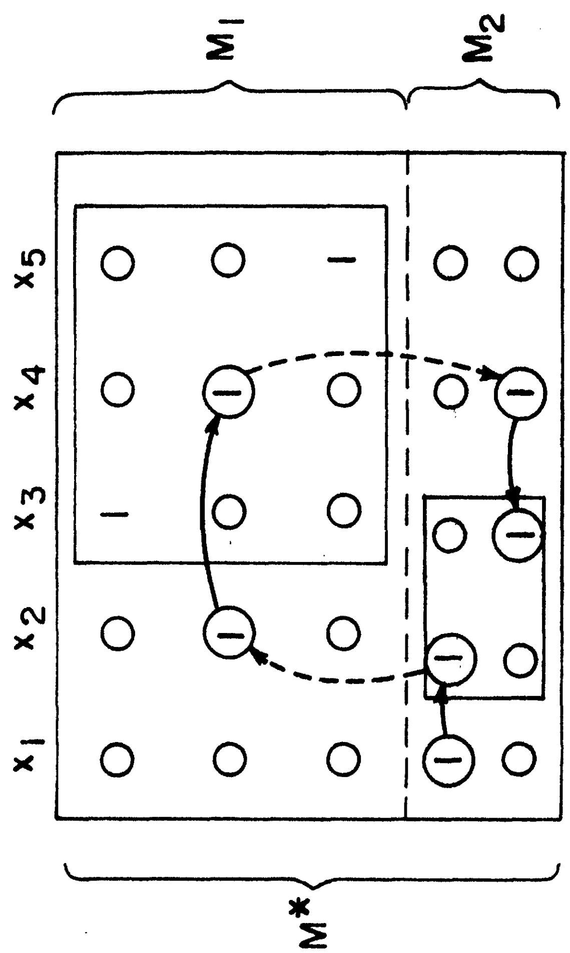

Fig. 1

Symbols appearing in text:

ϵ	set-membership (epsilon)
$\not\epsilon$	set-non-membership
$\subseteq$	set-inclusion
$\cup$	set-union (small)
$\cap$	set-intersection (small)
$\bigcap$	set-intersection (large)
$\bigcup$	set-union (large)
$\longrightarrow$	arrow [Note: all arrows are intended to be approximately the same length]