XVI. NETWORK SYNTHESIS

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A. CANONIC REALIZATIONS OF RC DRIVING-POINT ADMITTANCES

It is well known that the Foster and Cauer networks provide canonic realizations of two-element-kind driving-point admittances. There are, however, other canonic realizations of such admittances, if it is understood that the term "canonic realization" means any realization involving a minimum number of circuit elements. Work is now being done to ascertain the exact nature of these additional canonic networks. This report presents what is probably the essential property of these networks.

We start with some general considerations. Assume that the RC network N realizes the following admittance at a certain pair of its terminals:

$$Y(s) = \frac{a_r s^r \dots + a_1 s}{s^r \dots + b_1 s + b_o} \tag{1}$$

where $a_r, a_1, b_o \neq 0$. When N is excited by a current source I(s) acting at its driving terminals, and one of these terminals is taken as a datum, while the potentials e_1, e_2 $\dots$ e_N are assigned to the remaining nodes of N in such a way that the second driving terminal receives the potential e_1, then the matrix equilibrium equation for N has the form:

$$\begin{bmatrix} y_{11} & y_{12} & \cdots & y_{1N} \\ y_{21} & y_{22} & \cdots & y_{2N} \\ \vdots & \vdots & & \vdots \\ y_{n1} & \cdots & \cdots & y_{nn} \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_N \end{bmatrix} = \begin{bmatrix} I \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

The admittance Y(s) can be extracted from these equations.

$$Y(s) = \frac{I}{e_1} = \frac{|Y|}{Y_{11}} \tag{2}$$

where Y_{11} denotes the cofactor of y_{11}. Since $\frac{|Y|}{Y_{11}}$ reduces to Eq. 1, $|Y|$ and Y_{11} must reduce, respectively, to the numerator and denominator of Eq. 1 after their common factors are divided out. If P(s) denotes the product of all such common factors, then $|Y|$ and Y_{11} must be expressible as follows:

$$|Y| = P(s)\left[a_r s^r \dots + a_1 s\right] \tag{3a}$$

$$Y_{11} = P(s)\left[s^r \dots + b_1 s + b_o\right] \tag{3b}$$

As is well known, $|Y|$ and Y_{11} can be computed directly from the topology of N. If N_{oc} and N_{sc} denote the networks that result when the driving terminals of N are open-circuited or short-circuited, the rules for this computation are

$$|Y| = \Sigma \text{ Trees of } N_{oc}$$

$$Y_{11} = \Sigma \text{ Trees of } N_{sc}$$

Here, the value of any given tree is defined as the product of its branch admittances. Because N is an RC network, the value of each of its trees involves a non-negative power of s. Accordingly, one may expect $|Y|$ and Y_{11} both to be polynomials. If, under short-circuit conditions, the tree of N_{sc} involving the fewest capacitors contains q capacitors, then the term of lowest degree in Y_{11} will have degree q. This being the case, Eqs. 1 and 3b require that P(s) contain q factors of the form (s+0). All remaining factors of P(s) must be of the form $(s+a_i)$, $a_i \neq 0$; if there are i such factors, then P(s) may be written

$$P(s) = K s^q \prod_i (s+a_i) \qquad q, i \geqslant 0 \tag{4}$$

From Eqs. 3b and 4 it is clear that Y_{11} must have the form

$$Y_{11} = K s^q \prod_i (s+a_i)\left[s^r \dots + b_o\right]$$

$$= K s^{r+q+i} \dots + \left(K b_o \prod_i a_i\right) s^q \tag{5a}$$

where

$$q, i \geqslant 0 \tag{5b}$$

The highest degree term in Eq. 5a arises from one or from several trees in N_{sc}. Clearly, each tree that contributes to this term must contain r + q + i capacitors, and accordingly the number of branches per tree in N_{sc}, n_{sc}, must equal or exceed r + q + i. That is,

$$C_{sc} \geqslant r + q + i \tag{6a}$$

and

$$n_{sc} \geqslant r + q + i \tag{6b}$$

where C_{sc} denotes the number of capacitors in N_{sc}. The lowest degree term in Eq. 5a arises from one or from several trees that contain q capacitors and $n_{sc} - q$ resistors. If R_{sc} denotes the number of resistors in N_{sc}, this means that

$$R_{sc} \geq n_{sc} - q \geq i + r \tag{7}$$

From Eqs. 5b, 6a, and 7 it follows that

$$C_{sc} \geq r \tag{8a}$$

$$R_{sc} \geq r \tag{8b}$$

The equality sign applies only when $q = i = 0$. This concludes the general discussion.

There exist networks that realize Y(s), so that the lower bounds on C_{sc} and R_{sc} given by Eqs. 8a and 8b are realized (the Foster and Cauer networks, for instance). Thus the definition of a canonic network, given at the beginning of this report, implies that all networks which canonically realize Y(s) have:

$$C_{sc} = r \tag{9a}$$

$$R_{sc} = r \tag{9b}$$

$$q = i = 0 \tag{9c}$$

When conditions 9c are imposed upon Eq. 5a it is seen that for a canonical realization of Y(s), Y_{11} must have the form

$$Y_{11} = K\left[s^r \ldots +b_1 s+b_o\right] \tag{10}$$

Assume, now, that N realizes Y(s) canonically so that Eqs. 9a, 9b, 9c, and 10 hold. The Kb_o term of Eq. 10 can only arise from a tree of resistors selected from the r total resistors of N_{sc}; this implies that

$$n_{sc} \leq r \tag{11}$$

This condition, together with Eqs. 9c and 6b, shows that $n_{sc} = r$. Thus the Kb_o term of Eq. 10 is due to a resistor tree of N_{sc} which contains $n_{sc} = r$ resistors. But N_{sc} contains only r resistors. Therefore <u>the resistors of N_{sc} must form a tree.</u> Similarly, the Ks^r term of Eq. 10 arises from a tree containing r capacitors (hence all of the capacitors of N_{sc}) and possibly some resistors. But as all trees of N_{sc} contain r branches, this tree cannot involve resistors. Thus <u>the capacitors of N_{sc} must form a tree.</u> Thus we find that if a network N canonically realizes Y(s) at a pair of terminals, then N_{sc} has the following properties:

I. N_{sc} contains a total of r resistors which form a tree.

II. N_{sc} contains a total of r capacitors which form a tree.

It is not hard to show that for any N_{sc} having properties I and II, the numerator and denominator polynomials of any pliers type of admittance have degree less than r, when N_{sc} is hinged. Thus if N canonically realizes Y(s) at a pair of terminals, we must find additionally that

III. N_{sc} is unhinged.

On the basis of several examples that have been worked out in detail, the properties mentioned above appear to be essential characteristics of canonic topologies. That is, it seems probable that on any network constructed in accordance with these three properties, the element values can be so adjusted as to canonically realize any RC admittance having the form of Eq. 1, at any pliers type of entry. Work is now being carried out which should provide a definite answer to this conjecture.

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