IX. NOISE IN ELECTRON DEVICES[*]

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A. OPTIMUM NOISE PERFORMANCE OF MULTITERMINAL AMPLIFIERS

The problem to be considered is the optimization of the output signal-to-noise ratio of a multiterminal pair linear noisy amplifier driven by a multiterminal pair linear noisy source. We shall begin by making some assumptions about the properties of the source and amplifier networks.

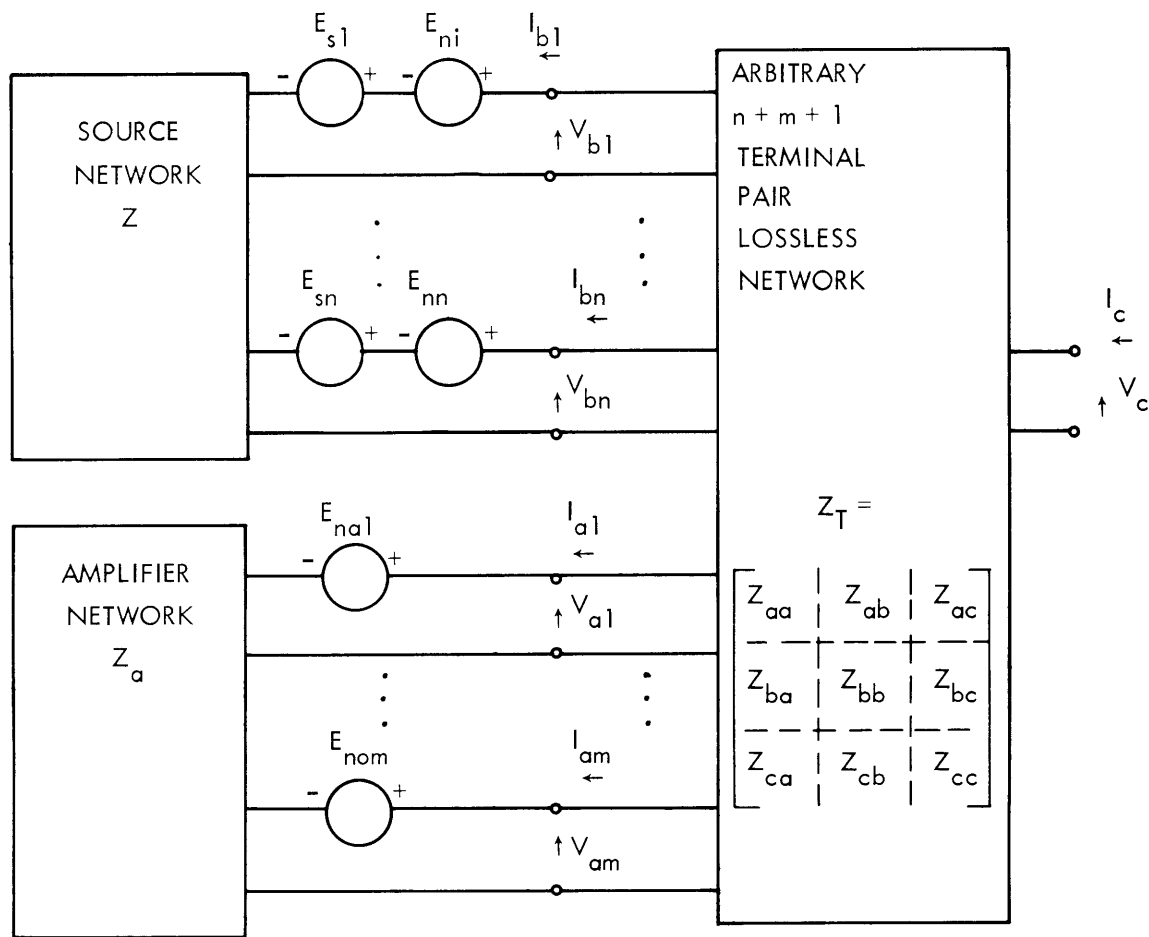The source network (Fig. IX-1) has n terminal pairs and its terminal properties

Fig. IX-1. Arbitrary lossless imbedding of source and amplifier networks.

will be described on an impedance basis. It is characterized by an impedance matrix, Z, and by the complex Fourier amplitudes of two sets of independent open-circuit voltages, the source voltages and the noise voltages. The voltage sources are completely characterized by their cross-power matrices[1] $\overline{E_s E_s^\dagger}$ and $\overline{E_n E_n^\dagger}$, where E_n is a column vector of the complex amplitudes of the open-circuit noise voltages, $E_n^\dagger$ is the Hermitian transpose of this vector, and the bar indicates an ensemble average. We use similar definitions for the signal voltages; rms values will be used for voltage amplitudes throughout this report. Both of the voltage matrices are positive definite or positive semidefinite. We shall assume that the Hermitian part of the source impedance matrix is positive definite. A positive definite impedance matrix implies that only a finite amount of power may be extracted from the source.

The amplifier network (Fig. IX-1) will also be described on an impedance basis with an impedance matrix Z_a and an open-circuit noise voltage vector E_{na}. The only require- ment that we shall make for these quantities is that the characteristic noise matrix[1]

$$N = -\frac{1}{2}\left(Z_a + Z_a^\dagger\right)^{-1} \overline{E_{na} E_{na}^\dagger}$$

have at least one positive eigenvalue. This is tantamount to assuming that the amplifier network is an active network that is capable of providing power gain.

We now wish to drive this amplifier with the source network in the most general way. To achieve this, we imbed the n-terminal pair source and the m-terminal pair amplifier in an n + m + 1 terminal pair lossless network (Fig. IX-1) and ask for the best signal- to-noise ratio that can be obtained at the single-output terminal pair. We require that the m-terminal pair amplifier provide net gain.

The equations characterizing this combined network are:

$$V_b = ZI_b + E_s + E_n \tag{1}$$

$$V_a = Z_a I_a + E_{na} \tag{2}$$

$$
\begin{bmatrix} V_a \\ \cdots \\ V_b \\ \cdots \\ V_c \end{bmatrix} =
\begin{bmatrix}
Z_{aa} & \vdots & Z_{ab} & \vdots & Z_{ac} \\
\cdots & & \cdots & & \cdots \\
Z_{ba} & \vdots & Z_{bb} & \vdots & Z_{bc} \\
\cdots & & \cdots & & \cdots \\
Z_{ca} & \vdots & Z_{cb} & \vdots & Z_{cc}
\end{bmatrix}
\begin{bmatrix} -I_a \\ \cdots \\ -I_b \\ \cdots \\ +I_c \end{bmatrix} =
\begin{bmatrix} Z_T \end{bmatrix}
\begin{bmatrix} -I_a \\ \cdots \\ -I_b \\ \cdots \\ I_c \end{bmatrix} \tag{3}
$$

where $Z_T + Z_T^\dagger = 0$. Through algebraic manipulation we can obtain a relation involving only V_c, I_c, and the sources, and from this we find the exchangeable power at the output terminals which is due to all of the sources

$$P_{eo} = \frac{x_1^\dagger \overline{E_s E_s^\dagger} x_1 + x_1^\dagger \overline{E_n E_n^\dagger} x_1 + x_2^\dagger \overline{E_{na} E_{na}^\dagger} x_2}{2x_1^\dagger \left(Z+Z^\dagger\right) x_1 + 2x_2^\dagger \left(Z_a+Z_a^\dagger\right) x_2} , \tag{4}$$

where

$$\left\{ \begin{bmatrix} Z & \vdots & 0 \\ \cdots & \vdots & \cdots \\ 0 & \vdots & Z_a \end{bmatrix} + \begin{bmatrix} Z_{aa} & \vdots & Z_{ab} \\ \cdots & \vdots & \cdots \\ Z_{ba} & \vdots & Z_{bb} \end{bmatrix} \right\}^{\dagger -1} \begin{bmatrix} Z_{ac} \\ \cdots \\ Z_{bc} \end{bmatrix} = \begin{bmatrix} x_1 \\ \cdots \\ x_2 \end{bmatrix}.$$

Where, x_1 is an n-dimensional vector and x_2 is an m-dimensional vector. Both of these vectors may be varied arbitrarily by varying the lossless network.

From Eq. 4 we see that the signal power that is exchangeable at the output is

$$P_{eo}^S = \frac{x_1^\dagger \overline{E_s E_s^\dagger} x_1}{2x_1^\dagger \left(Z+Z^\dagger\right) x_1 + 2x_2^\dagger \left(Z_a+Z_a^\dagger\right) x_2} , \tag{5}$$

and the ratio of the exchangeable signal power to the exchangeable noise power is

$$\left(\frac{S}{N}\right)_o = \frac{x_1^\dagger \overline{E_s E_s^\dagger} x_1}{x_1^\dagger \overline{E_n E_n^\dagger} x_1 + x_2^\dagger \overline{E_{na} E_{na}^\dagger} x_2}. \tag{6}$$

The problem may now be restated in terms of Eqs. 5 and 6. We wish to vary x_1 and x_2 (by varying the lossless network) to optimize $(S/N)_o$ and keep P_{eo}^S constant. Gain will be obtained by varying x_2 only through those values for which $x_2^\dagger \left(Z_a+Z_a^\dagger\right) x_2$ is negative. Then, by adjusting only the relative lengths of the vectors x_1 and x_2, we may obtain any negative values of P_{eo}^S and any positive values of P_{eo}^S for which

$$P_{eo}^S > \frac{x_1^\dagger \overline{E_s E_s^\dagger} x_1}{2x_1^\dagger \left(Z+Z^\dagger\right) x_1} . \tag{7}$$

Solving Eq. 5 for $2x_2^\dagger \left(Z_a+Z_a^\dagger\right) x_2$ and multiplying and dividing the second term in the denominator of Eq. 6 by this quantity, we obtain

$$\left(\frac{S}{N}\right)_o = \frac{x_1^\dagger \overline{E_s E_s^\dagger} x_1}{x_1^\dagger \overline{E_n E_n^\dagger} x_1 + \left[-\dfrac{x_2^\dagger \overline{E_{na} E_{na}^\dagger} x_2}{2x_2^\dagger \left(Z_a+Z_a^\dagger\right) x_2} \right] \left[2x_1^\dagger \left(Z+Z^\dagger\right) x_1 - \dfrac{1}{P_{eo}^S} x_1^\dagger \overline{E_s E_s^\dagger} x_1 \right]} , \tag{8}$$

in which, because of the preceding assumptions, both of the quantities in brackets are positive. If we now begin our optimization by varying x_2 and holding x_1 and P_{eo}^S constant, we see that our constraints require that we vary $x_2^\dagger \overline{E_{na} E_{na}^\dagger} x_2$ while we keep $x_2^\dagger \left(Z_a + Z_a^\dagger \right) x_2$ at a constant negative value in order to have P_{eo}^S constant. But this is just the problem of finding the stationary values of the quantity $-x_2^\dagger \overline{E_{na} E_{na}^\dagger} x_2 / 2 x_2^\dagger \left(Z_a + Z_a^\dagger \right) x_2$. The stationary value of this quantity which gives the best signal-to-noise ratio is the least positive stationary value, which, in turn, is the least positive eigenvalue of the characteristic noise matrix, $N = -1/2 \left(Z_a + Z_a^\dagger \right)^{-1} \overline{E_{na} E_{na}^\dagger}$. We shall denote this eigenvalue by λ_1, and its corresponding eigenvector by $x_2^{(1)}$.

Using this notation in Eq. 8, we have

$$\left(\frac{S}{N} \right)_o = \frac{x_1^\dagger \overline{E_s E_s^\dagger} x_1}{x_1^\dagger \overline{E_n E_n^\dagger} x_1 + \lambda_1 \left[2 x_1^\dagger \left(Z + Z^\dagger \right) x_1 - \frac{1}{P_{eo}^S} x_1^\dagger \overline{E_s E_s^\dagger} x_1 \right]} . \tag{9}$$

We may vary $x_1^\dagger$ in a completely arbitrary fashion and keep λ_1 and P_{eo}^S constant. P_{eo}^S may be held constant by simultaneously varying the length of x_2; these variations have no effect on λ_1. Then the stationary values of the signal-to-noise ratio are found to be the eigenvalues σ_i of the equation

$$\overline{E_s E_s^\dagger} x_1 - \sigma \left[\overline{E_n E_n^\dagger} + 2 \lambda_1 \left(Z + Z^\dagger \right) - \frac{\lambda_1}{P_{eo}^S} \overline{E_s E_s^\dagger} \right] x_1 = 0. \tag{10}$$

Or, if we let

$$\mu = \frac{\sigma}{1 + \dfrac{\lambda_1 \sigma}{P_{eo}^S}} , \tag{11}$$

then

$$\overline{E_s E_s^\dagger} x_1 - \mu \left[\overline{E_n E_n^\dagger} + 2 \lambda_1 \left(Z + Z^\dagger \right) \right] x_1 = 0. \tag{12}$$

The eigenvectors $x_1^{(i)}$ of Eq. 12 are also eigenvectors of Eq. 10. Therefore, for each eigenvalue μ_i there is a corresponding σ_i, which may be determined from μ_i by use of the equation

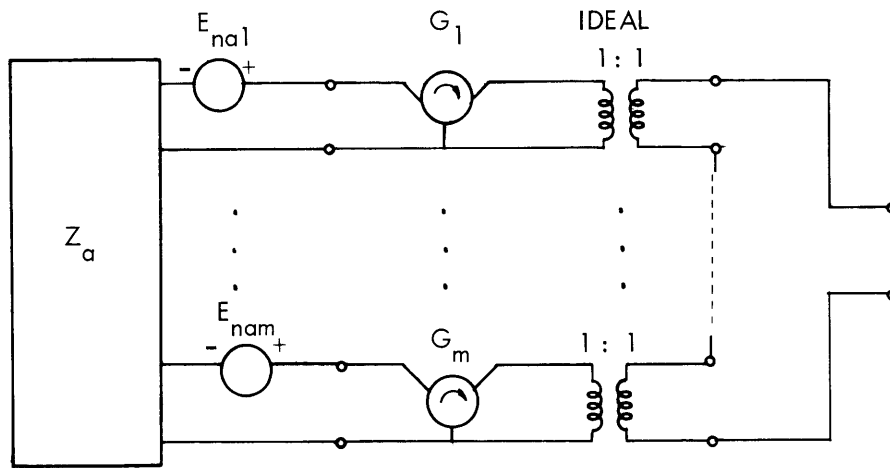$$\sigma = \frac{\mu}{1 - \dfrac{\lambda_1 \mu}{P_{eo}^S}} . \tag{13}$$

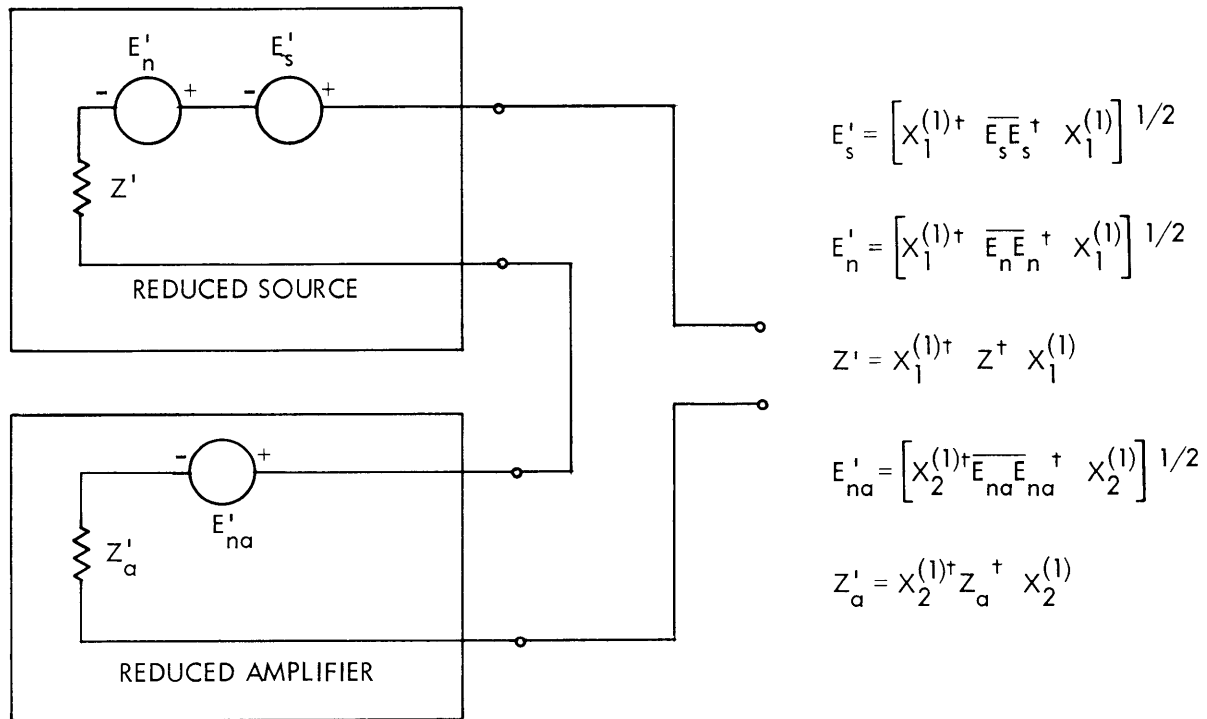Fig. IX-2. Reduction of amplifier to a one-terminal pair device.

Fig. IX-3. Realization of the optimal network.

$$E_s' = \left[X_1^{(1)\dagger} \; \overline{E_s E_s}^\dagger \; X_1^{(1)} \right]^{1/2}$$

$$E_n' = \left[X_1^{(1)\dagger} \; \overline{E_n E_n}^\dagger \; X_1^{(1)} \right]^{1/2}$$

$$Z' = X_1^{(1)\dagger} \; Z^\dagger \; X_1^{(1)}$$

$$E_{na}' = \left[X_2^{(1)\dagger} \overline{E_{na} E_{na}}^\dagger \; X_2^{(1)} \right]^{1/2}$$

$$Z_a' = X_2^{(1)\dagger} Z_a^\dagger \; X_2^{(1)}$$

We can show that the maximum signal-to-noise ratio, σ_1, corresponds to the largest eigenvalue μ_1 of Eq. 12, by taking the derivative of Eq. 13 with respect to μ. Since

$$\frac{d\sigma}{d\mu} = \frac{1}{\left(1 - \dfrac{\mu\lambda_1}{P^s_{eo}}\right)^2} \quad , \tag{14}$$

σ is an increasing function of μ everywhere.

By referring to Eq. 13, we can make several statements concerning the optimum noise performance of multiterminal pair amplifiers. If two amplifiers that have the same best value λ_1 of their characteristic noise matrices are driven by the same positive definite source, and if under these conditions one of the amplifiers has a positive output impedance (positive exchangeable power) and the other has a negative output impedance (negative exchangeable power), then the best signal-to-noise ratio obtainable with the latter cannot be better than that obtainable with the former.

Along these same lines, we can state that since large amounts of output power are available only for large positive values of output exchangeable power or for negative values of exchangeable power, the best signal-to-noise ratio that can be obtained at large values of available output signal power is approximately μ_1, which is the largest eigenvalue of the matrix

$$\left[\overline{E_n E_n^\dagger} + 2\lambda_1\left(Z + Z^\dagger\right)\right]^{-1}\overline{E_s E_s^\dagger} \quad .$$

The foregoing optimization can be generalized somewhat to include more than one amplifier and lossy imbeddings. In particular, it can be demonstrated that an arbitrary passive dissipative interconnection of any number of independently noisy amplifiers with a given positive definite noisy source cannot produce a higher signal-to-noise ratio at large values of available power than the optimal lossless connection of the best amplifier with that source. This is most readily proved by regarding the imbedding network as a lossless interconnection of its canonic form with each of the amplifiers. We may then regard all of the amplifiers and the set of noisy resistors arising from the canonic representation of the imbedding network as a new amplifier. This new amplifier has as the eigenvalues of its characteristic noise matrix all of the eigenvalues of the characteristic noise matrices of each of the amplifiers and the resistors. Here, as before, we pick the least positive one of these eigenvalues, which proves the statement made above.

Realization of the Optimal Network

The simplest realization of the optimal amplifying device may be derived in a straightforward manner. We may reduce the amplifier to a one-terminal pair network with a lossless network consisting of ideal gyrators and ideal transformers as shown in

Fig. IX-2. If we adjust the gyrator coefficients so that $G_n = x_{2,n}^{(1)}$, where $x_{2,n}^{(1)}$ is the nth element of the vector $x_2^{(1)}$, we obtain the one-terminal pair amplifier shown in Fig. IX-3. A similar reduction may be performed on the source network with $G_n = x_{1,n}^{(1)}$. The optimal network is then the series connection shown in Fig. IX-3. Its signal-to-noise ratio at large values of available power is given by μ_1.

An optimal unilateral device may be obtained by using the reduced source and amplifier networks shown in Fig. IX-3 in conjunction with an ideal lossless circulator.

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References

1. H. A. Haus and R. B. Adler, Circuit Theory of Linear Noisy Networks (The Technology Press of Massachusetts Institute of Technology, Cambridge, Mass., and John Wiley and Sons, Inc., New York, 1959).