VIII. PHYSICAL ACOUSTICS*

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A. STABILITY OF PARALLEL FLOWS

Heisenberg's discovery of unstable asymptotic solutions for the separated perturbation equations for plane viscid Poiseuille flow,[1] together with Rayleigh's proof of the nonexistence of unstable separating solutions in the inviscid case,[2] has encouraged the conjecture that viscous forces can be a cause of instability. When the rarity of separating solutions among the solutions of the full initial value problem for the perturbed flow is properly taken into account, however, this apparent disconformity with the expected stabilizing role of viscosity disappears. The absence of separating inviscid solutions corresponding to the viscous ones found by Heisenberg is to be viewed as implying not that the inviscid flow is stable, but only that the equations of perturbation fail to separate in the inviscid case.

The equations of perturbation do seperate in general under perturbations that are homogeneous in the direction of the basic flow; otherwise, in the case of inviscid flows without points of inflection, they do not. In the author's thesis,[3] the initial value problem restricted to such perturbations was worked out for an arbitrary basic flow in the inviscid case, and for plane and cylindrical flows in the viscid case. A somewhat less transparent approach was used earlier to discuss the general inviscid and plane viscid cases.[4] It is shown that every inhomogeneous inviscid flow is unstable under such perturbations. In the viscid case, the solutions follow the corresponding inviscid ones in time until growth by a factor of the order of the Reynolds number is achieved, after which rapid viscosity-controlled decay sets in. Comparison with the exact equations of motion indicates that qualitatively different results are not to be expected when perturbations of arbitrary amplitude are admitted.[3,4]

In the author's thesis,[5] it is also shown in the plane Couette case, and proposed on physical grounds in the general case, that the inviscid flow is actually not stable under any perturbation at all. Asymptotic separating solutions are found for the viscid Couette case; and these, in contrast with their inviscid counterparts, are all stable. In the absence of boundaries, viscid Couette flow is shown, moreover, to be stable under every sufficiently smooth perturbation.

A brief outline of the work on plane Couette flow[5] is given here.

The fundamental perturbation equation for a plane parallel flow $W(x_1)$ in the x_3 direction has been given[4] in the form

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$$\frac{\partial}{\partial t}\nabla^2 u_1 = \nu\nabla^4 u_1 - W\frac{\partial}{\partial x_3}\nabla^2 u_1 + \frac{d^2W}{dx_1^2}\frac{\partial u_1}{\partial x_3}. \tag{1}$$

For the plane Couette flow $W(x_1) = x_1 V$, Eq. 1 simplifies to

$$\left(\frac{\partial}{\partial t} - \nu\nabla^2 + Vx_1\frac{\partial}{\partial x_3}\right)\nabla^2 u_1 = 0. \tag{2}$$

The exact free-field propagator for (2) has been found to be

$$K(\vec{x},\vec{y},t) = \frac{1}{(4\pi\nu t)^{3/2}\left(1+\frac{V^2t^2}{12}\right)^{1/2}} \cdot \exp\left\{-\frac{1}{4\nu t}\left[(x_1-y_1)^2 + (x_2-y_2)^2 + \frac{\left(x_3 + Vt\frac{x_1+y_1}{2} - y_3\right)^2}{1+\frac{V^2t^2}{12}}\right]\right\}, \tag{3}$$

so that, formally, for $t > 0$,

$$\nabla^2 u_1(\vec{x},t) = \iiint K(\vec{x},\vec{y},t)\,\nabla^2 u_1(\vec{y},0)\,d^3y. \tag{4}$$

The propagator (3) is similar to the familiar Poisson propagator for the heat equation, the result of the extra terms in (3) being ultimately, if anything, an increase in the damping of the initial disturbance. Thus it is shown readily that in the absence of boundaries the plane viscid Couette flow is stable under every twice-differentiable perturbation $u_1(\vec{x},0)$ that tends to zero, along with its second derivative, for $x^2 \to \infty$. When periodic boundary conditions are imposed in the x_2 and x_3 directions, the same result holds for perturbations that are periodic rather than those that tend to zero in those directions.

In the presence of boundaries, we require, not the free-field propagator, but a propagator maintaining viscous boundary conditions. The term $Vt(x_1+y_1)/2$ in the exponential in (3) rules out the use of the image methods usually employed in constructing bounded-field propagators from free-field ones.

In the absence of a conclusive demonstration of the stability of plane viscid Couette flow in the presence of boundaries, we have sought unstable separating solutions for (2) of the type

$$\nabla^2 u_1(\vec{x},t) = a(x_1)\,e^{st}\,\exp(ik_2x_2+ik_3x_3) \tag{5}$$

for which (2) reduces to

$$\left(s + \nu k^2 - \nu \frac{d^2}{dx_1^2} + ik_3 V x_1\right) a(x_1) = 0, \tag{6}$$

where $k^2 = k_2^2 + k_3^2$.

Equation 6 is a form of Airy's equation, the two basic solutions of which can be written as contour integrals by means of the Laplace method. Accurate asymptotic representations have been found for these integrals through the use of the saddle-point method of integration. The requirement that u_1 vanish, along with its first derivative, at given boundaries, say $x_1 = 0$ and $x_1 = 1$, establishes an eigenvalue problem for s. The problem has solutions, the number and accuracy of which increase without limit as $k_3 V/\nu = k_3 R \to \infty$. All such solutions for $u_1(x,t)$ are stable, and, in fact, decay with time at least as rapidly as

$$\exp\left[-\left(\nu k^2 + \frac{|k_3 V|}{4}\right)t\right]. \tag{7}$$

When $\nu = 0$, the basic equations of perturbation[6] contain no derivatives of the u_i higher than the first, so that $\nabla^2 u_1$ need not exist. When the Laplacian of the initial perturbation $u_1^{(0)}$, as well as $\frac{\partial}{\partial x_3} \nabla^2 u_1$, does exist, however, they will satisfy the inviscid form of (2):

$$\left(\frac{\partial}{\partial t} + V x_1 \frac{\partial}{\partial x_3}\right) \nabla^2 u_1 = 0. \tag{8}$$

Under the initial conditions $\nabla^2 u_1^{(0)} = a(x_1, x_2, x_3)$, the relevant solution of (8) will have the familiar form

$$\nabla^2 u_1(x_1, x_2, x_3, t) = a(x_1, x_2, x_3 - V x_1 t), \tag{9}$$

which is verifiable by direct substitution.

Equation 9 can be solved for u_1 with the boundary conditions that u_1 vanish at the walls and either be periodic in x_2 and x_3 or tend to zero as x_2 and x_3 become infinite.

In order to use (8) and (9) even when $\nabla^2 u_1$ does not exist, we expand $u_1^{(0)}$ in a Fourier series or Fourier integral in x_2 and x_3, our choice depending upon the boundary conditions. Provided that $\frac{d^2 u_1^{(0)}}{dx_1^2}$ exists, the individual Fourier components[7]

$$u_1^k(x_1, t) \exp(ik_2 x_2 + ik_3 x_3) \tag{10}$$

for u_1 can be computed from (8) whether or not the full series or integral is twice differentiable with respect to x_2 and x_3. The expansion is then a solution for the first of the basic equations of perturbation,[6] provided that it converges along with its first derivative with respect to x_3.

Substituting (10) in (9), we obtain the equation

$$\left(\frac{\partial}{\partial t}+ik_3Vx_1\right)(\partial^2-k^2)\, u_1^k(x_1,t) = 0, \tag{11}$$

where $\partial = \frac{\partial}{\partial x_1}$ and $k^2 = k_2^2 + k_3^2$, so that

$$(\partial^2-k^2)\, u_1^k(x_1,t) = \exp(-ik_3Vx_1t)\, (\partial^2-k^2)\, u_1^k(x_1,0). \tag{12}$$

Note that in case $k_3 = 0$, (11) and (12) become trivial, and $u_1^{k_2}(x_1,t) = u_1^{k_2}(x_1,0)$. This is precisely the case of perturbations that are homogeneous in the direction of flow that was studied in general.[3] The material[5] discussed here thus involves the behavior of the modes with $k_3 \neq 0$.

It can be verified by direct substitution that a particular inversion integral for (12) is

$$u_p^k(x_1,t) = e^{-kx_1} \int_0^{x_1} e^{2kx'} \int_{\epsilon(t)}^{x'} e^{-kx''} \exp(-ik_3Vx''t)\, (\partial^2-k^2)\, u_1^k(x'',0)\, dx''dx'. \tag{13}$$

Equation 13, as written, vanishes identically for $x_1 = 0$, and it is clear that, for each t, $1 \geqslant \epsilon \geqslant 0$ can be chosen so that (13) will also vanish at $x_1 = 1$ and thus satisfy the boundary conditions for u_1.

By means of estimates of the integrals in (13), certain total integrability conditions on $\frac{\partial^2}{\partial x_1^2} u_1(x_i,0)$ are derived which suffice for the uniform boundedness of $\sum_k |u_1^k(x_1,t)|$ and therefore of $u_1(x_i,t)$. For example, either

$$\int dx_2 \int dx_3 \left(\int_0^1 \left|\frac{\partial^2}{\partial x_1^2} u_1(x_i,0)\right| dx_1\right)^2 < \infty$$

or (14)

$$\int dx_2 \int dx_3 \left\|\frac{\partial^2}{\partial x_1^2} u_1(x_i,0)\right\|_{\infty(x_1)} < \infty$$

suffices.

As a counterexample to much weaker sufficient conditions than those of the character of (14), a perturbation $u_1^*(x_i,0)$ has been given,[5] which possesses derivatives of all orders with respect to x_2 and x_3, but only one continuous derivative with respect to x_1, for which the solution $u_1(x_i,t)$ is not bounded in time, but rather oscillates with linearly increasing amplitude for large t.

Under certain conditions $u_1(x_i,t)$ does itself tend to zero for large t, but it does not follow that the flow is stable under these modes of perturbation. The inhomogeneous term $-Vu_1$ in the equation[6] for u_3 yields a contribution to u_3 which fails to vanish for large t even when u_1 does tend to zero, unless $u_1(x_i,0)$ vanishes identically.

The equations of perturbation[6] for u_2 and u_3, for an arbitrary plane parallel flow $W(x_1)$, are

$$\frac{\partial u_i}{\partial t} = -W\frac{\partial u_i}{\partial x_3} - \delta_{i3}\frac{dW}{dx_1}u_1 - \frac{\partial p}{\partial x_i}. \tag{15}$$

One may easily verify that the solution for (15) with given initial conditions $u_i^{(0)}$ can be formally written

$$u_i(x_1,x_2,x_3,t) = u_i^{(0)}(x_1,x_2,x_3-W(x_1)t) - \int_0^t f(x_1,x_2,x_3-W(x_1)(t-t'),t')\,dt' \qquad i = 2, 3, \tag{16}$$

where f represents the inhomogeneous terms on the right-hand side of (15).

It is clear from (16) that the u_i cannot tend uniformly to zero for large t unless $u_2^{(0)}$ and $u_3^{(0)}$ vanish identically. In general, then, we conclude that plane inviscid Couette flow is stable under no nontrivial perturbations whatsoever.

Actually, in view of the absence of apparent means of dissipation, the only surprising element in this result should be the vanishing of u_1 for large t in certain cases. This effect can be "explained" in the following way: as a result of the shearing action of the basic flow, as $t \to \infty$ every neighborhood of a given point (x_1,x_2,x_3) will contain elements of the initial perturbation field from a continuum of points along a tubular neighborhood extending from $(x_1,x_2,-\infty)$ to (x_1,x_2,∞) along the streamlines of the basic flow. One might expect that the vanishing of $u_1(x_i,t)$ at (x_1,x_2,x_3) as $t \to \infty$ could result from the vanishing of the average of the initial perturbation $u_1(x_i,0)$ along streamlines passing near (x_1,x_2,x_3).

The result of the mathematical analysis is essentially the same as that given above. It is clear that, according to the Riemann-Lebesgue theorem, the inner integral in (13), and therefore $u_1^k(x_1,t)$, will vanish for $t \to \infty$, provided that $k_3 \neq 0$ and $\frac{\partial^2}{\partial x_1^2}u_1^k(x_1,0)$ is integrable in x_1.

Because of the uniformity of convergence of $\sum_k |u_1^k(x_1,t)|$, $u_1(x_i,t)$ itself will tend to zero for large t, provided that all of the $u_1^k(x_1,0)$ for $k_3 = 0$ vanish identically. The latter condition is clearly equivalent to

$$\int_{-\infty}^{\infty} u_1(x_i,0)\, dx_3 = 0 \qquad \text{for almost all } x_1 \text{ and } x_2,$$

which is the same condition that was stated in words on the basis of the physical argument.

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References

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5. H. L. Willke, Jr., Stability Problems in Parallel Flow, op. cit., pp. 40-77.

6. H. L. Willke, Jr., Stability of parallel flows, op. cit., p. 45, Eq. 1a.

7. It is hoped that no confusion will result from the use of k in superscripts to stand for (k_2,k_3).

B. SOUND EMISSION FROM KARMAN VORTICES

The instability of flow past a cylinder resulting in the familiar Karman vortex street generally represents a very weak sound source. However, we have found that when the cylinder is placed inside a tube and perpendicular to the axis, an intense oscillation is produced whenever the frequency of the Karman vortex shedding coincides with one of the cross-mode resonances in the tube. At these frequencies the standing wave established across a tube stimulates the instability of the flow in the tube, and through this feedback mechanism a self-sustained fluid oscillation of considerable magnitude results. The frequency of shedding of the Karman vortices is known empirically to be $f_o = \alpha V/d$, where V is the flow velocity and d the cylinder diameter. The Strouhal number α is a weak function of Reynolds number R and is approximately a constant $\alpha \approx 0.18$ in the region $10^2 < R < 10^5$. In the case of a rectangular tube, the relevant n^{th} cross-mode resonance occurs at a wavelength $\lambda_n = 2D/n$, where D is the transverse horizontal dimension of the tube, which is perpendicular to the cylinder. Strong coupling between the Karman vortex street and the transverse acoustic modes, then, occurs when $f_o = f_n = nc/2D$, where c is the speed of sound and n an integer. Using the value for f_o given above, we see that the condition for resonance can also be written

$$V = \left(\frac{n}{2\alpha}\right) \frac{d}{D} c \approx 2.8n \frac{d}{D} c.$$

Although in free space the spectrum of the sound from the Karman vortices contains practically only the fundamental frequency f_o, the resonance oscillations in the tube contain a large number of harmonics.

The feedback between the sound field and the vortex street is produced by the velocity field in the sound field. This has been demonstrated by measuring the intensity of the sound field as a function of position of the cylinder in the direction perpendicular to the tube axis. Maximum intensity is obtained when the cylinder is placed in a maximum velocity of the sound field.

A similar feedback oscillator involving two interacting cylindrical rods has also been studied. When two rods are placed one after the other and perpendicular to the flow, a resonance oscillation occurs when the time of convection of a vortex from one cylinder to the other is equal to the period of the Karman tone. Since the drift velocity of a vortex is approximately 80 per cent of the mean flow speed in the tube, it follows that the condition for such a resonance is simply $0.8 \frac{L}{V} = \frac{1}{f_o}$, which corresponds to a separation of the cylinders which is equal to approximately four times the cylinder diameter.

A more detailed account of these studies is being prepared for publication.

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C. GENERATION OF SHOCK WAVES BY MEANS OF EXPLODING METALLIC FILMS

Shock waves produced by discharging a condenser bank through film of aluminized Mylar have been studied in air at various pressures. High-frequency electrostatic transducers have been used as probes for the determination of shock speed and the measurement of shock strength and reflection coefficients. A preliminary analysis of the data indicates that the shock speed is proportional to the square root of discharged energy and, at least at low pressures, proportional to the fourth power of the ambient density. Also at low pressures electrical effects attributed to plasma flow behind the shock were observed. A detailed analysis of the data is now under way.

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