XX. PROCESSING AND TRANSMISSION OF INFORMATION*

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RESEARCH OBJECTIVES

1. Picture Processing

Work is continuing on the processing of pictures by means of computers. The broad objective of this work is to elucidate the fundamental properties of vision as they apply to image transmission and reproduction. Among the more specific objectives are the design of efficient image-transmission systems, and the development of devices capable of performing some "human" operations, such as noise reduction, image detection, and quality improvement.

Progress during the past year toward reaching these objectives has been made in studies pertaining to the visibility of noise of known spectral content, the improvement of image quality through linear filtering, and the synthesis of motion-picture sequences from a small portion of the information of each frame. Studies are continuing on two-dimensional image synthesis techniques for improving transmission efficiency.

W. F. Schreiber

2. Communications

Preliminary work on the transmission of vocoded speech has emphasized the need for an experimental vocoder suitable for use in conjunction with various modulation and coding schemes. Such a vocoder must be flexible in its configuration. Accordingly, the IBM 7090 is being programmed, through use of the BTL BLD-DI compiler, to provide a general-purpose speech-processing facility. Various techniques for the reduction of redundancy will be used, together with error-correcting codes, in an effort to improve voice communication over noisy channels.

Adequate analytical treatment of the performance of encoding and decoding schemes over noisy time-variant channels is very difficult. Theoretical investigations of this work continue, but it is certain that experimental evaluation of the ideas generated by such studies will be necessary. To this end, a channel characterized by acoustic reflection from air bubbles in water has been constructed in the laboratory, and is now being evaluated. It appears that most of the communication difficulties inherent in channels with a time-bandwidth product greater than unity are exhibited by this acoustic channel.

*This research was supported in part by the National Science Foundation (Grant G-16526), the National Institutes of Health (Grant MH-04737-03), the National Aeronautics and Space Administration (Grant NsG-496); and in part by Purchase Order DDL BB-107 with Lincoln Laboratory, a center for research operated by Massachusetts Institute of Technology, with the support of the U.S. Air Force under Contract AF 19(628)-500.

In order to complete the laboratory facilities needed to support experimental communication study, a flexible encoder-decoder is required. Special-purpose adjuncts, added to a general-purpose digital computer, would provide an efficient compromise between speed and flexibility. The conceptual design of these ancillary devices, and the evaluation of the over-all performance resulting from their use, is under way.

In addition to the experimental program outlined above, fundamental theoretical work in the processing and transmission of information continues on a broad front. Significant progress has been made recently on the bounding of the error performance achievable by means of coding. One of the most important results is new insight into the interrelation of coding and modulation, and the solution of certain related problems has already been forthcoming. A considerable amount of further work in this direction is now in progress.

J. M. Wozencraft

3. Digital Machines and Automata

Work continues on the basic capabilities of digital machines and automata. Two primary objectives of this work are: to gain a better understanding of the relationship between a given processing problem and the amounts of equipment and computation time that it requires; and to achieve description and synthesis of digital processors in the form of arrays of identical building blocks.

F. C. Hennie III

A. PICTURE PROCESSING

1. THE SUBJECTIVE EFFECTS OF PICTORIAL NOISE

A study has been made of the subjective effects of the class of independent additive rectangular lowpass Gaussian noises.[1] Three original pictures, varying in the amount of detail, were used. The general shapes of the isopreference surfaces in σ-k_1-k_2 space, where σ is the rms value, and k_1 and k_2 are the bandwidths of the noise in the horizontal and vertical directions, respectively, were found to be similar for all three pictures. In particular: (a) If we keep σ constant and go radially outward from the origin in the k_1-k_2 plane, the objectionable noise will increase, reach a maximum, then fall off. (b) Noises with vertical streaks are more objectionable than those with horizontal streaks.

The details of the isopreference surfaces, however, depend very much on the original picture. Generally speaking, noises that contain frequencies similar to those of the picture are less annoying.

The agreement among the observers was rather good. They were more in agreement as to the effect of changes in noise power than in noise bandwidths.

If objectionable noise is additive, then we may deduce that for the class of noises whose power density spectra are symmetrical with respect to both horizontal and vertical frequencies, the weighting function in the integral representing objectionable noise is similar in shape to the isopreference surface mentioned above.

T. S. Huang

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2. CODING COLOR PICTURES

A computer-simulation study of efficient coding for color pictures has been undertaken.[1] Two typical color transparencies were resolved into three primaries, sampled in a square array and recorded digitally on magnetic tape. The computer program transformed these data into luminance and chrominance quantities, performed certain parameter modifications, reconverted them into primary-color quantities and wrote them on an output tape. The parameters that were modified were the effective number of samples per picture and the number of quantum values each for the luminance and for the chrominance. The output tape was played back through the recorder-reproducer to produce images of the coded pictures on the face of the cathode-ray tube, which were photographed through appropriate filters on color film. The resultant transparencies were later viewed and compared by a number of observers to determine the absolute and relative quality achievable with the various codes (as affected by the variously modified parameters). Also, a test was run with a large number of observers to determine the relative recognizability of objects in one of the pictures when variously coded in color or monochromatically.

The results show that while the best monochromatic reproduction achievable in the experimental system requires a transmission rate of 5 bits per sample (with logarithmic quantization used), the best color reproduction in the same system (with the same luminance sample density) requires an average of 5.55 bits per sample. This is achieved by quantizing chrominance to approximately 1000 values and reducing the spatial density of chrominance samples to 1/18 of that of luminance. The results also indicate that the luminance sample density of a color picture can be reduced by a factor of from 1.5 to 18, or more, and still be equal in quality to the monochromatic reproduction, the amount depending on the subject matter and on the criterion used for comparison.

Two major conclusions were drawn from this study. (i) A normal monochromatic picture can be converted into a full color picture of the same apparent sharpness by transmitting additionally only a fraction of a bit per sample. (ii) For many purposes, inclusion of color may result in an over-all lower transmission rate requirement than would the same picture coded monochromatically; for some purposes, such as recognizing objects, this reduction can be substantial.

U. F. Gronemann

References

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B. EXPERIMENTAL FACILITY FOR SEQUENTIAL DECODING

This report presents progress on a study concerning the advisability of constructing special-purpose digital equipment to work in conjunction with an IBM 7094 computer for the purpose of aiding the investigation of sequential decoding processes. We hope that suitable equipment can be developed which will be adaptable to a large class of channels and modulation processes and will increase the speed with which information bits may be decoded by a factor of 10, or more, over that possible with computer-simulation programs.

1. Motivation

The need for an experimental facility for the study of sequential decoding is quite obvious. For any but the simplest channel, analytic results are hard to obtain because of the mathematical difficulties that arise. Experimental results can be used to great advantage in combination with the analytic results that are now available to extend our understanding of sequential decoding processes to the more complicated channels found in the real world.

Questions of interest at present are: What and how much information should be saved at the receiver for each use of the channel? For example, if one of M orthogonal signals is sent each time that the channel is used, then there will be M signal values (from the M matched filters) available at the receiver. If all of these values were saved for use later in the tree-search algorithm, an excessive amount of storage would be required, and computations involving this many numbers would be complicated. Various alternatives suggest themselves: (a) Save only the information as to which of the M signals was largest. (b) Save an ordered list of the ℓ largest signals (which one was largest, which was second largest, and so forth). (c) Save an ordered list as in (b), but also include the values of the signals. (d) Save the largest signal, plus the sum of the squares of all of the other signal values. Clearly, the less information retained about each use of the channel, the lower will be the effective $R_{comp.}$. Experimental results would determine just how much $R_{comp.}$ is lowered by each of the suggested decoding procedures.

2. Machine Organization

From consideration of the sequential decoding process as described by Fano,[1] we can break up the proposed special-purpose machine into several parts. (See Fig. XX-1.)

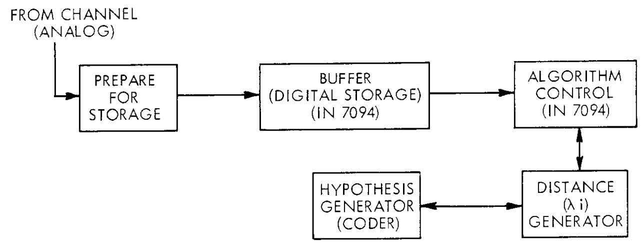

Fig. XX-1. Block diagram of sequential decoder.

Assume that the machine is to be working with a real channel, or at least that input information is in analog form. One section of the machine must abstract from the analog input signal a certain amount of digital information about each use of the channel (that is, for each baud). The proposed machine (as the design is now formulated) would prepare an ordered list of the 16 largest of the M channel symbols, together with the values of these 16 signals and the sum of the squares of all M signals. (Remember that the larger the output of the matched filter, the greater the probability that symbol was sent, for orthogonal signals on a memoryless channel.) Any part of the prepared list could be saved in the buffer memory for use later in the tree-search algorithm. A list of the type just described would allow all of the previously suggested decoding methods to be tested. We would place a limit of 256 on M.

The next question is the required size of the buffer memory, which would be part of the memory of the IBM 7094 computer. If the decoder is to work with a channel that produces bauds "on call," then it need only be large enough to store the information associated with a number of bauds corresponding to approximately 3 constraint lengths (in information bits), so as to allow the machine to search back that far. The experimental facility would normally run with an "on call" channel, since this would result in the fastest decoding rate. Waiting-line behavior could be simulated very easily with such a channel.

The execution of a search algorithm like Fano's requires two quantities that we may think of as being supplied by two different sections of the machine.

First is the hypothesis generator, which is a replica of the convolutional encoder; constraint lengths of up to 100 bits could be handled. The hypothesis generator would generate the signals corresponding to all of the branches stemming from a node in the tree. It would be able to handle up to 4 information bits/baud (corresponding to 16 branches/node) and be able to put out up to 16 check bits/node. Information bits plus check bits then specify one of the M channel symbols, or a series of channel symbols.

The second piece of equipment must generate a number (for each hypothesis signal at a node) which is the "distance" between the received signal and that particular hypothesis signal. This "distance" (λ_i in Fano's notation) is related to Pr $(\rho|X)$, where ρ is the information saved about this baud and X is the particular hypothesis signal. This equipment must return to the search algorithm control the λ of the n^{th} most likely hypothesis, where n is a parameter specified by the search algorithm.

The last section of the sequential decoder is the search-algorithm control. This section is concerned with the decision making and bookeeping required for the execution of the algorithm. We have left this part of the machine as a program in the IBM 7094 computer to anticipate changes in the algorithm, changes to make use of two-way strategies, and changes to allow various amounts of information to be printed out concerning the behavior of the decoder, the amount depending on the particular experiment being run.

The sequential decoding system described briefly above has been designed in detail, and it has been estimated that it would be able to decode approximately 3500 nodes per second. This speed is calculated for the case of an "on call" channel and a tree structure of one baud per branch. It is almost independent of alphabet size and number of branches per node. This speed is in excess of 10 times the speed of an entirely programmed decoder working with the same input channel.

3. Channel Simulation

For the equipment that has been designed, the channel is required to deliver one baud in M microseconds (M is the number of channel symbols). The outputs of the matched filters are converted to 7-bit binary numbers. Thus, in effect, the channel delivers 7 megabits/second to the decoder. This rate is substantually larger than IBM tape machines can handle, so we must eliminate the possibility of generating a channel output at some distant location, recording it on tape, and then at a later time playing it back into the decoder. We are lead to the necessity of designing "on-line" channel simulators. Investigations are being made into the equipment necessary to simulate channels of interest. Designs for coherent and incoherent detection of orthogonal signals in white Gaussian noise on constant and Rayleigh-fading channels have been completed thus far.

C. W. Niessen

References

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C. AN ERROR BOUND FOR FIXED TIME-CONTINUOUS CHANNELS WITH MEMORY

The author has previously demonstrated[1] that the channel in Fig. XX-2 can be represented by the vector equation

$$\underline{y} = [\sqrt{\lambda}]\,\underline{x} + \underline{n}, \tag{1}$$

where $\underline{x}$, $\underline{y}$, and $\underline{n}$ are column vectors representing the channel input, output, and additive noise, respectively, the matrix $[\sqrt{\lambda}]$ is diagonal, and the components of $\underline{n}$ are statistically independent and identically distributed Gaussian random variables. This report

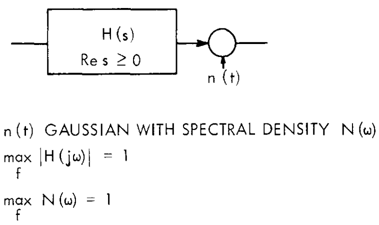

Fig. XX-2. Gaussian channel with memory.

presents an upper bound to the probability of error for this channel based on the representation of Eq. 1 and a slight generalization of a bound derived previously by Gallager.[2]

1. Vector Dimensionality Problem

Before proceeding with the derivation of the error bound, it is necessary to consider in detail the dimensionality of the vectors involved. In deriving the representation of Eq. 1 it was shown[1] that the basis functions used in defining $\underline{x}$ are complete in the space of all $\mathscr{L}_2(0,T)$ signals, that is, in the space of all finite-energy signals defined on the interval $[0,T]$. Since it is well known that this space is infinite-dimensional,[3] it follows that, in general, the vectors, as well as the matrix, of Eq. 1 must be infinite-dimensional. In many cases this infinite dimensionality is of no concern and mathematical operations can be performed in the usual manner. An attempt to define a "density function" for an infinite-dimensional random vector, however, leads to conceptual, as well as mathematical, difficulties. Consequently, problems in which this situation arises are usually approached[4] by assuming initially that all vectors are finite-dimensional. The analysis is then performed and an attempt is made to show that a limiting form of the answer is obtained as the dimensionality becomes infinite. (If such a limiting result exists, it is asserted to be the desired solution.) This approach is used in the following derivations in which all vectors are initially assumed to be d-dimensional. If desired, the number d can be considered to be arbitrarily large but finite. For this case, however, it will be shown that for minimum probability of error, the vectors will be constrained to be finite-dimensional. This constraint arises because the λ_i approach zero for large "i", and it gives an indication of the useful dimensionality of the channel.

2. Random-Coding Bound

Let $\underline{x}_1, \dots, \underline{x}_M$ be a set of M d-dimensional code words (that is, $\underline{x}_1 \dots \underline{x}_M$ are the vector representations, with respect to the set of basis functions defined by the channel and noise, of a set of M signals of T-sec duration) for use with the channel of Fig. XX-2. Let the a priori probability of each code word be 1/M and assume that maximum-likelihood detection[4] is used. Then, given that $\underline{x}_j$ is transmitted, the probability of error is given by

$$P_j(e) = \int_{\underline{Y}} P(\underline{y}|\underline{x}_j)\, C(\underline{y},\underline{x}_j)\, d\underline{y}, \tag{2}$$

where

$$C(\underline{y},\underline{x}_j) = \begin{cases} 0 & \{\underline{y}:\ P(\underline{y}|\underline{x}_j) > P(\underline{y}|\underline{x}_i) \quad \text{all } i \neq j\} \\ 1 & \{\underline{y}:\ P(\underline{y}|\underline{x}_j) \leq P(\underline{y}|\underline{x}_i) \quad \text{some } i \neq j\}. \end{cases}$$

Equation 2 as given is mathematically intractable. A useful upper bound to Eq. 2 is obtained by first upper-bounding $C(\underline{y},\underline{x}_j)$, and then averaging the result over a suitable ensemble of code words.

An obvious inequality is

$$C(\underline{y},\underline{x}_j) \leq \left\{ \sum_{\substack{k=1\\k\neq j}}^{M} \left[\frac{P(\underline{y}|\underline{x}_k)}{P(\underline{y}|\underline{x}_j)}\right]^{\alpha} \right\}^{\rho} \qquad \alpha,\ \rho \geq 0,$$

since the right-hand side is always greater than zero, and is not less than 1 when $P(\underline{y}|\underline{x}_j) \leq P(\underline{y}|\underline{x}_i)$ for some $i \neq j$. Thus

$$P_j(e) \leq \int_{\underline{Y}} P(\underline{y}|\underline{x}_j)^{1-\alpha\rho} \left\{ \sum_{\substack{k=1\\k\neq j}}^{M} P(\underline{y}|\underline{x}_k)^{\alpha} \right\}^{\rho} d\underline{y}. \tag{3}$$

Let each code word be chosen according to a probability measure $P(\underline{x})$ and average both sides of Eq. 3 over this ensemble of codes. Now, let

$$\overline{P_j(e)} = \overline{P}_e \leq \int_{\underline{Y}} \overline{P(\underline{y}|\underline{x}_j)^{1-\alpha\rho}}\ \overline{\left\{ \sum_{\substack{k=1\\k\neq j}}^{M} P(\underline{y}|\underline{x}_k)^{\alpha} \right\}^{\rho}}\ d\underline{y}. \tag{4}$$

Here, the bar denotes averaging with respect to the ensemble of codes. Equation 4 can be further upper-bounded by noting that $\overline{z^{\rho}} \leq \overline{z}^{\rho}$ for $0 \leq \rho \leq 1$.[5] Introducing this inequality into Eq. 4, and recalling that the average of a sum of random variables equals the sum of the individual averages, gives

$$\overline{P}_e < M^{\rho} \int_{\underline{Y}} \overline{P(\underline{y}|\underline{x})^{1-\alpha\rho}}\ \overline{P(\underline{y}|\underline{x})^{\alpha}}^{\rho}\, d\underline{y} \qquad 0 \leq \rho \leq 1. \tag{5}$$

By straightforward but tedious differentiation, it can be shown that with ρ fixed the right-hand side of Eq. 5 has an absolute minimum for variation of α when $\alpha = 1/(1+\rho)$. Thus

$$\overline{P}_e < e^{-TE(R,\rho)} \qquad 0 \leq \rho \leq 1, \tag{6}$$

where

$$E(R,\rho) = E_o(\rho) - \rho R$$

$$E_o(\rho) = -\frac{1}{T} \ln \int_{\underline{Y}} \left[\int_{\underline{X}} P(\underline{y}|\underline{x})^{1/(1+\rho)} P(\underline{x})\, d\underline{x} \right]^{1+\rho} d\underline{y}$$

$$R = \frac{\ln M}{T}.$$

This bound will now be applied to the channel of Fig. XX-2. For convenience, let the noise variance in Eq. 1 be normalized to unity. Then

$$P(\underline{y}|\underline{x}) = \prod_{i \in I} \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2} (y_i - \sqrt{\lambda_i}\, x_i)^2 \right], \tag{7}$$

where the set I is, at this point, an arbitrary collection of d non-negative integers. Furthermore, let $P(\underline{x})$ be chosen to be

$$P(\underline{x}) = \prod_{i \in I} \frac{1}{\sqrt{2\pi\sigma_i^2}} \exp\left[-\frac{1}{2} \left(x_i^2/\sigma_i^2 \right) \right]. \tag{8}$$

The reasons for this choice of $P(\underline{x})$ are several.

(a) This form of $P(\underline{x})$ results in a mathematically tractable expression for the error exponent of Eq. 6.

(b) It is known[6] that this choice for $P(\underline{x})$ leads to maximum average mutual information between the $\underline{x}$ and $\underline{y}$ vectors when the values of $\overline{x_i^2}$ are specified. Furthermore, maximization of the resulting mutual information with respect to the $\overline{x_i^2}$ yields a meaningful definition of capacity for this channel.

(c) When the resulting exponent is specialized to the case considered by Shannon,[7] it is within a few per cent of his exponent, which is the best known.

Finally, assume an average power constraint on the ensemble of codes of the form

$$\sum_{i \in I} \sigma_i^2 = ST. \tag{9}$$

Substituting Eqs. 7 and 8 in Eq. 6 gives, after evaluation of the integrals,

$$E(R, \rho, \underline{\sigma}) = \frac{\rho}{2T} \sum_{i \in I} \ln \left(1 + \frac{\lambda_i \sigma_i^2}{1+\rho}\right) - \rho R, \tag{10}$$

where

$$\underline{\sigma} = \left(\sigma_1^2, \sigma_i^2, \ldots \sigma_k^2, \ldots\right).$$

For fixed R, maximization of Eq. 10 over ρ, $\underline{\sigma}$, and the set I gives the desired random-coding error exponent. For convenience, let this maximization be performed in the order I, $\underline{\sigma}$, ρ.

Maximization over the set I is easily accomplished by recalling[1] that the λ_i are by assumption ordered so that $\lambda_0 \geq \lambda_1 \geq \lambda_2 \ldots$. Thus, the monotonic property of $\ln x$ for $x \geq 1$ implies that $E(R, \rho, \underline{\sigma})$ is maximized over the set I by choosing $I = \{0, 1, \ldots, d-1\}$.

The maximization over $\underline{\sigma}$ is most readily accomplished by using the properties of convex functions[8] defined on a vector space. For this purpose, the following definitions and a theorem of Kuhn and Tucker[9] (in present notation) are presented.

DEFINITION 1: A region of vector space is defined as convex if for any two vectors $\underline{\alpha}$ and $\underline{\beta}$ in the region and for any λ, $0 \leq \lambda \leq 1$, the vector $\lambda\underline{\alpha} + (1-\lambda)\underline{\beta}$ is also in the region.

DEFINITION 2: A function $f(\underline{\alpha})$ whose domain is a convex region of vector space is defined as concave if, for any two vectors $\underline{\alpha}$ and $\underline{\beta}$ in the domain of f and for any λ, $0 < \lambda < 1$,

$$\lambda f(\underline{\alpha}) + (1-\lambda) f(\underline{\beta}) \leq f[\lambda\underline{\alpha} + (1-\lambda)\underline{\beta}].$$

From these definitions is follows that the region of Euclidean d-space defined by the vector $\underline{\sigma}$, with

$$\sigma_i^2 \geq 0, \text{ and } \sum_{i=0}^{d-1} \sigma_i^2 = TS,$$

is a convex region of vector space, that ln x is a concave function for $x \geq 1$, that a sum of concave functions is concave, and thus that $E(R, \rho, \underline{\sigma})$ is a concave function of $\underline{\sigma}$.

THEOREM 1 (Kuhn and Tucker): Let $f(\underline{\sigma})$ be a continuous differentiable concave function in the region in which $\underline{\sigma}$ satisfies $\sum_{i=0}^{d-1} \sigma_i^2 = TS$ and $\sigma_i^2 \geq 0$, $i = 0, 1, \ldots, d-1$. Then a necessary and sufficient condition for $\underline{\alpha}$ to minimize f is

$$\left.\frac{\partial f(\underline{\sigma})}{\partial \sigma_i^2}\right|_{\underline{\sigma}=\underline{\alpha}} \leq A \qquad \text{for all } i \text{ with equality if and only if } \sigma_i^2 \neq 0.$$

Here, A is a constant independent of i whose value is adjusted to satisfy the constraint $\sum_{i=0}^{d-1} \sigma_i^2 = TS$. It follows that the $\underline{\sigma}$ maximizing Eq. 10 must satisfy

$$\frac{\partial E(R,\rho,\underline{\sigma})}{\partial \sigma_i^2} = \frac{\rho}{2T} \frac{(-\lambda_i)/(1+\rho)^2}{1+\left(\lambda_i\sigma_i^2\right)/(1+\rho)} \leqslant A \qquad \text{all } i = 0, \ldots, d-1$$

with equality if and only if $\sigma_i^2 > 0$.

Thus

$$\sigma_i^2 = \begin{cases} (1+\rho)\left\{\dfrac{1}{B_T(\rho)} - \dfrac{1}{\lambda_i}\right\} & i = 0, 1, \ldots, N-1 \\ 0 & i = N \ldots d-1 \end{cases} \tag{11}$$

where N is defined by

$$\lambda_{N-1} > B_T(\rho) \geqslant \lambda_N$$

and

$$\frac{1}{B_T(\rho)} \triangleq \frac{-\rho}{2TA(1+\rho)^2}.$$

The value of $B_T(\rho)$, and thus N, is chosen to satisfy the constraint

$$\sum_{i=0}^{N-1} \sigma_i^2 = TS,$$

which yields

$$\frac{1}{B_T(\rho)} = \frac{\dfrac{ST}{1+\rho} + \displaystyle\sum_{i=0}^{N-1} \lambda_i^{-1}}{N}. \tag{12}$$

Substituting Eqs. 11 and 12 in Eq. 10 gives

$$E(R,\rho) = \frac{\rho}{2T} \sum_{i=0}^{N-1} \ln \frac{\lambda_i}{B_T(\rho)} - \rho R. \tag{13}$$

Maximization over ρ is accomplished by using standard techniques of differential calculus. Since N is an implicit function of ρ in Eq. 13, there is a possibility that $E'(R,\rho)$ might not exist for values of ρ and N, such that $B_T(\rho) = \lambda_N$. It can be shown, however, that $E'(R,\rho^-) = E'(R,\rho^+)$ for all ρ. Thus the final result is

$$E_T(\rho) = \left(\frac{\rho}{1+\rho}\right)^2 \frac{S}{2} B_T(\rho) \qquad R(1) \leqslant R \leqslant R(0) = C_T \tag{14}$$

where

$$R(\rho) = \frac{1}{2T} \sum_{i=0}^{N-1} \ln \frac{\lambda_i}{B_T(\rho)} - \frac{\rho}{(1+\rho)^2} \frac{S}{2} B_T(\rho) \qquad 0 \leq \rho \leq 1 \tag{15}$$

and

$$E_T(R) = \frac{1}{2T} \sum_{i=0}^{N-1} \ln \frac{\lambda_i}{B_T(1)} - R \qquad 0 \leq R \leq R(1). \tag{16}$$

A bound that is in some cases more useful, and in all cases more readily evaluated, can be derived by considering Eqs. 14-16 for $T \to \infty$. It can be shown by a generalization of a result derived by Jordan,[10] and other arguments too long to be presented here, that the resulting form for the exponent is

$$E(\rho) = \left(\frac{\rho}{1+\rho}\right)^2 \frac{S}{2} B(\rho) \qquad R_c \leq R \leq C \tag{17}$$

$$R(\rho) = \int_W \ln \frac{|H(j\omega)|^2}{N(\omega)\, B(\rho)}\, df - \frac{\rho}{(1+\rho)^2} \frac{S}{2} B(\rho) \qquad 0 \leq \rho \leq 1 \tag{18}$$

$$E(R) = \int_W \ln \frac{|H(j\omega)|^2}{N(\omega)\, B(1)}\, df - R \qquad 0 \leq R \leq R_c, \tag{19}$$

where

$$C = R(0)$$

$$R_c = R(1)$$

$$\frac{1}{B(\rho)} = \frac{\dfrac{S}{2(1+\rho)} + \displaystyle\int_W \frac{N(\omega)}{|H(j\omega)|^2}\, df}{W}$$

$$W = \left\{ +f: \frac{|H(j\omega)|^2}{N(\omega)} \geq B(\rho) \right\}.$$

A convenient method for interpreting the significance of $B(\rho)$ and W is illustrated in Fig. XX-3. This is the well-known "water-pouring" interpretation discussed by Fano[6] and others for the special case of channel capacity. Pertinent properties of the exponents of Eqs. 14-16 and that of Eqs. 17-19 are presented in Fig. XX-4 in which the notation of the latter exponent is used.

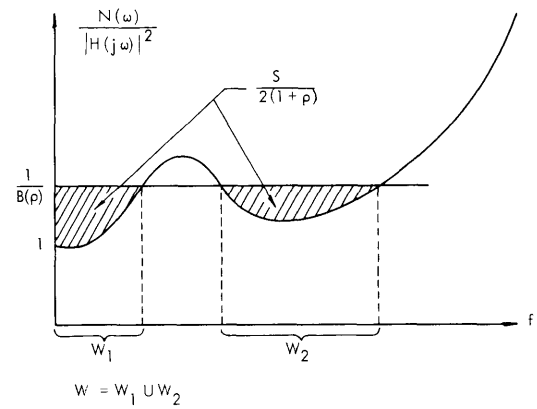

Fig. XX-3. Concerning the interpretation of B(ρ) and w.

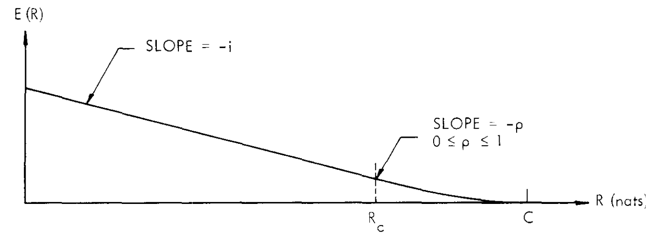

Fig. XX-4. Error exponent for channel of Fig. XX-2.

3. Random-Coding Bound for $S \ll 1$

In this section an asymptotic form for the bound of Eqs. 17-19 is determined for the condition $S \to 0$. For convenience, it is assumed that $K(f) \overset{\Delta}{=} |H(j\omega)|^2/N(\omega)$ has a lowpass characteristic. It will be clear from the derivation, however, that an identical result holds for the general case. Consider the bound for $0 \leq \rho \leq 1$. By expanding Eq. 17 in a Taylor series about $S = 0$, it follows that $E(\rho)$ becomes approximately

$$E(\rho) \cong E(\rho)\Big|_{S=0} + \frac{dE(\rho)}{dS}\Big|_{S=0} S = \left(\frac{\rho}{1+\rho}\right)^2 \frac{S}{2}. \tag{20}$$

Likewise,

$$R(\rho) \cong R(\rho)\Big|_{S=0} + \frac{dR(\rho)}{dS}\Big|_{S=0} S.$$

Using the lowpass assumption for $K(f)$ and also assuming that $K(0) = 1$, $K(f) < 1$ for $|f| > 0$, gives

$$\frac{dR(\rho)}{dS}\Big|_{S=0} = - W\frac{K'(W)}{K(W)} \frac{dW}{dS}\Big|_{S=0} - \frac{\rho/2}{(1+\rho)^2}. \tag{21}$$

The assumptions on $K(f)$ imply, however, that for $S \to 0$

$$B(\rho) = K(W)$$

and

$$\frac{S}{2(1+\rho)} = \int_0^W \left[\frac{1}{B(\rho)} - \frac{1}{K(f)}\right] df,$$

which leads to

$$\frac{dW}{dS} = -\frac{1}{2(1+\rho)} \frac{B^2(\rho)}{WK'(W)}.$$

Combining this with Eq. 21 gives

$$R(\rho) \cong \frac{1}{2(1+\rho)} S. \tag{22}$$

Finally, solving for ρ from Eq. 22 and substituting in Eq. 20 gives the desired exponent

$$E(R) = C\left[1 - \sqrt{\frac{R}{C}}\right]^{1/2} \qquad R_c \leq R \leq C \tag{23}$$

where

$$C = S/2$$

$$R_c = S/8.$$

A similar analysis gives

$$E(R) = C\left[\frac{1}{2} - \frac{R}{C}\right] \qquad 0 \leq R \leq R_c. \tag{24}$$

This exponent is presented in Fig. XX-5 and has several noteworthy features.

(i) It is independent of both the channel filter characteristics and the shape of the

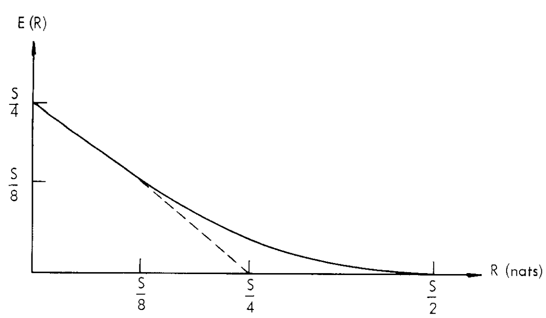

Fig. XX-5. Error exponent for S « 1.

noise spectrum. This fact may be interpreted physically in terms of Fig. XX-3 in the following manner. When S is "sufficiently small," the water-pouring interpretation shows that the ratio $|H(j\omega)|^2/[B(\rho)N(\omega)]$ is "almost" unity and, furthermore, that $|H(j\omega)|^2/N(\omega)$ is "almost" constant for $f \in W$. Thus to a first-order approximation Eqs. 17 and 18 become

$$E(\rho) \cong \left(\frac{\rho}{1+\rho}\right)^2 \frac{S}{2} \qquad R_c \leq R \leq C$$

and

$$R(\rho) \cong \int_W \left[\frac{|H(j\omega)|^2}{N(\omega)\ B(\rho)} - 1\right] df - \frac{\rho}{(1+\rho)^2}\frac{S}{2} = \frac{S}{2(1+\rho)^2} \qquad 0 \leqslant 1 \leqslant \rho$$

which, when combined, yield Eq. 24.

(ii) It agrees precisely with that found by Shannon[7] for the analogous case in his problem, and is also identical to a bound found by Gallager[11] for "very noisy" discrete memoryless channels. Thus this bound could in some sense be considered to be a universal bound for "very noisy" channels.

The application of Theorem 1 and the theory of convex functions to this problem was brought to the author's attention by Professor R. G. Gallager.

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