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A. BOUNDS ON MULTIPLE-THRESHOLD FUNCTIONS

This report presents some preliminary results regarding multiple-threshold functions.[1,2] A lower bound on the number of thresholds required to realize all functions of n variables will be derived.

Multiple-threshold functions will be defined as follows.

DEFINITION 1: A Boolian function $f(x_1, \ldots, x_n)$ is k-threshold threshold realizable iff there exists a set of real numbers $w_1, \ldots, w_n, T_1, \ldots, T_k$ such that

$$\prod_{j=1}^{k} \left(\sum_{i=1}^{n} w_i x_i - T_j \right) > 0 \Longleftrightarrow f(x_1, \ldots, x_n) = q$$

$$\prod_{j=1}^{k} \left(\sum_{i=1}^{n} w_i x_i - T_j \right) < 0 \Longleftrightarrow f(x_1, \ldots, x_n) = \bar{q}, \tag{1}$$

where q = 0 or 1. Thus a given set of w_i and T_j define one function with q = 1 and the complement of that function with q = 0. It is also clear that if a function is realizable with k thresholds it is realizable with m thresholds for m greater than k.

It is of substantial theoretical and practical interest to determine the minimum number of thresholds required to realize any function of n variables. We shall give a lower bound for this minimum number. To the author's knowledge no one has exhibited an n-variable function that requires more than n thresholds. We will show that for sufficiently large n such functions must exist.

We see that

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$$\prod_{j=1}^{k} \left(\sum_{i=1}^{n} w_i x_i - T_j \right) = 0 \tag{2}$$

will be satisfied iff Eq. 3 holds.

$$\sum_{i=1}^{n} w_i x_i - T_j = 0 \qquad \text{for some } j,\ 1 \leq j \leq k. \tag{3}$$

Consider an (n+k)-dimensional space (called the realization space) with axes labeled $w_1, \ldots, w_n, T_1, \ldots, T_k$. Each point in this space corresponds to multiple-threshold realizations of a function and its complement. Both these realizations require k or fewer thresholds. Using the vectors $\overline{W} = (w_1, \ldots, w_n)$ and $\overline{X} = (x_1, \ldots, x_n)$, we can write Eq. 3 as

$$\overline{W} \cdot \overline{X} - T_j = 0. \tag{4}$$

For any particular $\overline{X}$, Eq. 4 is the equation of a hyperplane passing through the origin of the realization space.

For a given $\overline{X}$, the k hyperplanes defined by Eq. 5 below divide the realization space into a finite number of regions, the exact number depending on the relative orientations of the hyperplanes.

$$\overline{W} \cdot \overline{X} - T_j = 0 \qquad 1 \leq j \leq k \tag{5}$$

The coordinates of any point on any hyperplane are such that

$$\prod_{j=1}^{k} (\overline{W} \cdot \overline{X} - T_j) = 0. \tag{6}$$

The coordinates of a point that is not on any hyperplane (internal to a region) are such that either

$$\prod_{j=1}^{k} (\overline{W} \cdot \overline{X} - T_j) > 0$$

or (7)

$$\prod_{j=1}^{k} (\overline{W} \cdot \overline{X} - T_j) < 0.$$

Furthermore, the coordinates of <u>all</u> points internal to a given region will yield the same sign for the product in Eq. 7.

Now let $\overline{X}$ be a vector in n-dimensional switching space. Each of the 2^n possible $\overline{X}$'s generates k hyperplanes. Thus all 2^n $\overline{X}$ vectors generate $k2^n$ hyperplanes, which divide the realization space into a finite number of regions. The coordinates of a point internal to a given region specify a Boolian function and its complement, both of which require k or fewer thresholds for their realizations. The coordinates associated with all points in a given region correspond to realizations of the <u>same</u> two functions. It is possible, however, that different regions of the realization space may correspond to the same two functions.

Let S(k, n) be the maximum number of regions into which the realization space can be divided by $k2^n$ hyperplanes, all passing through the origin. Then 2S(k, n) is an upper bound to T(k, n), the number of n-variable Boolian functions that are realizable with k or fewer thresholds. Using a result of Cameron,[3] we have

$$S(k,n) = 2 \sum_{\ell=0}^{n+k-1} \binom{k2^n-1}{\ell} \tag{8}$$

This gives

THEOREM 1:

$$T(k,n) \leqslant 4 \sum_{\ell=0}^{n+k-1} \binom{k2^n-1}{\ell}. \tag{9}$$

Employing a bound of Winder[4] and then using Stirling's approximation, we have

$$T(k,n) < \frac{4(k2^n)^{n+k-1}}{(n+k-1)!} < \frac{2}{\sqrt{\pi}} \left(\frac{ek2^n}{n+k-1}\right)^{n+k-1}. \tag{10}$$

Let K(n) be the smallest number of thresholds required to realize all 2^{2^n} functions of n variables. K(n) must be such that

$$\frac{2}{\sqrt{\pi}} \left(\frac{eK(n)\ 2^n}{n+K(n)-1}\right)^{n+K(n)-1} > T(K(n),n) \geqslant 2^{2^n} \tag{11}$$

Using the fact[2,5] that $K(n) \geqslant n$ and $K(n) \leqslant 2^n$ and a series of manipulations on the left-most term of Eq. 11, we can establish

THEOREM 2:

$$K(n) > \frac{2^{n-2}}{n} \qquad \text{for } n \geqslant 2. \tag{12}$$

Thus for values of $n \geqslant 8$, $K(n) > n$, and hence there must exist functions of 8 variables that require more than 8 thresholds.

Also, with reference to Spann[6] we have shown the following.

THEOREM 3: For $n \geq 10$ the class of Modular Threshold functions does not contain all functions.

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