VI. NOISE IN ELECTRON DEVICES*

Prof. H. A. Haus
Prof. P. L. Penfield, Jr.
Prof. R. P. Rafuse
J. L. Doane
H. J. E. H. Pauwels
V. K. Prabhu

A. SIGNAL-TO-NOISE RATIO OF PHOTOMULTIPLIER SPECTRUM MEASUREMENT AND COUNTING EXPERIMENT

Intensity fluctuations of a narrow-band light source such as an optical maser can be observed experimentally by letting the light emitted from the source impinge upon a photomultiplier.[1-4] One may either observe the spectrum of the photomultiplier anode current or connect the anode to a counter and record the photoelectron counts in a set of fixed time intervals of duration T. The spectral density of the photomultiplier anode current is given[5] by

$$\Phi(\omega) = \frac{AeI_o}{2\pi}\left[\Gamma + 2\pi \frac{\xi}{h\nu} \frac{\Phi_p(\omega)}{\overline{p}}\right] \tag{1}$$

where A is the photomultiplier gain; e, the electron charge; I_o, the anode current; Γ, the secondary-emission shot-noise enhancement factor; ξ, the quantum efficiency; h, Planck's constant; ν, the frequency of the light; $\Phi_p(\omega)$, the spectral density of the light power (intensity); and $\overline{p}$, the average power. The first term is the enhanced shot noise. The second term gives the excess noise resulting from time variation of the light intensity and contains the information on the spectral density of the incident light power.

The second-order factorial moment $\overline{n(n-1)}$ of the photoelectron count n in a time interval of duration T contains the same information as the spectral measurement. One can show[2,6] that

$$\frac{\overline{n(n-1)} - \overline{n}^2}{\overline{n}} = 2\,\frac{n}{T^2}\int_0^T (T-\tau)\,\rho_p(\tau)\,d\tau, \tag{2}$$

where $\rho_p(\tau)$ is the normalized time-dependent part of the autocorrelation function of the light power p(t)

$$\overline{p(t)\,p(t+\tau)} = \overline{p}^2[1+\rho_p(\tau)]. \tag{3}$$

Because $\Phi_p(\omega)/\overline{p}^2$ and $\rho_p(\tau)$ are related by a Fourier transform, the second factorial moment indeed yields the same information as the spectral measurement.[7]

The purpose of this report is to evaluate the signal-to-noise ratio of these two

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experiments and compare them with the Brown and Twiss correlation measurement[8] and coincidence counting experiment.[9] Before we do this, we shall consider briefly the advantages and disadvantages of these various methods – aside from their respective signal-to-noise ratios (which will be found to be comparable to each other except for the coincidence counting experiment). In order to obtain the full spectral information in the Brown and Twiss correlation measurement and the coincidence experiment, it is necessary to introduce delays into one of the two photomultiplier outputs used in the experiments. The delays must be of the order of the inverse bandwidth of the incident light. If the light is of narrow bandwidth, such as the light from a gaseous laser, the delays required are prohibitively long. Thus, the Brown and Twiss experiments are suited for the measurement of light spectra of bandwidths greater than, say, 1 Mc. The spectral measurement and counting experiment discussed here take preference for bandwidths less than that.

The counting experiment, as opposed to the spectral measurement, gives more information. Indeed, if enough samples are taken, it is possible to find the complete probability distribution $P(n)$ of counting exactly n photoelectrons within a time interval of duration T; however, it is more laborious. Furthermore, the photoelectron rate cannot exceed the resolution rate of single photoelectron pulses, whereas the spectral measurement does not impose the same stringent restriction. Thus, if the source used is capable of producing a photoelectron rate higher than the rate that can be resolved, attenuation must be used at the expense of signal-to-noise ratio.

1. Signal-to-Noise Ratio of the Spectral Measurement

In the experiments on the fluctuations of the light emitted by a gaseous maser,[1-4] the operation was sufficiently near threshold, so that the modulation of the light was strong and it was not difficult to distinguish the excess noise from the shot noise. In experiments farther away from threshold, this becomes increasingly more difficult and it is necessary to study the question of signal-to-noise ratio.

If it were possible to determine experimentally the shot-noise term in Eq. 1 with perfect accuracy, one could subtract it from the observed total spectrum $\Phi(\omega)$, and thus it would be possible to discern the signal with no attendant uncertainties. In fact, however, the shot-noise level cannot be determined with certainty by a spectral measurement of finite bandwidth B and observation time T_o. We shall define the signal-to-noise ratio of the spectral measurement by the ratio of the excess noise observed in a bandwidth B to the uncertainty in the shot-noise level[7]

$$\frac{\text{Signal}}{\text{Noise}} = \frac{2AeI_oB\frac{\xi}{h\nu}\, 2\pi\Phi_p(\omega)/\overline{p}}{\text{Uncertainty of shot-noise level}}. \tag{4}$$

Here, the uncertainty of shot-noise level is

$$\left\{\overline{\left[\frac{1}{T_o}\int_0^{T_o} i^2(t)\,dt\right]^2} - \overline{\left[\frac{1}{T_o}\int_0^{T_o} i^2(t)\,dt\right]}^2\right\}^{1/2}, \tag{5}$$

where i(t) is the current passing the filter of bandwidth B. We shall evaluate the uncertainty of the shot-noise level in the limit of a negligible signal, an assumption that is legitimate in the limit of a small signal-to-noise ratio. In this case the current i(t) in (5) is a random time function with a Gaussian amplitude distribution. Assuming that the filter characteristic is square, one may represent i(t) as a superposition of sinusoids of random amplitudes, N in number:

$$i(t) = \sum_{i=1}^{N} (a_i \sin\omega_i t + b_i \cos\omega_i t), \tag{6}$$

in which, according to the sampling theorem, N is given by

$$N = BT_o. \tag{7}$$

The random amplitudes of the sinusoids satisfy the conditions

$$\overline{a_i b_i} = 0$$

$$\overline{a_i a_j} = \overline{a^2}\,\delta_{ij} = \overline{b_i b_j} \tag{8}$$

where we have assumed stationarity and symmetry of the current spectrum. The ratio of the uncertainty of the experimental determination of the shot-noise level normalized to shot noise is given by

$$\frac{\text{Uncertainty of shot-noise level}}{\text{Shot noise}} = \frac{\left\{\overline{\left[\frac{1}{T_o}\int_0^{T_o} i^2(t)\,dt\right]^2} - \overline{\left[\frac{1}{T_o}\int_0^{T_o} i^2(t)\,dt\right]}^2\right\}^{1/2}}{\frac{1}{T_o}\int_0^{T_o} \overline{i^2(t)}\,dt} \tag{9}$$

The shot-noise level in terms of N and $\overline{a^2}$ may be found immediately by using (6), (7), and (8):

$$\frac{1}{T_o}\int_0^{T_o} \overline{i^2(t)}\,dt = N\overline{a^2}. \tag{10}$$

On the other hand, we know that

$$\frac{1}{T}\int_0^T \overline{i^2(t)}\, dt = 2BAeI_o\Gamma. \tag{11}$$

The numerator of (9)

$$\overline{\left[\frac{1}{T_o}\int_0^{T_o} i^2(t)\, dt\right]^2} - \overline{\left[\frac{1}{T_o}\int i^2(t)\, dt\right]}^2 = \overline{\left[\sum_{i=1}^{N}\frac{1}{2}\left(a_i^2+b_i^2\right)\right]^2} - (N\overline{a^2})^2$$

$$= \frac{1}{4}\left(\sum_i \overline{a_i^4} + \sum_i \overline{b_i^4} + 2\sum_{i,j}\overline{a_i^2}\,\overline{b_j^2} + \sum_{i\neq j}\overline{a_i^2}\,\overline{a_j^2} + \sum_{i\neq j} b_i^2\, b_j^2\right) - N^2\,\overline{a^2}^2$$

$$= N(N+1)\,\overline{a^2}^2 - N^2\,\overline{a^2}^2 = N\,\overline{a^2}^2. \tag{12}$$

The second expression in (12) is obtained by introducing (6) and integrating over the time interval T_o. The third expression is obtained by replacing the square of the sum by a double summation and using the statistical independence of a_i and a_j, $j\neq i$, and a_i and b_j; the fourth expression results by noting that the single summations contain N terms and the summations over unequal indices contain N(N−1) terms, and further using the relationship applicable to the Gaussian variables a_i and b_i:

$$\overline{a_i^4} = 3\overline{a^2}^2 = \overline{b_i^4}. \tag{13}$$

Using (11) and (12) in (9), one finally obtains

$$\frac{\text{Uncertainty of shot-noise level}}{\text{Shot noise}} = \frac{1}{\sqrt{N}} = \frac{1}{\sqrt{BT_o}}. \tag{14}$$

Assuming that the spectral density of the light power, $\Phi_p(\omega)$, is that of Gaussian light with a Lorentzian line shape of bandwidth $\Delta\omega$, so that

$$\frac{\Phi_p(\omega)}{\bar{p}^2} = \frac{1}{\pi\Delta\omega}\,\frac{1}{1+\dfrac{\omega^2}{\Delta\omega^2}},$$

one finally obtains from (4), (9), (11), and (14) for the maximum signal-to-noise ratio at $\omega \to 0$:

$$\frac{\text{Signal}}{\text{Noise}} = \frac{2\bar{r}}{\Gamma\Delta\omega}\sqrt{BT_o}, \tag{15}$$

in which we have used the fact that the photoelectron rate $\bar{r}$ is related to $\bar{p}$ by

$$\bar{r} = \frac{\xi}{h\nu}\,\bar{p}.$$

The signal-to-noise ratio increases with the square root of the bandwidth and of the observation time, with the photoelectron rate, and decreases with increasing bandwidth of the incident light. A correction factor would have to be included in (15) to account for other than Gaussian light.

2. Signal-to-Noise Ratio for the Counting Experiment

The signal of the counting experiment may be defined as

$$\text{Signal} = \frac{\overline{n(n-1)} - \bar{n}^2}{\bar{n}} \tag{16}$$

This quantity would vanish if the process were Poisson and an infinite number of samples were taken so that the ensemble averages may be equated to the experimental averages. Because of the finite number of samples taken in an experiment, however, (16) would not yield zero even for a Poisson process. It is meaningful to define as the "noise" in this experiment the mean-square deviation (from zero) of (16) for a pure Poisson process, because of the finite number of samples taken. In this case one may evaluate the noise by using Poisson statistics for the photoelectron counts. Assuming that N samples are taken, we have

$$\text{Noise} = \frac{1}{\bar{n}} \overline{\left\{\frac{1}{N}\sum_{i=1}^{N} n_i(n_i-1) - \left(\frac{1}{N}\sum_{i=1}^{N} n_i\right)^2\right\}^2}^{\,1/2}. \tag{17}$$

Replacing the higher powers of the sums in (17) by multiple sums, one obtains

$$\text{Noise} = \frac{1}{\bar{n}} \left\{\frac{1}{N^2}\sum_{i,j} \overline{n_i(n_i-1)\,n_j(n_j-1)} + \frac{1}{N^4}\sum_{i,j,k,\ell} \overline{n_i n_j n_k n_\ell} - 2\,\frac{1}{N^3}\sum_{i,j,k} \overline{n_i(n_i-1)\,n_j n_k}\right\}^{1/2}. \tag{18}$$

In the first sum we have to be concerned with terms of equal indices i and j, and with terms of unequal indices. There are N terms of the former type, and N(N-1) terms of the latter type. In the second summation, there are N terms in which all subscripts are the same, 4N(N-1) terms in which three subscripts are the same and one is different; 3N(N-1) terms in which two pairs have equal subscripts; 6N(N-1)(N-2) terms in which two subscripts are the same and the others are different; and N(N-1)(N-2)(N-3) terms in which all subscripts are different. A similar study of the third summation in (18) gives N terms in which all subscripts are alike, N(N-1) terms in which $j = k$, but $j \neq i$, 2N(N-1) for which $j = i$ or $k = i$, but $j \neq k$, and N(N-1)(N-2) for which all subscripts are different. Further, using the expressions for the moments n^k for a Poisson process, one may calculate (18) retaining only terms of 0^{th} and 1^{st} order 1/N: this is legitimate because in all experiments, the number of samples N would be large. One finds that the

$$\text{Noise} = \sqrt{\frac{2}{N}} \tag{19}$$

Again, assuming that the "signal" is produced by Gaussian light of Lorentzian line shape and bandwidth $\Delta\omega$, one has

$$\rho_p(\tau) = e^{-\Delta\omega\tau}$$

and therefore, from (2) and (16) for the maximum signal attained in the limit $T \gg \Delta\omega$

$$\text{Signal} = \frac{2\bar{r}}{\Delta\omega}. \tag{20}$$

Introducing this expression for the signal, one obtains with the aid of (20)

$$\frac{\text{Signal}}{\text{Noise}} = \sqrt{2}\ \frac{\bar{r}}{\Delta\omega} \sqrt{N}. \tag{21}$$

This expression has to be multiplied by the same correction factor as (15) to account for other than Gaussian light. Note the similarity of the signal-to-noise ratio of this experiment and the signal-to-noise ratio of the spectrum measurement, (15), which is even enhanced by the fact that the product T_oB stands for the number of samples necessary to describe the time function i(t) of bandwidth B in the observation time T.

3. Comparison with Brown and Twiss Experiments

We shall now compare the results obtained here with the corresponding expressions obtained by Brown and Twiss.[8,9] The correlation experiment yields in their case, for a square filter characteristic, the result [Eq. (3.62) of Brown and Twiss[7]]

$$\frac{S}{N} = \frac{\sqrt{2}}{\Gamma} \frac{\bar{r}}{\Delta\nu} \sqrt{BT_o} = \frac{2\sqrt{2}\pi\bar{r}}{\Gamma\Delta\omega} \sqrt{BT_o}, \tag{22}$$

where we have set

$$\frac{1}{\Gamma} = \left(1 - \frac{1}{\mu}\right),\ \eta \cong 1,\ A = A_1 = A_2$$

and

$$\frac{A\int_0^\infty a^2(\nu)\, n^2(\nu)\, d\nu}{\int_0^\infty a(\nu)\, n(\nu)\, d\nu} = \frac{\bar{\bar{r}}}{\Delta\nu}.$$

Except for the factor $\sqrt{2}\,\pi$, this is the same expression as (15).

Next, compare the signal-to-noise ratio of the coincidence experiment with the expression obtained thus far. Brown and Twiss point out[8] that the signal-to-noise ratio for the coincidence experiment is given by (22) as well, if one interprets $B = 1/4\,\tau_c$, where τ_c is the resolving time of the counter. But their analysis applies to the case for which the inverse resolving time of the counter is much smaller that the light bandwidth $\Delta\omega$. If one develops an expression for the signal-to-noise ratio for the case $\Delta\omega \ll 1/\tau_c$, one finds[10] [Eq. (5.23) of Brown and Twiss[9]]

$$\frac{\text{Signal}}{\text{Noise}} = \bar{\bar{r}}\sqrt{T_o\tau_c}$$

$$= \frac{\bar{\bar{r}}}{\Delta\omega}\sqrt{T_o\Delta\omega}\,\sqrt{\Delta\omega\tau_c}\,.$$

Insofar as the bandwidth of the spectral measurement can be made comparable to the light bandwidth (say, $2\pi B \sim \Delta\omega/4$), the expression above looks like the signal-to-noise ratio of the spectral experiment, except for the factor $\sqrt{\Delta\omega\tau_c}$. The resolving time must be made short enough to accommodate the rate $\bar{\bar{r}}$. Thus $\Delta\omega\tau_c$ is usually much less than unity; and, accordingly, the signal-to-noise ratio of the coincidence counting experiment is smaller than that of the other measurements discussed here.

H. A. Haus

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B. QUANTUM ANALYSIS OF NOISE IN THE LASER OSCILLATOR

1. Introduction and Summary

Spontaneous emission noise is essentially a quantum phenomenon. It can only be described by an analysis in which the field is quantized. For the laser amplifier and the laser oscillator below threshold, which are both linear devices, such an analysis is known.[1] In this report we shall give the general outline[2] of a quantum analysis of the nonlinear laser oscillator above threshold. We make use of the concept of quantum noise sources. These are operators whose first-order moments are zero and whose second-order moments are nonzero. They drive Van der Pol equations whose variables are operators. We linearize these equations in the noise, and solve for the first- and second-order Glauber functions,[3] $G^{(1)}$ and $G^{(2)}$, and for the expectation value of the commutator of the field variables. These three results refer to the field inside the cavity.

These results will be compared with the results of an earlier theory[4] in which the semiclassical equations are considered to be driven by the linear noise sources. We shall call this theory "semiclassical." Our results contain "saturation corrections" caused by the fact that the correlation functions of our quantum noise sources differ slightly from the corresponding quantities in the "semiclassical" theory. Our results also contain "quantum corrections" caused by the fact that our variables are operators. Both of these corrections are small. Because our results refer to the fields inside the cavity, and because experiments[5] are performed on the fields of the laser beam outside the cavity, we cannot yet give an exact discussion of the experimental meaning of these corrections.

Furthermore, it can be shown that any particular moment of the field can be rederived from an equivalent classical problem consisting of the semiclassical equations driven by appropriate noise sources. For different field moments one needs different noise sources. It turns out that $G^{(1)}$ and $G^{(2)}$ need the same noise sources. These noise sources differ slightly from the linear sources of the "semiclassical" theory.

2. Fundamental Equations

We shall consider one field mode in interaction with N two-level systems (particles, material) in resonance with the field mode. The particles undergo collisions and we restrict ourselves in this report to one type with collision time T. The field also interacts with a loss system consisting of an infinite set of harmonic oscillators, originally in thermal equilibrium at temperature T_L, and with a flat spectral distribution. We shall concentrate here on the interaction between field and material; the effect of the loss will be mentioned without proof.[1,2] In between collisions the system is described by the Hamiltonian

$$H = \hbar\omega_o a^+ a + \sum_j \hbar\omega_o w_j + \sum_j i\hbar\kappa_j (a^+ p_j^- - p_j^+ a) + \text{Loss} \tag{1}$$

in which a, a^+ are the annihilation and creation operators of the field mode, and w_j, p_j^-, p_j^+ are the energy operator and the negative and positive frequency components of the polarization operator of the j^{th} two-level system. They are adequately normalized so that

$$\left[p_j^+, p_j^-\right] = 2w_j; \quad \left[w_j, p_j^+\right] = p_j^+; \quad \left[p_j^-, w_j\right] = p_j^- \tag{2}$$

This Hamiltonian leads to the following equations of motion:

$$\frac{da(t)}{dt} = \sum_j \kappa_j p_j^-(t) + \text{Loss}; \quad \text{and Hermitian conjugate (h. c.)} \tag{3}$$

$$\frac{dp_j^-(t)}{dt} = 2\kappa_j w_j(t)\, a(t); \quad \text{and h. c.} \tag{4}$$

$$\frac{dw_j(t)}{dt} = -\kappa_j\left[p_j^+(t)\, a(t) + a^+(t)\, p_j^-(t)\right]. \tag{5}$$

We adopt the following model for collision. When there is no field in the cavity, the material is in a randomized equilibrium state characterized by a given inversion $\rho_+ - \rho_-$ (or, equivalently, a given negative temperature $-T_m$). When the field is excited, a particle j interacts for some time t_j with the field, whereby both field and particle develop components in each other's Hilbert space. At the collision the interaction stops, the field retains its components in the j^{th} particle Hilbert space, but the j^{th} particle is kicked back to its original randomized equilibrium state and becomes independent of all of its previous states. It is now in fact a "new" j^{th} particle with a new Hilbert space, and during the next interaction the field will develop additional components in this new space. The material operators immediately after such a collision will be denoted by $p_j^{\pm}(0)$, $w_j(0)$. They have the properties

$$\left\langle p_j^+(0)\, p_k^-(0)\right\rangle = 2\left\langle w_j(0)\right\rangle (1+\beta_m)\, \delta_{jk}; \quad \left\langle p_j^-(0)\, p_k^+(0)\right\rangle = 2\left\langle w_j(0)\right\rangle \beta_m \delta_{jk}$$

$$\left\langle p_j^{\pm n}(0)\right\rangle = 0; \quad 2\left\langle w_j(0)\right\rangle = \rho_+ - \rho_- \tag{6}$$

in which $j \neq k$ for a different particle or collision, and $\beta_m = [(\rho_+/\rho_-)-1]^{-1} = [\exp(\hbar\omega_o/kT_m)-1]^{-1}$.

3. Solution

Consider a time interval $t_i, t_i + \tau$ of order a few times T, and a particular interaction of particle j with duration t_j, somewhere in that interval. We put $a^{\pm}(t) = a^{\pm}(t_i)$ in Eqs. 4 and 5 during τ. These equations are then solved during this interaction for $p_j^{\pm}(t)$ in terms of $p_j^{\pm}(0)$, $w_j(0)$ and $a^{\pm}(t_i)$ to third order in $\kappa_j t$. These solutions are then used to integrate Eq. 3 in the interval $t_i, t_i + \tau$ to fourth order in $\kappa_j t_j$. If τ is considered a differential dt, one may cast the result in the form of a differential equation

$$\frac{da}{dt} - (\gamma-\mu-\alpha\gamma a^+ a)\, a = x^-(t); \qquad \frac{da^+}{dt} - a^+(\gamma-\mu-\alpha\gamma a^+ a) = x^+(t) \tag{7}$$

in which $\gamma\, dt = \sum_j \sum_c \kappa_j^2 t_j^2 w_j$, $\alpha\gamma\, dt = \sum_j \sum_c (1/3)\, \kappa_j^4 t_j^4 w_j$, $\langle\gamma\rangle = 2N\kappa^2 T\langle w\rangle$, $\langle\alpha\gamma\rangle = 8N\kappa^4 T^4 \langle w\rangle$ ($\sum_j$ means summation over the particle index, $\sum_c$ summation over the collisions in dt, the argument (0) has been dropped), $x^-(t) = x_L^-(t) + x_m^-(t)$, and h. c. The quantities μ and x_L are caused by the loss.[1,2] The loss noise sources x_L are independent of the material noise sources x_m and for $t_1 = t_2$

$$\left\langle x_L^+(t_1)\, x_L^-(t_2)\right\rangle = 2\mu\beta_L(1/dt); \qquad \left[x_L^-(t_1),\, x_L^+(t_2)\right] = 2\mu/dt, \tag{8}$$

where $\beta_L = [\exp(\hbar\omega_o|kT_L)-1]^{-1}$; for $|t_1-t_2| > dt$ these expressions are zero. The material noise sources are given by

$$x_m^-(t)\, dt = \sum_j \sum_c \left\{\kappa_j t_j p_j^- - (1/3)\, \kappa_j^3 t_j^3 \left[p_j^+ a^2 + p_j^- a^+ a\right]\right\}$$

$$x_m^+(t)\, dt = \sum_j \sum_c \left\{n_j t_j p_j^+ - (1/3)\, n_j^3 t_j^3 \left[p_j^- a^{+2} + p_j^+ a^+ a\right]\right\}. \tag{9}$$

It can be shown that these noise sources are Gaussian (operators u, v, w, x, ... are defined to be Gaussian in some ensemble if $\langle uvwx\rangle = \langle uv\rangle\langle wx\rangle + \langle uw\rangle\langle vx\rangle + \langle ux\rangle\langle vw\rangle$); the errors made in Eq. 7 by replacing τ by a differential are negligible if $\gamma T < 1$ or, in experimental terms, if the cold-cavity bandwidth is smaller than the collision-broadened linewidth; Eq. 7 conserves the field comutator $[a, a^+] = 1$; if we consider the field operators in $x_m^{\pm}$ as c-numbers, we must consider γ as a c-number (because of the large number of particles and collisions in dt, this c-number is obviously $\langle\gamma\rangle$).

We use the substitution

$$a(t) = [R_o+\Delta(t)]\, e^{-i\theta_t}; \qquad a^+(t) = [R_o+\Delta^+(t)]\, e^{i\theta_t} \tag{10}$$

in which Δ, Δ^+ are operators with $[\Delta, \Delta^+] = 1$, and R_o and θ_t are c-numbers. By putting $\alpha\gamma R_o^2 = \gamma - \mu$, we have adjusted R_o^2 so that it is equal to the steady-state photon number in the cavity, n_o, as predicted by the semiclassical theory without noise sources. We linearize Eqs. 7 in Δ, Δ^+, and θ_t; we replace the field operators in $x_m^{\pm}$ by their main terms, which are c-numbers; and consistently consider γ as a c-number. Furthermore, defining $2in_s = x^- \exp(i\theta_t) - x^+ \exp(-i\theta_t)$, $2n_c = x^- \exp(i\theta_t) + x^+ \exp(-i\theta_t)$, we obtain

$$i\theta_t' 2R_o + \frac{d(\Delta^+-\Delta)}{dt} = -2in_s; \qquad \frac{d(\Delta^++\Delta)}{dt} + 2(\gamma-\mu)(\Delta^++\Delta) = 2n_c. \tag{11}$$

Equations 11 can now be solved for the correlation functions of θ_t, Δ, and Δ^+. The third unknown, θ_t, can be chosen freely as an independent Gaussian, but its correlation function is uniquely defined by the condition that the correlation function of $(\Delta^+-\Delta)$ should stay finite. These correlation functions are then used to calculate the moments of the field. Consistency with the linearization approximation requires that all moments of $\Delta^{\pm}$ higher than the second be neglected.

From Eqs. 6, 8, and 9 we obtain

$$\left\langle n_s(t+\tau)\ n_s(t)\right\rangle = A_s\delta(\tau) = \left[\gamma\left(\frac{1}{2}+\beta_m\right) + \mu\left(\frac{1}{2}+\beta_L\right)\right]\delta(\tau)$$

$$\left\langle n_c(t+\tau)\ n_c(t)\right\rangle = A_c\delta(\tau) = \left[A_s - 4(\gamma-\mu)\left(\frac{1}{2}+\beta_m\right)\right]\delta(\tau)$$

$$i\left\langle\left[n_c(t+\tau), n_s(t)\right]\right\rangle = (\gamma-\mu)\ \delta(\tau). \tag{12}$$

This leads to the following results for $G^{(1)} = \langle a^+(t+\tau)\ a(t)\rangle$, $G^{(2)} = \langle T^+(a^+(t)\ a^+(t+\tau))\ T(a(t+\tau)\ a(t))\rangle$ and the field commutator, respectively.

$$G^{(1)} = \exp\left[-\frac{A_s}{2n_o}|\tau|\right]\left[n_o + \frac{1}{4}\frac{A}{\gamma-\mu}\, e^{-2(\gamma-\mu)|\tau|}\right] \tag{13a}$$

$$\frac{G^{(2)} - n_o^2}{n_o} = \frac{A}{(\gamma-\mu)}\, e^{-2(\gamma-\mu)|\tau|} \tag{13b}$$

$$\left\langle\left[a(t+\tau), a^+(t)\right]\right\rangle = \exp\left[-\frac{A_s}{2n_o}|\tau|\right]\left[\frac{1}{2}+\frac{1}{2}e^{-2(\gamma-\mu)|\tau|}\right]. \tag{14}$$

Here, we have introduced the time-ordering operators T (which puts the later time first) and T^+ (which puts the earlier time first); these were needed in the definition of $G^{(2)}$ because $[a(t-\tau), a(t)] \neq 0$. We have also introduced the average number of photons in the cavity $\langle n(t)\rangle \approx R_o^2 = n_o$, and the parameter A, defined by

$$A = A_c - (\gamma-\mu) = A_s - (\gamma-\mu)(3+4\beta_m). \tag{15}$$

We note that all parameters in Eqs. 12-15 have experimental meaning: 2μ is the cold-cavity bandwidth ($\Delta\omega_o$ in Haus[5]), $2(\gamma-\mu)$ is the hot-cavity bandwidth ($\Delta\omega$ in Haus[5]), $2\gamma = \omega_o/|Q_m^o|$ with Q_m^o the negative cavity Q,[5] and n_o is related to the power, P_o, transmitted in the laser beam by $P_o = 2\mu n_o \hbar\omega_o$.

The field commutator (Eq. 14) is 1 for $\tau = 0$, decays to 1/2 with time constant $(1/2(\gamma-\mu))$ for $|\tau|$ small, and to zero with the time constant $(2n_o|A_s)$ for $|\tau|$ large. The terms having A in $G^{(1)}$ and $G^{(2)}$ describe the influence of the amplitude fluctuations on $G^{(1)}$ and $G^{(2)}$. The influence on $G^{(1)}$ is small and if we neglect it, the spectrum of $G^{(1)}$ becomes Lorentzian with full half-power width, $\Delta\omega_1$,

$$\Delta\omega_1 = \frac{A_s}{n_o} = \frac{\hbar\omega_o}{2P_o}(2\mu)^2\left[\frac{\gamma}{\mu}\left(\frac{1}{2}+\beta_m\right) + \left(\frac{1}{2}+\beta_L\right)\right]. \tag{16}$$

Apart from the factor [ ], this is the double of the Townes width.[7] The influence on $G^{(2)}$ is essential: the semiclassical meaning of $G^{(2)}$ tells us that Eq. 13b gives us the relative correlation function of the photon number, and for $\tau = 0$ it is

$$\frac{A}{\gamma-\mu} = \frac{\mu}{\gamma-\mu}\left[\frac{\gamma}{\mu}\left(\frac{1}{2}+\beta_m\right) + \left(\frac{1}{2}+\beta_L\right)\right] - (3+4\beta_m). \tag{17}$$

As we have mentioned, we still cannot translate this result into the experimentally important power correlation in the laser beam.

We shall now interpret the quantities A and A_s in Eqs. 13 in the light of an equivalent classical problem. In Fig. VI-1 we put $G_m = G_m^o - (\alpha C|2\hbar\omega_o)\, G_m^o |V(t)|^2$, $\omega_o^2 = 1/LC$, $V = |V(t)| \cos(\omega_o t+\theta_t)$. The noise source i_n has positive and negative frequency components i_n^+ and i_n^- such that $i_n = i_n^+ \exp(i\omega_o t) + i_n^- \exp(-i\omega_o t)$. From these we derive the in-phase component $i_c = (1/2)[i_n^- \exp(i\theta_t) + i_n^+ \exp(-i\theta_t)]$ and the quadrature component $i_s = (1/2)[i_n^- \exp(i\theta_t) - i_n^+ \exp(-i\theta_t)]$, so that $i_n = 2i_c \cos(\omega_o t+\theta_t) + 2i_s \sin(\omega_o t+\theta_t)$.

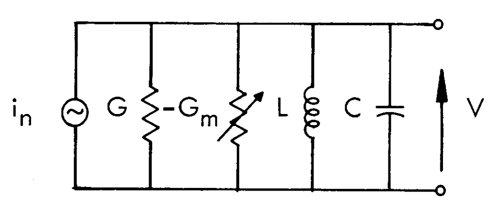

Fig. VI-1. Equivalent circuit of the noisy laser oscillator.

The components i_c and i_s are supposed to be independent, stationary processes, "white" with respect to the "hot" cavity bandwidth but narrow-band with respect to ω_o. One can show that the circuit of Fig. VI-1 gives rise to the equations

$$\frac{dV^+}{dt} + \frac{G - G_m}{2C} V^+ = \frac{i_n^+}{2C}; \quad \frac{dV^-}{dt} + \frac{G - G_m}{2C} V^- = \frac{i_n^-}{2C}. \tag{18}$$

The correspondence with our analysis is made by taking the following scaling factors into account: $V^+ = (\hbar\omega_o/2C)^{1/2} a^+$, $G/2C = \mu$, $G_m^o/2C = \gamma$, $(\alpha C/2\hbar\omega_o) G_m^o |V(t)|^2 = \alpha\gamma a^+ a$, $i_n^+ = (2\hbar\omega_o C)^{1/2} x_{eq}^+$, $i_c = (2\hbar\omega_o C)^{1/2} n_{ceq}$. By putting $\Delta = \Delta^+ = R_1$ = c-number in Eq. 10, we obtain from Eq. 18

$$R_o\theta_t' = -n_{seq}; \quad \frac{dR_1}{dt} + 2(\gamma-\mu) R_1 = n_{ceq} \tag{19}$$

The "semiclassical" circuit[4] is obtained by postulating $\langle n_{seq}(t+\tau)\, n_{seq}(t)\rangle = \langle n_{ceq}(t+\tau)\, n_{ceq}(t)\rangle = A_s\delta(\tau)$ and $\langle n_{ceq}(t+\tau)\, n_{seq}(t)\rangle = 0$, where $\delta(\tau)$ is a delta function on time scales of the inverse "hot" cavity bandwidth, but certainly not on time scales of $1/\omega_o$. This leads to a stationary i_n with spectrum $S_{i_n}(f)$ around ω_o: $\overline{i_n^2} = 2S_{i_n}(f)\, df = 4S_{i_c}(f)\, df = 4A_s(2\hbar\omega_o C)\, df = \left[4G_m^o\left(\frac{1}{2}+\beta_m\right) + 4G\left(\frac{1}{2}+\beta_L\right)\right]\hbar\omega_o\, df$. This is the well-known "linear voltage source" (i.e., it predicts the exact voltage fluctuations below threshold[7]). These noise sources would follow from our theory if we dropped the nonlinear terms in Eqs. 9 for x_m^- and x_m^+. Equation 19 now leads to the results (13) but with A replaced by A_s. Therefore, in the "semiclassical" theory one predicts correctly the width $\Delta\omega_1$ (Eq. 16) but because A_s differs from A (Eq. 15), one makes an error of $(3+4\beta_m)$ photons in the relative photon number fluctuations at $\tau = 0$ (Eq. 17). Close to the threshold $(\gamma \approx \mu)$, $\mu/(\gamma-\mu)$ is large, so that this error is relatively small compared with the main term of Eq. 17. Higher above threshold $\gamma - \mu$ increases and the error becomes relatively more important, but absolutely it is independent of $\gamma - \mu$ and is always small.

The exact equivalent problem instead is obtained by postulating $\langle n_{seq}(t+\tau)\, n_{seq}(t)\rangle = A_s\delta(\tau)$, $\langle n_{ceq}(t+\tau)\, n_{ceq}(t)\rangle = A\delta(\tau)$ and $\langle n_{ceq}(t+\tau)\, n_{seq}(t)\rangle = 0$. This leads then to the exact results (13). Because $A \neq A_s$ the new source i_n is nonstationary. It is interesting to investigate the cause of $A \neq A_s$. First, A is different from A_c (Eq. 15). This is a "pure quantum" effect because it is caused by the operator character of Δ and Δ^+, and by $\langle [n_c, n_s]\rangle \neq 0$. This effect corrects the relative photon number fluctuation by exactly 1 photon. It is also interesting to note that it is not present in the exact expression for $\langle a^+(t)\, a(t)\, a^+(t+\tau)\, a(t+\tau)\rangle$, which for $\tau = 0$ equals $G^{(2)} + n_o$. Second, A_c is different from A_s (Eq. 12). This is a "saturation" effect and can be explained classically. This difference is indeed caused by the terms containing a^2 and a^{+2} in Eqs. 9. These terms are phase-dependent and they make x^- and x^+ nonstationary, although n_s and n_c are stationary. This corresponds to the classical statement that i_n is nonstationary if $\langle i_c^2\rangle \neq \langle i_s^2\rangle$. This effect corrects the relative photon number fluctuation by $(2+4\beta_m)$

photons. These two effects add and that leads ultimately to $A \neq A_s$, and the correction $(3+4\beta_m)$ in Eq. 17.

H. J. Pauwels

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C. SPECTRAL ANALYSIS OF LASER OSCILLATOR BY MEANS OF HIGHER AUTOCORRELATION FUNCTIONS

Reported here is the theoretical basis for an experimental confirmation of the supposed Gaussian property of the noise caused by spontaneous emission in a cavity-type laser oscillator. This noise in the semiclassical analysis appears as a random source in a van der Pol equation describing the oscillation of the electric field of a laser operating somewhat above threshold.[1] It has been shown experimentally that in that region the noise is due mainly to spontaneous emission.[2]

The variation $R_1(t)$ in the electric field amplitude about its steady-state value R_o above threshold is assumed to satisfy a linearized equation derivable from the van der Pol equation. In this region of operation, it is found that information about the third-order autocorrelation function $\overline{R_1(t)\,R_1(t+\tau_1)\,R_1(t+\tau_2)\,R_1(t+\tau_3)}$ can be extracted from the spectrum of the square of the anode current in a photomultiplier placed in the laser beam. This is accomplished with the aid of a direct current and lowpass filter before squaring.

We assume that the effect of the filters is to produce the following form for the transform of the deterministic current pulse:

$$F_i(\omega) = \begin{cases} \dfrac{eA}{2\pi}, & 0 < |\omega| < \omega_f \\ 0, & \omega = 0 \text{ or } |\omega| \geq \omega_f \end{cases} \tag{1}$$

where ω_f is the filter bandwidth, A is the photomultiplier gain, and e is the unit of electronic charge. Each pulse is the result of the emission of one photoelectron, if the effect of secondary emission is neglected. With no lowpass filter present, ω_f can be interpreted as the photomultiplier bandwidth, which can be 10^8-10^9 rad/sec.

In the region where the linearized theory is valid, the modulation coefficient $m \equiv R_1^2(t)/R_o^2$ is much less than unity. Furthermore, if the noise source is Gaussian, then $R_1(t)$ must be Gaussian in this region. Using these facts and the assumptions of the preceding paragraphs, we obtain for the squared current spectrum $S_{I^2}(\omega)$ when $\omega \ll \omega_f$, and when the modulation process bandwidth ω_o is much less than ω_f

$$S_{I^2}(\omega) = \frac{8A^4e^4}{\pi^3} \omega_f^2 r^2 m \frac{\omega_o}{\omega_o^2 + \omega^2} + \frac{128A^4e^4}{\pi} r^4 m^2 \frac{\omega_o}{4\omega_o^2 + \omega^2}. \tag{2}$$

Here, r is the average rate of emission of photoelectrons, which ranged around 10^{10} sec^{-1} in the measurements made above threshold on the ordinary spectrum as reported by Haus.[2] The first term in Eq. 2 arises from the first-order autocorrelation $\overline{R_1(t) R_1(t+\tau)}$ and is basically the same as the term measured in the unsquared spectrum. The second term arises from the third-order autocorrelation of $R_1(t)$. All other terms, including shot-noise terms and contributions from even higher autocorrelation functions of $R_1(t)$, are negligible in the region where the linearized theory is valid, and not too far above threshold.

By increasing the attenuation in front of the photomultiplier, r is decreased and the first term of (2) will predominate. Increasing r or decreasing ω_f will make the second term predominate. It is shown elsewhere[3] that the assumption of typical realizable DC and lowpass filters instead of the ideal filter represented by (1) only increases the first term of (2) by $\pi^2/4$. The second term, which is independent of ω_f, remains unchanged as long as ω lies within the passband of the filter.

If the predictions of (2) are verified by experiment, then we may conclude that the spontaneous emission noise source of the semiclassical analysis is Gaussian. This assumption could not be checked with measurements of the ordinary spectrum. Furthermore, as m increases and threshold is approached, the linearized theory will become invalid. Measurements of the squared current spectrum with the filters used as discussed above should show deviations from the linearized theory before measurements of the ordinary spectrum. Because of the possibility of using filters and operating with a higher photoelectron emission rate, we also conclude that measurements of the squared current spectrum could yield more information about the third-order autocorrelation of $R_1(t)$ than could counting experiments. Higher speed and thus less sensitivity to drift in laser operation would also be achieved.

J. L. Doane

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