IMMERSIONS OF SYMMETRIC SPACES

by

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SUBMITTED IN PARTIAL FULFILLMENT

OF THE REQUIREMENTS FOR THE

DEGREE OF DOCTOR OF

PHILOSOPHY

at the

MASSACHUSETTS INSTITUTE OF

TECHNOLOGY

June 1970

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Department of Mathematics, June 1, 1970

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A formula for the second fundamental form of equivariant immersions of compact symmetric spaces G/K in Euclidean spaces is derived in terms of the Lie algebra and the classical properties of the immersions are treated using this formula.

It is shown that if G/K is locally symmetric and has a 0-tight equivariant immersion in Euclidean space then it is in fact a tight imbedding and G/K is a symmetric R-space.

Finally an inequality between the eigenvalues of the Laplacian and the Betti numbers of G/K is derived.

Thesis supervisor: Sigurdur Helgason

Title: Professor of Mathematics

ACKNOWLEDGEMENT

I am deeply indebted to Professor Sigurdur Helgason, my advisor, for a mathematical education and his constant help and encouragement.

I would like to thank my wife for making the "downs" in my work bearable and the "ups" enjoyable.

For the first three years, 1965-68, at M.I.T. I was supported by a Northern Ireland Ministry of Education Post-graduate Studentship.

A very special word of thanks is due my typist Gladys Borkowski.

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INTRODUCTION.

Interest in equivariant immersions of homogeneous spaces has increased with revival of interest in the general theory of immersions. The most interesting class of homogeneous spaces is, of course, symmetric spaces. In this work we classify all locally symmetric homogeneous spaces with tight (minimal absolute curvature) equivariant immersions. We show that the class of immersions exhibited by Kobayashi and Takeuchi [12] are in effect the only tight equivariant immersions.

In a slightly different vein is the problem of finding to what symmetric spaces can the work of Frankel [6] be extended. One can describe Frankel's method as "Immerse the space and examine the critical manifolds for non-degenerate height functions." This work shows that the extension of Frankel's results to the exceptional groups for instance will require some modification of method.

An outline of the work follows.

CHAPTER 1. THE SECOND FUNDAMENTAL FORM.

In this chapter the second fundamental form is calculated and used to study some classical properties of the immersion. Although most of the results are not very deep the (classical) machinery developed is indespensible for the rest of the work.

CHAPTER 2. TIGHTNESS.

We use the term tightness for what most geometers call minimal absolute curvature to avoid confusion with the classical meaning of minimal and because most of the recent developments in the field have been due to topologists who introduced the term. We develop some interesting properties of immersions which are tight and using them classify all equivariant tight immersions.

CHAPTER 3. EIGENVALUES OF THE LAPLACIAN.

Using a formula of Chern-Lashof for total curvature a weak inequality between eigenvalue of the Laplacian and Betti numbers is proven. This formula suggests a method for searching for tight immersions; this is borne out by an examination of eigenvalues for R-spaces.

The Lie theory used is based on [7] and as I owe such an enormous debt to it I do not cite it throughout but all nonproven Lie results can be found there. For geometric definitions [11] was my guide.

I wish to thank Professor Sigurdur Helgason for his encouragement and many helpful suggestions.

CHAPTER 1. THE SECOND FUNDAMENTAL FORM.

§0. Introduction.

Let M be an n-dimensional manifold immersed in a Riemannian manifold $\overline{M}$ of dimension N ; for convenience we shall not differentiate between a point $x \in M$ and its image in $\overline{M}$ as long as there is no danger of confusion. At any point $x \in M$ the tangent space $\overline{M}_x$ has the decomposition

$$\overline{M}_x = M_x \oplus M_x^\perp$$

where $M_x^\perp$ is the orthogonal complement of M_x with respect to the Riemannian metric on $\overline{M}$. A vector field $\tilde{Y}$ on an open set $U \subset \overline{M}$ with $U \cap M \neq \emptyset$ has the decomposition $\tilde{Y} = \tilde{Y}_T + \tilde{Y}_N$ where $(\tilde{Y}_T)_x \in M_x$ and $(\tilde{Y}_N)_x \in M_x^\perp$ for all $x \in U \cap M$. $\tilde{Y}$ is called a normal vector field if $\tilde{Y}_T = 0$ and tangential if $\tilde{Y}_N = 0$.

Let $\underset{\sim}{Y}$ be a vector field on M . Locally we can extend $\underset{\sim}{Y}$ to a tangential vector field, on $\overline{M}$, also denoted by $\underset{\sim}{Y}$. For this reason we shall talk freely about vector fields on M when in fact we mean the extension.

Let $\overline{\nabla}$ denote the Riemannian connection on $\overline{M}$, then if $\underset{\sim}{X}$ and $\underset{\sim}{Y}$ are vector fields on M we write at $x \in M$

$$(\overline{\nabla}_{\underset{\sim}{X}}\tilde{Y})_x = (\overline{\nabla}_{\underset{\sim}{X}}\tilde{Y})_x + \alpha_x(\tilde{X},\tilde{Y})$$

where $(\nabla_{\tilde{X}}\tilde{Y})_x \in M_x$ and $\alpha_x(\tilde{X},\tilde{Y}) \in M_x^{\perp}$.

Theorem 0.1. (i) The vector field $\nabla_{\tilde{X}}\tilde{Y}$ which assigns to each point $x \in M$ the tangent vector $(\nabla_{\tilde{X}}\tilde{Y})_x$ is differentiable, and $\nabla_{\tilde{X}}\tilde{Y}$ is the Riemannian connection on M given by the Riemannian structure induced from $\overline{M}$.

(ii) The normal vector field $\alpha(\tilde{X},\tilde{Y})$ which assigns to each point $x \in M$ the vector $\alpha_x(\tilde{X},\tilde{Y})$ is differentiable, symmetric in $\tilde{X}$ and $\tilde{Y}$ and is bilinear over $C^{\infty}(M)$. Hence $\alpha_x(\tilde{X},\tilde{Y})$ depends only on $\tilde{X}_x$ and $\tilde{Y}_x$ and is a map

$$M_x \times M_x \to M_x^{\perp} .$$

Proof. This is standard, cf. [11], page 12.

Theorem 0.1 part (ii) allows us to make the following definition.

Definition 0.1. The second fundamental form, α , of the immersion $M \to \overline{M}$ is the assignment to each point $x \in M$ of the map α_x given in Th. 0.1, part (ii).

Remark. Where there is no danger of confusion we write α for α_x .

Let $\tilde{X}$ be a vector field on M and ξ a normal vector field; then we write

$$-(A_{\tilde{\xi}}(\tilde{X}))_x = \text{the tangential component of } (\bar{\nabla}_{\tilde{X}}\tilde{\xi})_x \ .$$

Theorem 0.2. (1) The vector field $A_{\xi}(\tilde{X})$ which assigns to each $x \in M$ the vector $(A_{\tilde{\xi}}(\tilde{X}))_x$ is differentiable and bilinear over $C^{\infty}(M)$, hence $(A_{\tilde{\xi}}(\tilde{X}))_x$ depends only on $\tilde{X}_x$ and $\tilde{\xi}_x$ and gives a bilinear map

$$M_x \times M_x^{\perp} \to M_x^{\perp}$$

(2) If we denote by $(\ , \ )_{\bar{M}_x}$ the inner product on $\bar{M}_x$ and by $(\ , \ )_{M_x}$ the restriction of $(\ , \ )_{\bar{M}_x}$ to M_x then $(\ , \ )_{M_x}$ is called the structure induced by $\bar{M}$ and

$$(A_{\xi}(X),Y)_{M_x} = (\xi,\alpha(X,Y))_{\bar{M}_x} \ .$$

Hence A_{ξ} may be regarded as a symmetric linear operator on M_x .

Proof. Cf. [11], page 14.

Remark. When there is no danger of confusion we write $(\ , \ )$ for both $(\ , \ )_{\bar{M}_x}$ and $(\ , \ )_{M_x}$.

Definition 0.2. For $\xi \in M_x^{\perp}$, A_{ξ} will denote the symmetric operator on M_x given in Th. 0.2. part (ii).

<u>Definition 0.3</u>. If M is a Riemannian manifold the immersion is called an isometry if the Riemannian structure induced by $\overline{M}$ coincides with the structure on M.

If $\overline{M}$ is $\mathbb{R}^N$ we have further

<u>Definition 0.4</u>. The immersion $f: M \to \mathbb{R}^N$ is substantial if $f(M)$ is not contained in any hyperplane of $\mathbb{R}^N$.

<u>Definition 0.5</u>. If $f: M \to \mathbb{R}^N$ is an immersion then the height function, φ_a, associated with $a \in \mathbb{R}^N$ is the function on M given by

$$\varphi_a(x) = (f(x),a) .$$

<u>Definition 0.6</u>. If a group G acts on M and $\overline{M}$ and $f: M \to \overline{M}$ is an immersion such that

$$f(g \cdot x) = g \cdot f(x)$$

for all $g \in G$ and $x \in M$, then the immersion is said to be G-equivariant.

§1. The Immersion.

Let G/K be an n-dimensional, compact, irreducible symmetric space with

$$\text{Lie Algebra of } G = \mathfrak{g} = \mathfrak{k} + \mathfrak{p}$$

the standard decomposition. We shall always assume that G/K carries the G-invariant Riemannian structure induced by -B on $\mathfrak{p}$, where B is the killing form on $\mathfrak{g}$.

Let π: G -> End(E^N) be a real representation with a non-zero K fixed vector e. (We can and will assume that π is an orthogonal representation.) Then π induces a map, also denoted by π, from G/K into E^N by

$$\pi(gK) = \pi(g)e .$$

Lemma 1.1. The map gK -> $\pi(g)\cdot e$ gives an equivariant immersion.

Although this is well known we sketch a proof since it is a "sine qua non" of the subject.

Proof. By the G-equivariance of the map gK -> $\pi(g)e$ we need only consider what happens at O (the origin of G/K).

Given any $X \in \mathfrak{g}$ we get a vector field $\tilde{X}$ on E^N as follows

$$(\tilde{X}f)(x) = \frac{d}{dt}\,(f(\pi(\exp\, tX)\cdot x))|_{t=0}$$

for all $x \in E^N$ and $f \in C^\infty(E^N)$.

If we also denote by π the corresponding representation of $\mathfrak{g}$ a simple calculation yields

$$\tilde{X}_x = \pi(X)x$$

where we make the usual identification of E^N with its tangent space at any point.

Now if we consider X as a vector field on G/K in the usual manner, i.e.

$$(Xg)(p) = \frac{d}{dt}\,(g(\exp\, tX\cdot p))|_{t=0}$$

all $p \in G/K$ and $g \in C^\infty(G/K)$, we have

$$d\pi(X_o) = \tilde{X}_e = \pi(X)e$$

where $d\pi$ is the differential of the map $\pi\colon G/K \to E^N$.

Consider the inner product on $\mathfrak{p}$ given by

$$\langle\langle X,Y\rangle\rangle = (\pi(X)e,\pi(Y)e)$$

where $(\ ,\ )$ is the Euclidean inner product on E^N .

$\langle\langle\ ,\ \rangle\rangle$ is K invariant so irreducibility of G/K implies $\langle\langle\ ,\ \rangle\rangle = -cB$ where $c \geq 0$. But $c > 0$

otherwise the representation would be trivial. Q.E.D.

As of now we shall assume the vector "e" is chosen in such a way that the constant "c" in the proof of Lemma 1.1 is in fact = 1.

<u>Remark</u>. This leads immediately to the following lower bound on the dimensions of representations which I imagine is well known although I have not seen it remarked in the literature.

<u>Lemma 1.2</u>. If G/K is an irreducible symmetric space and $\pi: G \to E^N$ is a real, class-one representation then

$$N \geq \dim(G/K) + \operatorname{rank}(G/K) \quad .$$

<u>Proof</u>. An immediate consequence of the following theorem of Chern and Kuiper [1] and Otsuki [19].

"Let M be an n-dimensional compact Riemannian manifold isometrically immersed in $\mathbb{R}^{n+p}$. If at every point $x \in M$ the tangent space M_x contains an n-dimensional subspace with the sectional curvature of any plane in the subspace ≤ 0, then $p \geq m$." Q.E.D.

We now calculate the second fundamental form of the immersion $\pi: G/K \to E^N$. To do so we shall use two lemmas. Lemma 1.3 which gives a local coordinate expression for the second fundamental form, is well known.

Lemma 1.4 is algebraic and although relatively simple is very important to the discussion in the next chapter.

Lemma 1.3. Let $f: M \to \mathbb{R}^N$ be an immersion. Suppose $\{x_1, \ldots, x_n\}$ is a local coordinate system on a neighborhood U of m in M. Then $\alpha((\frac{\partial}{\partial x_i})_m, (\frac{\partial}{\partial x_j})_m)$ is the normal component of $(\frac{\partial^2 f}{\partial x_i \partial x_j})_m$ under the usual identification of $\mathbb{R}^N$ and its tangent space.

Proof. Cf. [11], pages 17 and 18. Q.E.D.

Lemma 1.4. Let π be an orthogonal representation of G with non-zero K fixed vector. Then

$$(\pi(X)e, \pi(Y)\pi(Z)e) = 0 \quad \text{for all } X, Y, Z \in \mathfrak{p} .$$

Proof. First assert that we need only prove

$$(\pi(X)e, \pi(Y)\pi(Y)e) = 0 \qquad \text{all } X \text{ and } Y \in \mathfrak{p} .$$

Indeed since $\pi[Y,Z]e = 0$ we have

$$\pi(Y)\pi(Z)e = \pi(Z)\pi(Y)e \tag{a}$$

So we can write

$$\pi(Y)\pi(X)e = \frac{1}{2}[\pi(X+Y)\pi(X+Y)e - \pi(X)\pi(X)e - \pi(Y)\pi(Y)e]$$

which proves the assertion.

We know $(\pi(Y)e, \pi(X)\pi(Y)e) = 0$ since the representation is orthogonal but by (a) this is equivalent to $(\pi(Y)e, \pi(Y)\pi(X)e) = 0$ or $(\pi(Y)\pi(Y)e, \pi(X)e) = 0$.
Q.E.D.

<u>Theorem 1.1.</u> Let α be the second fundamental form of the immersion $\pi: G/K \to E^N$ then at the origin of G/K

$$\alpha(X,Y) = \pi(X)\pi(Y)e \quad \text{for all } X, Y \in \mathfrak{p}$$

<u>Proof.</u> Let $X_1,\ldots,X_n$ be an orthonormal basis of $\mathfrak{p}$. Let U be a normal neighborhood of 0 in G/K so that

$$\mathrm{Exp}_p(x_1X_1 + \ldots + x_nX_n) \to (x_1,\ldots,x_n)$$

is a coordinate system about 0 in G/K with

$$\left(\frac{\partial}{\partial x_i}\right)_0 = X_i \quad .$$

Now

$$\mathrm{Exp}(x_1X_1 + \ldots + x_nX_n) = \exp(x_1X_1 + \ldots + x_nX_n)$$

$$\pi(\mathrm{Exp}(x_1X_1 + \ldots + x_nX_n) = \pi(\exp(x_1X_1 + \ldots + x_nX_n)e$$

$$= \exp(x_1\pi(X_1) + \ldots + x_n\pi(X_n))e$$

To compute $\frac{\partial^2\pi}{\partial x_i\partial x_j}\Big|_{(0,0\ldots)}$ we need only consider $\exp(x_i\pi(X_i) + x_j\pi(X_j))e$.

But if A and B are $r \times r$ matrices then

$$\frac{\partial^2}{\partial t_1 \partial t_2} (\exp(t_1 A + t_2 B))\big|_{(0,0)} = \frac{1}{2} (AB + BA)$$

(This can be proven by expansion in series.) Hence

$$\frac{\partial^2 \pi}{\partial x_i \partial x_j}\Big|_0 = \frac{1}{2} (\pi(X_i)\pi(X_j)e + \pi(X_j)\pi(X_i)e)$$

which by the proof of Lemma 1.4 equals $\pi(X_i)\pi(X_j)e$.

But Lemma 1.4 shows $\pi(X_i)\pi(X_j)e$ is perpendicular to (G/K_0) so

$$\alpha(X_i, X_j) = \pi(X_i)\pi(X_j)e \; .$$

For any X and $Y \in \mathfrak{p}$ the bilinearity of α gives

$$\alpha(X, Y) = \pi(X)\pi(Y)e \; .$$

Q.E.D.

Since the classical information about an immersion is contained in the second fundamental form and since the form has a simple expression for our immersions one might expect that the study of the classical properties would be relatively easy.

To show that this is indeed the case we digress a little from our main theme and consider the following two concepts first introduced by Chern-Kuiper [1].

Let $f: M^n \to E^N$ be an immersion of a compact manifold in Euclidean space.

Definition 1.1. X in M_x is an asymptotic vector if $\alpha(X,X) = 0$.

Definition 1.2. If X and Y in M_x are such that $\alpha(X,Y) = 0$ then X and Y are said to be conjugate.

To completely describe these concepts for our immersion we have

Theorem 1.2. For the immersion $\pi: G/K \to E^N$ constructed above

(i) There are no asymtotic vectors at any point.

(ii) If $X \in \mathfrak{p}$ then the set of vectors in $\mathfrak{p}$ which are conjugate to X form a Lie triple system.

Remark. By way of illustration of Th. 1.2 part (ii) it is instructive to consider the sphere S^n.

If we consider S^n as imbedded in $\mathbb{R}^{n+1}$ in the usual way then if X is a vector in S^n_p then the tangent space to the sphere S^{n-1} contains all tangent vectors conjugate to X.

In [5] the following is shown. Consider $S^n = SO(n+1)/SO(n)$. For each positive integer S choose an orthonormal basis $f_o,\dots,f_m$ for the space V^s of spherical harmonics of degree s on S^n and define

$$F_s(X) = \frac{1}{n+1}(f_o(X),\ldots,f_m(X)) \qquad X \text{ in } S^n$$

Then F_s gives an equivariant immersion of S^n in E^m.

If we define

$$k(s) = \frac{n}{s(s+n-1)}$$

$$\lambda_s = \frac{(1-k(s))2n}{n+2}$$

and take X_i and X_j orthonormal vectors in the orthogonal complement of $SO(n)$ in $SO(n+1)$ then

$$(\pi(X_i)\pi(X_j)e,\pi(X_i)\pi(X_j)e) = \lambda/2$$
$$= 0$$

if and only if $s = 1$, i.e. F_s is the standard immersion. Thus part (ii) of Th. 1.2 shows that the immersions F_s have no conjugate vectors for $s \geq 2$.

<u>Proof of Theorem 1.2</u>. (i) By Lemma 1.1 $(\pi(X)\pi(X)e,e) \neq 0$ all X in $\mathfrak{p}$.

(ii) Perpendicularity is an immediate consequence of Lemma 1.1 and the fact that $d\pi(X_o) = \pi(X)e$ for $\pi(X)\pi(Y)e = 0$ implies $(\pi(X)e, \pi(Y)e) = 0$.

For the second part we prove the following stronger result.

Let $X \in \mathfrak{p}$ define

$$\mathfrak{p}_X = \{Y \in \mathfrak{p} \mid \pi(Y)\pi(X)e = 0\}$$
$$\mathfrak{k}_X = \{Z \text{ in } \mathfrak{k} \mid [Z,X] = 0\}$$
and
$$\mathfrak{J}_X = \{W \text{ in } \mathfrak{p} \mid \pi(W)\pi(X)e = 0\}$$

We assert that $\mathfrak{J}_X = \mathfrak{k}_X \oplus \mathfrak{p}_X$.

Let $Z \in \mathfrak{k}$ be such that $\pi(Z)\pi(X)e = 0$. Then $\pi[Z,X]e = 0$. Thus by Lemma 1.1 $d\pi([Z,X]_o) = 0$ and $Z \in \mathfrak{k}_X$.

Thus $\mathfrak{J}_X \cap \mathfrak{k} \subset \mathfrak{k}_X$; the converse inclusion follows by reversing the above argument so $\mathfrak{J}_X \cap \mathfrak{k} = \mathfrak{k}_X$.

Choose $W \in \mathfrak{J}_X$. We can write $W = Z + Y$, $Z \in \mathfrak{k}$, Y in $\mathfrak{p}$.

$$\pi(W)\pi(X)e = \pi(Z)\pi(X)e + \pi(Y)\pi(X)e$$
$$= \pi[Z,X]e + \pi(Y)\pi(X)e = 0$$

But by Lemma 1.4 the terms on the right are mutually perpendicular.

So $\pi(W)\pi(X)e = 0$ if and only if $Z \in \mathfrak{k}_X$, $Y \in \mathfrak{p}_X$.

Hence $\mathfrak{J}_X = \mathfrak{k}_X \oplus \mathfrak{p}_X$.

The fact that $\mathfrak{p}_X$ is a Lie triple system is now obvious. Q.E.D.

We now turn to

<u>Definition 1.3</u>. Let M^n be immersed in the

Riemannian manifold N . Then the mean curvature normal ξ_x at a point $x \in M^n$ is given as follows: Let α be the second fundamental form and $e_1 \ldots e_n$ an orthonormal basis for M_x . Then

$$\xi_x = \sum_{i=1}^{n} \alpha(e_i, e_i)$$

Remark. If $v \in M_x^{\perp}$ then $(v, \xi) = \mathrm{Tr} A_v$ showing ξ_x is independent of choice of basis.

Definition 1.4. M is minimal in N if $\xi_x = 0$ for all x in M .

To show that the type of immersion we are considering is minimal in the sphere Do-Carmo and Wallach [14] used a result of Takahashi [21] which we state since we refer to it again in Ch. 3.

"A submanifold M^n of $S^N(r)$ (where r is the radius) is minimal if and only if every height function (see Def. 0.5) is an eigenfunction of the Laplacian with eigenvalue $-n/r^2$."

However as can be expected from the foregoing, this can be given a very direct and simple algebraic proof.

Let $\pi: G \to U(V^N)$ be a unitary class one irreducible representation. Make V^N into Euclidean 2N-space E^{2N} as follows: let $\langle\langle\ ,\ \rangle\rangle$ be the inner product on V^N . Consider the Euclidean inner product $(\ ,\ ) = \mathrm{Re}\langle\langle\ ,\ \rangle\rangle$. This gives us an orthogonal representation of G also denoted by π . Pick a K fixed vector "e" in E^{2N} and we have the immersion $\pi(gK) \to \pi(g)e$.

<u>Theorem 1.3</u>. The immersions $\pi\colon G/K \to E^{2N}$ are minimal in the sphere $S^{2N-1}((n/\gamma)^{1/2})$ where γ is the eigenvalue of the Casimir operator of the representation π of $\mathfrak{g}$ on V^N .

<u>Proof</u>. The fact that $\pi\colon G/K \to E^{2N}$ can also be regarded as an immersion in a sphere is obvious.

Let $\bar{\bar{\nabla}}$ be the affine connection on E^{2N} , $\bar{\nabla}$ the affine connection on $S^{2N-1}(r)$ and ∇ the affine connection on G/K .

If $\tilde{X}$ and $\tilde{Y}$ are vector fields on G/K then locally they can be considered as vector fields on S^{2N-1} and on E^{2N} . Thus we have

$$(\bar{\bar{\nabla}}_{\tilde{X}}\tilde{Y})_x = (\bar{\nabla}_{\tilde{X}}\tilde{Y})_x + (\bar{\bar{\alpha}}(X,Y))_x$$

where $\bar{\bar{\alpha}}$ is the second fundamental form of the immersion $S^{2N-1}(r) \to E^{2N}$. But

$$(\bar{\nabla}_{\tilde{X}}\tilde{Y})_x = (\nabla_{\tilde{X}}\tilde{Y})_x + (\bar{\alpha}(X,Y))_x$$

where $\bar{\alpha}$ is the second fundamental form of the immersion $G/K \to S^{2N-1}(r)$.

Thus if α is the second fundamental form of the immersion $G/K \to E^{2N}$ then

$$(\alpha(\tilde{X},\tilde{Y}))_x = (\overline{\alpha}(\tilde{X},\tilde{Y}))_x + (\overline{\overline{\alpha}}(\tilde{X},\tilde{Y}))_x$$

Hence we need only show that the mean curvature normal of the immersion $\pi: G/K \to E^{2N}$ is perpendicular to the sphere.

By equivariance we need only consider what happens at 0. Let ξ_0 be the mean curvature normal at 0.

$$\xi_0 = \sum_{i=1}^{n} \pi(X_i)\pi(X_i)e$$

where $\{X_i\}$ is an orthonormal basis for $\mathfrak{p}$.

$$(\xi_0,v) = \mathrm{Re} \sum_{i=1}^{n} \langle\langle\pi(X_i)\pi(X_i)e,v\rangle\rangle \quad \text{all } v \in E^{2N}.$$

Let $\{Y_s\}$ be an orthonormal basis of $\mathfrak{k}$ w.r.t. $-B$. Then if Γ is the Casimir operator

$$\Gamma = -\sum_{i=1}^{n} \pi(X_i)\pi(X_i) - \sum_{s} \pi(Y_s)\pi(Y_s)$$

$$\Gamma e = \gamma e = -\sum_{i=1}^{n} \pi(X_i)\pi(X_i)e$$

in V^N where γ is real, cf. [23], pg. 247. So $(\xi_0,v) = \mathrm{Re}\langle\langle-\gamma e,v\rangle\rangle = -\gamma(e,v)$. Hence $\xi_0 = -\gamma e$ which is perpendicular to $S^{2N-1}(r_0)$ where $r = (e,e)$.

So the immersion is minimal in $S^{2n-1}(r)$. A trivial calculation now yields r.

$$- \gamma(e,e) = (\sum_{i=1}^{n} \pi(X_i)\pi(X_i)e,e)$$

$$= - \sum_{i=1}^{n} (\pi(X_i)e,\pi(X_i)e) = -n$$

$$r = (e,e)^{1/2} = (n/\gamma)^{1/2} \quad . \qquad \text{Q.E.D.}$$

Remark. Before closing out the chapter perhaps we should mention that we have not assumed that the orthogonal representation $\pi: G \to O(N)$ is irreducible in this chapter, as this is unnecessary and in fact much too restrictive. For instance we have the following immersion of G/K . Let F_λ be the space dim N of all eigenfunctions of the Laplacian with eigenvalue $-\lambda$. Let φ_λ be the map $M \to \mathbb{R}^N$ given by $\varphi_\lambda(x) = (f_1(x),\ldots f_N(x))$ where $\{f_i\}$ is some orthonormal basis of F_λ w.r.t. the unique G invariant inner-product. That this gives a minimal immersion into a sphere is obvious from Takahashi's result. Let $\psi_\lambda(x)$ be another minimal substantial immersion of G/K in a sphere such that coordinate functions form an irreducible subspace of F_λ . Then although φ_λ and ψ_λ can be regarded as immersions in the same sphere they are equivalent (i.e. differ by an isometry of the sphere) if and only if F_λ is irreducible cf. [4]. However we will see that for the discussion of absolute curvature there is no loss of generality in restricting to irreducible representations.

CHAPTER 2. TIGHTNESS.

§1. Introduction.

Let $f: M^n \to \mathbb{R}^N$ be an immersion of a compact manifold. Let B be the unit normal sphere bundle of the immersion, i.e.

$$B = \{(x,v) \mid x \in M \text{ , } v \in M_x^{\perp} \text{ , } \|v\| = 1\}$$

We have the map $\nu: B \to S^{N-1}$ where $\nu((x,v)) = v$. If $d\sigma$ is the volume element of S^{N-1} and C_{N-1} is the volume of S^{N-1} we have ([2] or [13]).

Definition 2.1. The total absolute curvature of the immersion is

$$\tau(M^n,f,\mathbb{R}^N) = \frac{1}{C_{N-1}} \int_B \nu^*(d\sigma)$$

Remark. If M^n is orientable Chern-Lashof [2] showed the formula

$$\tau(M^n,f,\mathbb{R}^N) = \frac{1}{C_{N-1}} \int_M \left(\int_{S_m} |\det A_\xi| d\sigma\right) dm$$

where S_m is the unit sphere in $M_m^{\perp}$ and A_ξ is the operator on M_m given in Definition 0.2.

Remark. Although the formula in [2] is in a somewhat different form in formula (21) of [2], an expression is given which is our A_ξ in terms of moving frames. See [24] for instance.

We shall use the following terminology.

$$\Phi(M) = \{C^\infty \text{ functions on } M \text{ with no degenerate critical points}\}$$

$$\beta_k(\varphi) = \# \text{ of critical points of index } k \text{ of } \varphi \in \Phi(M)$$

$$\beta(\varphi) = \sum_{k=0}^{n} \beta_k(\varphi)$$

$$\beta_k(M) = \min_{\varphi \in \Phi(M)} \{\beta_k(\varphi)\}$$

$$\beta(M) = \min_{\varphi \in \Phi(M)} \{\beta(\varphi)\}$$

For any coefficient field K set

$$b_K(M,K) = \dim_K H_K(M,K)$$

$$b(M,K) = \sum_{0}^{n} b_K(M,K) .$$

Then we have the Morse Inequalities, cf. [15]

$$\beta_k(M) \geq b_k(M,K) \quad \text{any field } K .$$

<u>Definition 2.2</u>. A function $\varphi \in \Phi(M)$ is called k-tight if $\beta_k(\varphi) = \beta_k(M)$.

<u>Definition 2.3</u>. A function $\varphi \in \Phi(M)$ is called

tight if $\beta(\varphi) = \beta(M)$.

All this is related by means of

Theorem 2.1. Let $\varphi \in \Phi(M)$ then

(i) If φ is k-tight all k then φ is tight and if the Morse inequalities are equalities for some coefficient field K and some function $\psi \in \Phi(M)$ then this conclusion can be reversed.

(ii) If φ is tight then it is 0-tight and n-tight.

Proof. (i) Is obvious

(ii) See [16]. Q.E.D.

Theorem 2.2. Let I be the set of all immersions of M^n in Euclidean space. Then

$$\inf_{f \in I} \tau(M,f,\mathbb{R}^N) = \beta(M) \geq \sum_K \beta(M) .$$

Proof. See [13]. Q.E.D.

Definition 2.4. An immersion in $\mathbb{R}^N$ has minimal total curvature if $\tau(M,f,\mathbb{R}^N) = \beta(M)$; such an immersion will be called tight.

Remark. It is well known that not all manifolds have tight immersions, e.g. the exotic sphere does not have one [13].

We shall need the following well known lemma.

Lemma 2.1. (Kuiper [13]) An immersion M^n in $\mathbb{R}^N$ is tight if φ_a is tight for all height functions φ_a with non-degenerate critical points.

This leads to

Definition 2.5. An immersion f is k-tight if $\varphi_a(x) = (f(x),a)$ is k-tight whenever it is non-degenerate.

Theorem 2.1 shows that an immersion which is tight is also 0-tight, but Banchoff, cf. [14], has shown that the reverse is not necessarily the case; however we will see that for equivariant immersions of symmetric spaces they are equivalent.

A theorem of Chern-Lashof [3] shows that if $f\colon M \to \mathbb{R}^N$ is an immersion and $i \circ f\colon M \to \mathbb{R}^{N+1}$ is the immersion induced by the inclusion $i\colon \mathbb{R}^N \to \mathbb{R}^{N+1}$ then $\tau(f,M,\mathbb{R}^N) = \tau(i\, f,M,\mathbb{R}^{N+1})$. Also total curvature is invariant under affine transformation [13] so the search for tight immersions can be restricted to the study of substantial immersions.

§2. Reduction of the Problem for Homogeneous Spaces.

We now prove two theorems the second of which proves a conjecture of Wilson [22]. In fact [22] contains a particular case of the theorem proved in a very different manner.

Theorem 2.3. If G/H is a compact homogeneous space and π is a class-one orthogonal representation of G ; if π is reducible $\pi = \rho \oplus \mu$ and gives an immersion $\pi: G/H \to E_\pi$ which is 0-tight and substantial then if $\rho(\mu)$ gives an immersion $\rho: G/H \to E_\rho$, then the immersion ρ is 0-tight.

Remark. ρ need not give an immersion; for instance ρ could be the trivial representation of G .

Before proceeding with the proof of Th. 2.3 we recall the two-piece property, cf. [14].

Definition 2.6. Let $f: M \to E^N$ be an immersion. Then f has the two piece property if given any hyperplane $H \subset E^N$. $\{m \in M | f(m) \notin H\}$ has at most two components.

Lemma 2.1. [14]. Let $f: M \to E^N$ be an immersion. The f is 0-tight if and only if it has the two-piece property.

Proof of Theorem 2.3. Let E_ρ and E_μ be the representation spaces for ρ and μ respectively. Since $\pi: G/H \to E_\pi$ is substantial then the H fixed vector e can be written

$$e = e_\rho + e_\mu \qquad \begin{array}{l} 0 \neq e_\rho \text{ H-fixed in } E_\rho \\ 0 \neq e_\mu \text{ H-fixed in } E_\mu \end{array}$$

Suppose ρ gives an immersion of G/H into E_ρ. We show ρ satisfies the two-piece property.

Since π satisfies the two piece property given any $v \in E_\pi$, $\{p \in G/H \mid (\pi(p),v) \neq c\}$ has at most two components, for any constant c.

Write $v = v_\rho + v_\mu$. $v_\rho \in E_\rho$ and $v_\mu \in E_\mu$, and $p = g \cdot 0$. Then $(\pi(g)e,v) = (\rho(g)e_\rho,v_\rho) + (\mu(g)e_\mu,v_\mu)$. In particular if we consider $v_\mu = 0$, then for any $v_\rho \in E_\rho$, $\{gH \in G/H \mid (\rho(g)e_\rho,v_\rho) \neq c\}$ has at most two components which is the two piece property for the immersion $\rho: G/H \to E_\rho$. Hence the immersion ρ is 0-tight by Lemma 2.1. Q.E.D.

<u>Corollary</u>. Suppose G/K is an irreducible symmetric space and π an orthogonal class-one representation of G such that the immersion $\pi: G/K \to E^N$ is 0-tight. Then there is an irreducible orthogonal class one representation π' of G such that $\pi': G/K \to E^N$ is 0-tight.

<u>Proof</u>. There is no loss of generality in assuming

$\pi: G/K \to E^N$ substantial. For if not then there is i.e. a v with $(v,\pi(g)e) \equiv 0$. If e is K-fixed vector so there is a G invariant space E_v with $(E_v, G/K) = 0$.

Suppose $E^N = E_\rho \oplus E_\mu$. Then since immersion is substantial, we can write as in Th. 2.3 $e = e_\rho + e_\mu$. Then by proof of Lemma 1.1 $\pi(X)e_\rho = 0$ all $X \in \mathfrak{p}$ or $\pi(X)e_\rho \neq 0$ all $X \in \mathfrak{p}$; so we get immersion from either ρ or μ. Assume ρ gives immersion. We repeat the process for E_ρ and eventually we get immersion $\pi': G/H \to E_{\pi'}$ where π' is irreducible. Then repeated applications of Th. 2.3 show π' is 0-tight. Q.E.D.

Remark. Henceforth we assume all representations are irreducible unless explicitly stated otherwise.

Theorem 2.4. Let G/H be a compact homogeneous space and π a class-one orthogonal representation of G (not necessarily irreducible). If the map $\pi: G/H \to E^N$ by $\pi(gH) = \pi(g)e$ where e is the H-fixed vector, gives a 0-tight immersion, then it is in fact an imbedding.

Proof. Let e be the H-fixed vector in $\mathbb{R}^N$. Then if $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ is the direct sum decomposition $\pi(X)e \neq 0$ for any $X \in \mathfrak{m}$ since π is an immersion.

Suppose π is not an imbedding. Then there is $g \notin H$ with $\pi(g)e = e$, i.e. if H_e is the subgroup leaving e fixed then

$$H \subset H_e$$

properly.

But since π is an immersion the Lie algebra of H_e is also $\mathfrak{h}$; thus since both groups are compact H_e/H is finite. If index $[H,H_e]$ is m say, then G/H is an m-fold covering of G/H_e . Denote this covering by τ .

We can regard G/H_e as imbedded in $\mathbb{R}^N$ by $\tilde{\pi}$ where $\tilde{\pi}(gH_e) = \pi(g)e$. So we can factor the map $\pi\colon G/H \to \mathbb{R}^N$ by $\pi = \tilde{\pi}_o\tau$.

Now consider the height function φ_a on G/H .

$$\begin{aligned}\varphi_a(x) &= (\pi(x),a) \qquad x \in G/H \\ &= (\tilde{\pi}_o\tau(x),a) \\ &= \tilde{\varphi}(\tau(x))\end{aligned}$$

where $\tilde{\varphi}_a$ is the height f_n on G/H_e . Thus $d\varphi_a = d\tilde{\varphi}_a \circ d\tau$. So the singularities of φ_a occur "above" the singularities of $\tilde{\varphi}_a$, cf. Fig. (i).

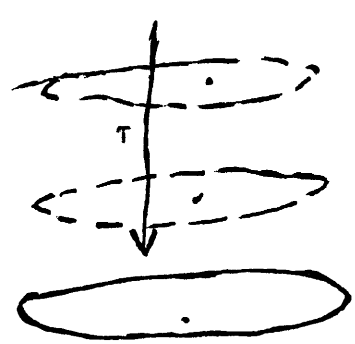

FIGURE (i)

Since a singularity is a local phenomenon the singularities have the same type and index on both G/H_e and G/H .

But G/H_e is compact. Thus for any non-degenerate $\tilde{\varphi}_a$ there is at least one critical point of index - 0 (namely the minimum). So φ_a has m-points of index-0. Contradicting 0-tightness.

Hence $H = H_e$ and π is an imbedding. Q.E.D.

§3. The Second Fundamental Form of 0-Tight Immersions.

The following theorem is a useful improvement of Theorem 4 in [13] and represents more of a change in point of view than anything else.

Theorem 2.5. If $f: M \to \mathbb{R}^N$ is a substantial 0-tight immersion then there is an open subset U of M such that $\alpha: M_m \times M_m \to M_m^{\perp}$ is onto for all $m \in U$.

Proof. Let $a \in \mathbb{R}^N$ be such that the height function φ_a is non-degenerate. Assume $\varphi_a(x)$ attains its maximum at $x_o \in M^n$. Since tightness is translation invariant we can assume $f(x_o) = 0$. Then the function

$$\varphi_{-a}(x) = -(a, f(x))$$

has a non-degenerate critical point of index 0 at x_o.

If α_{x_o} is not onto then by Lemma 1.3 we can choose $0 \neq z \in M_{x_o}^{\perp}$ such that

$$F_z(x) = -(a-z, f(x))$$

has a non-degenerate critical point of index - 0 at x_o.

Assert $\exists \lambda \ni F_\lambda(x) = -(a-\lambda z, f(x))$ assumes both +ve and -ve values.

The function $h(x) = \frac{(z, f(x))}{(a, f(x))}$ is not constant since f is a substantial immersion. Thus there is λ such that h takes values $> \frac{1}{\lambda}$ and values $< \frac{1}{\lambda}$. Thus $F_\lambda = -(a-\lambda z, f(x))$ assumes +ve and -ve values.

Let $w = \lambda z - a$.

Then $\varphi_w(x) = (w, f(x))$ has a non-degenerate critical point of index - 0 at x_o.

Assert we can choose w' in $\mathbb{R}^N$ such that $\varphi_{w'}(x)$ has non-degenerate critical points and $\varphi_{w'}$ has a critical point of index - 0 near x_o which is not a true minimum.

Since $\varphi_w(x)$ has a non-degenerate critical point of index - 0 at x_o there is a local coordinate system, $(u_1, \ldots, u_n)$ on an open neighborhood U of $x_o = (0, \ldots, 0)$ such that

$$\varphi_w(x) = u_1^2 + \ldots + u_n^2 \quad \text{on} \quad U.$$

Consider the "sphere" $S(r) \subset U$ given by $u_1^2 + \ldots + u_n^2 = r^2$. Then $\varphi_w(x) = r^2$ on $S(r)$. We can choose w' in any neighborhood of w such that $\varphi_{w'}$ has only non-degenerate critical points

$$\|\varphi_w - \varphi_{w'}\|_M = \|(f(x), w - w')\|_M$$
$$\leq \Lambda \|w - w'\|$$

where
$$\Lambda = \max_{x \in M} \|f(x)\|$$

Choose w' with $\|w-w'\| < \frac{r^2}{2\Lambda}$. Then $\|\varphi_w - \varphi_{w'}\| \leq \frac{r^2}{2}$.

$\varphi_{w'}$ has a minimum in the closed ball $\overline{S}(r)$. We assert that this minimum does not occur on the sphere $S(r)$.

But this is easy since

$$\varphi_{w'}(x_o) = 0$$

and
$$\varphi_{w'}(x) > \frac{r^2}{2} \quad \text{for} \quad x \in S(r)$$

So the minimum on $\overline{S}(r)$ is in fact a critical point of index - 0 of $\varphi_{w'}$.

Since φ_w takes +ve and -ve values it is clear that we can choose w' such that $\varphi_{w'}$ takes +ve and -ve values. Hence the point will not be the absolute minimum of $\varphi_{w'}$.

So $\varphi_{w'}$ is a non-degenerate height function with two critical points of index - 0, contradicting 0-tightness. Hence α must be onto at x_o .

The fact that α is onto in an open neighborhood U of x_o is a trivial consequence of the differentiability of α . Q.E.D.

<u>Corollary 1.</u> Cf. Th. 4 in [13].

$$N - n \leq \frac{1}{2} n(n+1) .$$

<u>Proof.</u> Trivial since α is a symmetric map from $M_x \times M_x$ to $M_x^{\perp}$. Q.E.D.

<u>Corollary 2.</u> Let $M = G/H$ be a homogeneous space

and π a real class one representation of G on E^N such that the imbedding $\pi: G/H \to E^N$ is 0-tight and substantial. The α is onto everywhere.

Proof. Obvious.

§4. Symmetric R-Spaces.

The theory of R-spaces, as we need it, is scattered throughout the literature so in this section what we need is organized with outlines of the main proofs. We do not define a general R-space but give a somewhat ad-hoc definition of symmetric R-spaces since that is all we need.

Let $\mathcal{L}$ be a real semi-simple Lie algebra and $Z \in \mathcal{L}$ such that ad Z is semi-simple with real eigenvalues $0, \pm 1$.

Theorem 2.6. There is a Cartan decomposition $\mathcal{L} = \mathcal{J} + \beta$ such that $Z \in \beta$.

Proof. Cf. [10], Ths. 2 and 3.

Let $\mathcal{L} = \mathcal{L}_{-1} + \mathcal{L}_0 + \mathcal{L}_1$ be the eigenspace decomposition of $\mathcal{L}$ and define $\alpha: \mathcal{L} \to \mathcal{L}$ a linear map by

$$\alpha(X+Y+W) = -X+Y-W \;, \quad X \in \mathcal{L}_{o_1} \;, \quad Y \in \mathcal{L}_o \;, \; Z \in \mathcal{L} \;.$$

Then α is an involutive automorphism.

Let $\mathcal{L} = \mathcal{J}' + \beta'$ be any Cartan decomposition with involution σ' . Then if

$$B^{\sigma'}(X,Y) = -B_{\mathcal{L}}(X,\sigma' Y) \quad X \text{ and } Y \in \mathcal{L} \quad .$$

$B^{\sigma'}$ is a symmetric, positive definite, bilinear form on

$\mathcal{L}$ and $\alpha\sigma'$ is self-adjoint w.r.t. $B^{\sigma'}$. Now it is almost standard that there is a Cartan decomposition $\mathcal{L} = \mathcal{J} + \beta$ with involution σ such that α and σ commute.

Thus $\mathcal{L}_0 = \mathcal{L}_0 \cap \mathcal{J} \oplus \mathcal{L}_0 \cap \beta$ direct sum. To see that $Z \in \beta$ it suffices to show that ad Z is symmetric w.r.t. B^{σ}. Q.E.D.

Now let (L,G) be a pair associated with $(\mathcal{L},\sigma)$ such that L has no center.

Theorem 2.7. Let $K = \{g \in G | \mathrm{ad}_{\mathcal{L}} g(Z) = Z\}$. Then
(i) G/K is symmetric
(ii) The immersion $\varphi: G/K \to \beta$ by

$$\varphi(gK) = \mathrm{ad}_{\mathcal{L}} g\, Z$$

is tight, equivariant.

Proof. (i) Since L has no center, $\mathrm{Ad}_{\mathcal{L}}: L \to \mathrm{Int}(\mathcal{L})$ is an analytic isomorphism onto so we shall assume $L = \mathrm{Int}(\mathcal{L})$. Let L^c be the complex Lie group $\mathrm{Int}(\mathcal{L}^c)$ where $\mathcal{L}^c$ is the complexification of $\mathcal{L}$. Then $L \subset L^c$ and $\exp(i\pi Z) \in L^c$ where $i = \sqrt{-1}$.

Let θ denote the inner automorphism of L^c defined by $\exp(i\pi Z)$. Then $\theta^2 = \mathrm{Id}$. We assert G is θ-stable.

First we show $\mathcal{L}$ is stable in $\mathcal{L}^c$ under $\mathrm{Ad}_{L^c}(\exp i\pi Z)$.

Let $W \in \mathscr{L}$, $W = W_{-1} + W_o + W_1$, $W_{-1} \in \mathscr{L}_{-1}$, $W_o \in \mathscr{L}_o$ and $W_1 \in \mathscr{L}_1$ in the notation of Th. 2.6.

$$\begin{aligned} \mathrm{Ad}(\exp i\pi Z)W &= \mathrm{Ad}(\exp i\pi Z)W_{-1} + \mathrm{Ad}(\exp i\pi Z)W_o \\ &\qquad + \mathrm{Ad}(\exp i\pi z)W_1 \\ &= e^{-i\pi}W_{-1} + W_o + e^{i\pi}W_1 \\ &= W_o - W_{-1} - W_1 \in \mathscr{L} . \end{aligned}$$

Now

$$\begin{aligned} \sigma\, \mathrm{Ad}(\exp i\pi Z) &= \mathrm{Ad}(\exp i\pi\sigma Z) \circ \sigma \\ &= \mathrm{Ad}(\exp - i\pi Z) \circ \sigma \\ &= \mathrm{Ad}(\exp i\pi Z) \circ \sigma . \end{aligned}$$

So $\mathfrak{J}$ is stable under $\mathrm{Ad}(\exp i\pi Z)$. Thus G is θ-stable.

We let $\theta|G$ be also denoted by θ .

Let K_θ be the fixed point set of θ . Then $(K_\theta)_o \subset K \subset K_\theta$ where $(K_\theta)_o$ is connected component of the identity of K_θ . So G/K is Riemannian symmetric.

(ii) The equivariance of φ is obvious.

Tightness is proven in [12] Th. 3.1 for an even more general type of space. Q.E.D.

<u>Definition</u>. A symmetric homogeneous space G/K is a symmetric R-space if it can be constructed as in Th. 2.6 and Th. 2.7.

§5. The Fundamental Lemma.

We now examine the implications of Th. 2.5 for equivariant immersions of symmetric spaces.

Let G/K be an irreducible symmetric space, π an irreducible class-one representation of G on E^N with a K-fixed vector e. Denote by π the immersion $\pi(gK) = \pi(g)e$, of G/K in E^N. Then if the second fundamental form is onto we have

$$E^N = T_o + T_o^{\perp}$$

where

$$T_o = \{\pi(X)e \ ; \ X \in \mathfrak{p}\}$$
$$T_o^{\perp} = \text{linear hull of } \{\pi(X)\pi(X)e : X \in \mathfrak{p}\}$$

We shall need the following lemma.

Lemma 2.2. If π is a real orthogonal representation of $\mathfrak{g}$ with a vector $e \neq 0$ annihilated by $\mathfrak{k}$ then

$$(\pi(X)\pi(X)e \ , \ \pi(Z)\pi(Y)\pi(Y)e) = 0$$

for all X, Y and Z in $\mathfrak{p}$.

Proof. We know $(\pi(X)\pi(Y)e, \pi(Z)\pi(X)\pi(Y)e) = 0$. We can rewrite $\pi(Z)\pi(X)\pi(Y)e = \pi[Z,X]\pi(Y)e + \pi(X)\pi(Z)\pi(Y)e$. So $(\pi[Z,X]\pi(Y)e, \pi(X)\pi(Y)e) + (\pi(X)\pi(Z)\pi(Y)e, \pi(X)\pi(Y)e) = 0$. But the first term is zero by Lemma 1.4 So

$$0 = (\pi(X)\pi(Z)\pi(Y)e,\ \pi(X)\pi(Y)e)$$

$$= -\ (\pi(X)\pi(X)\pi(Y)e,\ \pi(Z)\pi(Y)e)$$

$$= -\ (\pi(X)\pi(Y)\pi(X)e,\ \pi(Z)\pi(Y)e)$$

since

$$\pi(Y)\pi(X)e = \pi(X)\pi(Y)e$$

$$= (\pi(X)\pi(X)e,\ \pi(Y)\pi(Z)\pi(Y)e)$$

as above

$$= (\pi(X)\pi(X)e,\ \pi(Z)\pi(Y)\pi(Y)e) \quad . \qquad \text{Q.E.D.}$$

Now let $\mathcal{L} = \mathfrak{G} \oplus E^N$.

Give $\mathcal{L}$ the following algebraic structure.

(i) X,Y in $\mathfrak{G}$: [X,Y] as in $\mathfrak{G}$.

(ii) X in $\mathfrak{G}$ u in E^N : $[X,u] = -[u,X] = \pi(X)u$

(iii) u,v in E^N then [u,v] is in $\mathfrak{G}$ where

$-B([u,v],X) = -\ (v,\pi(X)u)$ for all X in $\mathfrak{G}$.

<u>Lemma 2.3</u>. If G/K is a symmetric space and π a class-one orthogonal representation of G giving imbedding $\pi\colon G/K \to E^N$ then if the second fundamental form is onto the above operations make $\mathcal{L} = \mathfrak{G} + E^N$ into a Lie algebra.

<u>Proof</u>. Anti-commutativity. For X and Y in $\mathfrak{G}$ then anti-commutativity is inherited from $\mathfrak{G}$. For $X \in \mathfrak{G}$ and $u \in E^N$ then it is defined. For u and v in E^N just note $(u,\pi(X)v) = -\ (v,\pi(X)u)$.

So now we need only check the Jacobi identity. Unfortunately this must be done case by case.

Case (i). X, Y and $Z \in \mathfrak{g}$ then it is inherited from $\mathfrak{g}$.

Case (ii). $X, Y \in \mathfrak{g}$ and $u \in E^N$ then

$$\begin{aligned}&[X,[Y,u]] + [u,[X,Y]] + [Y,[u,X]]\\ &= \pi(X)\pi(Y)u - \pi[X,Y]u - \pi(Y)\pi(X)u\\ &= 0\end{aligned}$$

Case (iii). $X \in \mathfrak{g}$, u and $v \in E^N$. Let Y be in $\mathfrak{g}$. Then

$$\begin{aligned}&([X,[u,v]],Y) + ([v,[X,u]],Y) + ([u,[v,X]],Y)\\ &= - ([u,v],[X,Y]) + ([v,\pi(X)u],Y) - ([u,\pi(X)v],Y)\\ &= (v,\pi[X,Y]u) - (\pi(X)u,\pi(Y)v) + (\pi(X)v,\pi(Y)u)\\ &= (v,\pi[X,Y]u) + (\pi(Y)\pi(X)u,v) - (\pi(X)\pi(Y)u,v)\\ &= 0 .\end{aligned}$$

Now before we consider u, v and w in E^N we develop a few preliminary results.

If X and Y are in $\mathfrak{p}$ consider $([\pi(X)e,\pi(Y)e],Z)$ for $Z \in \mathfrak{g}$.

$$([\pi(X)e,\pi(Y)e],Z) = - (\pi(Y)e,\ \pi(Z)\pi(X)e)$$

Thus $([\pi(X)e,\pi(Y)e],\mathfrak{p}) = 0$ by Lemma 1.4. So we need only consider $Z \in \mathfrak{k}$.

$$-(\pi(Y)e,\pi(Z)\pi(X)e) = (-\pi(Y)e,\pi[Z,X]e)$$
$$= -\ (Y,[Z,X])$$
$$= -\ ([X,Y],Z)$$

Thus $$[\pi(X)e,\pi(Y)e] = -\ [X,Y] \qquad (a)$$

Now we can consider

Case (iv). u, v, w in T_o , $u = \pi(X)e$, $v = \pi(Y)e$, $w = \pi(Z)e$.

$$[\pi(X)e,[\pi(Y)e,\pi(Z)e]] + [\pi(Z)e,[\pi(X)e,\pi(Y)e]]$$
$$+ [\pi(Y)e,[\pi(Z)e,\pi(X)e]]$$
$$= \pi[Y,Z]\pi(X)e + \pi[X,Y]\pi(Z)e + \pi[Z,X]\pi(Y)e \qquad \text{by (a)}$$
$$= -\ \pi[X,[Y,Z]]e - \pi[Z,[X,Y]]e - \pi[Y,[Z,X]]e$$
$$= 0 \qquad \text{by Jacobi identity on } \mathcal{G}$$

Case (v). u, v in T_o and w in $T_o^{\perp}$.

By Lemma 1.4 we need only consider w of the form $\pi(Z)\pi(Z)e$ with $Z \in \mathcal{P}$ so

$$[u,[v,w]] + [w,[u,v]] + [v,[w,u]]$$
$$= [\pi(X)e,[\pi(Y)e,\pi(Z)\pi(Z)e]]$$
$$+ [\pi(Z)\pi(Z)e,[\pi(X)e,\pi(Y)e]]$$
$$+ [\pi(Y)e,[\pi(Z)\pi(Z)e,\pi(X)e]]$$

where $u = \pi(X)e$, $v = \pi(Y)e$

$$= -\ \pi([\pi(Y)e,\pi(Z)\pi(Z)e])\pi(X)e$$
$$+ \pi[X,Y]\pi(Z)\pi(Z)e$$
$$+ \pi([\pi(X)e,\pi(Z)\pi(Z)e])\pi(Y)e \qquad (b)$$

Assert $[\pi(X)e, \pi(Z)\pi(Z)e]$ is in $\mathfrak{p}$ all X and Z in $\mathfrak{p}$. For any $W \in \mathfrak{g}$

$$([\pi(X)e, \pi(Z)\pi(Z)e], W) = - (\pi(Z)\pi(Z)e, \pi(W)\pi(X)e)$$
$$= 0 \quad \text{if} \quad W \in \mathfrak{k}$$

by Lemma 1.4 since $\pi(W)\pi(X)e = \pi[W,X]e \in T_o$. So we can write

$$\pi([\pi(X)e, \pi(Z)\pi(Z)e])\pi(Y)e = \pi(Y)\pi([\pi(X)e, \pi(Z)\pi(Z)e)e$$

Now let $\{X_i\}$ be an orthonormal basis for $\mathfrak{p}$. Then

$$[\pi(X)e, \pi(Z)\pi(Z)e] = \sum_{i=1}^{n} ([\pi(X)e, \pi(Z)\pi(Z)e], X_i)X_i \quad .$$

Thus

$$\pi([\pi(X)e, \pi(Z)\pi(Z)e])e$$
$$= \sum_{i=1}^{n} ([\pi(X)e, \pi(Z)\pi(Z)e], X_i)\pi(X_i)e$$
$$= -\sum_{i=1}^{n} (\pi(Z)\pi(Z)e, \pi(X_i)\pi(X)e)\pi(X_i)e$$
$$= \sum (\pi(X)\pi(Z)\pi(Z)e, \pi(X_i)e)\pi(X_i)e$$
$$= \pi(X)\pi(Z)\pi(Z)e$$

Since by Lemma 2.2 $\pi(X)\pi(Z)\pi(Z)e \in T_o$ and $\pi(X_i)e$ is a basis. Substituting in (b) we get

$$[u,[v,w]] + [w,[u,v]] + [v,[w,u]]$$

$$= - \pi(X)\pi(Y)\pi(Z)\pi(Z)e + \pi[X,Y]\pi(Z)\pi(Z)e$$

$$+ \pi(Y)\pi(X)\pi(Z)\pi(Z)e$$

$$= 0 \text{ .}$$

Case (vi). u in T_o , v and w in $T_o^\perp$. Let $u = \pi(X)e$, $v = \pi(Y)\pi(Y)e$, $w = \pi(Z)\pi(Z)e$. Then let $J = [u,[v,w]] + [w,[u,v]] + [v,[w,u]]$. Assert $J \in T_o$

$$J = [\pi(X)e,[\pi(Y)\pi(Y)e,\pi(Z)\pi(Z)e]]$$

$$+ [\pi(Z)\pi(Z)e, [\pi(X)e,\pi(Y)\pi(Y)e]]$$

$$+ [\pi(Y)\pi(Y)e, [\pi(Z)\pi(Z)e, \pi(X)e]]$$

We saw in the course of the proof of case (v) that $[\pi(X)e, \pi(Y)\pi(Y)e] \in \mathfrak{p}$. Thus Lemma 2.2 implies the second and third terms of J are in T_o .

Let W be in $\mathfrak{g}$. Then

$$([\pi(Y)\pi(Y)e, \pi(Z)z(Z)e],W) = - (\pi(Z)\pi(Z)e, \pi(W)\pi(Y)\pi(Y)e)$$

$$= 0 \quad \text{if} \quad W \in \mathfrak{p} \quad \text{by Lemma 2.1.}$$

Thus $[\pi(Y)\pi(Y)e,\pi(Z)\pi(Z)e] \in \mathfrak{k}$ which proves that the first term of J is in T_o .

Let $W \in \mathfrak{p}$. We shall consider $(J,\pi(W)e)$ term by term.

$$([\pi(X)e,\ [\pi(Y)\pi(Y)e,\ \pi(Z)\pi(Z)e]],\ \pi(W)e)$$

$$= -\ (\pi([\pi(Y)\pi(Y)e,\ \pi(Z)\pi(Z)e])\pi(X)e,\ \pi(W)e)$$

$$= ([\pi(Y)\pi(Y)e,\pi(Z)\pi(Z)e],\ [\pi(X)e,\pi(W)e])$$

$$= -\ ([\pi(Y)\pi(Y)e,\pi(Z)\pi(Z)e],\ [X,W])$$

$$= (\pi(Z)\pi(Z)e,\ \pi[X,W]\pi(Y)\pi(Y)e) \qquad (c)$$

Now consider

$$([\pi(Z)\pi(Z)e,\ [\pi(X)e,\pi(Y)\pi(Y)e]],\ \pi(W)e)$$

$$= -\ (\pi([\pi(X)e,\ \pi(Y)\pi(Y)e])\pi(Z)\pi(Z)e,\ \pi(W)e)$$

$$= -\ ([\pi(X)e,\pi(Y)\pi(Y)e],\ [\pi(W)e,\ \pi(Z)\pi(Z)e])$$

$$= (\pi(Y)\pi(Y)e,\ \pi([\pi(W)e,\ \pi(Z)\pi(Z)])\pi(X)e)$$

$$= (\pi(Y)\pi(Y)e,\ \pi(X)\pi(W)\pi(Z)\pi(Z)e) \qquad (d)$$

Substituting (c) and (d) along with the equivalent expression for the third term gives

$$\begin{aligned}(J,\pi(W)e) &= (\pi(Z)\pi(Z)e,\ \pi[X,W]\pi(Y)\pi(Y)e) \\ &\quad + (\pi(Y)\pi(Y)e,\ \pi(X)\pi(W)\pi(Z)\pi(Z)e \\ &\quad - (\pi(Z)\pi(Z)e,\ \pi(X)\pi(W)\pi(Y)\pi(Y)e) \\ &= 0\end{aligned}$$

Case (vii). u, v, w in $T_0^{\perp}$.

Essentially this is done by reducing it to cases (i) through (vi). First consider

$$[\pi(X)\pi(X)e,\ [\pi(Y)\pi(Y)e,\ \pi(Z)\pi(Z)e]]$$

$$= [[X,\pi(X)e],\ [\pi(Y)\pi(Y)e,\ \pi(Z)\pi(Z)e]]$$

$$= [\pi(X)e,\ [[\pi(Y)\pi(Y)e,\ \pi(Z)\pi(Z)e],\ X]]$$

$$+ [X,\ [\pi(X)e,\ [\pi(Y)\pi(Y)e,\ \pi(Z)\pi(Z)e]]]$$

by case (ii)

$$= [\pi(X)e,\ [\pi(Z)\pi(Z)e,\ [X,\pi(Y)\pi(Y)e]]$$

$$+ [\pi(X)e,\ [\pi(Y)\pi(Y)e,\ [\pi(Z)\pi(Z)e,X]]]$$

$$- [X,\ [\pi(Z)\pi(Z)e,\ [\pi(X)e,\ \pi(Y)\pi(Y)e]]]$$

$$- [X,\ [\pi(Y)\pi(Y)e,\ [\pi(Z)\pi(Z)e,\ \pi(X)e]]]$$

by cases (iii) and (v)

$$= [\pi(X)e,\ [\pi(Z)\pi(Z)e,\ \pi(X)\pi(Y)\pi(Y)e]]$$

$$- [\pi(X)e,\ [\pi(Y)\pi(Y)e,\ \pi(X)\pi(Z)\pi(Z)e]]$$

$$- [X,\ [\pi(Z)\pi(Z)e,\ [\pi(X)e,\ \pi(Y)\pi(Y)e]]$$

$$- [X,\ [\pi(Y)\pi(Y)e,\ [\pi(Z)\pi(Z)e,\ \pi(X)e]]]$$

Now consider

$$[\pi(Z)\pi(Z)e,\ [\pi(X)\pi(X)e,\ \pi(Y)\pi(Y)e]]$$

$$= -\ [\pi(Z)\pi(Z)e,\ [\pi(Y)\pi(Y)e,\ [X,\ \pi(X)e]]]$$

$$= -\ [\pi(Z)\pi(Z)e,\ [\pi(X)e,\ \pi(X)\pi(Y)\pi(Y)e]]$$

$$+ [\pi(Z)\pi(Z)e,\ [X,\ [\pi(X)e,\ \pi(Y)\pi(Y)e]]$$

by case (iii)

$$= [\pi(X)\pi(Y)\pi(Y)e,\ [\pi(Z)\pi(Z)e,\ \pi(X)e]]$$
$$+ [\pi(X)e,\ [\pi(X)\pi(Y)\pi(Y)e,\ \pi(Z)\pi(Z)e]]$$
$$+ [[\pi(X)e,\ \pi(Y)\pi(Y)e],\ \pi(X)\pi(Z)\pi(Z)e]$$
$$- [X,\ [[\pi(X)e,\ \pi(Y)\pi(Y)e]\ ,\ \pi(Z)\pi(Z)e]]$$

by cases (iv) and (ii)

If we write the corresponding expression for $[\pi(Y)\pi(Y)e,\ [\pi(Z)\pi(Z)e,\ \pi(X)\pi(X)e]]$ and combine we get

$$[\pi(X)\pi(X)e,\ [\pi(Y)\pi(Y)e,\ \pi(Z)\pi(Z)e]]$$
$$+ [\pi(Z)\pi(Z)e,\ [\pi(X)\pi(X)e,\ \pi(Y)\pi(Y)e]]$$
$$+ [\pi(Y)\pi(Y)e,\ [\pi(Z)\pi(Z)e,\ \pi(X)\pi(X)e]]$$
$$= 0\ .$$

So we have proven Jacobi Identity. Hence $\mathcal{L}$ is a Lie algebra. Q.E.D.

We can now prove

Theorem 2.8. Let G/K be a symmetric space and π an irreducible class-one orthogonal representation of G giving the imbedding $\pi\colon G/K \to E^N$. If the second fundamental form of the imbedding is onto then $\mathcal{L} = \mathfrak{g} \oplus E^N$ is a semi-simple non-compact Lie algebra with $\mathcal{L} = \mathfrak{g} \oplus E^N$ a Cartan decomposition.

If σ is the Cartan involution then in fact $(\mathcal{L}, \sigma)$ is irreducible orthogonal symmetric.

Proof. We define a representation ρ of G on $\mathcal{L}$ by $\rho(G)|_{\mathfrak{g}} = Ad_G$, $\rho(G)|_{E^N} = \pi(G)$.

Consider $u \in E^N$ and let $ad_{\mathcal{L}}u$ be the adjoint action of u on $\mathcal{L}$.

Assert $ad_{\mathcal{L}}(\rho(g)u) = \rho(g)ad_{\mathcal{L}}u\ \rho(g)^{-1}$.

Let X be in $\mathfrak{g}$.

$$\begin{aligned}[\pi(g)u,X] &= -\pi(X)\pi(g)u \\ &= -\pi(g)\pi(Ad_G g^{-1}X)u \\ &= \rho(g)\circ ad\ u\circ\rho(g^{-1})\ (X)\end{aligned}$$

Let v be in E^N , X in $\mathfrak{g}$:

$$\begin{aligned}([\pi(g)u,v],\ X) &= -\ (v,\ \pi(X)\pi(g)u) \\ &= -\ (\pi(g^{-1})v,\ \pi(Ad_G g^{-1}X)u) \\ &= \quad ([u,\ \pi(g^{-1})v],\ Ad_G g^{-1}X) \\ &= \quad (Ad_G g[u,\ \pi(g^{-1})v],\ X)\end{aligned}$$

So $$[\pi(g)u,v] = \rho(g)\circ ad\ u\circ\rho(g^{-1})\ v$$

Assertion is thus proven.

Now let $B_{\mathcal{L}}$ = killing form on $\mathcal{L}$. Then by above $B_{\mathcal{L}}$ is G invariant hence is a constant multiple of the Euclidean inner product on E^N . We now show that this constant is ≥ 0 .

Consider $ad\ e \cdot ad\ e$.

$ade|_k = 0$ since e is K-invariant. Assert $ade|_{T_o^\perp} = 0$. Also let $W \in \mathcal{J}$. Then

$$([e,\pi(X)\pi(X)e],W) = -(\pi(X)\pi(X)e,\pi(W)e)$$

$$= 0 \text{ if } W \text{ is in}$$

$$\text{but also } = 0 \text{ if } W \text{ is in } \mathcal{P}$$

by Lemma 1.4.

So to find $B_{\mathcal{L}}(e,e)$ we need only consider $(ade)^2$ on $\mathcal{P}$ and T_o.

Let $X \in \mathcal{P}$

$$([e,[e,X]],X) = -([e,X],[X,e])$$

$$= (\pi(X)e,\pi(X)e)$$

$$= (X,X)$$

So $$tr(ade)^2|_{\mathcal{P}} = n$$

Let X and Y be in $\mathcal{P}$.

$$([e,\pi(X)e],Y) = -(X,Y)$$

by above. Hence

$$[e,\pi(X)e] = -X$$

Hence

$$([e,[e,\pi(X)e],\pi(Y)e) = (\pi(X)e,\pi(Y)e)$$

So $$\operatorname{tr}\,(\operatorname{ad}e)^2|_{T_o} = n$$

But $$B_{\mathcal{L}}(e,e) = \operatorname{tr}(\operatorname{ad}e)^2|_{\mathcal{J}} + \operatorname{tr}(\operatorname{ad}e)^2|_{T_o}$$
$$= 2n \,.$$

We have thus shown that if $B_{\mathcal{L}}$ is the killing form on $\mathcal{L}$, $B_{\mathcal{L}}(X,X) > 0$ if X is in E^N .

If X is in $\mathcal{J}$

$$B_{\mathcal{L}}(X,X) = \operatorname{tr}(\operatorname{ad} X\,)^2 + \operatorname{tr}(\pi(X))^2$$
$$< 0$$

So $\mathcal{J}$ and E^N being orthogonal under $B_{\mathcal{L}}$, $\mathcal{L}$ is semi-simple.

Define $s\colon \mathcal{L} \to \mathcal{L}$ a linear map by

$$s(X+v) = X-v\ , \quad X \text{ in } \mathcal{J}, \quad v \text{ in } E^N \,.$$

Clearly s^2 = Identity. Assert s is a Lie algebra automorphism.

$$s[X+v,Y+w] = s[X,Y] + s\pi(X)w - s\pi(Y)v + s[v,w]$$
$$= [X,Y] - \pi(X)w + \pi(Y)v + 4v,w]$$
$$= [s(X+v),s(Y+w)]$$

So s is an involutive automorphism and $\mathcal{L} = \mathcal{G} \oplus E^N$ is a Cartan decomposition. The fact that $\mathcal{L}$ is irreducible orthogonal symmetric is easily seen from the irreducibility of the representation of $\mathcal{G}$ on E^N and the fact representation must be faithful. Q.E.D.

§2.5. Geometric Results.

We now apply the results we have obtained to the problem of classifying those locally symmetric homogeneous spaces which have equivariant tight immersions.

We have the situation G/K locally symmetric, π a real class-one representation of G giving 0-tight immersion $\pi: G/K \to E^N$. By the corollary to Th. 2.3 we can assume π is in fact irreducible and we get the following classification theorems.

Theorem 2.9. Let G/K be an irreducible locally symmetric space and π an irreducible class-one orthogonal representation of G giving the immersion $\pi: G/K \to E^N$. Then the following are equivalent.

(i) π is 0-tight.

(ii) G/K is a symmetric R-space and π is in fact one of the imbeddings constructed in [12].

(iii) π is tight (has minimal total curvature).

Theorem 2.10. Let G/K be a locally symmetric space. Then the following are equivalent.

(i) G/K covers a symmetric R-space.

(ii) There is an irreducible class-one representation π of G such that the second fundamental form of the immersion $\pi: G/K \to E^N$ is an onto map.

Proof of Theorem 2.9. (i) => (ii).

Since π is an irreducible representation of G, the immersion $\pi: G/K \to E^N$ is substantial and thus since the immersion is 0-tight Th. 2.5 shows the second fundamental form is onto; so Th. 2.8 shows $\mathcal{L} = \mathfrak{g} \oplus E^N$ is a semi-simple Lie algebra with G the compact subgroup of $\mathrm{Int}(\mathcal{L})$ with Lie algebra $\mathfrak{g}$. Thus G is maximal compact in $\mathrm{Int}(\mathcal{L})$. Th. 2.4 shows that π is an imbedding so K is the subgroup of G leaving e fixed, and e has eigenvalues $0, \pm 1$; so by definition G/K is a symmetric R-space and the imbedding is one of the class considered in [12].

(ii) => (iii). Shown in Th. 3.1, [12].

(iii) => (i). See Th. 2.1, part (ii).

Proof of Theorem 2.10. (i) => (ii). Suppose $f: G/K \to M'$ is the covering. Let π be the imbedding constructed in [12]. Then $\pi \circ f$ gives the required immersion.

(ii) => (i). As above G is maximal compact in $\mathrm{Int}(\mathcal{L})$ where $\mathcal{L} = \mathfrak{g} + E^N$. Thus K is a subgroup of the isotropy subgroup of e, K_e, both are compact and have the same Lie algebra. Hence G/K covers G/K_e which is by definition a symmetric R-space. Q.E.D.

Theorem 2.11. Suppose G/K is a locally symmetric space which admits a motion group L such that L is simple, non-compact and properly contains G as a subgroup. Then L is locally equivalent to an R-space.

Proof. If such a group L exists then by [17] Theorem 3.1 a space G'/K' equivalent to G/K can be immersed in $ad(L)/ad(G)$. Hence G'/K' covers an R-space by Th. 2.11. Q.E.D.

Remark. We have confined our considerations to irreducible spaces but the extension to reducible spaces is easy as given by

Theorem 2.12. If G/K is a locally symmetric space which has a tight immersion then we can write

$$G/K = M_1 \times M_2 \times \ldots \times M_n$$

where the M_i are irreducible symmetric R-spaces.

Proof. We can write $G/K = M_1 \times M_2 \times \ldots \times M_n$ where the M_i are irreducible locally symmetric spaces.

Shall consider the case where $G/K = M_1 \times M_2$. The general case follows easily.

The theorem follows from

Lemma 2.4. Let $f\colon M \to \mathbb{R}^N$ and $g\colon M' \to \mathbb{R}^{N'}$ be immersions. Then the immersion

$$f \times g\colon M \times M' \to \mathbb{R}^{N+N'}$$

by $$f \times g\,(x,y) = (f(x), g(y))$$

is 0-tight if and only if f and g are 0-tight.

Proof of Lemma. Let $\varphi: M \to \mathbb{R}^1$, $\chi: M' \to \mathbb{R}^1$ be functions with non-degenerate critical points. Then the fn $\varphi+\chi: M \times M' \to \mathbb{R}^1$ by $(\varphi+\psi)(x,y) = \varphi(x) + \psi(y)$ has a critical point at (x,y) if and only if x is a critical point of φ and y is a critical point of χ.

If χ is a critical point of index i and if y is a critical point of index j then (x,y) is a critical point of index I+j so $\varphi+\chi$ has only one critical point of index-0 iff φ, ψ have only one critical of index-0, which gives Lemma and Theorem. Q.E.D.

CHAPTER 3. EIGENVALUES OF THE LAPLACIAN.

§1. An Inequality on the Betti Numbers.

In this section we work with the formula given in the remark following Def. 2.1. So we are tacitly assuming G/K is orientable throughout this section.

Definition 3.1. If $f\colon M \to \mathbb{R}^N$ is an immersion then the absolute curvature at the point $m \in M$ is defined by

$$\tau_m(f) = \int_{S_m} |\det A_\xi| d\sigma$$

where S_m is unit sphere in $M_m^\perp$. We have if $V(G/K)$ is volume of G/K

Lemma 3.1. For the immersions $\pi\colon G/K \to \mathbb{R}^N$

$$\tau(G/K,\pi,E^N) = \frac{V(G/K)}{C_{N-1}} \quad \tau_o(\pi)$$

Proof. Let $u \in T_o^\perp$. Then $\pi(g)u \in T_{\pi(g)e}^\perp$ since $Y \in (G/K)_{g\,0}$ if and only if there is $X \in \mathfrak{p}$ with $Y = (\mathrm{Ad}\ g\ X)$. Thus $\pi(Y)\pi(g)e = \pi(g)\pi(X)e$. Thus $(\pi(g)u,\pi(Y)\pi(g)e) = (\pi,\pi(X)e) = 0$. Consider the endomorphism A_u of $\mathfrak{p}$. Then

$$(A_u X, Y) = (u,\pi(X)\pi(Y)e)$$

Thus $(A_{\pi(g)u} \mathrm{Ad}\, gX, \mathrm{Ad}\, g\, Y) = (u, \pi(X)\pi(Y)e)$. So

$$\det A_u = \det A_{\pi(g)u} \quad . \qquad \text{Q.E.D.}$$

So to calculate the total absolute curvature we need only calculate the absolute curvature at 0 and this leads to

<u>Theorem 3.1.</u>

$$\Sigma\, b_i(G/K, *) \leq V(G/K) \, \frac{C_{2N-n-1}}{C_{2N-1}} \, \frac{\gamma^{n/2}}{n}$$

where γ is any eigenvalue of the Laplacian and N is the complex dimension of any irreducible subspace of the eigenfunctions with eigenvalue γ and C_r = volume of sphere S^r .

<u>Proof</u>. Consider once again the situation described before Th. 1.3. We have $\pi: G \to V^N$ a class one unitary representation and an immersion $\pi: G/K \to E^{2N}$.

Let $\xi \in T_o^{\perp}$ and X and $Y \in \mathcal{P}$.

$$(A_\xi X, Y) = (\pi(X)\pi(Y)e, \xi) \quad .$$

Now A_ξ is a symmetric operator so we can choose an orthonormal basis of $\mathcal{P}$ such that A_ξ is diagonal. Let basis be $\{X_i\}$. Then

$$\text{Det } A_\xi = \prod_{i=n}^{n} (\pi(X_i)\pi(X_i)e, \xi)$$

Squaring both sides

$$|\text{Det } A_\xi|^2 = \Pi \ (\pi(X_i),\pi(X_i)e,\xi)|^2$$

$$\leq \Pi \ \|\pi(X_i)\xi\|^2 \qquad \text{by Schwartz' inequality}$$

$$\leq \left(\frac{\Sigma \ \|\pi(X_i)\xi\|^2}{n}\right)^n \qquad \text{arithmetic, geometric means}$$

$$\leq \left(\frac{\gamma(\xi,\xi)}{n}\right)^n = \left(\frac{\gamma^n}{n}\right)^n$$

Thus $$|\text{Det } A_\xi| \leq \left(\frac{\gamma}{n}\right)^{n/2}$$

So $$\tau_o(f) \leq C_{2N-n-1} \left(\frac{\gamma}{n}\right)^{n/2}$$

$$\tau(G/K,\pi,E^N) \leq V(G/K) \frac{C_{2N-n-1}}{C_{2N-1}} \left(\frac{\gamma}{n}\right)^{n/2}$$

The other inequality is merely a Morse inequality. Q.E.D.

Remark. Although this inequality is very weak it would suggest that to find a tight imbedding the optimal method would be to immerse in an irreducible subspace of the eigenfunctions of the Laplacian and procede to "eliminate unnecessary critical points." This is strengthened by the next section.

§3. Minimal Eigenvalues for R-Spaces.

We shall prove

Theorem 3.2. For many symmetric R-spaces the tight immersion is in fact an immersion in a space of eigenfunctions of the Laplacian with least eigenvalues.

(The term "many" will be clearer as we proceed.)

Something of this nature is done in [12]. But the authors seemed to have ignored to a certain extent the results of [18] especially that $\lambda \geq \frac{n+1}{2n}$ for any eigenvalue λ of the Laplacian.

Proof. We break the proof into a series of Lemmas.

First let us recall the idea of scalar curvature. For convenience we shall define the scalar curvature at a point $m \in M$ of a Riemannian manifold by

$$\rho(m) = - \sum_{i,j} (R(X_i, X_j)X_i, X_j)$$

where R is the curvature tensor and $\{X_i\}$ is an orthonormal basis of M_m.

Lemma A. Let G/K be an irreducible symmetric space of dimension n. Then

$$\rho(m) = \frac{n}{2} \quad \text{at all points.}$$

Proof. See e.g. [12].

<u>Lemma B.</u> For the isometric immersion $\pi\ G/K \to E^N$

$$(R(X,Y)Z,W) = (\pi(Y)\pi(Z)e,\pi(X)\pi(W)e)$$
$$- (\pi(X)\pi(Z)e,\pi(Y)\pi(W)e) \qquad X, Y, Z, W \in \mathfrak{p}$$

<u>Proof.</u> This is of course just the classical Gauss curvature equation but in our case has a very simple proof.

$$(R(X,Y)Z,W) = -([[X,Y],Z],W)$$
$$= -(\pi[[X,Y],Z]e,\pi(W)e)$$
$$\pi[[X,Y],Z]e = \pi(X)\pi(Y)\pi(Z)e - \pi(Y)\pi(X)\pi(Z)e$$
$$(R(X,Y)Z,W) = (\pi(Y)\pi(Z),\pi(X)\pi(W)e) - (\pi(X)\pi(Z)e,\pi(Y)\pi(W)e)$$

Q.E.D.

Now for $\mathbb{R}$ spaces we have

$$\mathcal{L} = \mathfrak{g} + E^N$$

where

$$E^N = T_o + T_o^{\perp}$$
$$T_o = \{\pi(X)e\} \qquad T_o^{\perp} = \text{lin. hull } \{\pi(X)\pi(X)e\}$$

<u>Lemma C.</u> The immersion $G/K \to E^N$ is minimal in the sphere.

<u>Proof.</u> Cf. [12].

<u>Lemma D.</u> Let $Y, X \in \mathfrak{p}$. Then

$$\text{trace } \pi(X)\pi(Y)|_{T_0} = \frac{K}{2} B(X,Y) \qquad \text{for some constant } K.$$

<u>Proof</u>. First let $\Pi(X,Y) = \text{trace } \pi(X)\pi(Y)$. But this is K invariant so $\Pi = kB$. Now by Lemmas 1.4 and 2.2 we can choose an orthonormal basis of E^N by first choosing one for T_0 then one for $T_0^{\perp}$ so that we have in matrix form

$$\pi(X) = \begin{pmatrix} 0 & A(X) \\ -A\ (X) & 0 \end{pmatrix}$$

Thus
$$\pi(X)\pi(Y) = 2 \text{ tr } A(X)A(Y)$$
$$= 2 \text{ tr } \pi(X)\pi(Y)|_{T_0}$$

Thus
$$\text{tr } \pi(X)\pi(Y)|_{T_0} = \frac{k}{2} B \quad . \qquad \text{Q.E.D.}$$

If we now let ξ_0 be mean curvature normal at 0

$$\xi_0 = \sum_i \pi(X_i)\pi(X_i)e \quad .$$

<u>Lemma E</u>.

$$\rho(0) = (\xi_0, \xi_0) - \frac{Kn}{2} \quad .$$

Proof.

$$\rho(0) = \sum_{i,j} R((X_i,X_j)X_i,X_j)$$

$$= -\sum_{i,j} (\pi(X_j)\pi(X_i)e,\pi(X_j)\pi(X_i)e)$$

$$+\sum_{i,j} (\pi(X_i)\pi(X_i)e,\pi(X_j)\pi(X_j)e)$$

$$\sum_{i,j} (\pi(X_i)\pi(X_i)e,\pi(X_j)\pi(X_j)e) = (\xi_o,\xi_o) .$$

Now consider

$$(\pi(X_i)\pi(X_j)e,\pi(X_i)\pi(X_j)e) = - (\pi(X_i)\pi(X_i)\pi(X_j)e,\pi(X_j)e$$

but $\{\pi(X_j)e\}$ form orthonormal basis for T_o. Thus

$$\sum_{ij} (\pi(X_i)\pi(X_j)e,\pi(X_i)\pi(X_j)e) = \frac{Kn}{2} .$$

Lemma F. $(\xi_o,\xi_o) = n\lambda$ where $-\lambda$ is an eigenvalue of the Laplacian.

Proof. By Lemma C $\varphi_e(gK) = (\pi(g)e,e)$ is an eigenfunction of the Laplacian and by Takahashi's Result cited in Chapter I

$$\Delta\varphi_e = -\frac{n}{(e,e)}\varphi_e = -\lambda\varphi_e \qquad \text{say}$$

$$(e,e) = n/\lambda$$

But $$\Delta\varphi_e(0) = \Sigma\ (\pi(X_i)\pi(X_i)e,e)$$
$$= (\xi_o,e)$$

So $$(\xi_o,e) = -\ \lambda(e,e)$$
$$\xi_o = -\ \lambda e$$
$$(\xi_o,\xi_o) = \lambda^2(e,e)$$
$$= n\lambda\ .$$ Q.E.D.

Lemma G.

$$\lambda = \frac{K+1}{2}\ .$$

Proof. Direct from Lemmas A, E and F. Q.E.D.

But K can be calculated easily for many R spaces. We shall do examples.

$B_{\mathcal{L}}$ is G invariant so $B_{\mathcal{L}} = c\ B$. But

$$B_{\mathcal{L}}(X,Y) = B\ \ (X,Y) + \mathrm{tr}\ \pi(X)\pi(Y)$$
$$= (1+k)B\ .$$

The list can be found in [12] where credit is given to [8].

1. Hermitian Symmetric Spaces.

By [9] the immersion is given as follows. $E^N \simeq \mathcal{J}$ and π is equivalent to Ad_G . Thus $\mathrm{tr}\ \pi = \mathrm{tr\ ad}\ X\ \mathrm{ad}\ Y$

$X, Y \in \mathfrak{p}$. Thus $k = 1$ and $\lambda = 1$ which is the minimum value for eigenvalues of the Laplacian.

2. Sphere.

Let $f: G/K \to F_\gamma$ be the immersion constructed in Th. 1.3. We can project onto a sphere radius 1 changing the metric by a factor of $\frac{\gamma}{n}$ hence multiplying scalar curvature by $\frac{n}{\gamma}$. The property of minimality is unaltered so by a result of Simons [20]

$$\rho(m) = \frac{nc}{\gamma} \leq n(n-1)$$

with equality if and only if G/K is the sphere immersed in standard way

but $c = \frac{n}{2}$

$$\therefore \quad \gamma \geq \frac{n}{2(n-1)}$$

This can also be deduced from [18].

Thus for sphere immersed in standard way

$$\gamma = \frac{n}{2(n-1)}$$

3.

$$G/K = SO(2n)/U(n)$$

$$\mathcal{L} = SO(2n, C)$$

$$E^N = SO(2n)$$

$$1+k = 2$$

$\lambda = 1$ which is minimal by [18].

4.
$$G/K = Sp(p+q)/Sp(p) \times Sp(q)$$

$$\mathcal{L} = SU^*(2(p+q))$$

$$1+k = \frac{2(p+q)}{p+q+1}$$

$$\lambda = \frac{p+q}{p+q+1}$$ minimal by [18]. Q.E.D.

BIBLIOGRAPHY

1. Chern, S-S and Kuiper, N. Some theorems on the isometric imbeddings of compact Riemannian manifolds in Euclidean space, Annals of Math. 56(1952), 422-430.

2. Chern, S-S and Lashof, R. K. On the total curvature of immersed manifolds, I. Amer. J. Math. 79(1957), 306-318.

3. ________ ________ On the total curvature of immersed manifolds, II. Mich. Math. J. 5(1958), 5-12.

4. Do Carmo, M and Wallach, N. Representations of compact groups and minimal immersions into spheres, J. Diff. Geom. (to appear).

5. Do Carmo, M and Wallach, N. Sphere bundles over spheres (to appear).

6. Frankel, T. T. Critical submanifolds of classical groups and Stiefel manifolds, Symposium on Differential and Combinatorial Topology in Honor of M. Morse, Princeton University Press, 1965, 37-53.

7. Helgason, S. Differential Geometry and Symmetric Spaces, Academic Press, New York 1962.

8. Iwahori, N. Theory of Lie Groups, II (Japanese) Iwanami 1957.

9. Kobayashi, S. Imbeddings of homogeneous spaces with minimum total curvature, Tohoku Math. J. 19(1967), 63-74.

10. Kobayashi, S. and Nagano, T. On filtered Lie algebras and geometric structures, I. J. Math. Mech. 13(1964), 875-908.

11. Kobayashi S. and Nomizu, K. Foundations of Differential Geometry, Vol. II, Interscience Tract 15, 1969.

12. Takeuchi, M. and Kobayashi, S. Minimal imbeddings of R-spaces, J. of Diff. Geom., Vol. 2 1968, 203-215.

13. Kuiper, N. Immersions with minimal total absolute curvature, Colloque de Geometrie Differentielle Globale, Bruxelles 1958, 75-88.

14. Kuiper, N. Immersions with minimal absolute total curvature, Mimeographed Notes 1970.

15. Milnor, J. W. Morse theory, Ann. of Math. Studies No. 51, Princeton University Press 1963.

16. Morse, M. The existence of polar non-degenerate functions on differentiable manifolds. Ann. of Math. 71(1959), 352-383.

17. Nagano, T. Transformation groups on compact symmetric spaces, Trans. Amer. Math. Soc. 118(1965), 428-453.

18. Nagano, T. On the minimum eigenvalues of the Laplacians in Riemannian manifolds, Sci. Papers Coll. Gen. Ed., University of Tokyo 11(1961), 177-182.

19. Otsuki, T. On the existence of solutions of a system of quadratic equations and its geometrical application, Proc. Japan Acad. 29(1953), 99-100.

20. Simons, J. Minimal varieties in Riemannian manifolds, Ann. of Math. 88(1968), 62-105.

21. Takahashi, T. Minimal immersions of Riemannian manifolds, J. Math. Soc. of Japan 18(1966), 380-385.

22. Wilson, J. P. Some minimal imbeddings of homogeneous spaces, J. London Math. Soc. Second Series V I , Part 2 (1969), 335-340.

23. Jacobson, N. Lie algebras, Interscience Tracts No. 10.

24. Chern, S. S. Minimal submanifolds in a Riemannian manifold. Technical Report 19. University of Kansas, Lawrence, Kansas 1968.

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