DETERMINATION OF DIRECTIONAL SPECTRA

OF SEA SURFACE WAVE FIELD

BY SCANNING OBSERVATION

by

Piotr Koziol
Warsaw University, Poland
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Certified by...
Thesis Supervisor

Accepted by..
Chairman, Departmental Committee on
Graduate Students

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ABSTRACT

This paper considers the possibility of determining the directional energy spectrum of ocean surface gravity waves from a set of one-dimensional spectra. The one-dimensional spectra are in Doppler shifted frequency domain and they are obtained from the signal given by towing a measuring device in different directions across a wave field. An attempt to solve the integral equation involved approximating it by a set of simultaneous linear algebraic equations led to a singular matrix.

Thesis Supervisor: Erik Mollo-Christensen
Title: Professor of Meteorology

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1. INTRODUCTION

The knowledge about the directional energy spectrum of a wave field is of interest in many applied sciences. In the case of ocean surface gravity waves, one of the possible methods is measurements of a sea surface elevation at a set of points on the sea and then (assuming stationarity and homogeneity of the wave field), using the time correlation analysis of data, one can get some information about the directional spectrum. The results depend strongly on the spatial distribution of wave detectors. One can optimize their relative positions with respect to the studied wave phenomenon. Still, the directional resolving power of the optimum array is limited. In this paper, a possibility of determining the directional spectrum from a set of one-dimensional spectra (as opposed to the co-spectra method mentioned above) is considered.

One can avoid the difficulties caused by measurements at discrete points or at discrete times by a continuous observation of water elevation in time and in space. A measuring device in this case moves with a known velocity across the wave field. As a result, a one-dimensional spectrum in a Doppler shifted frequency domain is obtained. This time waves with different wavenumbers, frequencies and directions of propagation contribute to a value of energy corresponding to a given Doppler shifted frequency. By towing the measuring device in different directions and with several values of velocity, one can collect enough information to deduce the directional distribution of energy.

2. THEORY

The directional energy spectrum is, in general, a third order density:

$$\Phi\,(\vec{k},\sigma)$$

where $\vec{k}$ is the wavenumber vector and σ is the frequency. If the measuring device moves with a velocity $\vec{U}$, the signal obtained has the following spectrum:

$$\phi = \iiint_{S_{\vec{U}}} \Phi\,(\vec{k},\sigma)\, d\vec{k}\, d\sigma \qquad (\sigma > 0)$$

where $S_{\vec{U}}$ is the surface consisting of the points $(\vec{k},\sigma)$ that give the same value of a Doppler shifted frequency, the argument of the left-hand side. Because we cannot distinguish between positive and negative values of frequency, the equation of this surface is:

$$S_{\vec{U}} \equiv |\sigma_D| - |\sigma - \vec{U}\cdot\vec{k}| = 0$$

and

$$\phi = \phi\,(\,|\sigma_D| = \text{const}\,)$$

is the measured one-dimensional spectrum.

If we now assume a dispersion relationship between $\vec{k}$ and σ, the equation of the corresponding surface, valid for deep water gravity waves, is:

$$S \equiv \sigma^2 - g\,|\vec{k}| = 0$$

It implies that

$$\Phi(\vec{k}, \sigma) = \Psi(\vec{k})\,\delta(\sigma - \sigma(|\vec{k}|))$$

i.e. the energy lies on the dispersion relationship surface S and $\Psi(\vec{k})$ is the directional spectrum we are looking for.

The measured spectrum becomes:

$$\phi(|\sigma_D|) = \iint_{S_{\vec{U}} \cap S} \Psi(\vec{k})\, d\vec{k} \qquad (2.1)$$

where $S_{\vec{U}} \cap S$ is a projection of an intersection of the surfaces $S_{\vec{U}}$ and S on the $\vec{k}$ plane (Figs. 2A, 2B). Therefore, $\phi(|\sigma_D|)$ is now equal to a second order density $\Psi(\vec{k})$ integrated over a line $S_{\vec{U}} \cap S$.

In order to solve the integral equation (2.1) for Ψ, we can try approximating it with a set of simultaneous equations, i.e. we have to change the description of the problem from a continuous to a discrete one. In this approach, the data $\phi(|\sigma_D|)$ becomes the right-hand side

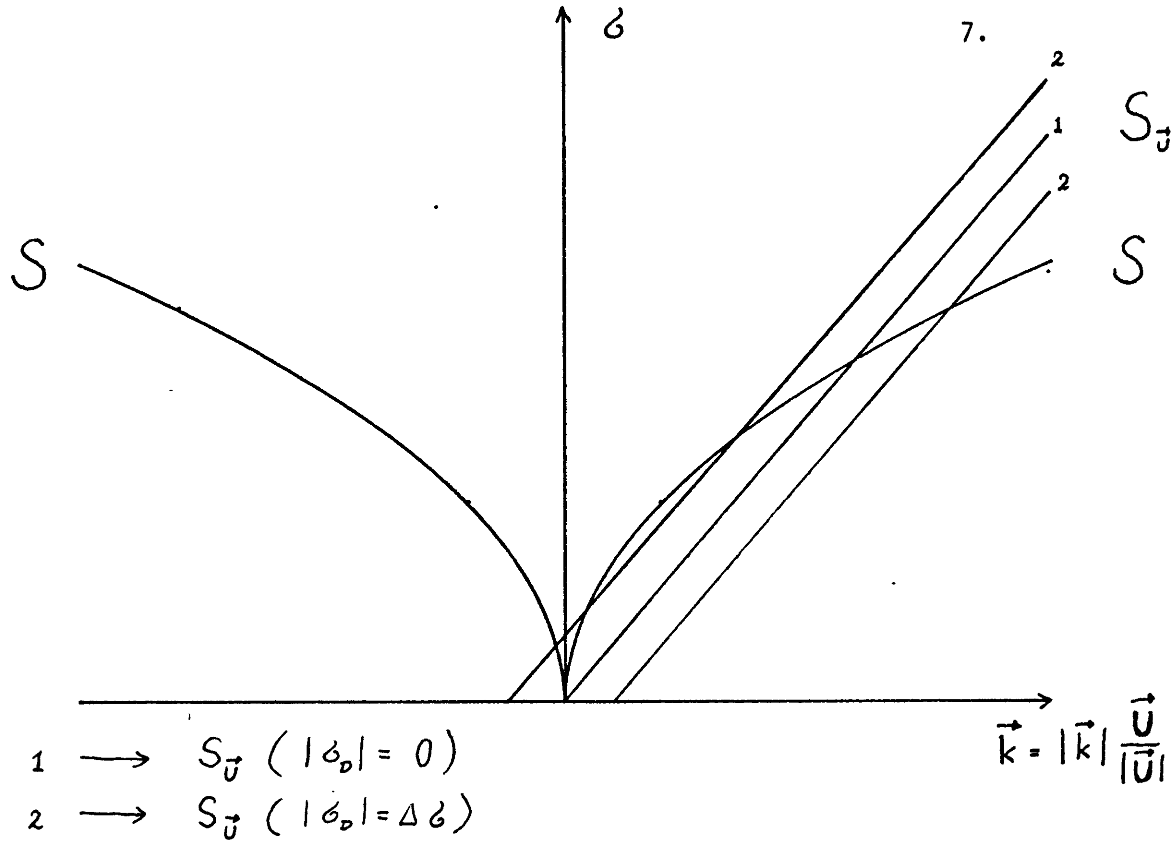

Fig. 2A Surfaces $S_{\vec{u}}$ and S

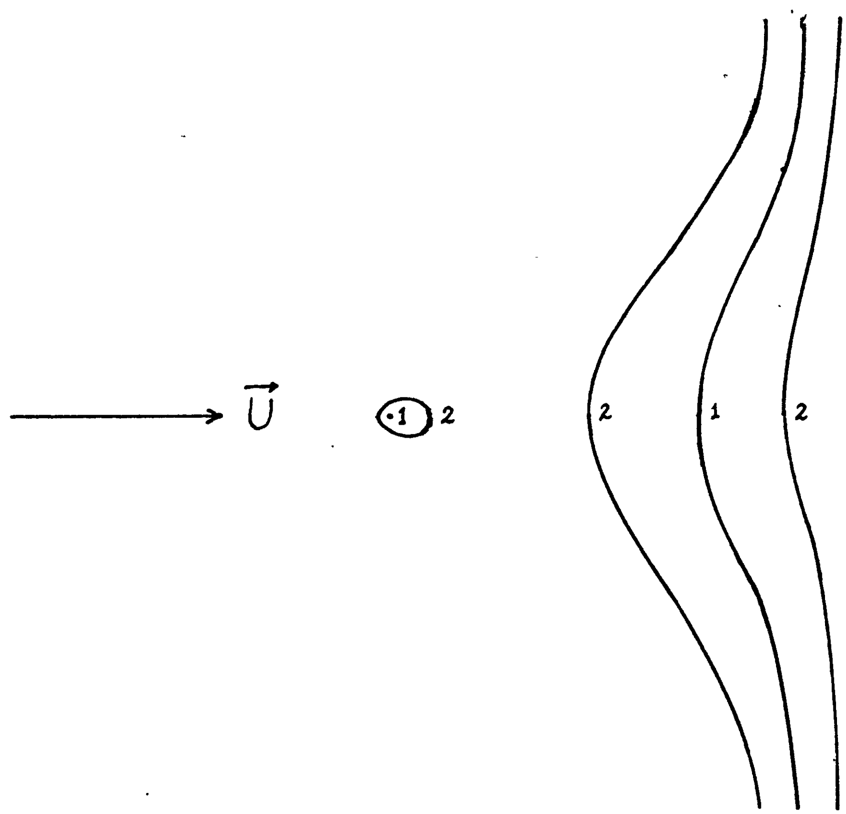

Fig. 2B Projection of $S_{\vec{u}} \cap S$ on $\vec{k}$ plane vector, the distribution of energy Ψ becomes the unknown vector and the integral operator

$$\iint_{S_{\vec{u}} \cap S} d\vec{k}$$

becomes the matrix that is determined by the geometry of the problem, i.e. by surfaces $S_{\vec{u}}$ and S.

It is now convenient to change the coordinates from cartesian to polar

$$\vec{k} \longrightarrow (k, \gamma)$$

In the discrete description, we have to specify a size of $\vec{k}$ space; in other words, we expect $\Psi(\vec{k}) \neq 0$ for $|\vec{k}| < K$. Then the maximum value of Doppler shifted frequency can be related to K by:

$$|\sigma_D|_{max} = \sqrt{gK} + |\vec{U}|K$$

Also, the data ϕ must be smoothed over some interval $\Delta\sigma$, and then the right-hand side values should be taken at points $|\sigma_D|$ = integer $\cdot\ \Delta\sigma$.

Concommitantly, one has to replace the densities of n^{th} order by the products:

$$\text{density}\,(\vec{x} = \vec{x}_0) \cdot \prod_{i=1}^{n} \Delta x_i$$

Let us determine $\Delta\sigma$ as:

$$\Delta\left(|\sigma_D|_{max}\right) = |\Delta\vec{U}|K$$

where $|\Delta\vec{U}|$ is the accuracy within which we can measure and maintain the velocity of the device.

Having specified the size of $\vec{k}$ space in which we are looking for energy distribution, we should fill it in with points. Their number has to be smaller than the amount of the right-hand side values, which for one graph $\phi(|\sigma_D|)$ is given by:

$$\frac{|\sigma_D|_{max}}{\Delta\sigma} + 1 \equiv n + 1$$

If one plans to use polar coordinates, the angle interval $\Delta\gamma$ should be larger than the accuracy within which one can determine the direction the device moves:

$$\Delta\gamma > |\Delta\alpha|$$

Now the values of $\Delta\gamma$, K and $|\sigma_D|_{max}$ specify the size of Δk which is the wavenumber interval. Δk should satisfy:

$$\left(\frac{2\pi}{\Delta\gamma}\right)\left(\frac{K}{\Delta k}\right) < \left(n+1\right)\left(\begin{array}{c}\text{number of}\\ \text{different } \vec{U}\end{array}\right)$$

Now there is a possibility to check the orders of magnitude of all the parameters by comparing $\Delta\sigma$ with:

$$\Delta\left(\sqrt{gk}\right)_{k=K} = \sqrt{\frac{g}{K}}\;\frac{\Delta k}{2}$$

The next step is to write down the set of simultaneous equations that corresponds to the integral equation (2.1). At first, one has to index all the points in $\vec{k}$ space, i.e. to a given pair

$$(k,\gamma) = (m\,\Delta k\,,\, m_\gamma\,\Delta\gamma) \qquad (m, m_\gamma - \text{integers})$$

we relate a subscript ℓ, so the values of energy

$$Z_\ell \equiv \Psi\left(k = m(\ell)\Delta k\,,\, \gamma = m_\gamma(\ell)\,\Delta\gamma\right)\Delta k\,\Delta\gamma$$

consist of the unknown vector (dimension $[Z_\ell] = cm^2$). Each value from the graphs $\phi(|\sigma_D|)$ becomes a component of the right-hand side vector. ℓj^{th} value ($1 \le \ell^j \le n+1$) on the j^{th} graph (corresponding to velocity $\vec{U}_j$) has a subscript

$$p = (n+1)(j-1) + \ell^j$$

and we write it as V_p:

$$V_p \equiv \phi\left(|\sigma_D| = (\ell^j - 1)\Delta\sigma\right)\Delta\sigma$$

(dimension $[V_p] = cm^2$).

The amount of experiments (number of different U) should assure the condition

$$p_{max} \geq \ell_{max}$$

and we will use the first ℓ_{max} of the P_{max} equations. The p^{th} equation is of the form

$$\sum_{\ell=1}^{\ell_{max}} A_{p\ell} Z_\ell = V_p \tag{2.2}$$

where $1 \leq p \leq \ell_{max}$ and the matrix element is equal to zero if a given point in $\vec{k}$ space does not contribute to the right-hand side value and is equal to one if it does. We define functions:

$$\sigma_+ \equiv +(\ell^j - 1)\Delta\sigma + U\, m(\ell)\, \Delta k \cos(\alpha_j - m_\gamma(\ell)\Delta\gamma)$$

$$\sigma_- \equiv -(\ell^j - 1)\Delta\sigma + U\, m(\ell)\, \Delta k \cos(\alpha_j - m_\gamma(\ell)\Delta\gamma)$$

$$\sigma \equiv \sqrt{g\, m(\ell)\, \Delta k}$$

where (Fig. 2C) :

$$|\vec{U}_j| \equiv U$$

$$|\vec{k}| = m(\ell)\, \Delta k$$

$$\gamma = m_\gamma(\ell)\, \Delta\gamma$$

and j = integer part $\left(\frac{p-1}{n+1}\right) + 1$

$\ell^j = p - (n+1)(j-1)$

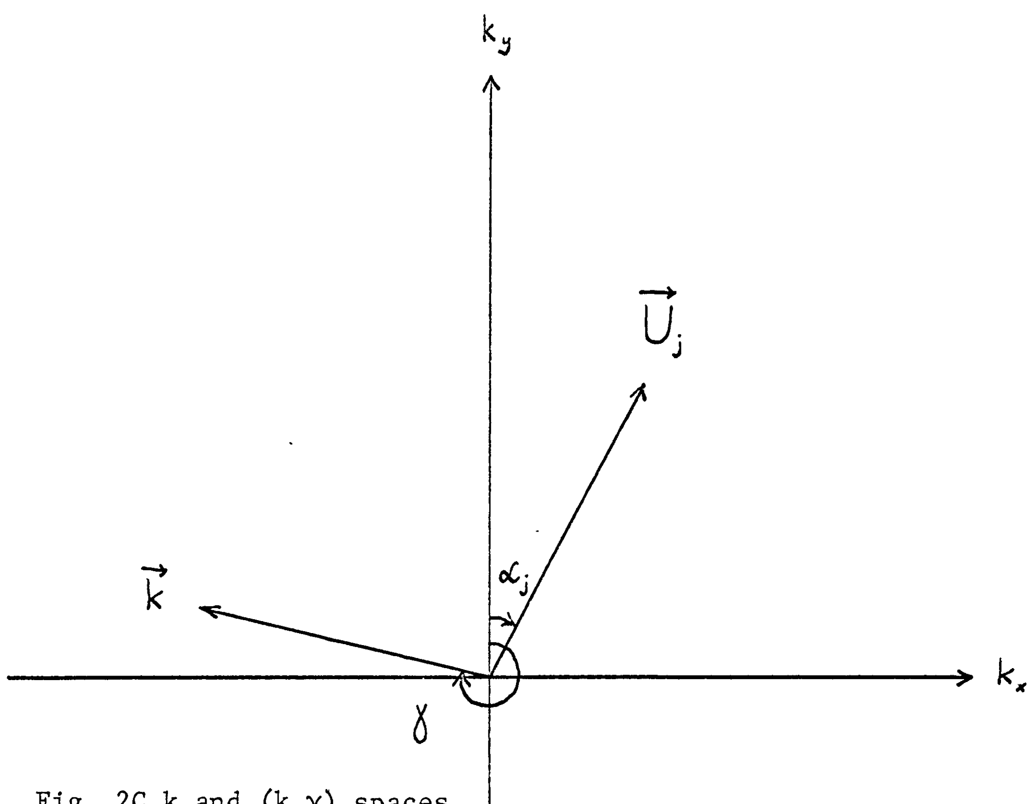

Fig. 2C k and (k,γ) spaces

Then we find the matrix as follows:

$$
A_{pl} = \begin{cases} = 1 & \text{if} \quad |\delta(l) - \delta_+(p,l)| < \dfrac{\Delta\delta}{2} \\ & \text{or} \quad |\delta(l) - \delta_-(p,l)| < \dfrac{\Delta\delta}{2} \\ = 0 & \text{elsewhere} \end{cases}
$$

3. EXPERIMENT AND INSTRUMENTS

A. Data Collection

The experiment took place at Quincy Bay on the sixth of April 1972 at six p.m. The wind was blowing from the South and its velocity was seven knots.

The experiment consisted of towing a wavegauge from a two-meter bowsprit mounted on the boat. The magnitude of ship velocity was kept constant (800 R.P.M. = 228 cm sec^{-1}) and data were recorded for different directions of scanning. An FM signal from the wavegauge was recorded on a closed-loop (time of one loop = 1 min 53 sec) by an eight channel tape recorder. A different channel was used to record the data from each direction of scanning.

The process of recording of one tape took less than twenty minutes. One had to select the optimum compromise between the length of a closed-loop (the longer it is, the better the quality of spectrum obtained) and the total time of the experiment. (It should not be too long if the data from different directions are to represent the same wave field).

B. Data Analysis

From the recorded signal, we want to obtain the energy density averaged over a frequency interval $\Delta\sigma$ (dimension = $\frac{cm^2}{Hz}$). The FM signal was converted to a voltage signal which was played back one hundred times faster than it was recorded. The signal was next passed through a QUAN-TECH wave analyzer, which gave as an output amplitude versus frequency. The parameters of the QUAN-TECH wave analyzer were as follows: Sweep width = SW = 5 kHz

Band Width = BW = 10 Hz

Averaging time = TC = 10 sec

The Sweep time = ST = 1800 sec was chosen to be much larger than the time of one closed-loop cycle = $\frac{1 \text{ min } 53 \text{ sec}}{100}$ in order to obtain a consistent spectrum. Additionally, all these parameters had to satisfy the relationship required by the properties of the wave analyzer:

$$\frac{1}{\sqrt{8 \cdot BW \cdot TC}} < \text{error} \quad , \quad ST > \frac{4\,SW}{(BW)^2}$$

where error was assumed to be = 2%.

The spectra obtained in this way have a long "tail" (that comes from a circuit noise) of an almost constant level. Having subtracted the noise level, one can determine the order of magnitude of $|\sigma_D|_{max}$ which is $\gtrsim 2\pi \cdot 30$ Hz. We solve next for K to satisfy:

$$|\sigma_D|_{max} \sim \sqrt{gK} + UK$$

and for U = 228 cm sec^{-1} one finds K $\simeq .75 \text{cm}^{-1}$. Assuming 3% accuracy in determining the value of U, one computes $\Delta\sigma \sim \Delta|\sigma_D|_{max} = \Delta U \cdot K \sim 2\pi \cdot .75$ Hz. The next step is to smooth the graphs over the chosen value of the interval $\Delta\sigma$. We need the values of energy at a discrete set of points $|\sigma_D|$ = integer $\cdot$ $\Delta\sigma$. One graph gives us

$$\frac{|\sigma_D|_{max}}{\Delta\sigma} + 1 = n + 1 \simeq 43 \text{ values}$$

so now we define

$$|\sigma_D|_{max} = 42 \cdot \Delta\sigma = 2\pi \cdot 31.5 \text{ Hz}$$

The wave analyzer output is the amplitude and we need the values of energy, so one has to square the values of amplitude

$$\eta_j \left(|\sigma_D| = (\ell^j - 1)\Delta\sigma \right)$$

for $\ell^j = 1, \ldots\ldots, n+1$ to obtain

$$\phi_j \left(|\sigma_D| \right) = \eta_j^2$$

j corresponds to one velocity $\vec{U}_j$ of the ship. The value of ϕ_j at $|\sigma_D| = 0$ is smoothed over $\Delta\sigma/2$ and values for $|\sigma_D| > 0$ are smoothed over an interval $\Delta\sigma$. Because all the graphs represent the same wave field, the energy content should be the same for all of them, i.e.:

$$\frac{\phi_j(|\sigma_D| = 0)}{2} + \sum_{\ell^j = 2}^{n+1} \phi_j \left(|\sigma_D| = (\ell^j - 1)\Delta\sigma \right) = \text{const (independent of j)}$$

This constant can be found by considering the spectra of the FM signal that are very similar to a normal distribution. Measuring the width at h/2 (Fig. 3A), one can determine the square root of the variance of the FM signal = $\frac{\overline{x}}{\sqrt{2 \ln 2}} \simeq 25.84$ Hz (where $\overline{x}$ is the average over all the experiments).

From the calibration of the wave gauge,

$$\eta = .36\,\nu + \text{const}$$

(where η - elevation in cm

ν - FM frequency in Hz)

one can then specify the energy content in the wave field:

$$\overline{\eta^2} = 86.5\ \text{cm}^2$$

(mean square amplitude $\sqrt{\overline{\eta^2}} = 9.3$ cm)

Now we are able to scale the graphs:

$$\frac{\phi_j(|\sigma_D| = 0)}{2} + \sum_{l^j=2}^{43} \phi_j\left(|\sigma_D| = (l^j - 1)\Delta\sigma\right) = \text{const} = \frac{86.5\ \text{cm}^2}{\Delta\sigma}$$

Scaled values are represented by Figs. 3B (1-6).

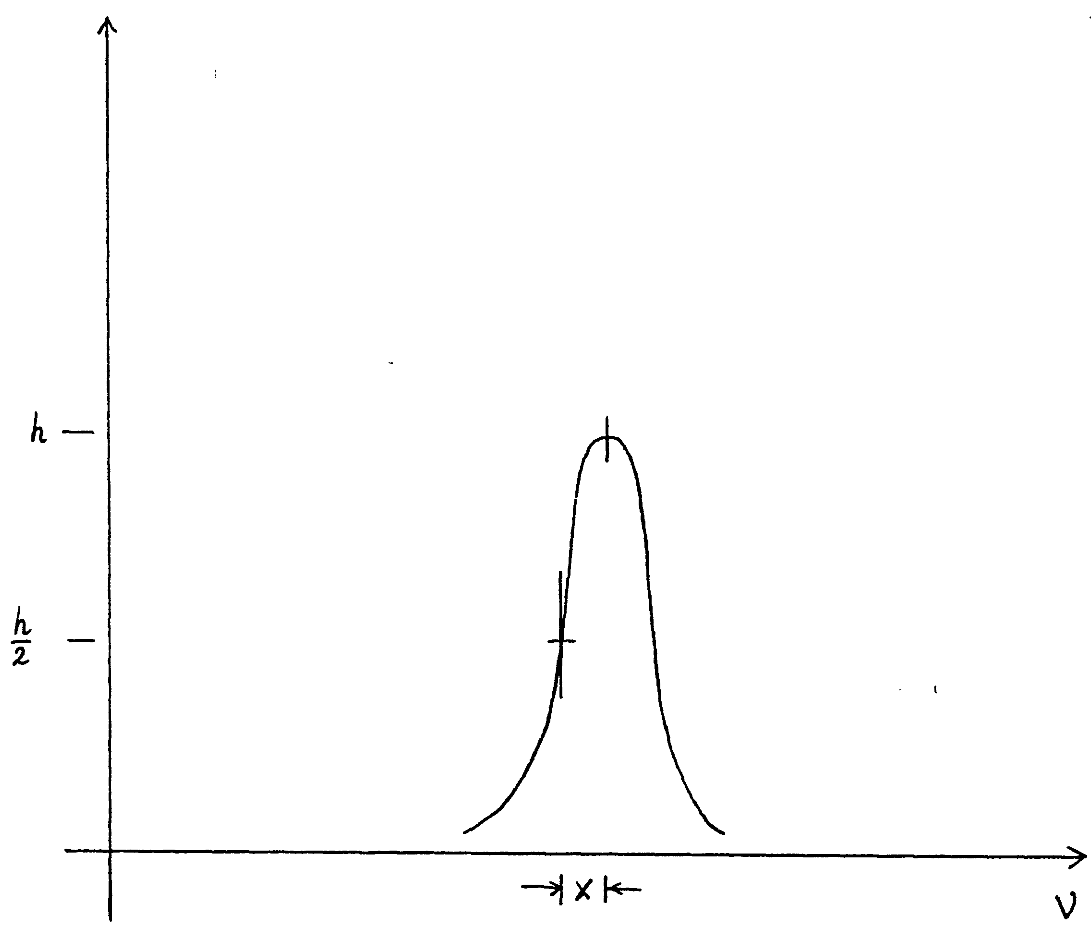

Fig. 3A Spectrum of the FM signal

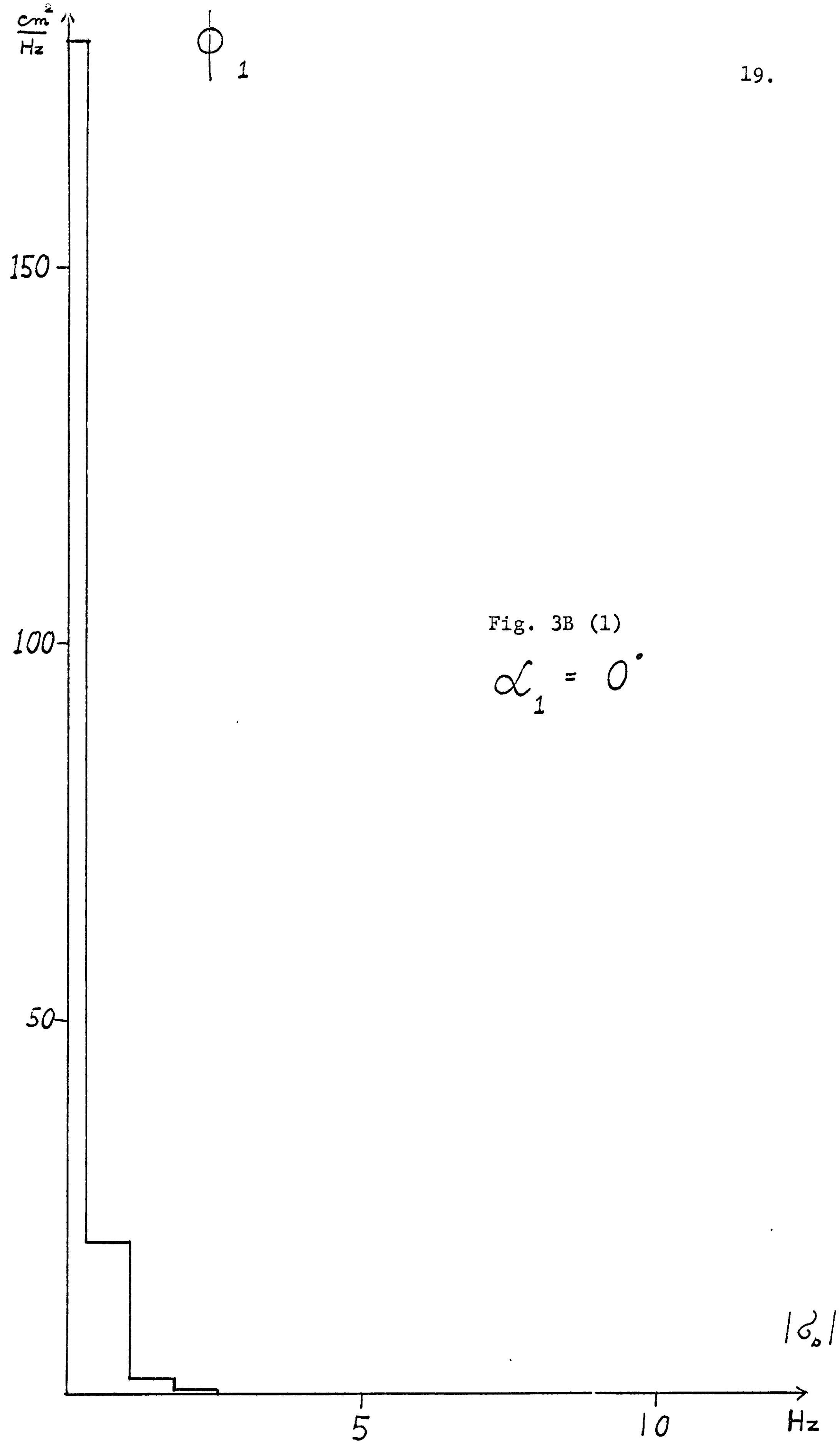

Fig. 3B (1)

$\alpha_1 = 0^\circ$

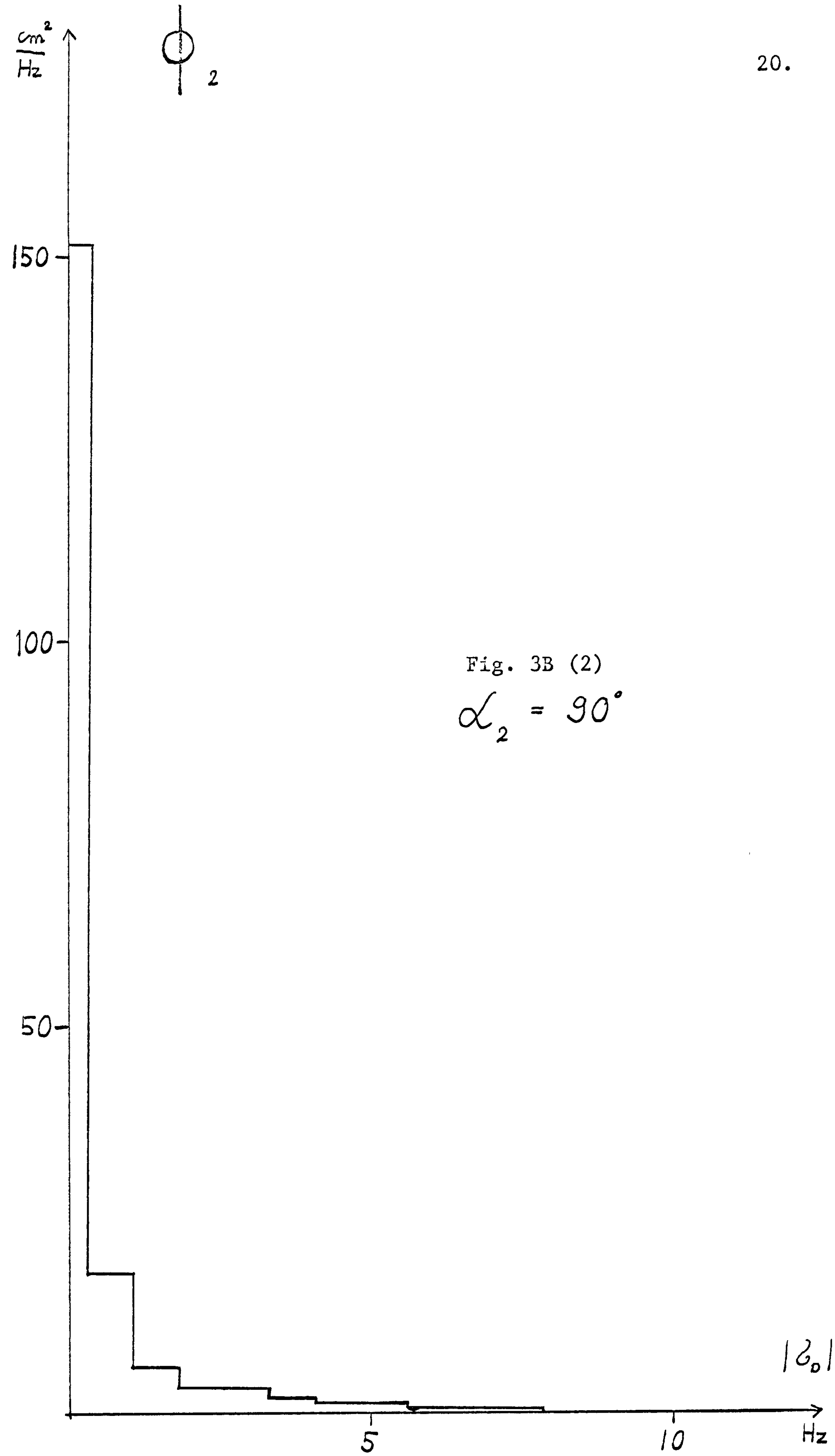

Fig. 3B (2)

$\alpha_2 = 90^\circ$

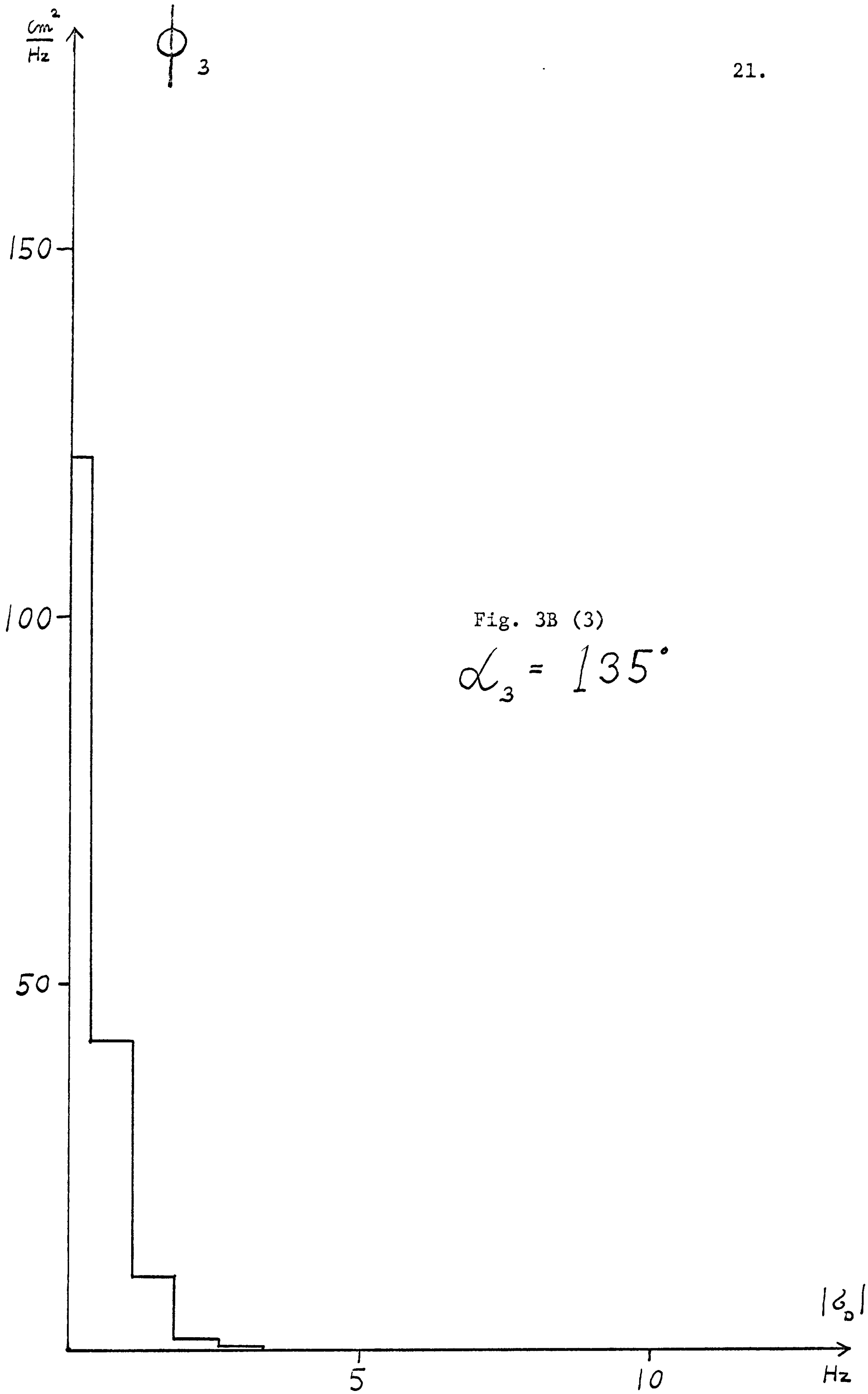

Fig. 3B (3)

$\alpha_3 = 135^\circ$

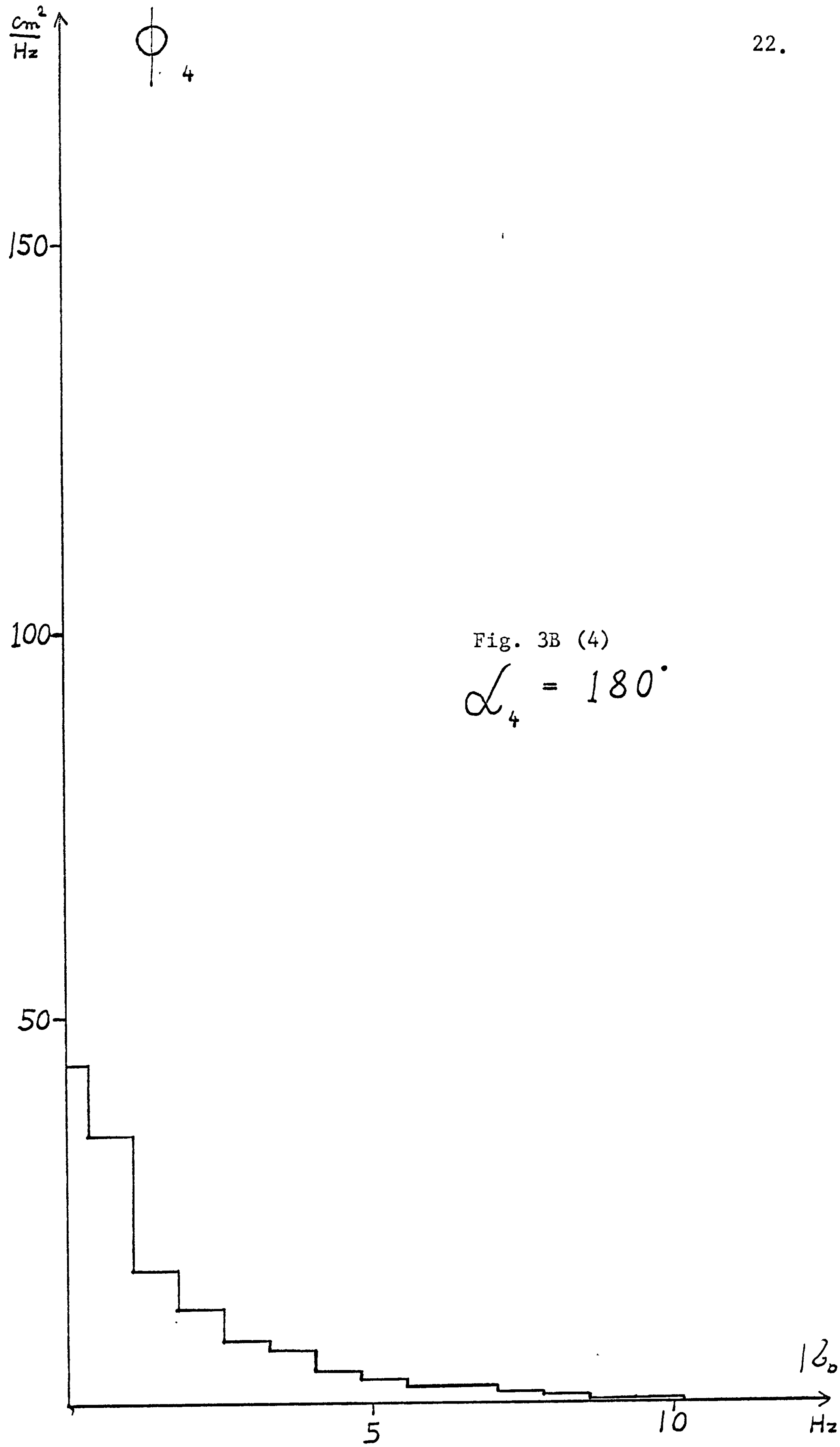

Fig. 3B (4)

$\alpha_4 = 180^\circ$

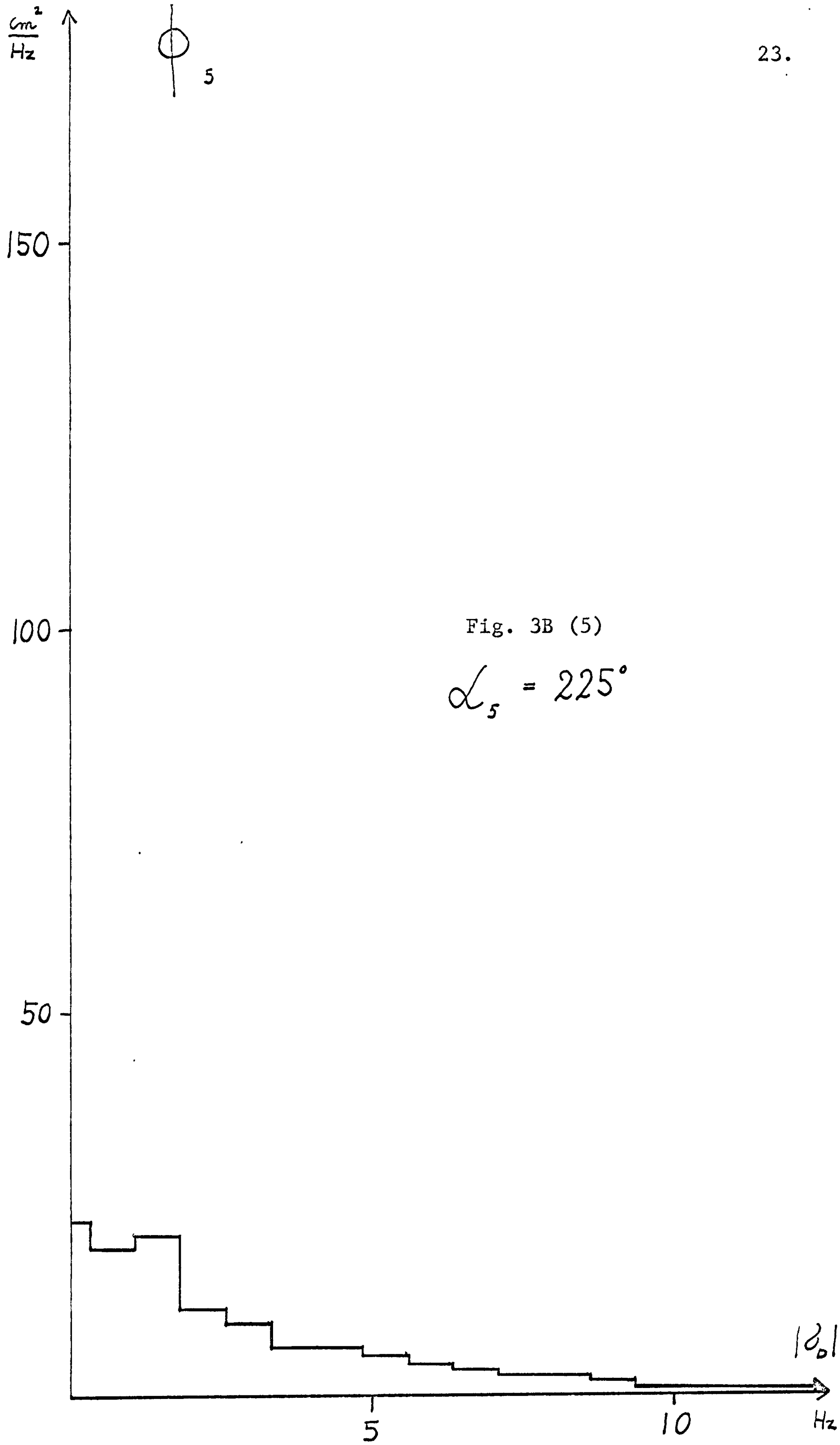

Fig. 3B (5)

$\alpha_5 = 225°$

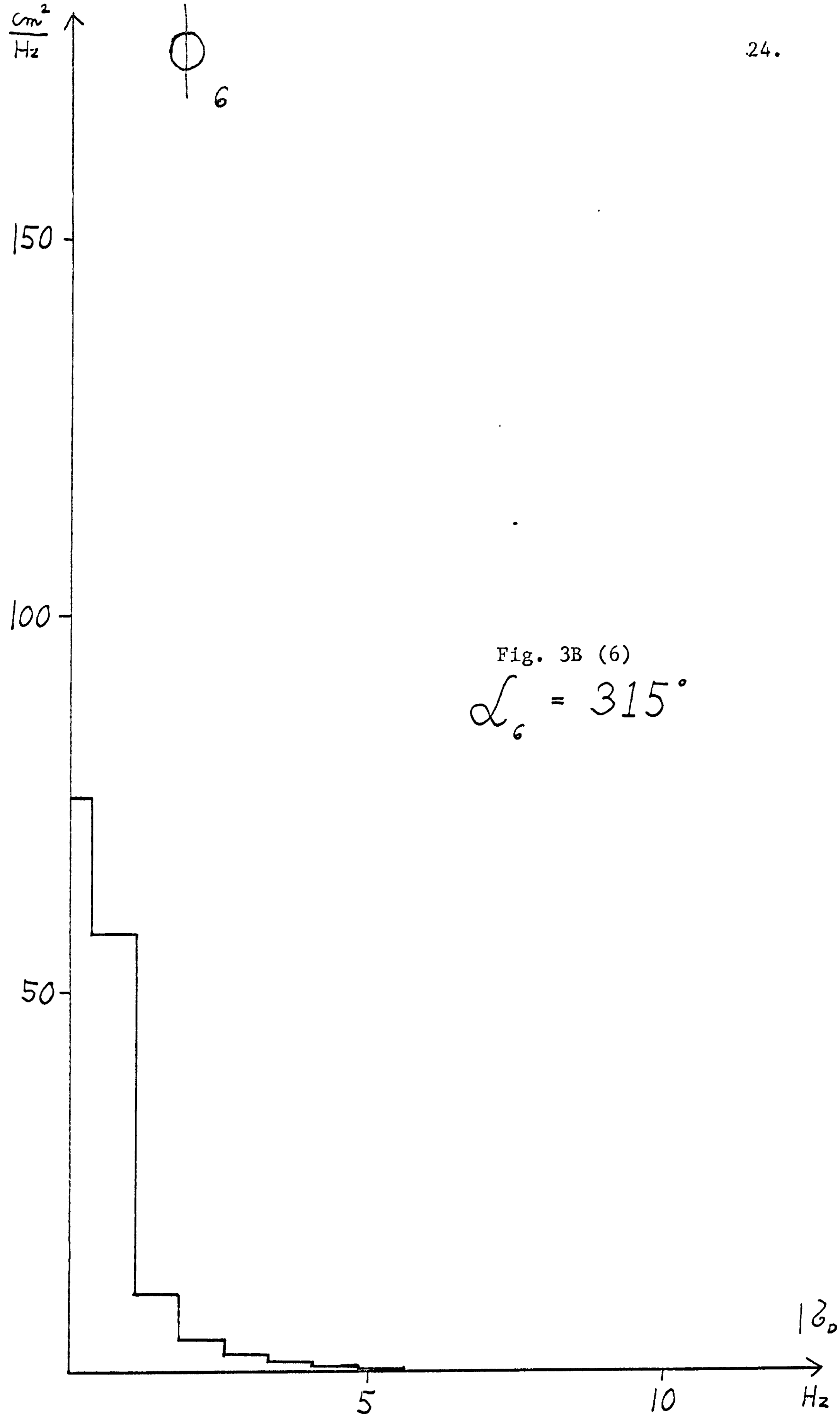

Fig. 3B (6)

$\alpha_6 = 315°$

They give the right-hand side values V_p for the set of equations (2.2). In order to find the elements of the matrix $A_{p\ell}$, one has to pick a value for Δk. The accuracy in determining and maintaining the direction of ship velocity was assumed to be $|\Delta\alpha_j| \simeq 4°$ $\simeq .07$ radian, so $\Delta\gamma$ has to be larger than 4°. Let us take for $\Delta\gamma$ a value 22.5°, i.e. we have in the discrete space (k,γ) 16 directions. Now Δk can be specified by considering the inequality:

$$\left(\frac{360°}{\Delta\gamma}\right)\left(\frac{K}{\Delta k}\right) < \left(n+1\right)\left(\begin{array}{l}\text{number of}\\ \text{experiments}\end{array}\right)$$

So we should have:

$$\Delta k > \frac{16}{43 \cdot 6} K$$

Let us take $\Delta k = .05\ \text{cm}^{-1}$, so we have 15 + 1 different values of the magnitude of the wavenumber vector (.0, .05,...,.75). The total number of points in (k,γ) space is now equal to:

$$\ell_{max} = 16 \cdot 15 + 1 = 241$$

The matrix of the coefficients of the set of simultaneous equations was found by a computer. A column ℓ of the matrix $A_{p\ell}$ corresponds to a point in (k,γ) space and row p corresponds to the ℓj^{th} value of ϕ on the j^{th} graph. For a given pair of integers p,ℓ $(1 \leq p,\ell \leq \ell_{max} = 241)$ the computer checks the values of differences:

$$|\sigma(\ell) - \sigma_+(p,\ell)|$$

and
$$|\sigma(\ell) - \sigma_-(p,\ell)|$$

and if at least one of them is less than $\Delta\sigma/2$, an adequate matrix element $A_{p\ell}$ is assigned a value 1; otherwise the value of $A_{p\ell}$ is 0. This method is equivalent to expressing the lines $S_{\vec{u}} \cap S$ on $\vec{k}$ plane by using the discrete points (k,γ); the matrix element is 1 if a given point belongs to a set in the discrete (k,γ) space that represents the $\bar{r}^{th}$ line $S_{\vec{u}} \cap S$ (it means the line corresponding to the j^{th} value of velocity and $|\sigma_D| = (\ell^j - 1)\Delta\sigma$) and if it does not belong, $A_{p\ell} = 0$.

4. RESULTS AND DISCUSSION

The matrix $A_{p\ell}$ was found to be singular. Let us consider once more the way the matrix was constructed. Two adjacent sets of lines, $S_{\vec{U}_j} \cap S$, (where:

$$S_{\vec{U}_j} \equiv |\sigma_D| - |\sigma - \vec{U}_j \cdot \vec{k}| = 0$$

and

$$|\sigma_D| = (\ell^j - 1)\Delta\sigma \quad)$$

correspond to two consecutive values of ℓ^j. This means that the representation of the line $S_{\vec{U}_j} \cap S$ in the discrete (k,γ) space is for ℓ^j = integer given by all the points which are in the area between the lines $\ell^j - 1/2$, $\ell^j + 1/2$. These are the dotted lines on Fig. 4A (little arrows show the directions corresponding to the increasing values of ℓ^j). It implies that for a chosen velocity $\vec{U}_j$, each point in (k,γ) space is used for such a representation exactly once. One point corresponds to one column of the matrix. Consequently, if we add all the rows that are computed for the same $\vec{U}_j$ (for a given $\vec{U}_j$ there are 43 of them) we will get a row that consists entirely of ones. The same can be obtained by adding rows corresponding to any other velocity $\vec{U}_j$. This is one reason why the matrix is singular.

There is another cause of singularity. For a given distribution of points in (k,γ) space, one can end up with rows which consist only of zeros. It happens with the $\ell^{j\,th}$ row when the area between the lines $\ell^j - 1/2$ and $\ell^j + 1/2$ (the dotted lines on Fig. 4A) does not contain any of the points.

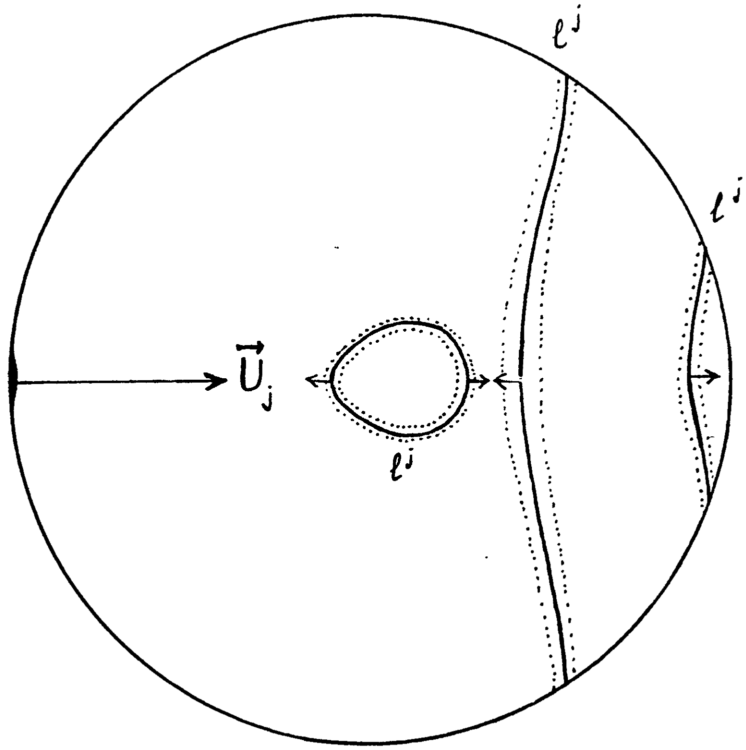

Fig. 4A $\ell^{j^{th}}$ projection of $S_{\vec{U}} \cap S$ on $\vec{k}$ plane

5. SUGGESTIONS

Let us look once more at the equation (2.1):

$$\iint\limits_{S_{\vec{U}} \cap S} \Psi(\vec{k})\, d\vec{k} = \phi(|\sigma_D|)$$

What we can get from an experiment are the values of $\phi(|\sigma_D|)$. The accuracy within which one can determine the velocity of the ship $\vec{U}$ as well as the accuracy of the wave analyzer limit the number of right-hand side values (the spectra obtained for the too-close directions of $\vec{U}$ can differ more because of the inaccuracy of the wave analyzer than because of the physics involved), in other words limit the amount of information we start with to some set of values of $\phi(|\sigma_D|)$(say to m different values).

If we want then to determine a continuous distribution of energy, that is to say the integrand $\Psi(\vec{k})$, we can assume a series representation of $\Psi(\vec{k})$ with constant coefficients. Next one can proceed to integrate over $S_{\vec{u}} \cap S$. As a result, the left-hand side of (2.1) becomes a known function of the coefficients mentioned above. The values of $\phi(|\sigma_D|)$ can then be used to determine the best fit of these parameters to the observations.

In other words, we replace the integral equation (2.1) by a set of linear equations. This time the number of coefficients we are to specify should be smaller than the number of equations (2.1)(=m). Then one picks the best values for the unknown parameters in terms of minimizing the distance in an m - dimensional space.

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