Finite Difference Seismic Wave Propagation Using Variable Grid Sizes

by

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ABSTRACT

Theoretical modeling has played an important role in understanding wave propagation in complex media. Finite difference is one of the most used methods to solve Partial Differential Equations numerically, and very often it requires enormous computational time and resources. In this thesis a variable finite difference method is developed, where a finer grid is used when model parameters are highly variable. In this scheme one can obtain accurate results with minimal computational requirements. In this study, a multigrid velocity-stress finite difference method is used to simulate the wave propagation across large models. The velocity-stress finite difference is formulated using a staggered grid, where a scheme is developed to relate the different-sized grids.

The variable grid scheme is first implemented in one dimension for the acoustic case. Different tests were carried out in order to obtain a validation of the method. Then, was developed a two-dimensional (2D) implementation of the multigrid finite difference method for elastic models. The (2D) implementation is tested using different models, both for acoustic and elastic media. The results obtained with the multigrid approach are in good agreement with the solutions obtained using the normal uniform grid finite difference.

Using the variable grid finite difference algorithm, we investigated the effect of interface irregularities on the reflection and scattering of elastic waves. We also examined the effects of interface roughness and the AVO (Amplitude Variation with Offset) analysis, commonly used in seismic exploration.

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Chapter 1

Introduction

1.1 Background and Motivation

Solutions to wave propagation problems by finite difference methods have received considerably attention in the last 25 years. It has greatly aided geophysicists in both forward modeling and the migration of seismic wave fields in complicated geologic media.

Most of the early works applying finite difference methods to seismologic problems deal with discrete solutions to the second-order elastic wave equations. For homogeneous media, see Alterman and Kornfeld (1968), Alterman and Karal (1968), Alterman and Rotenberg (1969), Alterman and Loewenthal (1970), Ottaviani (1971), and Ilan et al. (1975). For the heterogeneous case, see Boore (1972) and Kelly et al. (1976). An alternative means to modeling wave propagation in heterogeneous elastic media can be achieved by solving the first-order system of equations on two staggered grids. The second-order wave equation is reformulated to first-order hyperbolic equations using velocity and stress. The first of the currently popular staggered grid finite difference algorithms based on first-order equations was developed by Madariaga (1976). Virieux (1984) considers SH-wave propagation and Virieux (1986) considers P-SV wave propagation. More accurate solutions have been
obtained using higher-order finite difference schemes. Shubin and Bell (1987) developed fourth-order schemes by adding correction terms to second-order schemes. Levander (1988) extended the staggered grid scheme to the fourth-order finite difference for the P-SV problem.

The approximation of the continuous earth using meshes of both finite coarseness and finite extent leads to errors in accuracy that have to be well controlled to make the finite difference solution meaningful. Some of these errors are introduced by grid dispersion and inaccurate transmission. Another nagging problem that arises in the application of discrete solution methods for wave propagation calculations is the presence of artificial reflections from the boundaries of the numerical mesh. Many ideas have been developed to eliminate these boundary reflections. Absorbing boundary conditions that consist of a sponge or strip of nodes along the artificially truncated edges of the model are perhaps the simplest and most cost-effective means of eliminating undesirable edge reflections. An earlier method to solve the problem was developed by Lysmer et al. (1969). Other approaches have been developed by Smith (1974), Clayton and Engquist (1977), Reynolds (1978), Fuyuki and Matsumoto (1980), Emerman and Stephen (1983), Liao et al. (1984), Cerjan et al. (1985), Keys (1985), Kosloff and Kosloff (1986), and Higdon (1986, 1987, 1990).

For numerical practice it is important to realize that the accuracy of the approximate solution depends on the variation of the true solution from node to node. The larger the variation, the finer the grid has to be to obtain accurate results, and in almost all the cases this dependency leads to huge computational costs.

In the conventional finite difference scheme a uniform space grid is used, which means that all the grids cells are identical. In some cases, it can occur that the variations over the spatial domain are locally large at a particular location and relatively small everywhere else. Then, a fine grid size is needed for the whole area, even on areas where such high resolution is not necessary. This solution is computationally inefficient. First, the solution has to be computed at a very large number of nodes, increasing the CPU time needed to solve the problem. Second, storage for all the values has to be provided, and thus more
computer memory is needed. It is here where grid refinement methods prove to be very useful, because fine grid spacing is used only where it is needed (where large variations occur) and thereby using as few grid points as possible and providing the same level of accuracy.

A good preliminary compendium about the multigrid approach was presented by Hackbusch (1980). More recently, Trompert (1995) presented an extensive study about different techniques for static and dynamic grid refinement for the solution of time-dependent partial differential equations. Berger and Oliger (1984) and Berger and Colella (1989) developed an adaptive mesh refinement method for the solution of hyperbolic partial differential equations using finite difference techniques. Many of these multigrid approaches require an interpolation method to match the values at the boundary between component grids. Chesshire and Henshaw (1994) presented a scheme for conservative interpolation on overlapping grids. However, many researchers have successfully used nonconservative interpolation on overlapping grids to compute complex flows in two and three space dimensions (Berger and Oliger, 1984; Steger and Benek, 1987; Burning et al., 1988; Brown, 1991). In seismology, only a few studies have been reported that use the multigrid approach to improve the finite difference method. These include McLaughlin and Day (1994), who employed a multigrid scheme to seismic-wave simulations using 3D elastic velocity-displacement finite difference, and Falk et al. (1995), who used a varying grid spacing technique to model tube waves by finite difference methods. Falk et al. used grid spacing by power of three for the grid refinement.

In the present work, we studied a multigrid finite difference approach, and evaluated its benefits of accurately modeling the wave propagation and saving computational resources. We use a staggered stress-velocity finite difference approximation, which is valid for any Poisson’s ratio. The grid refinement can be done by any integer number. In addition, due to the simplicity of our implementation, the method could be easily extended to the three-dimensional (3D) case. This allows us to study models that have never before been considered.
Chapter 2

1D Multigrid Finite Difference

For simplicity, we begin by studying the one-dimensional (1D) acoustic wave equation. A second-order finite difference approximation of the first-order wave equation is implemented and the variable size multigrid option is incorporated. The method developed here is tested over different models. The results obtained from these tests are compared to similar results obtained using the normal uniform grid finite difference method in order to confirm the efficiency of the multigrid method.

2.1 Formulation

The one-dimensional wave propagation in an acoustic medium can be described by the following hyperbolic system of equations:

\[\rho \frac{\partial^2 u_z}{\partial t^2} = \frac{\partial \tau_{zz}}{\partial z}\]  \hspace{1cm} (2.1)

\[\tau_{zz} = \lambda \frac{\partial u_z}{\partial z}\]  \hspace{1cm} (2.2)

where \(\rho\) is the density of the media, \(u_z\) is the displacement vector, \(\tau_{zz}\) is the stress tensor, and \(\lambda\) and \(\mu\) are the Lamé constant. This system can be transformed into a first-order
hyperbolic system:

\[ \rho \frac{\partial v_z}{\partial t} = \frac{\partial \tau_{zz}}{\partial z} \quad (2.3) \]

\[ \frac{\partial \tau_{zz}}{\partial t} = \lambda \frac{\partial v_z}{\partial z} \quad (2.4) \]

where \( V_z \) is the velocity vector. This velocity and stress formula are the starting point for the finite difference method.

2.2 Finite Difference Approximation

The first-order hyperbolic equations are discretized on a 1D grid as shown in Figure 2-1. Velocities and stresses are interleaved in the one-dimensional space. The derivatives are discretized using centered finite difference. Therefore, the following explicit numerical scheme is equivalent to the system of equations 2.3 and 2.4:

\[ (V_z)_j^{k+1/2} = (V_z)_j^{k-1/2} + \left( \frac{1}{\rho} \right)_j \frac{\Delta t}{\Delta z} (T_{zz})_j^{k+1/2} - (T_{zz})_j^{k-1/2} \quad (2.5) \]

\[ (T_{zz})_j^{k+1/2} = (T_{zz})_j^{k+1/2} + \left( \lambda \right)_j \frac{\Delta t}{\Delta z} (V_z)_j^{k+1/2} - (V_z)_j^{k+1/2} \quad (2.6) \]

where \( k \) is the index for time discretization and \( j \) for the z-axis discretization. \( \Delta t \) is the grid step in time and \( \Delta z \) is the grid step in space. The numerical velocity \( V_z = v_z \) at time \( (k + \frac{1}{2})\Delta t \) and the numerical stress \( T_{zz} = \tau_{zz} \) at time \( (k + 1)\Delta t \) are computed explicitly from velocity at time \( (k - \frac{1}{2})\Delta t \) and stress at time \( k\Delta t \).

2.3 Stability and Boundary Conditions

For the finite difference calculation it is important to choose the values of time and space discretization, \( \Delta t \) and \( \Delta z \) for the case of 1D finite difference. The value of \( \Delta z \) is chosen using a rule of thumb that has been widely used, which establishes that \( \Delta z \) should be selected in
such a way that the number of grid points per wavelength is at least 10 (Stephen, 1983). In our case, we decided to use 20 grid points per wavelength to give better resolution to the results. Then, once having selected $\Delta z$, we have to select the value for $\Delta t$. In our finite difference calculations, we use the stability condition presented by Virieux (1986). For the case of propagation in homogeneous media, the stability criterion for the finite difference formulation is given by:

$$V_p \frac{\Delta t}{\Delta z} < \frac{1}{\sqrt{n}}$$

(2.7)

where $V_p$ is the P-wave velocity and $n$ is the dimensionality of the space, for the 1D case $n = 1$. Therefore, $\Delta t$ is selected in order to meet the stability condition:

$$\Delta t < \frac{\Delta z}{V_p}.$$  \hspace{1cm} (2.8)

Even artificial reflections result from the boundaries of the numerical mesh, no absorbing boundary condition was used in the 1D finite difference case. The simplicity of this scheme allows one to avoid these boundary effects by enlarging the numerical mesh, thus delaying the sides reflections longer than the times involved in the modeling.

### 2.4 Multigrid Implementation

#### 2.4.1 Description of the Method

The main idea for our implementation of wave propagation modeling using a finite difference scheme with variable grid size, consists in having multiple grids, each with a different grid size. The main grid, that we call the “Base Grid”, defines the complete area of the model being studied. At a particular region, where more resolution is needed or where the characteristics of the model require a smaller grid size, we add a new grid to satisfy the conditions. The grid size for this new grid needs to be a divisor of its predecessor’s grid size in order to obtain an integer value as grid ratio $r$ between them, ensuring that exactly $r$ fine grid cells fit in one coarse grid cell (Figure 2-2). Just for simplicity, in managing the
boundaries of the fine grid, an additional half grid cell is added at the left side of the grid, thus obtaining a symmetry that will let us handle both borders in the same way, since our stencil begins and ends with stresses $T_{zz}$. The finite difference stencil for the fine grid is located on top of the existing one, in such a way that each stress for the coarse grid ($T_{zz}^{\text{coarse}}$) matches together the stresses ($T_{zz}^{\text{coarse}}$ and $T_{zz}^{\text{fine}}$). When we use an even grid ratio, we can see in Figure 2-3a that the velocity $V_{zz}^{\text{coarse}}$ is centered between two velocities $V_{zz}^{\text{fine}}$. When the grid ratio is odd (Figure 2-3b), $V_z$ on both grids match at the same location.

The finite difference approximation presented in the previous section is calculated independently on each grid. However, at every time step we perform two actions to guarantee the continuity of the wave propagation across the two grids. The first action is to replace the values in the coarse grid at the overlapped region, with new values computed on the fine grid. This is done only when we have values for stress $T_{zz}$ and velocity $V_z$ on the fine grid that corresponds to the same time $t$ as on the coarse grid. This replacement is done:

for an even grid ratio $r$:

$$
(T_{zz}^{\text{coarse}})_j^t = (T_{zz}^{\text{fine}})_m+s1^t \\
(V_z^{\text{coarse}})_j^{t+\frac{dt}{2}} = \frac{1}{4} \sum_{p=s0}^{s0+1} \left[ (V_z^{\text{fine}})_{m+p}^{t+(r-1)\frac{dt}{2}} + (V_z^{\text{fine}})_{m+p}^{t+(r+1)\frac{dt}{2}} \right] 
$$

for an odd grid ratio $r$:

$$
(T_{zz}^{\text{coarse}})_j^t = (T_{zz}^{\text{fine}})_m+s1^t \\
(V_z^{\text{coarse}})_j^{t+\frac{dt}{2}} = \frac{1}{2} \sum_{p=s0}^{s0+1} (V_z^{\text{fine}})_{m+p}^{t+\frac{dt}{2}} 
$$

where $j$ is any index on the coarse grid within the overlapped region; $m$ is the index for the leftmost fine cell that corresponds to the same location of $j$; $s0 = \lfloor (r-1)/2 \rfloor$ and $s1 = r - 1$ represent the indexes for the middle and last fine cell relative to the coarse grid cell $j$ (Figure 2-3).

The second action performed consists of giving a continuity border to the fine grid in order to: (i) receive in any wave traveling from the coarse grid toward it, and (ii) act as
an absorbing boundary for the fine grid, to absorb the energy reaching its borders. This is done by replacing the values of the two stresses at both borders of the fine grid with values obtained at the same time and space from the coarse grid. It takes place before any integration step of the velocities on the fine grid. These values are calculated for the left side:

$$(T_{zz}^{\text{fine}})^h_{m_{\text{min}}} = \text{LinealInterpl1D}((T_{zz}^{\text{coarse}})^t_j, (T_{zz}^{\text{coarse}})^{t+dt}_j)$$

(2.13)

for the right side:

$$(T_{zz}^{\text{fine}})^h_{m_{\text{max}}} = \text{LinealInterpl1D}((T_{zz}^{\text{coarse}})^t_j, (T_{zz}^{\text{coarse}})^{t+dt}_j)$$

(2.14)

where \( t \) is the time in the coarse grid; \( h \) is the time in the fine grid; \( m_{\text{min}} \) and \( m_{\text{max}} \) are the indexes for the first and last stresses \( T_{zz}^{\text{fine}} \) in the fine grid; \( i \) is the index of the coarse grid that is overlapped by the fine grid being updated and \( \text{LinealInterpl1D}((T_{zz}^{\text{coarse}})^t_j, (T_{zz}^{\text{coarse}})^{t+dt}_j) \) is the interpolation in time from the two values \( (T_{zz}^{\text{coarse}})^t_j \) and \( (T_{zz}^{\text{coarse}})^{t+dt}_j \), obtained using a Lagrangian interpolating polynomial of degree 1.

### 2.4.2 Algorithm

The process of how the information is passed between the two grids is shown in Figure 2-4, where a grid size ratio of 2 (\( r = 2 \)) is considered. We assume that normal stresses for both grids start at time \( t \), while their velocities start a half space forward in time. Therefore, \( V_{coarse} \) starts at time \( t + dt/2 \), and \( V_{fine} \) at time \( t + dt/(2r) \). The algorithm is based on the following steps:

- Stresses \( (T_{zz}^{\text{coarse}})^t_j \) are computed to \( (T_{zz}^{\text{coarse}})^{t+dt}_j \).
- Stresses \( (T_{zz}^{\text{fine}})^t_m \) are computed to \( (T_{zz}^{\text{fine}})^{t+dt/2}_m \).
- Stresses \( T_{zz}^{\text{fine}} \) at the boundary (marked with circles in Figure 2-4) are replaced with values obtained by linear interpolation in time between \( (T_{zz}^{\text{coarse}})^t_j \) and \( (T_{zz}^{\text{coarse}})^{t+dt}_j \) (Equations 2.13 and 2.14).
• Velocities \((V_{z}^{\text{fine}})_{m}^{t+dt/4}\) are computed to \((V_{z}^{\text{fine}})_{m}^{t+3dt/4}\).

• Velocities \((V_{z}^{\text{coarse}})_{j}^{t+dt/2}\) overlapped by the fine grid are replaced with the velocity values from the fine grid. This is done by averaging the velocity values \(V_{z}^{\text{fine}}\) in space and time, as shown in equations 2.10 and 2.12 (see Figure 2-4 with the dashed square with the four \(V_{z}^{\text{fine}}\) at the corners and \(V_{z}^{\text{coarse}}\) at the center).

• Stresses \((T_{zz}^{\text{fine}})_{m}^{t+dt/2}\) are computed to \((T_{zz}^{\text{fine}})_{m}^{t+dt}\).

• Stresses \(T_{zz}^{\text{fine}}\) at the boundary (marked with squares in Figure 2-4) are replaced with values obtained by linear interpolation in time between \((T_{zz}^{\text{coarse}})_{j}^{t}\) and \((T_{zz}^{\text{coarse}})_{j}^{t+dt}\) (Equations 2.13 and 2.14).

• Velocities \((V_{z}^{\text{fine}})_{m}^{t+3dt/4}\) are computed to \((V_{z}^{\text{fine}})_{m}^{t+5dt/4}\).

• Stresses \((T_{zz}^{\text{coarse}})_{j}^{t+dt}\) overlapped by the fine grid, are updated with the values from \((T_{zz}^{\text{fine}})_{m}^{t+dt}\) using equations 2.9 and 2.11 (shown in 2-4 with a rhombus shape).

2.5 Test and Analysis of the 1D Multigrid Finite Difference Method

2.5.1 Homogeneous Case

To test the method previously described, we built an acoustic homogeneous model (Model 2-1) to be used with variable grid size and uniform grid size finite difference schemes, to compare the results obtained in both cases, and to verify its continuity, amplitudes and arrival times along the two grids' boundary. The homogeneous model was chosen for its simplicity, and because there is no reflection from any interface, so any artificial reflection created by the multigrid method can be easily noticed.

The velocity model used for this comparative study is shown in Figure 2-5. The total model is 1500 m long. The velocity of the media is 2513 m/s. The source was a point explosion,
simulated with a Kelly wavelet source time function, with a center frequency wavelet of 250 Hz (see Appendix A for more details about the Kelly wavelet). The first 135 m were mapped with the fine grid, while a coarse grid size was used for the remaining area. The source was located at 875 m from the top. Four receivers were used, the distance between the source and the first receiver was 30 m below, and the receivers' separation was 60 m.

The finite difference solution obtained using a uniform grid size is presented in the seismogram shown in Figure 2-6. The same finite difference solution was then calculated using the multigrid approach with several different grid ratios (2, 5, 10 and 20). The seismograms corresponding to all the cases are shown in Figures 2-7 to 2-10 (the seismograms are normalized).

As a general comparison, we can see that for all the different grid ratios, the arriving signal on each receiver is equivalent to the signal obtained using the uniform grid size. The arrival times are the same in all the cases. The amplitudes seem not to be affected by the change of the grid size; the normalized amplitudes agree well. However, there is an artificial reflection observed in the two receivers located on the same grid than the source. This reflection is smaller on the finite difference solution obtained using grid ratio 2, where the relation between the original signal and the artificial reflection is 0.5%, and for the other cases that ratio oscillate around 1%, even for grid ratios as large as 10 and 20. The relation between the original signal and the artificial one is presented in Figures 2-12 to 2-15, where the second trace is amplified to show in detail the amplitude of the artificial reflection obtained for the different grid ratios.

2.5.2 Heterogeneous Case

To complement the previous test, we repeated an experiment similar to the one described above, but this time using a two layer model (Model 2-2). A velocity of 1500 m/s was used on the top layer and 2513 m/s on the bottom (see Figure 2-16). All other parameters were kept equal.
The uniform finite difference scheme was used to compute the wave propagation over the new model, obtaining the solution shown in Figure 2-17. We used these results as a reference for comparisons with the multigrid scheme. We computed our method using grid ratios 2, 5, 10 and 20, obtaining the seismograms shown in Figures 2-18 to 2-21. These figures show that our method gives the same results as the original uniform approach. Additionally, the artificial reflection that we observed in the homogenous case does not seem to affect the results considerably.
Figure 2-1: Discretization of the medium on a 1D staggered grid.
Figure 2-2: Physical correspondence of two different grid sizes.
Figure 2-3: Overlapping of the staggered grid for two different grid cell sizes. The patterns in white correspond to the coarse grid, while the patterns in black correspond to the fine. (a) for an even grid ratio. (b) for an odd grid ratio.
Figure 2-4: Descriptive picture of the algorithm for 1D multigrid finite difference.
Figure 2-5: Model 2-1. One-dimensional homogeneous model used in the comparisons between uniform and variable grid size.
Figure 2-6: Finite difference solution using uniform grid size in Model 2-1 - Homogeneous media. (The amplitudes are normalized.)

Figure 2-7: Finite difference solution using grid size ratio 2 in model 2-1 - Homogeneous media. (The amplitudes are normalized.)
Figure 2-8: Finite difference solution using grid size ratio 5 in model 2-1 - Homogeneous media. (The amplitudes are normalized.)

Figure 2-9: Finite difference solution using grid size ratio 10 in model 2-1 - Homogeneous media. (The amplitudes are normalized.)
Figure 2-10: Finite difference solution using grid size ratio 20 (bottom) in model 2-1 - Homogeneous media. The figure can be compared with the solution obtained using uniform grid size (top). (The amplitudes are normalized.)
Figure 2-11: Amplified picture of trace 1 on Figure 2-6. Model 2-1 using uniform grid size. (The amplitude is normalized.)

Figure 2-12: Amplified picture of trace 1 on Figure 2-7. Model 2-1 using grid size ratio 2. (The amplitude is normalized.)
Figure 2-13: Amplified picture of trace 1 on Figure 2-8. Model 2-1 using grid size ratio 5. (The amplitude is normalized.)

Figure 2-14: Amplified picture of trace 1 on Figure 2-9. Model 2-1 using grid size ratio 10. (The amplitude is normalized.)
Figure 2-15: Amplified picture of trace 1 on Figure 2-10. (bottom) Model 2-1 using grid size ratio 20. The figure can be compared with the solution obtained using uniform grid size (top). (The amplitudes are normalized.)
Figure 2-16: Model 2-2. One-dimensional heterogeneous model used in the comparisons between uniform and variable grid size.
Figure 2-17: Finite difference solution using uniform grid size in model 2-2 - Heterogeneous media. (The amplitudes are normalized.)

Figure 2-18: Finite difference solution using grid size ratio 2 in model 2-2 - Heterogeneous media. (The amplitudes are normalized.)
Figure 2-19: Finite difference solution using grid size ratio 5 in model 2-2 - Heterogeneous media. (The amplitudes are normalized.)

Figure 2-20: Finite difference solution using grid size ratio 10 in model 2-2 - Heterogeneous media. (The amplitudes are normalized.)
Figure 2-21: Finite difference solution using grid size ratio 20 (bottom) in model 2-2 - Heterogeneous media. The figure can be compared with the solution obtained using uniform grid size (top). (The amplitudes are normalized.)
Figure 2-22: Amplified picture of trace 1 on Figure 2-17. Model 2-2 using uniform grid size. (The amplitude is normalized.)

Figure 2-23: Amplified picture of trace 1 on Figure 2-18. Model 2-2 using grid size ratio 2. (The amplitude is normalized.)
Figure 2-24: Amplified picture of trace 1 on Figure 2-19. Model 2-2 using grid size ratio 5. (The amplitude is normalized.)

Figure 2-25: Amplified picture of trace 1 on Figure 2-20. Model 2-2 using grid size ratio 10. (The amplitude is normalized.)
Figure 2-26: Amplified picture of trace 1 on Figure 2-10. (bottom) Model 2-2 using grid size ratio 20. The figure can be compared with the solution obtained using uniform grid size (top). (The amplitude is normalized.)
Chapter 3

2D Elastic Multigrid Finite Difference

3.1 Formulation

The wave propagation for a two dimensional space (2D) can be describe using the elastic-dynamic equations:

\[ \frac{\rho}{\partial t^2} = \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xz}}{\partial z} \]  
\[ (3.1) \]

\[ \frac{\rho}{\partial t^2} = \frac{\partial \tau_{zz}}{\partial x} + \frac{\partial \tau_{zz}}{\partial z} \]  
\[ (3.2) \]

\[ \tau_{xx} = (\lambda + 2\mu) \frac{\partial u_x}{\partial x} + \lambda \frac{\partial u_x}{\partial z} \]  
\[ (3.3) \]

\[ \tau_{zz} = (\lambda + 2\mu) \frac{\partial u_z}{\partial z} + \lambda \frac{\partial u_z}{\partial x} \]  
\[ (3.4) \]

\[ \tau_{xz} = \mu \left( \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} \right) \]  
\[ (3.5) \]
where \((u_x, u_z)\) is the displacement vector, \((\tau_{xx}, \tau_{zz}, \tau_{xz})\) is the stress tensor, \(\rho\) is the density, and \(\lambda\) and \(\mu\) are the Lamé coefficients. This system can be transformed in the following first-order hyperbolic system:

\[
\frac{\partial v_x}{\partial t} = \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xz}}{\partial z} \quad (3.6)
\]

\[
\frac{\partial v_z}{\partial t} = \frac{\partial \tau_{zz}}{\partial x} + \frac{\partial \tau_{xz}}{\partial z} \quad (3.7)
\]

\[
\frac{\partial \tau_{xx}}{\partial t} = (\lambda + 2\mu) \frac{\partial v_x}{\partial x} + \lambda \frac{\partial v_z}{\partial z} \quad (3.8)
\]

\[
\frac{\partial \tau_{zz}}{\partial t} = (\lambda + 2\mu) \frac{\partial v_z}{\partial x} + \lambda \frac{\partial v_z}{\partial z} \quad (3.9)
\]

\[
\frac{\partial \tau_{xz}}{\partial t} = \mu \left( \frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x} \right) \quad (3.10)
\]

where \((v_x, v_z)\) is the velocity vector.

### 3.2 Finite Difference Approximation

The first-order hyperbolic equations are discretized by using second-order centered finite difference on a staggered grid used by Virieux (1986) shown in Figure 3-1. The shear stress \((T_{xz})\) is located at the origin of the staggered grid, \(V_x\) is shifted half space in the \(Z\) direction, \(V_z\) is located half grid on the \(X\) direction and the normal stresses \((T_{xx} \text{ and } T_{zz})\) are shifted half grid in \(X\) and \(Z\) direction. The discretized numerical approximation, equivalent to the system represented by the equations 3.6 to 3.10, is:

\[
(V_x)^{k+1/2}_{i,j+1/2} = (V_x)^{k-1/2}_{i,j+1/2} + \frac{\Delta t}{\Delta x} \frac{(1)}{(1/\rho)}_{i,j+1/2} + \frac{\Delta t}{\Delta x} \frac{(T_{xx})^{k}_{i+1/2,j+1/2} - (T_{xx})^{k}_{i-1/2,j+1/2}}{2} + (3.11)
\]

\[
(V_z)^{k+1/2}_{i+1/2,j} = (V_z)^{k-1/2}_{i+1/2,j} + \frac{\Delta t}{\Delta x} \frac{(T_{zz})^{k}_{i+1,j+1/2} - (T_{zz})^{k}_{i+1,j-1/2}}{2} \quad (3.12)
\]
\[
\begin{align*}
(T_{xx})^{k+1}_{i+1/2,j+1/2} &= (T_{xx})^k_{i+1/2,j+1/2} + (\lambda_{i+1/2,j+1/2} + 2\mu_{i+1/2,j+1/2}) \frac{\Delta t}{\Delta z} \\
((V_z)^{k+1/2}_{i+1,j+1/2} - (V_z)^{k+1/2}_{i,j+1/2}) + \lambda_{i+1/2,j+1/2} \frac{\Delta t}{\Delta z} ((V_z)^{k+1/2}_{i+1,j+1} - (V_z)^{k+1/2}_{i+1,j}) 
\end{align*}
\]

\[3.13\]

\[
\begin{align*}
(T_{zz})^{k+1}_{i+1/2,j+1/2} &= (T_{zz})^k_{i+1/2,j+1/2} + (\lambda_{i+1/2,j+1/2} + 2\mu_{i+1/2,j+1/2}) \frac{\Delta t}{\Delta z} \\
((V_z)^{k+1/2}_{i+1/2,j+1} - (V_z)^{k+1/2}_{i+1/2,j}) + \lambda_{i+1/2,j+1/2} \frac{\Delta t}{\Delta z} ((V_z)^{k+1/2}_{i+1,j+1} - (V_z)^{k+1/2}_{i,j+1/2}) 
\end{align*}
\]

\[3.14\]

\[
\begin{align*}
(T_{xz})^{k+1}_{i,j} &= (T_{xz})^k_{i,j} + \mu_{i,j} \frac{\Delta z}{\Delta x} ((V_z)^{k+1/2}_{i+1/2,j} - (V_z)^{k+1/2}_{i-1/2,j}) \\
&+ \mu_{i,j} \frac{\Delta t}{\Delta z} ((V_z)^{k+1/2}_{i,j+1/2} - (V_z)^{k+1/2}_{i,j-1/2}) 
\end{align*}
\]

\[3.15\]

where \(k\) is the index for time discretization, \(i\) for x-axes discretization, and \(j\) for the z-axes discretization. \(\Delta t\) is the grid step in time, and \(\Delta x\) and \(\Delta z\) are the grid steps for the x-axis and z-axis, respectively, which are assumed equal. Numerical velocity \((V_x, V_z) = (v_x, v_z)\) at time \((k + 1/2)\Delta t\), and numerical stress \((T_{xx}, T_{zz}, T_{xz}) = (\tau_{xx}, \tau_{zz}, \tau_{xz})\) at time \((k + 1)\Delta t\) are computed from velocity at time \((k - 1/2)\Delta t\) and stress at time \(k\Delta t\).

### 3.3 Stability and Boundary Conditions

As in the 1D implementation, the same rule of thumb was used to select the values of space discretization \(\Delta x\) and \(\Delta z\). The same value is selected for both in such a way that the number of grid points per wavelength for the smallest velocity (shortest wavelength) is at least 10. The value for the time discretization \(\Delta t\) is again selected using the stability condition given by Virieux (1986), but this time for the 2D space. The stability condition is then given by:

\[
V_p \frac{\Delta t}{\Delta z} < \frac{1}{\sqrt{n}} \quad (3.16)
\]

where \(V_p\) is the P-wave velocity, \(n\) is the dimensionality of the space; for the 2D case \(n = 2\) and \(\Delta z\) is the spatial discretization and is equal to \(\Delta x\). Therefore, \(\Delta t\) is selected in order to meet the stability condition:

\[
\Delta t < \frac{\Delta z}{\sqrt{2V_p}} \quad (3.17)
\]

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In this equation $V_p$ is used because it is bigger than $V_s$ and thus gives us a more strict condition, so it can be used with both velocities. The main advantage of this condition is that it is independent of the S-wave velocity $V_s$, or of the Poisson’s ratio $\sigma$.

In order to minimize artificial reflections from the boundaries of the numerical mesh, Higdon's (1986, 1987, 1990) absorbing boundary condition was used in the 2D finite difference method. The discrete absorbing boundary condition is obtained directly from the discretized wave equation, rather than from the analytical boundary conditions. A detailed explanation of this approach is presented in Appendix B.

### 3.4 Multigrid Implementation

#### 3.4.1 Description of the Method

The implementation of our variable grid size finite difference scheme in two dimensions is based on the use of two independent uniform grids of different sizes (this can be easily extended to several meshes). The relation between these sizes is given by the number of points $n$ needed to represent the minimum possible wavelength in the area that each grid is covering. This number is usually $n = 5$ for a fourth-order finite difference scheme and $n = 10$ for second-order finite difference. Let us define our two grids as $G_0$ for the one with the coarse grid size and $G_1$ for the fine, where $G_1 \subset G_0$. The ratio in grid size between the two grids is given by an integer number $Gr_{0,1}$ (Grid Ratio) which defines how many times smaller the cell size is in $G_1$, compared with the cell size in $G_0$. This grid size variation is given in the same proportion in space as in time. Therefore for our two nested grids:

$$Gr_{0,1} = dx_0/dx_1 = dt_0/dt_1$$

(3.18)

this means that the finer grid ($G_1$) needs to execute $Gr_{0,1}$ time steps for every one on the coarser grid ($G_0$). Thus, the integration step on the different grids is interleaved, because the coarse grid is advanced to time $t + \delta t$ only if the finer grid has reached time $t$.  

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For simplicity, at the end of each grid an extra half grid cell is added in both directions (X and Z), in order to begin and end our staggered grid with \( T_{xx} \) and \( V_x \) on the horizontal direction and \( T_{xz} \) and \( V_z \) on the vertical direction. This gives us a symmetry to manage the borders and on the coarse grid, the absorbing boundary condition can be implemented with three variables instead of five (Figure 3-2).

The \( G_0 \) defines the true boundary of the model and the absorbing boundary condition is applied on all four sides. For the fine grid the absorbing boundary condition is only applied on those sides that match with the sides of \( G_0 \). Elsewhere, a continuity boundary condition is applied in order to eliminate any artificial reflection from that border and let in any wave coming from the coarse grid.

Each grid is defined separately (including its stability conditions), and the finite difference approximation is computed independently from the other, except for the overlapped region of the coarse grid which is not computed with the finite difference scheme. Additionally, special consideration is needed for two particular events:

(i) The inner border of the overlapped region on the coarse grid.

(ii) The boundary of the fine grid.

In event (i), the variables in the border of the overlapped region of the coarse grid are updated from the variables of the border of the fine grid. In event (ii), the information of the coarse grid is passed to the fine grid to create a continuity boundary for the wave traveling from the coarse grid into the fine, and to avoid artificial reflections from the boundary of the fine grid. For these purposes we need to create a way to associate the variables at the coarse grid with the variables at the fine grid, making it possible to interchange information between them. Figures 3-3 and 3-4 show how the two staggered grids are related, for the even and the odd grid size ratio. In both figures we can see that the normal stresses (\( T_{xx} \) and \( T_{xz} \)) for two staggered grids are located at the same position so that their values can be interchanged directly for any grid ratio value.

For an even grid ratio (Figure 3-3):

- The horizontal velocity (\( V_x \)) for the coarse grid is centered between two fine grid horizontal velocities (left and right).
- The vertical velocity \((V_z)\) at the coarse grid is centered between two fine grid vertical velocities (above and below).

- The coarse grid shear stress \((T_{zz})\) is located at the center of four shear stresses on the fine grid.

For an odd grid ratio (Figure 3-4):

All the variables from the coarse grid stencil match directly on their corresponding variable on the fine staggered grid.

Event (i) is accomplished whenever the fine grid \(G_1\) stresses and velocities reach the same time step \(t\) as its predecessor coarse grid \(G_0\). As we mentioned previously; the coarse grid has to wait until its finer grid reaches step \(t\) before making its integration step from \(t\) to \(t + \Delta t\), and is then when the information on the inner border of the overlapped region of \(G_0\) is replaced by its correspondent more accurate values from \(G_1\). Figures 3-5 (for an even grid ratio) and 3-6 (for an odd) show which cells on the fine grid are interpolated to update their correspondent coarse cell). This is done in the following way:

For an even grid ratio \(r\):

\[
(T_{\text{coarse}}^t)_{i,j} = (T_{\text{fine}}^t)_{m+s1,n+s1} \quad (3.19)
\]

\[
(T_{\text{coarse}}^t)_{i,j} = (T_{\text{fine}}^t)_{m+s1,n+s1} \quad (3.20)
\]

\[
(V_x^t)_{i,j} = \frac{1}{4} \sum_{p=s0}^{s0+1} \left[ (V_x^{\text{fine}})^t_{m+p,n+s1} + (V_x^{\text{fine}})^t_{m+p,n+s1} + (V_x^{\text{fine}})^t_{m+p,n+s1} + (V_x^{\text{fine}})^t_{m+p,n+s1} \right] \quad (3.21)
\]

\[
(V_x^t)_{i,j} = \frac{1}{4} \sum_{q=s0}^{s0+1} \left[ (V_x^{\text{fine}})^t_{m+s1,n+q} + (V_x^{\text{fine}})^t_{m+s1,n+q} + (V_x^{\text{fine}})^t_{m+s1,n+q} + (V_x^{\text{fine}})^t_{m+s1,n+q} \right] \quad (3.22)
\]

\[
(T_{zz}^t)_{i,j} = \frac{1}{8} \sum_{p=s0}^{s0+1} \sum_{q=s0}^{s0+1} \left[ (T_{zz}^{\text{fine}})^t_{m+p,n+q} + (T_{zz}^{\text{fine}})^t_{m+p,n+q} + (T_{zz}^{\text{fine}})^t_{m+p,n+q} + (T_{zz}^{\text{fine}})^t_{m+p,n+q} \right] \quad (3.23)
\]
For an odd grid ratio $r$:

$$(T_{xx}^{\text{coarse}})_{i,j}^t = (T_{xx}^{\text{fine}})_{m+s1,n+s1}$$  \hspace{1cm} (3.24)

$$(T_{xz}^{\text{coarse}})_{i,j}^t = (T_{xz}^{\text{fine}})_{m+s1,n+s1}$$  \hspace{1cm} (3.25)

$$(V_x^{\text{coarse}})_{i,j}^{t+dt/2} = (V_x^{\text{fine}})_{m+s0,n+s0}$$  \hspace{1cm} (3.26)

$$(V_z^{\text{coarse}})_{i,j}^{t+dt/2} = (V_z^{\text{fine}})_{m+s0,n+s0}$$  \hspace{1cm} (3.27)

$$(T_{xz}^{\text{coarse}})_{i,j}^t = ((T_{xz}^{\text{fine}})_{m+s0,n+s0}$$  \hspace{1cm} (3.28)

where $i$ and $j$ are, respectively, the horizontal and vertical coordinates of the border of the overlapped region on the coarse grid; $m$ and $n$ are, respectively, the indexes for the column and row, for the fine cell at the top-left corner of the coarse cell $(i,j)$; and $s0 = [(r-1)/2]$ and $s1 = r - 1$ represent the indexes for the middle and last fine cell relative to the coarse grid cell $(i,j)$ (See figures 3-3 and 3-4).

In equations 3.21 and 3.22, when the grid ratio is even, we have to do an interpolation in time between their respective velocities in the fine grid, between time $t + (r-1)dt/2r$ and $t + (r+1)dt/2r$, since the velocities on the fine grid never match the same corresponding time on the coarse grid.

The event (ii) takes place before any integration step on the velocities of the fine grid $G_1$. The idea is to compute all the stresses $(T_{xx}, T_{xz}$ and $T_{zz})$ on the boundary of $G_1$ from stresses taken from $G_0$, to be used as boundary values for the finite difference scheme on the fine grid. These values are computed by interpolations in space and time:

$$(T_{xx}^{\text{fine}})_{m,n}^h = \text{LinearInterpl2D}((T_{xx}^{\text{coarse}})_{i,j}^t, (T_{xx}^{\text{coarse}})_{i+p,j+q}^t, (T_{xx}^{\text{coarse}})_{i,j}^{t+dt}, (T_{xx}^{\text{coarse}})_{i+p,j+q}^{t+dt})$$  \hspace{1cm} (3.29)

$$(T_{zz}^{\text{fine}})_{m,n}^h = \text{LinearInterpl2D}((T_{zz}^{\text{coarse}})_{i,j}^t, (T_{zz}^{\text{coarse}})_{i+p,j+q}^t, (T_{zz}^{\text{coarse}})_{i,j}^{t+dt}, (T_{zz}^{\text{coarse}})_{i+p,j+q}^{t+dt})$$  \hspace{1cm} (3.30)
\[(T_{x}^{\text{fine}})^{h}_{m,n} = \text{LinearInterpl3D}((T_{x}^{\text{coarse}})^{t}_{i+f-1,j+g-1}, (T_{x}^{\text{coarse}})^{t}_{i+f,j+g-1}, (T_{x}^{\text{coarse}})^{t}_{i+f-1,j+g}, (T_{x}^{\text{coarse}})^{t}_{i+f,j+g})\]

(3.31)

where \(m\) and \(n\) are the indexes from the boundary of the fine grid; \(i\) and \(j\) are the coordinates of the coarse cell that include the fine grid cell \((m,n)\); \(h\) and \(t\) are the time variables, respectively, for the fine and the coarse grid; and the indexes \(p, q, f\) and \(g\) vary from different cases:

For left side \((m = 0)\):
- \(p = 0; q = 1; f = 0;\)
- \(g = 0\) for \(m <= [(r - 1)/2]\) and \(g = 1\) for \(m > [(r - 1)/2]\)

For right side \((m = \text{max}_{x}^{\text{fine}})\):
- \(p = 0; q = 1; f = 1;\)
- \(g = 0\) for \(m <= [(r - 1)/2]\) and \(g = 1\) for \(m > [(r - 1)/2]\)

For top side \((n = 0)\):
- \(p = 1; q = 0; g = 1;\)
- \(f = 0\) for \(n <= [(r - 1)/2]\) and \(f = 1\) for \(n > [(r - 1)/2]\)

For bottom side \((n = \text{max}_{x}^{\text{fine}})\):
- \(p = 1; q = 0; g = 1;\)
- \(f = 0\) for \(n <= [(r - 1)/2]\) and \(f = 1\) for \(n > [(r - 1)/2]\)

### 3.4.2 Algorithm

The algorithm used for 2D multigrid finite difference is based on the one used for the 1D in the previous chapter. The only additions are the new velocity \(V_{x}\) and the two stresses \(T_{xx}\) and \(T_{xz}\). Because of our staggered grid, scheme both stresses \(T_{xx}\) and \(T_{xz}\) are located in the same position. Therefore, they perform the same set of actions during the computation, hence we can treat them as one. Assume that we are using a grid size ratio of 2 \((r = 2)\). The normal and shear stresses for both grids start at time \(t\), while their velocities start half space forward in time. Therefore, \(V_{x}^{\text{coarse}}\) and \(V_{x}^{\text{coarse}}\) start at time \(t + \frac{dt}{2}\), and \(V_{x}^{\text{fine}}\) and \(V_{x}^{\text{fine}}\) at time \(t + \frac{dt}{2}\). The algorithm is based on the following steps:
• Stresses $(T_{xx}^{\text{coarse}})_{i,j}^t$, $(T_{xx}^{\text{coarse}})_{i,j}^t$ and $(T_{xx}^{\text{coarse}})_{i,j}^t$ are computed from time $t$ to time $t + dt$.

• Stresses $(T_{xx}^{\text{fine}})_{m,n}^t$, $(T_{xx}^{\text{fine}})_{m,n}^t$ and $(T_{xx}^{\text{coarse}})_{m,n}^t$ are computed from time $t$ to time $t + \frac{dt}{2} (t + \frac{dt}{2})$.

• The normal and shear stresses $T_{xx}^{\text{fine}}$ at the boundary of the fine grid are replaced with values obtained by linear interpolation in two or three dimensions (two dimensions in space and one in time) between the values of the stresses on the coarse grid in time $t$ and $t + dt$ (equations 3.29 to 3.31; see Figures 3-7 and 3-8).

• Velocities $(V_{x}^{\text{fine}})_{m,n}^{t+dt/4}$ and $(V_{x}^{\text{fine}})_{m,n}^{t+dt/4}$ are computed to time $t + \frac{3dt}{4}$.

• Velocities $(V_{x}^{\text{coarse}})_{i,j}^{t+dt/2}$ and $(V_{x}^{\text{coarse}})_{i,j}^{t+dt/2}$ at the border of the overlapped region of the coarse grid are replaced by the fine grid velocity values. This is done by averaging the velocity values $V_{x}^{\text{fine}}$ and $V_{x}^{\text{fine}}$ in space and time, as shown in equations 3.21, 3.22, 3.26 and 3.27 (Figures 3-5 and 3-6).

• The normal and shear stresses in the fine grid are computed from time $t + \frac{dt}{2}$ to $t + dt$.

• The stresses $T_{xx}^{\text{fine}}$ and $T_{xx}^{\text{fine}}$ at the boundary are replaced with values obtained by linear interpolation in time and space between the values of the stresses on the coarse grid in time $t$ and $t + dt$ (equations 3.29 to 3.31. see Figures 3-7 and 3-8).

• The velocities $(V_{x}^{\text{fine}})_{m,n}^{t+3dt/4}$ and $(V_{x}^{\text{fine}})_{m,n}^{t+3dt/4}$ are computed to $(V_{x}^{\text{fine}})_{m,n}^{t+5/4dt}$ and $(V_{x}^{\text{fine}})_{m,n}^{t+5/4dt}$.

• The stresses $(T_{xx}^{\text{coarse}})_{i,j}^{t+dt}$, $(T_{xx}^{\text{coarse}})_{i,j}^{t+dt}$ and $(T_{xx}^{\text{coarse}})_{i,j}^{t+dt}$, at the boundary overlapped by the fine grid, are replaced with their corresponding stresses from the fine grid using equations 3.19, 3.20, 3.23, 3.24, 3.25 and 3.28 (Figures 3-5 and 3-6).
3.5 Test and Analysis of the 2D Multigrid Finite Difference Method

As in the 1D case, we test the 2D multigrid approach over different models and compare them with the results obtained by using uniform grid size finite differences, in order to examine the fidelity of the solutions produced in the multigrid case. Once again, we start with the simplest case of a homogeneous model. Then, a two flat layers model is tested. Finally, a more realistic model is used to evaluate the usefulness of the method.

3.5.1 Homogeneous Model

The model (Model 3-1) is conformed for a whole medium with velocities $V_p = 4000\, m/s$ and $V_s = 2200\, m/s$ and density $\rho = 2.7\, g/cm^3$. The geometry of the experiment is presented in Figure 3-9 (Figure 3-10 shows the location of the fine grid in this model); the center frequency of the source wavelet is 30 Hz.

Even when the model is conformed for homogeneos media, two grids of different sizes were used in order to evaluate the accuracy of the method. The coarse grid has the size of the entire model, and the fine grid extends along the whole X direction, between 0 m and 627 m depth (Figure 3-10). This model is appropriate to test the passing of the wave through the boundary of both grids. In an ideal case, no additional events except for the passing of the original signal should be observed at any particular point.

The finite difference solution using conventional uniform grid size for this model is shown in Figure 3-11. Results for the multigrid finite difference approximation using different grid ratios (2, 5 and 8) are shown in Figures 3-12, 3-13 and 3-14 (bigger grid sizes were not used because of the excessive memory requirement for this particular case). In these results, we can see that solutions obtained for different grid ratios are very similar to the uniform case, and the artificial reflection obtained by the change in grid size is less than 1% of the original signal.
3.5.2 Heterogeneous Model

The second model (Model 3-2) corresponds to a low-velocity elastic half space over another elastic half space of higher velocity. This example is designed to test the accuracy of the multigrid finite difference scheme for the simplest possible layered medium. Figure 3-15 schematically shows the velocity model and the geometry used in this case. The velocities of the upper medium are \( V_p = 2800\text{m/s} \) and \( V_s = 1730\text{m/s} \), while the lower medium has velocities of \( V_p = 4000\text{m/s} \) and \( V_s = 2200\text{m/s} \). The density for each case is \( 2.5 \text{gm/cm}^3 \) and \( 2.7 \text{gm/cm}^3 \) respectively. The center frequency wavelet is again 30 Hz. The source is located in the low velocity medium at 14 m depth. The receivers were located horizontally at different spacing, depending on the incident angle, covering 2.5 to 45 degrees every 2.5 degrees. The model is shown in Figure 3-15.

Different grid size ratios 2, 4 and 8 were tested. Two different grids of different sizes were used over the model for each case. Both extend side-to-side of the model. However, the fine grid extends along the interface of the two half spaces, with a thickness of 100 m.

Grid ratios of 2, 4 and 8 were used to test our method and compare it to the traditional scheme. Figures 3-17 to 3-20, show the results from the uniform grid size and the different grid size ratios that were chosen. Note that the signal obtained for all the cases agree very well.

3.5.3 Thin Layer Model

Another test (Model 3-3) is the modeling of a very thin low velocity layer between two layers of higher velocities. The velocity model and geometry of the model is shown in Figure 3-21. The velocities of the thin layer are \( V_p = 3345\text{m/s} \) and \( V_s = 1580\text{m/s} \), and the density is \( \rho = 2.4\text{g/cm}^3 \). The velocities for the top layer are \( V_p = 4750\text{m/s} \) and \( V_s = 3345\text{m/s} \), and for the bottom layer \( V_p = 4994\text{m/s} \) and \( V_s = 2784\text{m/s} \). The densities are the same in both cases, \( \rho = 2.7\text{g/cm}^3 \). The center frequency wavelet for the source is 30 Hz. In this case,
we put the fine grid over the thin layer (see Figure 3-21), and define the rest of the model with a coarse grid size. In this case we use a grid size ratio 2, since the ratio between the maximum space sampling for both grids is less than two (1.76), however integer refinement values should be used. The results obtained for this case are presented in Figures 3-23 and 3-24. Once again, we can see a fairly good match between both results. The execution time using the multigrid approach was almost half of the time for the uniform case. However, as we move closer to integer values relation, more benefit can be obtained from our method.
Figure 3-1: Discretization of the medium on a 2D staggered grid.
Figure 3-2: Physical correspondence of two different grid sizes. The symbols in white correspond to the coarse grid, while the ones in black correspond to the fine.
Figure 3-3: Correspondence between one coarse grid and $r \times r$ fine cells. (Grid ratio $r$ is even.)
Figure 3-4: Correspondence between one coarse grid and $r \times r$ fine cells. (Grid ratio $r$ is odd.)
Values in the coarse grid are obtained from interpolation of values in the fine grid.

Figure 3-5: Fine grid cells used to update their closest coarse cell. (Even grid ratio.)
Values in the coarse grid are obtained from values in the fine grid.

Figure 3-6: Fine grid cells used to update their overlapped coarse cell. (Odd grid ratio.)
Interpolated normal stress in the fine grid

Normal stresses in the coarse grid used for the interpolation

Normal stress in the fine grid is taken from the value of the normal stress in the coarse grid

Figure 3-7: Coarse grid normal stresses used to update the normal stresses at the border of the fine grid.
Interpolated shear stress in the fine grid

Shear stresses in the coarse grid used for the interpolation

Figure 3-8: Coarse grid shear stresses used to update the shear stresses at the border of the fine grid.
Figure 3-9: Model 3-1. Two-dimensional homogeneous model used in the comparisons between uniform and variable grid size.
Figure 3-10: Two-dimensional homogeneous model 3-1. (Fine grid location.)
Figure 3-11: Finite difference solution using uniform grid size in model 3-1. Homogenous media. (the amplitudes are normalized.)

Figure 3-12: Finite difference solution using grid ratio of 2 in model 3-1. Homogenous media. (the amplitudes are normalized.)
Figure 3-13: Finite difference solution using grid ratio of 5 in model 3-1. Homogenous media. (the amplitudes are normalized.)

Figure 3-14: Finite difference solution using grid ratio of 8 in model 3-1. Homogenous media. (the amplitudes are normalized.)
Figure 3-15: Model 3-2. Two-dimensional heterogeneous media used in the comparisons between uniform and variable grid size.
Figure 3-16: Two-dimensional heterogeneous model 3-2. (Fine grid location).
Figure 3-17: Finite difference solution using uniform grid size in model 3-2. Heterogenous media. (the amplitudes are normalized).

Figure 3-18: Finite difference solution using grid ratio of 2 in model 3-2. Heterogenous media. (the amplitudes are normalized.)
Figure 3-19: Finite-difference solution using grid ratio of 5 in model 3-2. Heterogeneous media. (the amplitudes are normalized.)

Figure 3-20: Finite difference solution using grid ratio of 8 in model 3-2. Heterogenous media. (the amplitudes are normalized.)
Figure 3-21: Model 3-3. Two-dimensional model of a low velocity thin layer.
Figure 3-22: Two dimensional model of a low velocity thin layer. (Fine grid location.)
Figure 3-23: Finite difference solution using uniform size scheme in model 3-3. Thin layer model. (the amplitudes are normalized.)

Figure 3-24: Finite difference solution using grid ratio of 2 in model 3-3. Thin layer model. (the amplitudes are normalized.)
Chapter 4

Effect of Irregular Interfaces on Reflection and AVO

4.1 Introduction

Interface irregularities occur on layer boundaries due to geological processes such as faulting, folding and unconformities. These irregularities cause scattering of the seismic waves that can often be identified on seismograms. Many studies have been done to try to understand the effect of an incident wave at a rough interface. Studies by Asano (1960, 1966) have shown that interface irregularities with length smaller than the incident wavelength can affect reflected P and SV wave amplitudes and can generate diffracted waves whose amplitude and phase depend strongly on the angle of incidence. Prange and Toksöz (1990) demonstrated similar amplitude variation from random 3-D surfaces. Schultz (1994) worked on the scattering effect of seismic waves from highly irregular 2-D and 3-D elastic interfaces.

Amplitude Variation with Offset (AVO) is a widely used technique, which relates the reflection coefficient, the incident angle and the variation in the compressional and shear wave velocities across an interface (Castagna and Backus, 1993). AVO analysis assumes smooth
boundaries.

In this chapter we use our variable grid finite difference algorithm to model the reflections from irregular interfaces, and determine the reflection amplitudes dependent on the incident angles and the surface roughness characteristics.

4.2 Irregular Interface Modeling

To create the random irregular interface we used an algorithm developed at M.I.T's Earth Resource Laboratory based on a Gaussian distribution, where standard deviation controls the heights of the peaks and valleys of the corrugated surface, and the correlation length controls the separation between adjacent peaks and valleys.

Our model consists of two half spaces: a low-velocity elastic half space over another elastic half space of higher velocity. The dimensions of the model are 1100 m long and 1000 m deep, with the interface at 500 m depth. The velocities of the upper medium are \( V_p = 3300\text{m/s} \) and \( V_s = 1900\text{m/s} \), while the lower medium has velocities of \( V_p = 4150\text{m/s} \) and \( V_s = 1730\text{m/s} \). The densities are \( \rho_1 = 2.5\text{gm/cm}^3 \) and \( \rho_2 = 2.7\text{gm/cm}^3 \). The center frequency of the wavelet is 30 Hz. The source is located in the low-velocity medium at 14 m deep, and 100 receivers are located horizontally from the source at every 10 meters. This geometry covers up to 45 degrees of angle of incidence. The geometry of the sources and receivers is shown in Figure 4-1.

For finite difference modeling, a coarse grid (\( \Delta x = \Delta z = 2.88\text{m} \)) was used in each half space. Near the interface a fine grid, four times smaller (\( \Delta x = \Delta z = 0.72\text{m} \)) was used. The dimensions of the zone with fine grid were 72 m high and 1100 m wide.

We carried out modeling using seven different irregular interfaces using 30 Hz center frequency wavelet. For one case we used a 60 Hz wavelet. For the first three random interfaces we used a standard deviation of approximately 18 m and correlation lengths of 100 m (\( \approx \lambda \)),

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27 m (λ/4) and 13 m (λ/8). For interfaces 4 and 5 we used the same correlation lengths as for interfaces 2 and 3, but the standard deviation was set to around 37 m. For the sixth and seventh interface we used the same parameters as the second and third interfaces but setting the standard deviation to 9 m. For the last interface we used the same parameters as interface 2 with a 60 Hz wavelet. The profiles of these interfaces are shown at the end of this chapter (Figure 4-2). A summary of model parameters is shown in Table 4.1.

The “roughness parameter” listed on the table is the ratio of the standard deviation and correlation length.

<table>
<thead>
<tr>
<th>Model</th>
<th>Correlation length (meters)</th>
<th>Standard deviation (meters)</th>
<th>“Roughness parameter”</th>
<th>Center frequency (Hz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Model 1</td>
<td>100 (≈ λ)</td>
<td>18</td>
<td>0.18</td>
<td>30</td>
</tr>
<tr>
<td>Model 2</td>
<td>27 (λ/4)</td>
<td>18</td>
<td>0.67</td>
<td>30</td>
</tr>
<tr>
<td>Model 3</td>
<td>13 (λ/8)</td>
<td>18</td>
<td>1.38</td>
<td>30</td>
</tr>
<tr>
<td>Model 4</td>
<td>27 (λ/4)</td>
<td>37</td>
<td>1.27</td>
<td>30</td>
</tr>
<tr>
<td>Model 5</td>
<td>13 (λ/8)</td>
<td>37</td>
<td>2.85</td>
<td>30</td>
</tr>
<tr>
<td>Model 6</td>
<td>27 (λ/4)</td>
<td>9</td>
<td>0.33</td>
<td>30</td>
</tr>
<tr>
<td>Model 7</td>
<td>13 (λ/8)</td>
<td>9</td>
<td>0.69</td>
<td>30</td>
</tr>
<tr>
<td>Model 8</td>
<td>13 (λ/8)</td>
<td>18</td>
<td>1.38</td>
<td>60 Hz</td>
</tr>
</tbody>
</table>

Table 4.1: Parameters used for the random irregular interface models.

We modeled a flat interface to use as a reference and to compare with models of different irregular interfaces. The flat interface seismograms are shown in Figure 4-3. We calculated the seismograms for all the irregular interfaces listed in Table 4.1. The seismograms are shown in Figures 4-4 to 4-11. These show that for model 1, where correlation length is close to that of a wavelength, the scattering effect is very small. The scattering effect is also small when we used small standard deviations (models 6 and 7). For models 4 and 5 we note large scattering effects due to the high standard deviation used. The scattering is most pronounced when the wavelet center frequency is 60 Hz. Snapshots showing the scattering effect for the irregular interface used for model 5 are presented in Figures 4-12.
to 4-15 compared with snapshots obtained for a flat interface.

To study the effect of interface irregularities on AVO, we followed the commonly used approximation where $AVO = A + B\sin^2(\theta)$ (Shuey, 1985). To obtain the plots of the amplitude versus the offset, we picked the maximum amplitude of the first arrival for all the models, and we computed a best fitting straight line, using the “least squares” method. These fits are shown in Figures 4-16 to 4-18. The least square fit to the AVO data gives the term “A”, the gradient “B” and the variance. These values are shown in Table 4.2 for all the models, and for the flat interface. Note that the variance is not zero for the flat interface because of the above definition of AVO is a linear approximation where higher terms are dropped.

<table>
<thead>
<tr>
<th>“Roughness parameter”</th>
<th>A</th>
<th>B</th>
<th>Variance</th>
</tr>
</thead>
<tbody>
<tr>
<td>Flat Interface</td>
<td>0.00</td>
<td>0.1435</td>
<td>0.1948</td>
</tr>
<tr>
<td>Model 1</td>
<td>0.18</td>
<td>0.1511</td>
<td>0.1563</td>
</tr>
<tr>
<td>Model 2</td>
<td>0.67</td>
<td>0.1381</td>
<td>0.1424</td>
</tr>
<tr>
<td>Model 3</td>
<td>1.38</td>
<td>0.1293</td>
<td>0.1691</td>
</tr>
<tr>
<td>Model 4</td>
<td>1.27</td>
<td>0.1130</td>
<td>0.0731</td>
</tr>
<tr>
<td>Model 5</td>
<td>2.85</td>
<td>0.0901</td>
<td>0.1339</td>
</tr>
<tr>
<td>Model 6</td>
<td>0.33</td>
<td>0.1435</td>
<td>0.1734</td>
</tr>
<tr>
<td>Model 7</td>
<td>0.69</td>
<td>0.1410</td>
<td>0.1837</td>
</tr>
<tr>
<td>Model 8</td>
<td>1.38</td>
<td>0.0654</td>
<td>0.1152</td>
</tr>
</tbody>
</table>

Table 4.2: independent term “A”, gradient “B” and variance values for the different models used.

From these results we can deduce the effect of scattering. As one would expect for models with very low scattering, the independent term “A” is closer to the one obtained for the flat interface model, while for the highly scattering model “A” becomes smaller. The gradient “B” is smaller than the one obtained for the flat interface, one can notice that the three
smallest gradients belong to the models with the most scattering effects (models 4, 5 and 8). The variance is high for the models with high scattering. The variance is a good indicator of interface roughness.

Similar to P waves, amplitude variations with offset can be observed for P to SV converted waves reflected from a corrugated interface. To show this we calculated some examples using the same interface properties. However, we changed velocities in order to get a higher S-wave reflection coefficient ($V_{P1} = 3650 \text{ m/s}$, $V_{S1} = 1730 \text{ m/s}$, $V_{P2} = 4000 \text{ m/s}$ and $V_{S2} = 2200 \text{ m/s}$). The seismograms are shown in Figures 4-19 to 4-22, corresponding to the same profiles used for models 2, 3, 6 and 7. The SV arrival time is around 4.5 s for the first trace and about 0.5 s for the last.
Figure 4-1: Model used to compute the reflections on a random irregular interface.
Figure 4-2: Profiles used for our different models. (Dimensions of each profile are 1100 m wide and 72 m high.)
(a) Seismograms for the reference model.

(b) Traces 1 to 17 from sub-figure (a).
Figure 4-3: Seismograms corresponding to flat interface (reference model). (Vertical component.)
(a) Seismograms for model 1.

(b) Traces 1 to 17 from sub-figure (a).
(c) Traces 18 to 33 from sub-figure (a).

(d) Traces 34 to 99 from sub-figure (a).

Figure 4-4: Seismograms corresponding to model 1. (Vertical component.)
(a) Seismograms for model 2.

(b) Traces 1 to 17 from sub-figure (a).
(c) Traces 18 to 33 from sub-figure (a).

(d) Traces 34 to 99 from sub-figure (a).

Figure 4-5: Seismograms corresponding to model 2. (Vertical component.)
(a) Seismograms for model 3.

(b) Traces 1 to 17 from sub-figure (a).
(c) Traces 18 to 33 from sub-figure (a).

(d) Traces 34 to 99 from sub-figure (a).

Figure 4-6: Seismograms corresponding to model 3. (Vertical component.)
(a) Seismograms for model 4.

(b) Traces 1 to 17 from sub-figure (a).
(c) Traces 18 to 33 from sub-figure (a).

(d) Traces 34 to 99 from sub-figure (a).

Figure 4-7: Seismograms corresponding to model 4. (Vertical component.)
(a) Seismograms for model 5.

(b) Traces 1 to 17 from sub-figure (a).
Figure 4-8: Seismograms corresponding to model 5. (Vertical component.)

(c) Traces 18 to 33 from sub-figure (a).

(d) Traces 34 to 99 from sub-figure (a).
(a) Seismograms for model 6.

(b) Traces 1 to 17 from sub-figure (a).
Figure 4-9: Seismograms corresponding to model 6. (Vertical component.)
(a) Seismograms for model 7.

(b) Traces 1 to 17 from sub-figure (a).
(c) Traces 18 to 33 from sub-figure (a).

(d) Traces 34 to 99 from seismogram in sub-figure (a).

Figure 4-10: Seismograms corresponding to model 7. (Vertical component.)
(a) Seismograms for model 8.

(b) Traces 1 to 17 from sub-figure (a).
Figure 4-11: Seismograms corresponding to model 8. (Vertical component.)
(a) Times 0.125, 0.2 and 0.275 seconds.
(b) Times 0.35, 0.425 and 0.5 seconds.

Figure 4-12: Snapshots corresponding to flat interface model. (Vertical component.)
(a) Times 0.125, 0.2 and 0.275 seconds.
(b) Times 0.35, 0.425 and 0.5 seconds.

Figure 4-13: Snapshots corresponding to flat interface model. (Horizontal component.)
(a) Times 0.125, 0.2 and 0.275 seconds.
Figure 4-14: Seismograms corresponding to model 5 irregular interface. (Vertical component.)
(a) Times 0.125, 0.2 and 0.275 seconds.
Figure 4-15: Seismograms corresponding to model 5 irregular interface. (Horizontal component.)
Figure 4-16: Amplitude variation with offset, for flat interface and models 1, 2 and 3.
Figure 4-17: Amplitude variation with offset, for models 4, 5, 6 and 7.
Figure 4-18: Amplitude variation with offset, for model 8.
Figure 4-19: S-wave amplitude variation for model 1. (vertical component.)

Figure 4-20: S-wave amplitude variation for model 2. (vertical component.)
Figure 4-21: S-wave amplitude variation for model 6. (vertical component.)

Figure 4-22: S-wave amplitude variation for model 7. (vertical component.)
Chapter 5

Conclusions

We have presented a 2-D algorithm for the simulation of elastic wave propagation using the finite difference method with variable grid size. With this development wave propagation modeling can be modeled efficiently, using a finer grid in particular regions, rather than the whole area. Hence we can model problems with large contrast in scale lengths without the use of a large amount of memory and computational time where conventional methods could not be used because of computational resource limitations. The variable grid size method was implemented in one and two space dimensions. With several tests we showed that with our grid management method, we can do calculations with high accuracy using a fraction of the resources as compared with traditional uniform finite difference. Although the variable grid code was implemented in 2-D, it can be extended to three dimensional elastic wave propagation problems.

This method was also applied in to a practical case, where we modeled several irregular layer interfaces, using a very fine grid size compared with the rest of the model. These examples showed interesting results for the scattering. Relative effects of interface roughness parameters (correlation length and standard deviation) were investigated.
Appendix A

Source Time Function

The source time function used in this thesis is based on a Gaussian curve (Kelly et al., 1976; Stephen et al., 1985):

\[ f(t) = -2\xi T \exp^{-\xi T^2} \]  \hspace{1cm} (A.1)

where \( \xi \) is a pulse width parameter and \( T = t - t_s \). \( t_s \) is a time shift parameter.

It is straightforward to obtain the first derivative of \( f(t) \), which is given below:

\[ f'(t) = -2\xi(1 - 2\xi T^2) \exp^{-\xi T^2} \]  \hspace{1cm} (A.2)

For a pulse at a center frequency \( F_0 \) we chose pulse width parameter \( \xi = \frac{F_0^2}{0.1912t_s} \), selected such that \( f(0) \approx 0 \). Here we chose \( t_s = \frac{1.5}{F_0} \).

In the finite difference calculation when the source time function fed into the stress \( f'(t) \) is used to simulate a point explosion.
Appendix B

Absorbing Boundary Condition

For the purpose of application to elastic wave propagation problems, the Higdon’s (1986, 1987, 1990) absorbing boundary condition operator

\[ B = \prod_{j=1}^{m}(c_{j} \frac{\partial}{\partial t} - \alpha \frac{\partial}{\partial x}) \]  \hspace{1cm} (B.1)

is applied to each component of the displacement vector at \( x = x_l \). \( m \) is the order of the absorbing boundary condition. \( x_l \) is the left boundary along the \( X \) axis. For the right boundary along the \( X \) axis at \( x = x_r \), the minus sign in B.1 should be replaced by a plus sign. The coefficients \( c_{j} \) are positive constant for all \( j \). The similar operators can be used for the boundaries along the \( Y \) axis by replacing \( \frac{\partial}{\partial x} \) in Equation B.1 with \( \frac{\partial}{\partial y} \).

The \( jht \) operator in B.1 is perfectly absorbing for the P-wave traveling at angles of incidence \( \pm \cos^{-1} c_{j} \), and for the S-wave traveling at angles of incidence \( \pm \cos^{-1} c_{j} (\frac{\alpha}{\beta}) \). As an example, in case of \( m = 2 \), we can choose \( c_1 = 1 \) and \( c_2 = \frac{\alpha}{\beta} \) to absorb both the P as the S wave at zero angle perfectly.

Define operators \( E_x \) and \( E_t \) as a forward shift in \( x \) and \( t \):

\[ E_x f_{m,n,k}^i = f_{m+1,n,k}^i \]  \hspace{1cm} (B.2)

\[ E_t f_{m,n,k}^i = f_{m,n,k}^{i+1} \]  \hspace{1cm} (B.3)
The absorbing boundary condition operator in Equation B.1 can be approximated by the finite difference operator as

\[
D(E_x, E_t^{-1}) = \prod_{j=1}^{m} c_j \left( \frac{I - E_t^{-1}}{\Delta t} \right) \left[ (1 - a)I + aE_x \right] - \alpha \left( \frac{E_x - I}{\Delta x} \right) \left[ (1 - b)I + bE_t^{-1} \right]
\]

(B.4)

Parameters \(a\) and \(b\) give weighted space and time averages. Different \(a\) and \(b\) values result in different schemes. For example:

1. Forward Euler: \(a = 0, b = 1\). The stencil has an ‘L’ shape.
2. Backward Euler: \(a = 0, b = 0\). The stencil has an inverted ‘L’ shape.
3. Box scheme: \(a = 1/2\) and \(b = 1/2\).

If the boundary value of the displacement \(u\) is needed at \(x = x_0\), then the absorbing boundary condition is

\[
D(E_x, E_t^{-1}) u^{i+1}_{x=x_0} = 0
\]

(B.5)

We solve this equation for \(u^{i+1}\) using the previous time step values. In our staggered grid scheme, this condition is not only applied to the velocities but also to the stresses.

Higdon’s absorbing boundary condition can be applied directly to the corner of the grid. It only involves the differences perpendicular to the boundary, so it works well at the boundary with lateral discontinuity. The implementation is straightforward. Incompatibility can be removed by adding small positive constant \(\delta_j\), at least one \(\delta_j\) is non-zero, to the absorbing boundary condition operator. Thus it becomes

\[
B = \prod_{j=1}^{m} \left( c_j \frac{\partial}{\partial t} - \alpha \frac{\partial}{\partial x} + \delta_j \right)
\]

(B.6)

In the simple acoustic case the P-wave reflection coefficient has magnitude

\[
\prod_{j=1}^{m} \left| \frac{\cos \theta_j - \cos \theta}{\cos \theta_j + \cos \theta} \right|
\]

(B.7)

where \(\theta_j\) is the perfectly absorbing angle of incidence.
Bibliography


Steger, J. L. and J. A. Benek (1987). On the use of composite grids schemes in computa-


