Adelic Fourier-Whittaker Coefficients and the Casselman-Shalika formula

by

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Abstract

In their paper Metaplectic Forms, D. A. Kazhdan and S. J. Patterson developed a generalization of automorphic forms that are defined on metaplectic groups. These groups are non-trivial covering groups of usual algebraic groups, and the forms defined on them are representations that respect the covering. As in the case for automorphic forms, these representations fall into a principle series, indexed by characters on a torus of the metaplectic group, and there is an associated an L-function. In the final section of their paper, an equivalence is shown in the rank one case between this L-function and an Dirichlet series defined using Gauss sums, in order to demonstrate the arithmetic content. In this paper we reexamine this connection in the particular case that was discussed in Metaplectic Forms. By looking through the scope of twisted multiplicativity, a property of L-series, the computation is simplified and more easily generalized.

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This paper will use two constructions of the metaplectic group, an abstract one and an explicit one. The more abstract description will follow the construction of Matsumoto and is discussed in more detail in [7], [10], and [12]. The explicit construction is specific to the GL_2 case and follows Kubota [8]. For most cases, the abstract description is more useful, due to its generality and inherence. But for some calculations, having a simple, understandable cocycle to compute is advantageous, and for this we will turn to the explicit expression.

1. Useful subgroups

We are working with a split, simply connected algebraic group G over a number field k that contains the nth roots of -1, for a fixed n. We will reference several subgroups of G throughout our discussion, and here we take a section to describe the relevant notation. Because G is a split, simply connected algebraic group, the structure of its root system and the associated subgroups is at our disposal. We have Z the center of G and $H \supset Z$ a chosen maximal torus of G. Then we have Φ the set of roots, Φ_+ a choice of positive roots, and Δ the associated simple positive roots. From these choices we derive N_+, the unipotent subgroup of corresponding to Φ_+. Then we have $B = HN_+$, the Borel subgroup. Also, W will be the Weyl group associated to H. Another crucial subgroup of G is the maximal compact subgroup, unique up to conjugation. When we are working over a local field, we let K be the maximal compact subgroup. The Iwasawa decomposition $G = HNK$ will be well-used in our computation.

For some applications we need a subgroup of elements which has a trivial cover in the metaplectic group. The strategy to create these is to replace H with its of nth powers, which will be written with subscript n:

$$H_n = \{h^n : h \in H\}.$$

In the case that $G = GL_2$, these subgroups are chosen to be the following standard ones (note that matrix components which are omitted are zero):

$$Z_R = \left\{ \begin{pmatrix} a & \\ & a \end{pmatrix} \middle| a \in R^\times \right\}, \quad H_R = \left\{ \begin{pmatrix} a & \\ & b \end{pmatrix} \middle| a, b \in R^\times \right\}, \quad N_{+,R} = \left\{ \begin{pmatrix} 1 & c \\ & 1 \end{pmatrix} \middle| c \in R \right\},$$

$$B_R = H_R N_{+,R} = \left\{ \begin{pmatrix} a & b \\ & c \end{pmatrix} \middle| a, c \in R^\times, b \in R \right\}, \qquad W = \left\{ \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \right\}$$

where R will be either our global field k, one of its local completions k_v (in this case, the subscript of the group will simply be v, so H_v is the torus over k_v), or its adelic ring $\mathbb{A}$ (constructed below). Over a local field F, the form of K depends on F in the following way:

$$K = \begin{cases} O(2), & \text{if } F = \mathbb{R}, \\ U(2), & \text{if } F = \mathbb{C}, \text{ and} \\ GL_2(\mathfrak{O}_F), & \text{otherwise.} \end{cases}$$

Where $\mathfrak{O}_F$ is the ring of integers of F when F is p-adic.

2. Construction of the adeles

Given a global field k, its adelic ring $\mathbb{A}$ is a construction designed to combine all of the completions of k into one object so that they may be studied simultaneously. This way differences and similarities between the completions at k's places may be best observed. $\mathbb{A}$ is defined as the restricted direct product of k_v over the places v of k, with respect to the open sets $\mathfrak{O}_v$ of each nonarchimedean place of k. In other words, if S is the set of all the places of k,

$$\mathbb{A} = \{ (x_v)_{v \in S} | \forall v \in S, x_v \in k_v, \text{ and for almost all } v < \infty, x_v \in \mathfrak{O}_v \}.$$

In this paper, 'almost all' means 'for all but finitely many.' The operations of addition and multiplication are componentwise, which makes $\mathbb{A}$ a commutative ring with identity. We also have the norm $|\cdot|_{\mathbb{A}}$ defined on $\mathbb{A}^\times$ by

$$|(a_v)|_{\mathbb{A}} = \prod_v |a_v|_v.$$

One key property of $\mathbb{A}$ is the diagonal imbedding of the global field k into it, which maps $a \in k$ to the adele with a at each place. This imbedding is discrete and cocompact [5].

Once we have $\mathbb{A}$, we can construct $G_{\mathbb{A}}$ in a similar way. The adelic group is the restricted direct product of G_v with respect to the open subgroups K_v for each $v < \infty$. That is,

$$G_{\mathbb{A}} = \{ (g_v)_{v \in S} | \, g_v \in G_v \text{ and for almost all } v < \infty, g_v \in K_v \}.$$

Componentwise multiplication makes $G_{\mathbb{A}}$ a group, and the above imbedding of k into $\mathbb{A}$ induces an natural diagonal imbedding of G_k into $G_{\mathbb{A}}$.

3. Matsumoto's metaplectic group

We would like to construct a cocycle on G, over a local field, using only its algebraic group structure. This is done in several steps, by gradually expanding the cover from the torus to the full group. While we would like to be working abstractly, we will follow along our construction with the group GL_r because the formulas can be written explicitly in terms of matrix components.

We start with H, the maximal torus, which is the subgroup of diagonal matrices in GL_r. In this case, the cocycle is defined as

$$\sigma\left(\begin{pmatrix} h_1 & & & \\ & h_2 & & \\ & & \ddots & \\ & & & h_r \end{pmatrix}, \begin{pmatrix} h'_1 & & & \\ & h'_2 & & \\ & & \ddots & \\ & & & h'_r \end{pmatrix}\right) = \prod_{i<j}(h_i, h'_j)$$

for $(\cdot,\cdot)$ the nth-order local Hilbert symbol. This symbol is a nontrivial bilinear homomorphism from $k_v^\times$ to $\mu_n(k)$ defined in any text on local class field theory. There is a list of convenient identities for the Hilbert symbol in [1] that will be utilized without reference throughout this paper.

Now we can expand our cocycle by defining it on the Weyl group. The Weyl group is defined as the normalizer of H modulo the centralizer of H, so we expect that our cocycle is 'preserved' under conjugation by $w \in W$. By preserve, we can't ask that the values of σ are unchanged, but rather that the cohomology class of σ remains the same. This imlies that the coverings will be isomorphic.

On GL_r we have, for $w, w' \in W$ and $h \in H$,

$$\sigma(h, w) = \sigma(w, w') = 1, \text{ and } \sigma(w, h) = \prod_{i<j, w(j)<w(i)} (h_i, h_j)^{-1}.$$

The action of w on the indices is permutation by conjugation on the elements of H. Note that for a more general form, we actually are taking the product over positive roots which w takes to negative roots by conjugation.

After expanding the cocycle to HW, we have all the 'twisting' of the metaplectic group that we will get. Now we construct the pullback of G and $\widetilde{H}\widetilde{W}$, which will let the entire group inherit this twisting. The pullback only allows us to construct a variety, but the automorphisms of this variety will define the full metaplectic group. So our pullback is defined by this diagram:

$$\begin{array}{ccc} A & \xrightarrow{\pi_1} & G \\ \big\downarrow{\scriptstyle \pi_2} & & \big\downarrow{\scriptstyle R} \\ \widetilde{H}\widetilde{W} & \xrightarrow{p} & HW \end{array}$$

where R is defined using the Bruhat decomposition. We have

$$G = BWB = \coprod_{w\in W} BwB = \coprod_{w \in W} (HN_+)w(HN_+),$$

So we can fix the set of coset representatives in HW of $N_+\backslash G/N_+$, and R is defined to map G to HW by sending g to its coset representative.

Automorphisms of the variety A are defined on the coordinates in G and $\widetilde{H}\widetilde{W}$ and they will end up defining left-multiplication of our metaplectic group. For $g \in G$, $(l, \zeta) \in \widetilde{H}\widetilde{W}$, the automorphisms are first defined as:

$$\begin{aligned} (h, \zeta') \circ (g, (l, \zeta)) &= \big(p(h)g, (hl, \zeta\zeta'\sigma(h, l))\big) && \text{for } (h, \zeta) \in \widetilde{H}, \\ n \circ (g, (l, \zeta)) &= (ng, (l, \zeta)) && \text{for } n \in N_+, \text{ and} \\ s \circ (g, (l, \zeta)) &= \big(sg, (R(sg)R(g)^{-1}, 1) \cdot (l, \zeta)\big) && \text{for simple } s \in W. \end{aligned}$$

These define the left multiplication, and the entire metaplectic group generated by these three sets of elements. This group of automorphisms can be shown to act simply transitively on A. From this we get that the metaplectic group is in one-to-one correspondence with A, so our group of automorphisms can be thought of as defining the multiplication on A, which we know is an n-cover of G.

4. Kubota's explicit cocycle

In the case of GL_2, the cocycle defining the covering is fairly simple to write down, from [8]. Given k and n as above, let $G = GL_2$ we define the map $X : G_k \longrightarrow k^\times$ by

$$X\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{cases} c, & \text{if } c \neq 0, \\ d, & \text{otherwise.} \end{cases}$$

Then we define the cocycle $\sigma : G_v \times G_v \longrightarrow \mu_n$ by

$$\sigma(g_1, g_2) = \left(\frac{X(g_1g_2)}{X(g_1)}, \frac{X(g_1g_2)}{X(g_2)}\right)_v \cdot \left(\det(g_1), \frac{X(g_1g_2)}{X(g_1)}\right)_v,$$

where these parentheses are Hilbert symbols. This gives the direct definition of $\widetilde{G}_v$, and $\widetilde{G}_{\mathbb{A}}$ in the usual way, by letting the sets of each $\widetilde{G}$ be just $G \times \mu_n$ and defining the multiplication as

$$(g_1, \zeta_1)(g_2, \zeta_2) = (g_1g_2, \zeta_1\zeta_2\sigma(g_1, g_2)).$$

5. Metaplectic lifts

Now that we have the exact sequence

$$1 \longrightarrow \mu_n(k) \xrightarrow{i} \widetilde{G} \xrightarrow{p} G \longrightarrow 1,$$

we can naturally ask the following question: over which subgroups of G does the cover 'split?' In other words, for what subgroups X of G does there exist a homomorphism $s : X \longrightarrow \widetilde{G}$ which has $p \circ s = \mathrm{Id}_X$? If this s exists, it is called the *lift* of X, and its image in $\widetilde{G}$ will be written X^*. The choice of lift can be nonunique when there exists a nontrivial homomorphism from X to the nth roots of unity. In these cases we choose one lift to be the preferred one. In other cases, there is no lift of the subgroup. In these cases it may still be desired to consider X as a subgroup of $\widetilde{G}$. To do this, the only subgroup that can be used is the preimage of X under p, which will be a nontrivial n-cover of X (if it were trivial, the copy of X containing the identity would be a lift of X). We will always write $p^{-1}(X)$ as $\widetilde{X}$.

Going through the special subgroups defined above, let's quickly note which can be lifted to the metaplectic group and discuss other subgroups of $\widetilde{G}$ we will be considering. The cocycle is definitely not trivial on H or Z, so we only have their preimages under p, $\widetilde{H}$ and $\widetilde{Z}$. The center of $\widetilde{H}$ is $\widetilde{H}_n\widetilde{Z}$, and we fix a maximal abelian subgroup of $\widetilde{H}$ as $\widetilde{H}_*$. Because the definition of the cocycle isn't affected by the subgroup N_+, it is not surprising that this subgroup can be lifted to $\widetilde{G}$, over both the local fields and the adeles. Over k_v, for v a place with $|n|_v = 1$, we can construct a lift of K_v, $K_v^* \subset \widetilde{G}$ [11]. If v is one of the finite places for which $|n|_v < 1$, then there does exist a compact open subgroup of G_v that can be lifted to $\widetilde{G}_v$, but it may have to be chosen to be smaller than K_v.

Because these compact groups are contained in the open sets that the restricted direct product is taken with respect to, we may actually consider the *direct* product

$$K_{\mathbb{A}} = \prod_v K_v$$

as a subgroup of $G_{\mathbb{A}}$. If at the places with $|n|_v < 1$ we replace the K_v with the smaller compact groups (let us call these K_v' for now) that were liftable, we may lift the entire product to $\widetilde{G}_{\mathbb{A}}$. This construction is absolutely crucial for our eventual approach to studying these metaplectic objects. We use this compact subgroup to define our representation below by requiring our functions be invariant under right translation by it, so our representation will be smooth. But it is important that this compact group is the lift of $\prod K_v'$ rather than the preimage under p, because we want our functions to be *genuine*. A function is genuine if we have $f \circ i = \varepsilon$, i.e. it respects the metaplectic twist. If f were invariant under right translation by $p^{-1}(\prod K_v)$ rather than the lift, we would instead have $f \circ i = 1$ and so f would be an 'inflated' function on G, rather than a true function on $\widetilde{G}$.

Finally, we may consider the subgroup G_k as a subgroup of $G_{\mathbb{A}}$. This may actually be lifted to $\widetilde{G}_{\mathbb{A}}$. We will write s_0 for the lift of G_k to $\widetilde{G}_{\mathbb{A}}$.

For $G = GL_2$, the property that G_k lifts to $\widetilde{G}_{\mathbb{A}}$ is derived from Hilbert reciprocity, since the product over all local Hilbert symbols of any two elements of k gives one. Since the cocycle is written completely as the product of Hilbert symbols, it is trivial on the diagonal lift of G_k into $\widetilde{G}_{\mathbb{A}}$. So the lift can be defined as

$$s_0(g) = (g, 1).$$

The lift of $\prod_v K_v'$ isn't as simple, Kubota [8] has constructed the lift

$$\kappa_v\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{cases} \left(c, \dfrac{d}{ad-bc}\right)_v, & \text{if } 0 < |c|_v < 1, \text{ and} \\ 1, & \text{if } |c|_v = 1, 0. \end{cases}$$

6. Metaplectic representations

Now to construct our metaplectic forms. These are representations of the metaplectic group over a local field, constructed as right regular representations on particular function spaces. We begin by constructing these function spaces. For GL_r and any root $\alpha = (i, j)$, we can define a character μ_α on H (and inflated to B, $\widetilde{H}$, and $\widetilde{B}$) by

$$\mu_\alpha(h) = \left|\frac{h_i}{h_j}\right|_v.$$

Now this definition can be expanded to the root lattice by taking the sum of two roots to the product of their corresponding characters. Then, if we let ρ be half the sum of the positive roots, we can define the character $\mu(h) = \mu_\rho(h)$. In the case of $G = GL_2$, we get

$$\mu\left(\begin{pmatrix} a & \\ & b \end{pmatrix}\right) = \left|\frac{a}{b}\right|^{\frac{1}{2}}.$$

If instead we have a character ω on $\widetilde{H}_n$, we can define a character ω_α^n for any root $\alpha = (i, j)$ on k by $\omega_\alpha^n(c)$ being $\omega(a_{ll})$ for the diagonal matrix (a_{ll}) with

$$a_{ll} = \begin{cases} c^n, & \text{if } l = i, \\ c^{-n}, & \text{if } l = j, \text{ and} \\ 1 & \text{otherwise.} \end{cases}$$

For ε some fixed imbedding of $\mu_n(k)$ into $\mathbb{C}$, let ω be a quasicharacter of $\widetilde{H}_n\widetilde{Z}$ with $\omega \circ i = \varepsilon$. Then we can extend ω to ω', a quasicharacter on $\widetilde{H}_*$, because this is a maximal abelian subgroup of $\widetilde{H}$. Now, inflate ω' to be defined on $\widetilde{B}_*$ by making it trivial on N_+^*. Now we can define our function space. Let $V(\omega')$ be the set of functions $f : \widetilde{G} \mapsto \mathbb{C}$ such that

$$f(bg) = (\omega'\mu)(b)f(g)$$

for $b \in \widetilde{B}_*$, and for which there exists an open subgroup $K_f \subset \widetilde{G}$ such that

$$f(gk) = f(g)$$

for all $k \in K_f$. The representation we consider with this space is the right regular representation, that is, if $(\pi, V(\omega'))$ is our representation, then

$$(\pi(g)(f))(g') = f(g'g).$$

It is simple to see that this is a representation, in fact it is the induced representation from $\widetilde{B}_*$ to $\widetilde{G}$ with respect to the character $\omega'\mu$. The factor of μ is included to shift the set of characters so there is a symmetry from ω to ω^{-1}.

7. Calculation

Throughout we will employ the analogy with the nonmetaplectic case. A good reference for this case is chapter three of [3].

We have a number field k along with the above constructions of the adeles $\mathbb{A}$ and the n-cover metaplectic groups (both local $\widetilde{G}_v$ and adelic $\widetilde{G}_{\mathbb{A}}$). Furthermore, we choose an integral ideal of k, $\mathfrak{A}$, with the following properties. For any place v of k, we have the implication

$$|n|_v < 1 \Longrightarrow |\mathfrak{A}|_v < 1.$$

Second, for every v, if $x \in k_v$ has $x \equiv 1 \pmod{\mathfrak{A}}$, then x is an nth power in $k_v^{\times}$. Finally, if $S(\mathfrak{A})$ is the set of places v with $|\mathfrak{A}|_v < 1$, then the ring $r^{S(\mathfrak{A})}$ of $S(\mathfrak{A})$-integers is a principle ideal domain. This is the subring of k defined as

$$r^{S(\mathfrak{A})} = \{x \in k : |x|_v \leq 1, \forall v < \infty, v \notin S(\mathfrak{A})\}.$$

A suitable $\mathfrak{A}$ can always be chosen. The first condition only requires some finite number of divisors for each generator of $\mathfrak{A}$. The second condition requires that at some finite number of these places $\mathfrak{A}$ is small enough that $1+\mathfrak{A}$ lies inside the open subgroup $k_v^{\times n}$. For a uniformizer π_v of k_v, $\{\pi_v^i \mathfrak{A}\}_{i>0}$ forms a basis around 0, which implies that we can adjust $\mathfrak{A}$ at the places dividing n to satisfy this condition. Then the last condition can be satisfied by multiplying $\mathfrak{A}$ but the prime ideals that are coset representatives for the finite set of generators of the ideal class group. Once we have this $\mathfrak{A}$, we also have the set $S(\mathfrak{A})$, which we will sometimes write as S. Below 'bad' places refers to the (finite number of) places in S, and 'good' places are those that are not.

The strategy to writing down the L-series is inspired by the nonmetaplectic case. We construct a spherical element of the function space $V(\omega_*)$ of our representation, and then compute its Fourier-Whittaker coefficients and plug them into a series in ω_*.

For each ω, that is extended to ω_* uniquely to satisfy certain properties in II.3 of [7], the spherical element f of $V(\omega_*)$ is defined locally. At each place, we fix its values on a set of double coset representatives of its invariant groups. At the good places, we have $f_v(I) = 1$ for I the identity element of the lift, f_v is right K_v^*-invariant, and the support of f_v is $\widetilde{B}_{*,v}K_v^*$. From the invariance and $f_v \in V(\omega_{*,v})$, we have that this determines all of f's values at the good places. At the bad places, f_v must be a 'test function' with a smaller support; it only is supported on $\widetilde{B}_{*,v}L_v^*(\mathfrak{A})w_0$, for w_0 the lift of the long element of G. This is because the full compact group K_v doesn't lift at these places. Here f_v is well-defined by fixing the single value $f_v(w_0) = 1$. Just as in the nonmetaplectic case, this spherical section is a flat section with respect to the compact subgroup

$$\prod_{v \notin S} K_v^* \times \prod_{v \in S} L_v^*(\mathfrak{A})$$

over all ω_*.

We have the what we will refer to as the Fourier-Whittaker series

$$\Psi_f(\omega, f_*) = L_f(|\cdot|_{\mathbb{A}}\omega_\alpha^n) \cdot \sum_{\eta \in H_{n,k} \backslash H_k} (\omega_\infty \mu_\infty)(\eta) \prod_{v<\infty} G_v(\omega_{*,v}, s_v(\eta), h_v),$$

for

$$(\omega_\infty \mu_\infty)(\eta) = \prod_{v|\infty} (\omega_v \mu_v)(s_v(\eta)).$$

In [7], this function is shown to agree with an Dirichlet series whose coefficients are Gauss sums. The computation goes by the following straightforward and computational method, which will be reproduced here explicitly. They compute $G_v(\omega_*, \eta_v, h_v)$ for v good and bad. Then these values are plugged into $\Psi_f(\omega, f_*)$, and the products are distributed at the good places. Triviality conditions are noted, and the nonzero terms are matched with the Gauss sum series. My strategy is to exemplify the twisted multiplicativity of this construction directly, and then observe that the only series satisfying this property is one with Gauss sums as coefficients.

The Fourier-Whittaker series is defined as a product of two factors. The first factor,

$$L_f(|\cdot|_{\mathbb{A}}\omega_\alpha^n) = \prod_{v<\infty} L_v(|\cdots|_v \omega_{\alpha,v}^n) = \prod_{v<\infty}{}' (1 - |\pi_v|_v \omega_{\alpha,v}^n(\pi_v))^{-1},$$

a product over places of the local L-series (the prime on the second product just indicates that when the factor would be $(1-1)^{-1}$, i.e. when the character is ramified at v, remove it). The second factor is a sum of a product over places of local Fourier-Whittaker integrals. If we could interchange this sum and product then we would have our Fourier-Whittaker series written as an Euler product. As it turns out, this interchange isn't possible. The key to the failure of this 'factorization' is that the outer sum is global, and the inner product is local. The global lift does not factor into local lifts at each place; as we have seen, locally we cannot lift, but instead use a preferred section. This is why we need $s_v(\eta)$ in the argument of G_v.

It is important to note that the product of integrals can only be written as an integral over the adelic unipotent group because of the selection of the support of f. That is, we may write

$$\prod_{v<\infty} \int_{N^*_{+,v}} F_v(n)dn = \int_{N^*_{+,\mathbb{A}}} F(n)dn$$

for $F(n) = \otimes_{v<\infty} F_v(n_v)$, when F_v is supported on $\mathfrak{O}_v$ for almost all v. This is necessary because we have a direct product over the finite places of k on the left, while $\mathbb{A}$ is the *restricted* direct product of k_v on the right.

8. Local calculation

We need to compute $G_v(\omega_*, s_v(\eta), h_v)$. First we consider the good places, $v \notin S$. We let

$$p(\eta_v) = \begin{pmatrix} \eta_1 & \\ & \eta_2 \end{pmatrix}, \quad h_v = \left(\begin{pmatrix} h_1 & \\ & h_2 \end{pmatrix}, 1 \right), \quad n = \left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}, 1 \right) \in N^*_{+,v}, \quad \psi_v(x) = e_v \left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \right)$$

so ψ_v is the additive character of k_v corresponding to e_v. Also, because this entire section deals only with the local calculation, we will just write η and h for η_v and h_v. Then we have

$$G_v(\omega_*, \eta_v, h_v) = \int_{N^*_{+,v}} f^0_{*,v}(s_v(\eta) w_0 n h_v)\overline{e}_v(n)dn$$

$$= \int_{k_v} f^0_{*,v}\left(\left(\begin{pmatrix} \eta_1 & \\ & \eta_2 \end{pmatrix}, 1\right)\left(\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, 1\right)\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}, 1\right)\left(\begin{pmatrix} h_1 & \\ & h_2 \end{pmatrix}, 1\right)\right)\overline{\psi}_v(x)dx$$

$$= \int_{k_v} f^0_{*,v}\left(\left(\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, 1\right)\left(\begin{pmatrix} \eta_2 h_1 & \eta_2 h_2 x \\ & \eta_1 h_2 \end{pmatrix}, (\eta_2, h_2)_v\right)\right)\overline{\psi}_v(x)dx.$$

When combining all of the elements together, the only metaplectic 'twist' comes from multiplying η^{w_0} by h, which gives the Hilbert symbol $(\eta_2, h_2)_v$.

So, now we are computing an integral over k_v. One way to simplify the calculation is to determine conditions on which the integral is trivial. Because v is a good place, we know that the conductor of ψ_v is exactly $\mathfrak{O}_v$. Given some $y \in \mathfrak{O}_v$, consider the equality

$$\left(\begin{pmatrix} \eta_2 h_1 & \eta_2 h_2 x \\ & \eta_1 h_2 \end{pmatrix}, (\eta_2, h_2)_v\right) \cdot \left(\begin{pmatrix} 1 & y \\ & 1 \end{pmatrix}, 1\right) = \left(\begin{pmatrix} \eta_2 h_1 & \eta_2 h_2 (x + \frac{h_1}{h_2} y) \\ & \eta_1 h_2 \end{pmatrix}, (\eta_2, h_2)_v\right).$$

Since $f^0_{*,v}$ is right K^*_v-invariant, we see that if $\frac{h_1}{h_2} y \notin \mathfrak{O}_v$, for $y \in \mathfrak{O}_v$, then the integral is trivial. This follows because the substitution

$$x \mapsto x + \frac{h_1}{h_2} y$$

multiplies the entire integral by $\psi_v(\frac{h_1}{h_2} y) \neq 1$, while fixing the region of integration. So we have shown that

$$|h_1|_v > |h_2|_v \Longrightarrow \int f \cdot \psi dx = 0.$$

Now, for $|h_1|_v \leq |h_2|_v$, we have to break the region of integration into two parts based on this previous calculation. The first part is for

$$|x|_v \leq \left|\frac{h_1}{h_2}\right|_v.$$

By our nontriviality assumption, we have that $x \in \mathfrak{O}_v$, so $\psi_v(x) = 1$. Also, the inequality defining this region gives

$$\left|\frac{h_2}{h_1} x\right|_v \in \mathfrak{O}_v, \text{ and } \begin{pmatrix} \eta_2 h_1 & \eta_2 h_2 x \\ & \eta_1 h_2 \end{pmatrix} = \begin{pmatrix} \eta_2 h_2 & \\ & \eta_1 h_2 \end{pmatrix}\begin{pmatrix} 1 & \frac{h_2}{h_1} x \\ & 1 \end{pmatrix}.$$

So, by moving the long Weyl group element from left to right in the argument of $f^0_{*,v}$, and then using the K^*_v-invariance, we have

$$\int_{k_v} f^0_{*,v}\left(\left(\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, 1\right)\left(\begin{pmatrix} \eta_2 h_1 & \eta_2 h_2 x \\ & \eta_1 h_2 \end{pmatrix}, (\eta_2, h_2)_v\right)\right)\overline{\psi}_v(x)dx$$

$$= \int_{|x|_v \leq |\frac{h_1}{h_2}|_v} f^0_{*,v}\left(\left(\begin{pmatrix} \eta_1 h_2 & \\ & \eta_2 h_1 \end{pmatrix}, (\eta_2, h_2)_v\right)\right)dx.$$

For this to lie in the support of $f^0_{*,v}$, we need $\mathrm{ord}_v(\eta_1 h_2) \equiv \mathrm{ord}_v(\eta_2 h_1) \equiv 0 \pmod{n}$. When this is so, we obtain the value

$$(\omega_*\mu)\left(\left(\begin{pmatrix} \eta_1 h_2 & \\ & \eta_2 h_1 \end{pmatrix}, (\eta_2, h_2)\right)\right) \cdot \mathrm{Vol}(\{|x|_v \leq \left|\frac{h_1}{h_2}\right|_v\}) = \omega_{*,v}\mu_v(\eta_v h_v^{w_0}) \left|\frac{h_1}{h_2}\right|_v.$$

The rest of the integral is over the set $\{|x|_v > \left|\frac{h_1}{h_2}\right|_v\}$. For the argument

$$\left(\begin{pmatrix} & \eta_1 h_2 \\ \eta_2 h_1 & \eta_2 h_2 \end{pmatrix}, (\eta_2, h_2)_v\right)$$

to lie in the support of $f^0_{*,v}$, $\widetilde{B}_{*,v}K_v^*$, we need to be able to factor the matrix as $b'k'$, and we may assume that the determinant of k' is 1. Then we have, for some $y \in k_v^\times$ and $z \in k_v$,

$$\begin{pmatrix} & \eta_1 h_2 \\ \eta_2 h_1 & \eta_2 h_2 x \end{pmatrix} = \begin{pmatrix} \eta_1 h_2 y & z \\ & \eta_2 h_1 y^{-1} \end{pmatrix} k' \Longrightarrow k' = \begin{pmatrix} -\frac{z}{\eta_1 h_2} & (y^{-1} - \frac{xz}{\eta_1 h_1}) \\ y & \dfrac{h_2}{h_1} xy \end{pmatrix} \in K_v.$$

So we want to choose y and z so that each entry of k' is in fact an integer of k_v. From the bottom right component, we know $|y|_v \leq \left|\frac{h_1}{h_2 x}\right|_v < 1$ (we get the right part of this inequality by our region of integration, and this supercedes the inequality we get from the bottom left component of the matrix). So $y^{-1} \notin \mathfrak{O}_v$, so from the top right component of the matrix, we get the equality

$$|y^{-1}|_v = \left|\frac{xz}{\eta_1 h_1}\right|_v,$$

since their difference has to be an integer. But from the top left, we have $|z|_v \leq |\eta_1 h_2|_v$. Combining these, we get

$$|y^{-1}|_v \leq \left|\frac{h_2 x}{h_1}\right|_v \Longrightarrow |y|_v \geq \left|\frac{h_1}{h_2 x}\right|_v.$$

So we get that two inequalities actually fix the v–norm of y as $\left|\dfrac{h_1}{h_2 x}\right|_v$. We can choose a uniformizer of k_v, π_v, and let $t = \mathrm{ord}_v(\frac{h_1}{h_2 x})$ and $y = \pi_v^t$. Then we have to choose z so that

$$y^{-1} - \frac{xz}{\eta_1 h_1} \in \mathfrak{O}_v \Longrightarrow \frac{xz}{\eta_1 h_1} \equiv \pi_v^{-t} \pmod{\mathfrak{O}_v}.$$

The simplest choice for such a z is $u\eta_1 h_2$, for the $u \in (O)_v^\times$ that gives

$$u^{-1}\pi_v^{-t} = \frac{xh_2}{h_1}.$$

Before we plug our factorization into the integral, we need to consider the metaplectic twist of our group. First, we need to lift k' to K_v^*. By our construction, we know that we can lift it to

$$k' \mapsto (k', (c, \frac{d}{\det(k')})_v), \text{ for } k' = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

when c is not a unit or 0, and it lifts to $(k', 1)$ otherwise. Using our form of k', we have

$$k' \mapsto \left(k', \left(y, \frac{h_2}{h_1} xy\right)_v\right) \in K_v^*.$$

Note that $\left(y, \frac{h_2}{h_1}xy\right)_v = \left(y, \frac{h_2}{h_1}x\right)_v$ since $(y,y)=1$ (this comes from our assumption that -1 is an nth power).

The cocycle applied to this factorization gives

$$\left(\frac{\eta_2 h_1}{\eta_2 h_1 y^{-1}}, \frac{\eta_2 h_1}{y}\right)_v \left(\eta_1 h_2 \eta_2 h_1, \frac{\eta_2 h_1}{\eta_2 h_1 y^{-1}}\right)_v = (y, \eta_2 h_1 y^{-1})_v (\eta_1 h_2 \eta_2 h_1, y)_v = (\eta_1 h_2, y)_v.$$

When we use the factorization, we have to offset these twists. The result looks like

$$\int_{|x|_v \le |\frac{h_1}{h_2}|_v} f^0_{*,v}\left(\left(\begin{pmatrix} \eta_1 h_2 \pi_v^t & u\eta_1 h_2 \\ & \eta_2 h_1 \pi_v^{-t}\end{pmatrix} k', (\eta_2, h_2)_v (y, \eta_1 h_2)_v (y, \frac{h_2}{h_1}x)_v^{-1}\right)\right) \psi_v(x) dx.$$

The root of unity can be simplified to $(\eta_2, h_2)_v (y, \eta_1 h_1 x^{-1})_v$.

Now, any given x in the region of integration to contribute to the integral, we need the left factor to lie in $\widetilde{B}_{*,v}$, which means that we need

$$\mathrm{ord}_v(\eta_1 h_2 \pi_v^t) \equiv \mathrm{ord}_v(\eta_2 h_1 \pi_v^{-t}) \equiv 0 \pmod n.$$

From our choice of t, this implies that

$$\mathrm{ord}_v(x) \equiv \mathrm{ord}_v(\eta_1 h_1) \equiv -\mathrm{ord}_v(\eta_2 h_2) \pmod n.$$

So we have a nontriviality condition for this part of the integral, we need $\equiv \mathrm{ord}_v(\eta_1 h_1) \equiv -\mathrm{ord}_v(\eta_2 h_2) \pmod n$. Then the integral becomes

$$\begin{aligned}
&\sum_{i \equiv \mathrm{ord}(\eta_1 h_1), i < \mathrm{ord}(h_1 h_2^{-1})} \int_{\mathrm{ord}_v(x)=i} f^0_{*,v}\left(\left(\begin{pmatrix} \eta_1 h_2 \pi_v^t & u\eta_1 h_2 \\ & \eta_2 h_1 \pi_v^{-t}\end{pmatrix} k', (\eta_2, h_2)_v (y, \eta_1 h_1 x^{-1})_v\right)\right) \psi_v(x) dx \\
= &\sum_{i \equiv \mathrm{ord}(\eta_1 h_1), i < \mathrm{ord}(h_1 h_2^{-1})} \int_{\mathrm{ord}_v(x)=i} (\omega_{*,v}\mu_v)\left(\left(\begin{pmatrix} \eta_1 h_2 \pi_v^t & u\eta_1 h_2 \\ & \eta_2 h_1 \pi_v^{-t}\end{pmatrix}, (\eta_2, h_2)_v (\pi_v^t, \eta_1 h_1 x^{-1})_v\right)\right) \psi_v(x) dx \\
= &\sum_{i \equiv \mathrm{ord}(\eta_1 h_1), 0 \le i < \mathrm{ord}(h_1 h_2^{-1})} \int_{\mathrm{ord}_v(x)=i} (\omega_{*,v}\mu_v)\left(\left(\begin{pmatrix} \eta_1 h_2 \pi_v^t & u\eta_1 h_2 \\ & \eta_2 h_1 \pi_v^{-t}\end{pmatrix}, (\eta_2, h_2)_v (\pi_v^t, \eta_1 h_1 x^{-1})_v\right)\right) dx \\
&+ \int_{\mathrm{ord}_v(x)=-1} (\omega_{*,v}\mu_v)\left(\left(\begin{pmatrix} \eta_1 h_2 \pi_v^t & u\eta_1 h_2 \\ & \eta_2 h_1 \pi_v^{-t}\end{pmatrix}, (\eta_2, h_2)_v (\pi_v^t, \eta_1 h_1 x^{-1})_v\right)\right) \psi_v(x) dx
\end{aligned}$$

The last equality comes from the cancellation of ψ_v over the shells on which it is nontrival beyond the first, that is, for all $i < -1$. The only effect the value of x has within a shell is in the Hilbert symbol, factor depending on x is $(\pi_v^t, x^{-1})_v$. Now, inside any shell with $i < -1$, if x and x' are equivalent mod $\pi_v^{i+1}\mathfrak{O}_v$, it is simple algebra to show that their inverses are equivalent mod π_v^{-i+1}. This means that the ratio of the Hilbert symbols for x and x' is $(y, u')_v$ for $u' \in (1+\mathfrak{p}_v)$, but this implies that u' is an nth power, so the two Hilbert symbols give the same result. Over each of these equivalence classes ψ_v is nontrivial, so we have cancellation.

For $i = -1$, x runs over the nonzero equivalence classes of $\mathfrak{p}_v^{-1}/\mathfrak{O}_v$, with ψ_v constant on each class. Below we will see that this shell produces a Gauss sum, but first we will deal with the nonnegative i. Now, the volume of the ith shell is $q_v^{-i}(1-q_v^{-1})$, for $q_v = \#(\mathfrak{O}_v/\mathfrak{p}_v)$, so we can finally express the full answer. Let $l = \mathrm{ord}_v(h_1/h_2)$, so we have $t = \mathrm{ord}_v(\frac{h_1}{h_2 x}) = l - i$. For the sum over nonnegative i, we get

$$\sum_{i \equiv \mathrm{ord}(\eta_1 h_1), 0 \le i < \mathrm{ord}(h_1 h_2^{-1})} \left(\int_{\mathrm{ord}_v(x)=i} dx\right) (\omega_{*,v}\mu_v) \left(\left(\begin{pmatrix} \eta_1 h_2 \pi_v^t & u\eta_1 h_2 \\ & \eta_2 h_1 \pi_v^{-t} \end{pmatrix}, (\eta_2, h_2)_v (\pi_v^t, \eta_1 h_1 x^{-1})_v\right)\right)$$

$$= \sum_i (q_v^{-i}(1-q_v^{-1}))(\omega_{*,v}\mu_v) \left(\left(\begin{pmatrix} \eta_1 h_2 & \\ & \eta_2 h_1 \end{pmatrix}, (\eta_2, h_2)_v\right)\right) (\omega_{*,v}\mu_v) \left(\begin{pmatrix} \pi_v^{l-i} & \\ & \pi_v^{i-l} \end{pmatrix}, (\pi_v^t, \eta_1 h_1 x^{-1})_v\right)$$

$$= (\omega_{*,v}\mu_v)(\eta h^{w_0}) \sum_i (q_v^{-i}(1 - q_v^{-1})) \omega_{\alpha,v}^n(\pi_v)^{\frac{l-i}{n}} q_v^{i-l}$$

$$= (\omega_{*,v}\mu_v)(\eta h^{w_0}) \sum_{i \equiv \mathrm{ord}(\eta_1 h_1), 0 \le i < \mathrm{ord}(h_1 h_2^{-1})} (1 - q_v^{-1}) \omega_{\alpha,v}^n(\pi_v)^{\frac{l-i}{n}} \left|\frac{h_1}{h_2}\right|_v$$

$$= (\omega_{*,v}\mu_v)(\eta h^{w_0})(1 - q_v^{-1}) \frac{(\omega_{\alpha,v}^n(\pi_v) - \omega_{\alpha,v}^n(\pi_v)^{\lfloor \frac{l}{n} \rfloor + 1})}{(1 - \omega_{\alpha,v}^n(\pi_v))} \left|\frac{h_1}{h_2}\right|_v$$

Now, for the summand for $i = -1$, it is easier to use the factorization corresponding to $y = h_1 h_2^{-1} x^{-1}$ and $z = \eta_1 h_2$. This gives us the matrix factorization

$$p(\eta w_0 n h) = \begin{pmatrix} \eta_1 h_1 x^{-1} & \eta_1 h_2 \\ & \eta_2 h_2 x \end{pmatrix} \begin{pmatrix} 1 & \\ h_1 h_2^{-1} x^{-1} & 1 \end{pmatrix}.$$

Note that we are in the region with $|x|_v > |h_1 h_2^{-1}|_v$, so the second factor has integral entries, as required to evaluate $f_{*,v}^0$. This element of K_v lifts with 1 as its root of unity by the formula $(c, d\det(g)^{-1})_v$. Finally, the cocycle of the product of matrices in the factorization gives

$$\left(\frac{h_1\eta_2}{h_2\eta_2 x}, \frac{h_1\eta_2}{h_1 h_2^{-1} x^{-1}}\right)_v \left(\eta_1\eta_2 h_1 h_2, \frac{h_1\eta_2}{h_2\eta_2 x}\right)_v = \left(\frac{h_1}{h_2 x}, x\right)_v \left(\frac{\eta_1 h_1}{h_1 h_2^{-1} x^{-1}}\right)_v$$

$$= \left(\eta_1, \frac{h_1}{h_2 x}\right)_v \left(h_1, \frac{1}{h_2 x}\right)_v \left(\frac{1}{x}, h_1 h_2\right)_v = \left(\eta_1, \frac{h_1}{h_2 x}\right)_v \left(\frac{h_1}{x}, \frac{1}{h_2}\right)_v$$

$$= \left(\eta_1, \frac{h_1}{h_2 x}\right)_v \left(\frac{h_1}{x}, \frac{1}{h_2}\right)_v = \left(\eta_1, \frac{h_1}{h_2 x}\right)_v \left(\frac{h_1}{h_2 x}, \frac{1}{h_2}\right) = \left(\eta_1 h_2, \frac{h_2 x}{h_1}\right)_v^{-1}.$$

Since this comes from the product, we need to include the inverse when we factor, so in the metaplectic group we end up with

$$\eta w_0 n h = \left(\begin{pmatrix} \eta_1 h_1 x^{-1} & h_2\eta_1 \\ & \eta_1 h_2 x \end{pmatrix}, (\eta_1 h_2, h_1)_v \left(\eta_1 h_2, \frac{h_2 x}{h_1}\right)_v\right) \left(\begin{pmatrix} 1 & \\ h_1 h_2^{-1} x^{-1} & 1 \end{pmatrix}, 1\right)$$

$$= \left(\begin{pmatrix} \eta_1 h_1 x^{-1} & h_2\eta_1 \\ & \eta_1 h_2 x \end{pmatrix}, (\eta_1 h_2, h_2 x)_v\right) \left(\begin{pmatrix} 1 & \\ h_1 h_2^{-1} x^{-1} & 1 \end{pmatrix}, 1\right).$$

Applying this to the integral, we get

$$\int_{\mathrm{ord}_v(x)=-1} (\omega_{*,v}\mu_v)\left(\left(\begin{pmatrix} \eta_1 h_1 x^{-1} & h_2\eta_1 \\ & \eta_1 h_2 x\end{pmatrix}, (\eta_1 h_2, h_2 x)_v\right)\right)\psi_v(x)dx$$
$$= \sum_{c\in(\pi_v^{-1}\mathfrak{O}_v\setminus\mathfrak{O}_v)/\mathfrak{O}_v} (\omega_{*,v}\mu_v)\left(\left(\begin{pmatrix} \eta_1 h_1 c^{-1} & h_2\eta_1 \\ & \eta_1 h_2 c\end{pmatrix}, (\eta_1 h_2, h_2 c)_v\right)\right)\psi_v(c)$$
$$= (\omega_{*,v}\mu_v)\left(\eta h\begin{pmatrix}\pi_v & \\ & \pi_v^{-1}\end{pmatrix}\right)\sum_{u\in\mathfrak{O}_v\setminus\mathfrak{p}_v/\mathfrak{p}_v} \varepsilon((\pi_v,u)^{-l-1})\psi_v(u\pi_v^{-1})\varepsilon\left(\left(\frac{h_1}{h_2},\pi_v\right)_v\right).$$

The first equality breaks the integral up into the regions that are have equivalent x modulo $\mathfrak{O}_v$. For these regions, which have volume 1, the integrand is constant, so we are left with the finite sum over c, the representatives of $x \pmod{\mathfrak{O}_v}$. In the second equality, we change our index of summation to $u = c\cdot\pi_v \in \mathfrak{O}_v^\times$. We also rewrite the argument of the character as a product in the metaplectic group, which includes the $(\eta_1,h_2)_v$ and $(\eta_1,c)_v$ Hilbert symbols. Also, it requires a factor of $(h_1,c)_v$. This is why we have to multiply by its inverse, $\varepsilon((h_1,\pi_v))$. Then the only part left over is $\varepsilon((h_2,c)_v)$, which, when inverted twice, gives the last factor. We will consider this sum to be a local Gauss sum, and write it using the following notation:

$$g_v^m(\pi_v) = \sum_{u\in(\mathfrak{O}_v\setminus\mathfrak{p}_v)/\mathfrak{p}_v} \varepsilon((\pi_v,u)_v)^m\psi_v(u\pi_v^{-1}).$$

So, combining all of our results together, we have for v good,

$$G_v(\omega_{*,v},\eta,h) = (\omega_{*,v}\mu_v)(\eta_v h_v^{w_0})\left|\frac{h_1}{h_2}\right|_v + (\omega_{*,v}\mu_v)(\eta h^{w_0})(1-q_v^{-1})\frac{(\omega_{\alpha,v}^n(\pi_v)-\omega_{\alpha,v}^n(\pi_v)^{\lfloor\frac{l}{n}\rfloor+1})}{(1-\omega_{\alpha,v}^n(\pi_v))}\left|\frac{h_1}{h_2}\right|_v$$
$$+ (\omega_{*,v}\mu_v)\left(\eta h\begin{pmatrix}\pi_v & \\ & \pi_v^{-1}\end{pmatrix}\right) g_v^{-l-1}(\pi_v)\varepsilon\left(\left(\frac{h_1}{h_2},\pi_v\right)_v\right)$$
$$= (\omega_{*,v}\mu_v)(\eta h^{w_0})\frac{(1-q_v^{-1}\omega_{\alpha,v}^n(\pi_v)-(1-q_v^{-1})\omega_{\alpha,v}^n(\pi_v)^{\lfloor\frac{l}{n}\rfloor+1})}{(1-\omega_{\alpha,v}^n(\pi_v))}\left|\frac{h_1}{h_2}\right|_v$$
$$+ (\omega_{*,v}\mu_v)\left(\eta h\begin{pmatrix}\pi_v & \\ & \pi_v^{-1}\end{pmatrix}\right) g_v^{-l-1}(\pi_v)\varepsilon\left(\left(\frac{h_1}{h_2},\pi_v\right)_v\right).$$

Now for the bad places, which actually make for an easier computation. This is due to the smaller support of f_v for these places. First we want to restrict x to exactly the region that makes the argument of $f_{*,v}^0$ fall into its support, $\widetilde{B}_{*,v}L_v^*(\mathfrak{A})w_0$. We have the matrix decomposition

$$p(\eta w_0 n h) = \begin{pmatrix}\eta_1 h_2 & \\ \eta_2 h_2 x & \eta_2 h_1\end{pmatrix} w_0 = \begin{pmatrix}\eta_1 h_2 & \\ & \eta_2 h_1\end{pmatrix}\begin{pmatrix}1 & \\ \frac{h_2}{h_1}x & 1\end{pmatrix} w_0.$$

So we need $\eta_1 h_2 \in k_v^{\times n}$ for nontriviality, since otherwise the left matrix in the decomposition is not in $\widetilde{B}_{*,v}$. If we have this our integral is over the region

$$|x|_v \le \left|\frac{h_2}{h_1}\mathfrak{A}\right|_v$$

since the middle matrix in the decomposition must be equivalent to the identity matrix mod $\mathfrak{A}$.

It should be noted that adding the extra flexibility in the form of the variables y and z that we used in the decomposition at the good places does not uncover more nontrivial regions of this integral. If y is included, we would immediately derive that $y \equiv 1 \pmod{\mathfrak{A}}$, and by our choice of $\mathfrak{A}$ this implies that $y \in k_v^{\times n}$, so we have the same condition on $\eta_1 h_2$. Similarly, including z doesn't have any effect on the bottom left term of the middle matrix which gives us our second condition, so we may assume $y = 1$ and $z = 0$.

The integral is thus

$$\int_{|x|_v \leq |h_1 h_2^{-1}|_v |\mathfrak{A}|_v} f^0_{*,v}(\eta w_0 n h)\psi_v(x)dx.$$

Now, let $|\mathfrak{A}|_v = |\pi_v^i|_v$. Similarly to the calculation at good places, we can see that using the substitution

$$x \mapsto x + \frac{h_1}{h_2}\pi_v^i$$

fixes the value of $f^0_{*,v}$, so in order to have a nontrivial integral, we need

$$|\mathfrak{D}^{-1}|_v \geq \left|\frac{h_1}{h_2}\pi_v^i\right|_v,$$

where $\mathfrak{D}$ is the idele with $\psi(x) = \psi_0(\mathfrak{D}x)$ for ψ_0 a fixed additive character on the adeles that has conductor $\mathfrak{O}_v$ at every place. It is known [3] that the additive characters are indexed by these ideles. Then the requirement on h_i is most succinctly written

$$\left|\frac{h_1}{h_2}\right|_v \leq |\mathfrak{D}^{-1}\mathfrak{A}^{-1}|_v.$$

The decomposition gives

$$\int_{|x|_v \leq |h_1 h_2^{-1}|_v |\mathfrak{A}|_v} f^0_{*,v}(\eta w_0 n h)\psi_v(x)dx = (\omega_{*,v}\mu_v)(\eta h^{w_0}) \int_{|x|_v \leq |h_1 h_2^{-1}|_v |\mathfrak{A}|_v} \psi_v(x)dx,$$

and the condition we found in the previous paragraph gives that this integral is nontrivial, and from [13], we see that the integral gives the final result

$$(\omega_{*,v}\mu_v)(\eta h^{w_0})|\mathfrak{A}|_v \left|\frac{h_1}{h_2}\right|_v |\mathfrak{D}|_v^{\frac{1}{2}}.$$

So a summary of our calculations is:

(1) if $v \in S$, then $G_v(\omega_{*,v}, \eta_v, h_v) = 0$ unless $\eta_v h_v^{w_0} \in \widetilde{H}_{*,v}$ and $\left|\frac{h_1}{h_2}\right|_v \leq |(A)|_v^{-1}|\mathfrak{D}|_v^{-1}$, in which case

$$G_v(\omega_{*,v}, \eta_v, h_v) = (\omega_{*,v}\mu_v)(\eta h^{w_0})|\mathfrak{A}|_v \left|\frac{h_1}{h_2}\right|_v |\mathfrak{D}|_v^{\frac{1}{2}},$$

(2) if $v \notin S$, then $G_v(\omega_{*,v}, \eta_v, h_v) = 0$ unless $\left|\frac{h_1}{h_2}\right|_v \leq 1$, in which case

$$G_v(\omega_{*,v}, \eta_v, h_v) = (\omega_{*,v}\mu_v)(\eta h^{w_0}) \frac{(1 - q_v^{-1}\omega_{\alpha,v}^n(\pi_v) - (1-q_v^{-1})\omega_{\alpha,v}^n(\pi_v)^{\lfloor \frac{l}{n} \rfloor + 1})}{(1-\omega_{\alpha,v}^n(\pi_v))} \left|\frac{h_1}{h_2}\right|_v$$
$$+ (\omega_{*,v}\mu_v)\left(\eta h \begin{pmatrix} \pi_v & \\ & \pi_v^{-1} \end{pmatrix}\right) g_v^{-l-1}(\pi_v)\varepsilon\left(\left(\frac{h_1}{h_2}, \pi_v\right)_v\right),$$

where if the argument of $\omega_{*,v}$ is not an element of $H_{*,v}$, then the term is taken to be zero.

9. Dirichlet series

Now that we know the value of G_v for each place, we can combine them to evaluate of our series. These terms will be matched to the terms of a Dirichlet series, which we will now define.

For $c \in r^{S(\mathfrak{A})}$, we have the nth-power *residue symbol*

$$\left(\frac{\cdot}{c}\right) : (r^{S(\mathfrak{A})}/(c))^\times \longrightarrow \mu_n(k),$$

a homomorphism trivial on exactly the classes that are nth powers. The normal definition is to let d map to the element of $\mu_n(k)$ that is congruent to

$$d^{\frac{N(c)-1}{n}} \pmod{c}.$$

However, we will eventually be evaluating these roots of unity as elements of $\mathbb{C}$ via the map ε, which represents an arbitrary selection of primitive nth root of unity. Because of this lack of canonicality, we may choose any homomorphism of $(r^{S(\mathfrak{A})}/(c))^\times$ to $\mu_n(k)$, provided it still has kernel $(r^{S(\mathfrak{A})}/(c))^{\times n}$. A convenient choice is to let

$$\left(\frac{d}{c}\right) = \prod_{v \notin S(\mathfrak{A}), v|c} (c, d)_v.$$

This formulation is well-defined, because if $d \equiv d' \pmod{c}$, then the ratio of their residue symbols is

$$\prod_{v \notin S(\mathfrak{A}), v|c} (c, d^{-1}d')_v,$$

and since $d^{-1}d' \in (1+\mathfrak{p}_v) \subset k_v^{\times n}$ for each $v|c$, these Hilbert symbols are all 1. Also, it is clearly trivial on the nth powers, thanks to being a product of nth-power Hilbert symbols. Finally, consider the symbol for c a prime. Then c has exactly one v dividing it, so the residue symbol is a Hilbert symbol. Since the Hilbert symbol $(c, \cdot)_v$ is surjective onto the nth roots of unity because $\mathrm{ord}_v(c) = 1$, so does the residue symbol. By a count of cosets, we see that the largest the kernel of the residue symbol mod c could be is the set of nth powers. Finally, by construction, this formula is multiplicative in c, so we see that in fact this formula defines an nth-power residue symbol.

We define our Gauss sum as

$$g_{\mathfrak{A}}(\varepsilon, m, c) = \sum_d \varepsilon\left(\left(\frac{d}{c}\right)\right) \cdot e_\infty\left(\frac{md}{c}\right).$$

Then for ϕ a quasicharacter of $\prod_{v\in S_\infty(\mathfrak{A})} k_v^{\times n}$ that is trivial on the diagonal imbedding of

$$U_n(\mathfrak{A}) = \{u \in (r^{S(\mathfrak{A})})^\times | u \in k_v^{\times n} \forall v \in S(\mathfrak{A})\},$$

we can define our Dirichlet series as

$$\Psi^0_{\mathfrak{A}}(\phi,\varepsilon,m) = \left(\sum_c g_{\mathfrak{A}}(\varepsilon,m,c)\phi(c)\right) L_f(|\cdot|_{\mathbb{A}}\phi_1^n).$$

The sum is taken over $c \in (r^{S(\mathfrak{A})} \setminus \{0\})/U_n(\mathfrak{A})$ with $c \in k_v^{\times n}$ for all $v \in S_\infty(\mathfrak{A})$. We should show that this series is well-defined, because c is chosen up to a factor in $U_n(\mathfrak{A})$. The quasicharacter ϕ has been chosen to be trivial on this factor, and the effect on the Gauss sum of multiplying c by an element of $U_n(\mathfrak{A})$ is a factor of a Hilbert symbol. However, this Hilbert symbol will be trivial because any Hilbert symbol with one entry an element of $U_n(\mathfrak{A})$ is equal to 1, so the summands are in fact independent of the choice of c.

Also, L_f is the product over the finite places of the local L-series,

$$L_f(|\cdot|_{\mathbb{A}}\phi_1^n) = \prod_{v<\infty} L_v(|\cdot|_v\phi_{1,v}^n) = \prod_{v<\infty}(1-|\pi_v|_v\phi_{1,v}^n(\pi_v))^{-1}.$$

Here ϕ_1^n is the character of $\mathbb{A}^\times$ that takes

$$(x_v) \mapsto \prod_{v\in S_\infty(\mathfrak{A})} \phi_v((x_v^n))$$

and is unramified at the places outside $S_\infty(\mathfrak{A})$, that is, $\phi_v(\pi_v) = 1$ for all $v \notin S_\infty(\mathfrak{A})$.

It is easy to show that for a proper choice of ϕ, this sum converges. For example, if we have

$$|\phi((c_v))| < \prod_{v\in S_\infty(\mathfrak{A})} |c_v|_v^{-\frac{3}{2}-\epsilon},$$

for some $\epsilon > 0$ and with each $c_v \in k_v^{\times n}$, the series converges absolutely by bounding the Gauss sums by $O(c^{\frac{1}{2}})$, which is a well-known bound.

10. Global calculation

Now we evaluate the original series, with the assumption that $h \in H_k^*$ (once again $l_v = \mathrm{ord}_v(h_1/h_2)$), and match its terms to this Dirichlet series.

We have

$$\Psi_f(\omega,f_*) = L_f(||\cdot||_{\mathbb{A}}\omega_\alpha^n)\cdot \sum_{\eta\in H_{n,k}\setminus H_k} (\omega_\infty\mu_\infty)(\eta)\prod_{v<\infty} G_v(\omega_{*,v},s_v(\eta),h_v).$$

The product of G_v can now be computed by considering the calculation at each place. We will put the v subscripts back on η and h, since we are now considering them at multiple places in the different factors. We have

$$\prod_{v<\infty} G_v = \prod_{v\in S(\mathfrak{A})} (\omega_{*,v}\mu_v)(s_v(\eta)h_v^{w_0})|\mathfrak{A}|_v \left|\frac{h_1}{h_2}\right|_v |\mathfrak{D}|_v^{\frac{1}{2}}$$

$$\times \prod_{v\notin S(\mathfrak{A}),v<\infty} \left((\omega_{*,v}\mu_v)(s_v(\eta)h_v^{w_0}) \frac{(1-q_v^{-1}\omega_{\alpha,v}^n(\pi_v) - (1-q_v^{-1})\omega_{\alpha,v}^n(\pi_v)^{\lfloor \frac{l}{n}\rfloor+1})}{(1-\omega_{\alpha,v}^n(\pi_v))} \left|\frac{h_1}{h_2}\right|_v \right.$$

$$\left. + (\omega_{*,v}\mu_v)\left(s_v(\eta)h_v\begin{pmatrix}\pi_v & \\ & \pi_v^{-1}\end{pmatrix}\right) g_v^{-l-1}(\pi_v)\varepsilon\left(\left(\frac{h_1}{h_2},\pi_v\right)_v\right)\right).$$

For any given η to contribute to the series, each factor in the product must be nonzero. For a fixed h that we will assume satisfies the requirements of nontriviality, each place restricts what η can be in order for the product to be nonzero. At the bad places, we require $s_v(\eta)h_v^{w_0} \in \widetilde{H}_{*,v}$ in addition to the restriction on h that is $|h_1h_2^{-1}|_v \leq |\mathfrak{A}^{-1}\mathfrak{D}^{-1}|_v$. Therefore, we need $\eta_1 h_{2,v} \in k_v^{\times n}$.

At the good places, each factor is actually the sum of two different products. To distribute these, we can sum over the set of places that we choose one summand and at the other places choose the other summand. First, we will show that if the second summand in chosen at an infinite number of places, the term does not contribute to the series. This is because we need the product

$$s_v(\eta)h_v s_v\left(\begin{pmatrix}\pi_v & \\ & \pi_v^{-1}\end{pmatrix}\right)$$

to be in $\widetilde{H}_{*,v}$. But $h\in H_k^*$ and $\eta \in H_k$, so there is a finite number of places at which either of these has an entry that is not a unit. Therefore, at almost all places v, the product above has components with orders 1 and -1, which implies it is not an element of $\widetilde{H}_{*,v}$. At the same time, given any finite set of places, there is a choice of η that satisfies the condition. So if $\Sigma(\mathfrak{A})$ is the (infinite) set of finite places that are not in $S(\mathfrak{A})$, we can distribute at the good places by choosing finite subsets of this set of places:

$$\prod_v G_v = \sum_{\Sigma'\subset\Sigma(\mathfrak{A})} \prod_{v\in\Sigma'} \left((\omega_{*,v}\mu_v)\left(s_v(\eta)h_v s_v\left(\begin{pmatrix}\pi_v & \\ & \pi_v^{-1}\end{pmatrix}\right)\right) g_v^{-l_v-1}(\pi_v)\varepsilon\left(\left(\frac{h_1}{h_2},\pi_v\right)_v\right)\right)$$

$$\times \prod_{v\in(\Sigma(\mathfrak{A})\setminus\Sigma')} \left((\omega_{*,v}\mu_v)(s_v(\eta)h_v^{w_0})\frac{(1-q_v^{-1}\omega_{\alpha,v}^n(\pi_v)-(1-q_v^{-1})\omega_{\alpha,v}^n(\pi_v)^{\lfloor\frac{l_v}{n}\rfloor+1})}{(1-\omega_{\alpha,v}^n(\pi_v))}\left|\frac{h_1}{h_2}\right|_v\right)$$

$$\times \prod_{v\in S(\mathfrak{A})} |\mathfrak{A}|_v|\mathfrak{D}|_v^{\frac{1}{2}}(\omega_{*,v}\mu_v)(s_v(\eta)s_v(h^{w_0}))\left|\frac{h_1}{h_2}\right|_v.$$

This last product, at the bad places, can be computed for some of the factors. The simplified form is

$$\prod_{v\in S(\mathfrak{A})} |\mathfrak{A}|_v|\mathfrak{D}|_v^{\frac{1}{2}}(\omega_{*,v}\mu_v)(s_v(\eta)s_v(h^{w_0}))\left|\frac{h_1}{h_2}\right|_v = N(\mathfrak{A})^{-1}|\mathfrak{D}|^{-\frac{1}{2}}\prod_{v\in S(\mathfrak{A})}(\omega_{*,v}\mu_v)(s_v(\eta)h_v^{w_0})\left|\frac{h_1}{h_2}\right|_v$$

since $|\mathfrak{D}|_v = 1$ for all $v\notin S(\mathfrak{A})$.

We also remember that η is summed over $H_k/H_{k,n}$, so each component of η may be multiplied by an nth power of $k^\times$. So for a fixed choice of Σ', we may require

$$\mathrm{ord}_v(\eta_1 h_2) = 0 \text{ for all } v \in (\Sigma(\mathfrak{A}) - \Sigma')$$

and

$$\mathrm{ord}_v(\eta_1 h_1) = -1 \text{ for all } v \in \Sigma'$$

by multiplying by the proper nth power. Note that there is still the choice of a factor of a unit, modulo nth powers. We can also require that

$$\mathrm{ord}_v(\eta_2 h_1) = 0 \text{ for all } v \in (\Sigma(\mathfrak{A}) \setminus \Sigma')$$

and

$$\mathrm{ord}_v(\eta_2 h_2) = 1 \text{ for all } v \in \Sigma'.$$

So we can replace η with $\eta^{-1}h^{w_0}$, since the cosets over which η is summed are simply permuted by multiplying by inverting and multiplying by an element of H_k. Then the requirements above can be simplified to η_1 having order 0 at the good places not in Σ' and order $(-\mathrm{ord}_v(m) - 1)$ at the places in Σ', where $m = h_1 h_2^{-1}$. Similarly, the requirements on η_2 becomes the inverse of those for η_1, so we can just let $\eta_1 = u\eta_2^{-1}$ for u a unit of $r^{S(\mathfrak{A})}$. So for a fixed h and Σ', η_2 runs over unit multiples of a number, modulo nth powers of units, and then η_1 runs over the same unit multiples of η_2^{-1}.

$$\begin{aligned}
\Psi_f(\omega, f_*) &= L_f(|\cdot|_{\mathbb{A}}\omega_\alpha^n) \cdot \sum_{\eta \in H_k/H_{k,n}} (\omega_\infty \mu_\infty)(\eta(h^{w_0})^{-1}) \prod_{v<\infty} G_v(\omega_{*,v}, s_v(\eta), h_v) \\
&= L_f(|\cdot|_{\mathbb{A}}\omega_\alpha^n) N(\mathfrak{A})^{-1} |\mathfrak{D}|^{-\frac{1}{2}} \sum_{\Sigma' \subset \Sigma, \eta_2, u} (\omega_\infty \mu_\infty)(\eta h) \prod_{v \in \Sigma'} g_v^{-l_v - 1}(\pi_v) \varepsilon((\eta_2^{-1}, m\pi_v)_v) \\
&\times \prod_{v \in (\Sigma \setminus \Sigma')} (\omega_{*,v}\mu_v)\left(\begin{pmatrix} \eta_2^{-1} & \\ & \eta_2 \end{pmatrix}, 1 \right) \left(\frac{1 - q_v^{-1}\omega_{\alpha,v}^n(\pi_v) - (1 - q_v^{-1})\omega_{\alpha,v}^n(\pi_v)^{\lfloor \frac{l_v}{n} \rfloor + 1}}{1 - \omega_{\alpha,v}^n(\pi_v)} \right) \left| \frac{h_1}{h_2} \right|_v \\
&\qquad \times \prod_{v \in S(\mathfrak{A})} (\omega_{*,v}\mu_v) \left(s_v \begin{pmatrix} \eta_2^{-1} & \\ & u\eta_2 \end{pmatrix} \right) (\eta_2^{-1}, h_1)_v \left| \frac{h_1}{h_2} \right|_v .
\end{aligned}$$

We can simplify this sum the following observations.

(1) The sum over u can be factored out of the sum to form

$$\sum_u \left(\prod_{v \in S_\infty(\mathfrak{A})} (\omega_{*,v}\mu_v) \left(s_v \begin{pmatrix} 1 & \\ & u \end{pmatrix} \right) \right) = \sum_u (1),$$

where $S_\infty(\mathfrak{A})$ is just the set of places that are either infinite or in $S(\mathfrak{A})$. The equality comes from our choice of extension of $\omega_{*,v}$. Then the sum is just the number of u that we are summing over, or the index of the group of units that are nth powers in the group of units of $r^{S(\mathfrak{A})}$, $[U(\mathfrak{A}) : U_n(\mathfrak{A})]$.

(2) We can use Hilbert's product formula to take $|h_1/h_2|_v$ out of the last two products and square the term $\mu_\infty(h^{w_0})$, and balance this by inserting $|h_2/h_1|_v$ at the remaining places,

those in Σ'. The squared term at the infinite places becomes $N(h_1/h_2)^{-1} = N(m)^{-1}$, which can be distributed outside the sum.

(3) The factor $(\omega_{*,v}\mu_v)(\eta)$ for $v \in (\Sigma \setminus \Sigma')$ is trivial, by the fact that at these places η_2 has order zero.

(4) We can choose the unit multiple of π_v to be η_2/m at each of the places in Σ', by our conditions on the order of η_2 at these places.

(5) Finally, to simplify the sum, we note that although η_2 is chosen to depend on the places in Σ', we can switch this to choose η_2 first and then determine Σ' as the finite set of places v with $|\eta_2|_v < 1$. So the sum can actually be considered to be solely over η_2.

So using these ideas, we get

$$L_f(|\cdot|_{\mathbb{A}}\omega_\alpha^n)[U(\mathfrak{A}):U_n(\mathfrak{A})]N(\mathfrak{A})^{-1}|\mathfrak{D}|^{-\frac{1}{2}}N(m)^{-1}\sum_{\eta_2}\left(\prod_{v\in S_\infty(\mathfrak{A})}(\omega_v\mu_v)(s_v(\eta))\right)$$

$$\times \prod_{v\in\Sigma'(\eta_2)} g_v^{-l_v-1}(\eta_2/m)\varepsilon((h_1,h_2)_v)\left|\frac{h_2}{h_1}\right|_v \times \prod_{v\in(\Sigma\setminus\Sigma'(\eta_2))}\left(\frac{1-q_v^{-1}\omega_{\alpha,v}^n(\pi_v)-(1-q_v^{-1})\omega_{\alpha,v}^n(\pi_v)^{\lfloor\frac{l_v}{n}\rfloor+1}}{1-\omega_{\alpha,v}^n(\pi_v)}\right).$$

This sum over η_2 will be matched to $\Psi_{\mathfrak{A}}^0$. The conditions over which η_2 is summed will turn out to be the exact conditions for c that make $g_{\mathfrak{A}}(\varepsilon, m, c)$ nonzero. That is, we need either

$$\mathrm{ord}_v(c) = 1 + \mathrm{ord}_v(m),$$

in which case the local Gauss sum at v is nontrivial, or

$$n|\mathrm{ord}_v(c) \text{ and } \mathrm{ord}_v(m) \geq \mathrm{ord}_v(c),$$

in which case each summand in the local Gauss sum is 1, so the sum is just the number of terms, or $\#((r_v/(c))^\times)$. In all other cases, the Gauss sum is zero. So for any given c that gives a non-zero Gauss sum at every place in Σ, we define $\Sigma'(c)$ as the set of places v for which $\mathrm{ord}_v(m)+1 = \mathrm{ord}_v(c)$.

Now, we can show the following Gauss sum identity:

$$g_{\mathfrak{A}}(\varepsilon^{-1}, m, c) = \prod_{v\in\Sigma'(c)} g_v^{-l_v-1}(cm^{-1})|m|_v^{-1}\cdot \prod_{v\in\Sigma(\mathfrak{A})\setminus\Sigma'(c)}\#((r_v/(c))^\times).$$

The identity comes down to the Chinese Remainder Theorem for $r^{S(\mathfrak{A})}$, which we know holds by requiring that this ring be a PID. Let c be a unit multiple of $\prod_v p_v^{i_v}$. From our assumption of nontriviality, we have that for each v with $i_v \neq 0$, $i_v = 1 + l_v$ where l_v is $\mathrm{ord}_v(m)$. Then we have

$$g_{\mathfrak{A}}(\varepsilon^{-1}, m, c) = \sum_{d\in(r^{S(\mathfrak{A})}/(c))^\times}\varepsilon^{-1}\left(\left(\frac{d}{c}\right)\right)\cdot\psi_\infty\left(\frac{md}{c}\right) = \sum_{d\in(r^{S(\mathfrak{A})}/(c))^\times}\varepsilon^{-1}\left(\prod_{v|c, v\notin\mathfrak{A}}(c,d)_v\right)\cdot\psi_\infty\left(\frac{md}{c}\right).$$

Now, if we have $d \equiv d'$ mod π_v for all $v \in \Sigma'(c)$, we can show that the summands are equal. We need to show that the Hilbert symbol and the character are both equal. At the places $v \notin \Sigma'(c)$, we know $\mathrm{ord}_v(c) \equiv 0 \pmod n$, so these Hilbert symbols are trivial. At $v \in \Sigma'(c)$, the Hilbert symbols $(c,d)_v$ and $(c,d')_v$ differ by a factor of a Hilbert symbol with second entry congruent to 1 mod π_v, which is an nth power. The characters' arguments are md/c and md'/c, which differ by an integer, since d and d' differ by a multiple of π_v at every place at which $\mathrm{ord}_v(c) > \mathrm{ord}_v(m)$, and at these places

$$\mathrm{ord}_v(c) = \mathrm{ord}_v(m) + 1.$$

So we will simplify the sum by combining these terms under a single representative, but in doing so we must multiply by the number of terms we are combining. This factor is

$$\prod_{v \in \Sigma'(c)} |m|_v^{-1} \times \prod_{v \in \Sigma \backslash \Sigma'(c)} \#((r_v/(c))^\times).$$

So we continue the calculation:

$$g_{\mathfrak{A}}(\varepsilon^{-1}, m, c) = \sum_{d \in (r^{S(\mathfrak{A})}/(\prod_{v \in \Sigma'(c)} \pi_v))^\times} \prod_{v \in \Sigma'(c)} \left(\varepsilon^{-l_v - 1} \left(\left(\frac{c}{m}, d \right)_v \right) |m|_v^{-1} \right) \psi_\infty \left(\frac{dm}{c} \right) \times \prod_{v \in \Sigma \backslash \Sigma'(c)} \#((r_v/(c))^\times).$$

We may replace c by $\pi_v^{l_v+1}$ in the Hilbert symbol, since both their ratio and d are in $r_v^\times$. Now we use the Chinese Remainder Theorem to move the sum inside the product, which produces our local Gauss sum, once we note that $\psi_\infty = \overline{\psi}_f$ on k.

$$g_{\mathfrak{A}}(\varepsilon^{-1}, m, c) = \prod_{v \in \Sigma'(c)} \left(\sum_{d \in (r^{S(\mathfrak{A})}/(\pi_v))^\times} \varepsilon^{-l_v - 1} \left(\left(\frac{c}{m}, d \right)_v \right) \psi_v \left(\frac{dm}{c} \right) \right) |m|_v \times \prod_{v \in \Sigma \backslash \Sigma'(c)} \#((r_v/(c))^\times).$$

With the definition of our local Gauss sum, this gives the desired identity.

This identity allows us to break up the Gauss sums of our Dirichlet series into their local parts. In the Dirichlet series, the Gauss sums are the coefficients of a character ϕ. So we can collect the c's of the series for which $\Sigma'(c)$ is fixed. This clearly includes all of the $r^{S(\mathfrak{A})}$-unit multiples of c, but at the places $v \notin \Sigma'$, we have a choice of the order of c, it can be any multiple of n that is less than or equal to $\mathrm{ord}_v(m)$. These c's can be collected together into a finite geometric series which begins with a c_0 that has order 0 at every place not in Σ'. The simplification is then

$$\begin{aligned} \sum_{d: \Sigma'(d) = \Sigma'(c)} g_{\mathfrak{A}}(\varepsilon, m, d) \phi(d) &= \sum_d \left(\prod_{v \in \Sigma'(c)} g_v^{-l_v - 1} \left(\frac{c_0}{m} \right) |m|_v^{-1} \times \prod_{v \in (\Sigma \backslash \Sigma'(c))} \#((r_v/(d))^\times) \right) \phi(d) \\ &= \phi(c_0) \prod_{v \in \Sigma'(c)} g_v^{-l_v - 1} \left(\frac{c_0}{m} \right) |m|_v^{-1} \times \prod_{v \in (\Sigma \backslash \Sigma'(c))} \sum_{d_v} \#((r_v/(d_v))^\times) \phi(d_v/c_0). \end{aligned}$$

In this final line, we have used the Chinese remainder theorem to move the sum inside the product, and so our sum is locally over d_v, modulo units. This sum can be indexed by the v-order of d, which will be a nonnegative multiple of n. Now, we know, for $\mathrm{ord}_v(d) \geq 1$,

$$\#((r_v/(d))^\times) = q_v^{\mathrm{ord}_v(d) - 1}(q_v - 1),$$

while if $\mathrm{ord}_v(d) = 0$, we just get 1.

So at each place $v \in \Sigma \setminus \Sigma'(c)$, we have the sum

$$\phi_v(1) + q_v^{n-1}(q_v - 1)\phi_v(\pi_v^n) + q_v^{2n-1}(q_v - 1)\phi_v(\pi_v^{2n}) + \cdots + q_v^{\lfloor \frac{l_v}{n} \rfloor n-1}(q_v - 1)\phi_v(\pi_v^{\lfloor \frac{l_v}{n} \rfloor n})$$

$$= 1 + \frac{q_v^{n-1}(q_v - 1)\phi_v(\pi_v^n) - q_v^{(\lfloor \frac{l_v}{n} \rfloor + 1)n}(1 - q_v^{-1})\phi_v(\pi_v^n)^{\lfloor \frac{l_v}{n} \rfloor + 1}}{1 - q_v^n \phi_v(\pi_v^n)}$$

$$= \frac{1 - q_v^{n-1}\phi_v(\pi_v^n) - (1 - q_v^{-1})(q_v^n \phi_v(\pi_v^n))^{\lfloor \frac{l_v}{n} \rfloor + 1}}{1 - q_v^n \phi_v(\pi_v^n)}.$$

This finally gives us our relationship between the two series. Combining our results for the Gauss sums and putting them into the Dirichlet series, we get that

$$\Psi_{\mathfrak{A}}^0(\phi, \varepsilon^{-1}, m) = L_f(|\cdot|_{\mathbb{A}}\phi_1^n) \sum_c \phi(c) \prod_{v \in \Sigma'(c)} g_v^{-l_v - 1}\left(\frac{c_0}{m}\right) |m|_v^{-1}$$

$$\times \prod_{v \in (\Sigma \setminus \Sigma'(c))} \frac{1 - q_v^{n-1}\phi_v(\pi_v^n) - (1 - q_v^{-1})(q_v^n \phi_v(\pi_v^n)^{\lfloor \frac{l_v}{n} \rfloor + 1})}{1 - q_v^n \phi_v(\pi_v^n)}.$$

Comparing this result to Ψ_f, we find that, if we choose ϕ such that

$$\phi(c) = \prod_{v \in S_\infty(\mathfrak{A})} (\omega_v \mu_v)\left(s_v \begin{pmatrix} c_v^{-1} & \\ & c_v \end{pmatrix}\right) = \prod_{v \in S_\infty(\mathfrak{A})} \omega_{\alpha,v}(c_v)^{-1} |c_v|_v^{-1}$$

then we also get, by letting $c = \pi_v^n$,

$$q_v^n \phi_v(\pi_v^n) = \omega_{\alpha,v}^n(\pi_v), \text{ or } (|\cdot|_v \phi_v)^n(\cdot) = \omega_{\alpha,v}^n(\cdot).$$

These terms both appear in the Fourier-Whittaker series, so we can show, for $h_1/h_2 = m$ chosen inside the ideal $\mathfrak{A}^{-1}\mathfrak{D}^{-1}$,

$$\Psi_f(\omega, f_*) = N(\mathfrak{A})^{-1} |\mathfrak{D}|^{-\frac{1}{2}} [U(\mathfrak{A}) : U_n(\mathfrak{A})] (\omega_\infty \mu_\infty)(h^{w_0})^{-1} \Psi_{\mathfrak{A}}^0(\omega^{(1)}, \varepsilon^{-1}, m).$$

11. Twisted multiplicativity

In multiple Dirichlet series, the coefficients are functions $H(\vec{c})$ for $\vec{c}$ an r-tuple of coordinates that are summed over the integers of the global field modulo integral units. There is an Euler product of the series exactly when H is multiplicative. When $r > 1$, this means

$$H(c_1 d_1, c_2 d_2, \ldots, c_r d_r) = H(c_1, c_2, \ldots, c_r) H(d_1, d_2, \ldots, d_r)$$

whenever $\gcd(c_i, d_j) = 1$ for all i, j. These Euler product forms exist exactly as they do for rank one series, except of course that each local factor is itself a multiple series.

Now twisted multiplicativity exists when we do not quite have the equality above. We can think of multiplicativity as the triviality of the cocycle

$$\sigma(\vec{c}, \vec{d}) = \frac{H(\vec{c}\vec{d})}{H(\vec{c})H(\vec{d})},$$

for $\vec{c}$ and $\vec{d}$ relatively prime. For more general series, σ is a nontrivial cocycle that defines the 'twist' of the multiplicativity. The key fact is if you are given the prime power series, that is, the coefficients $H(\pi^{i_1}, \pi^{i_2}, \ldots, \pi^{i_r})$ for each fixed prime π, and the cocycle σ, the series can be reconstructed uniquely.

12. Example: Dirichlet series

The Dirichlet series

$$\Psi^0_{\mathfrak{A}}(\phi, \varepsilon, m) = \left(\sum_c g_{\mathfrak{A}}(\varepsilon, m, c)\phi(c) \right) L_f(|\cdot|_{\mathbb{A}} \phi_1^n)$$

as defined in [7] has twisted multiplicativity. We look at the Gauss sum coefficients, and in particular, how to write the sum over $c_1 c_2$ in terms of the Gauss sums for c_1 and c_2, when c_1 and c_2 are coprime. This last condition implies that no place divides both c_1 and c_2. Also, for ease of writing, in any of the products over places below, we only consider places outside $S(\mathfrak{A})$.

$$g_{\mathfrak{A}}(\varepsilon, m, c_1c_2) = \sum_{d \in \left(r^{S(\mathfrak{A})}/(c_1c_2)\right)^\times} \varepsilon\left(\prod_{v|c_1c_2, v\notin S(\mathfrak{A})} (c_1c_2, d)_v\right) \psi_\infty\left(\frac{md}{c_1c_2}\right)$$

$$= \sum_{e \in \left(r^{S(\mathfrak{A})}/(c_1c_2)\right)^\times} \varepsilon\left(\prod_{v|c_1}(c_1, d)_v\right) \varepsilon\left(\prod_{w|c_2}(c_2, d)_w\right) \psi_\infty\left(\frac{md}{c_1c_2}\right)$$

$$= \sum_{d_i \in \left(r^{S(\mathfrak{A})}/(c_i)\right)^\times, d=d_1c_2+d_2c_1} \varepsilon\left(\prod_{v|c_1}(c_1, d_1c_2+d_2c_1)_v\right) \varepsilon\left(\prod_{w|c_2}(c_2, d_1c_2+d_2c_1)_w\right) \psi_\infty\left(\frac{m(d_1c_2+d_2c_1)}{c_1c_2}\right)$$

$$= \sum_{d_i \in \left(r^{S(\mathfrak{A})}/(c_i)\right)^\times, d=d_1c_2+d_2c_1} \varepsilon\left(\prod_{v|c_1}(c_1, d_1c_2)_v\right) \varepsilon\left(\prod_{w|c_2}(c_2, d_2c_1)_w\right) \psi_\infty\left(\frac{md_1}{c_1}+\frac{md_2}{c_2}\right)$$

$$= \varepsilon\left(\prod_{v|c_1}(c_1, c_2)_v\right) \varepsilon\left(\prod_{w|c_2}(c_2, c_1)_w\right) \left(\sum_{d_1 \in \left(r^{S(\mathfrak{A})}/(c_1)\right)^\times} \prod_{v|c_1} \varepsilon\left((c_1, d_1)_v\right) \psi_\infty\left(\frac{md_1}{c_1}\right)\right)$$

$$\times \left(\sum_{d_2 \in \left(r^{S(\mathfrak{A})}/(c_2)\right)^\times} \prod_{w|c_2} \varepsilon\left((c_2, d_2)_w\right) \psi_\infty\left(\frac{md_2}{c_2}\right)\right) = \varepsilon\left(\left(\frac{c_2}{c_1}\right)\right) \varepsilon\left(\left(\frac{c_1}{c_2}\right)\right) g_{\mathfrak{A}}(\varepsilon, m, c_1) g_{\mathfrak{A}}(\varepsilon, m, c_2).$$

This shows that the twisted multiplicativity of this Dirichlet series is governed by the cocycle

$$\tau(c, d) = \frac{H(cd)}{H(c)H(d)} = \varepsilon\left(\left(\frac{c}{d}\right)\left(\frac{d}{c}\right)\right) \text{ for } \gcd(c, d) = 1.$$

This is a very nice property, because the twist of the series only changes the values of the coefficients up to an nth root of unity from what the series would be if it had an Euler product. We can see that the twist will be invariant under multiplication by nth powers, because the cocycle composed entirely of nth power residue symbols.

13. Example: Fourier-Whittaker series

Next we consider the Fourier-Whittaker coefficient series, with the objective to show that it possesses an identical twisted mulitplicativity. The series is defined

$$\Psi_f(\omega, f_*) = L_f(|\cdot|_{\mathbb{A}}\omega_\alpha^n) \sum_{\eta \in H_k/H_{n,k}} \prod_{v<\infty} G_v(\omega_{*,v}, s_v(\eta), h_v).$$

The first factor (L_f) is defined as an Euler product, and it demonstrates the multiplicativity of the nth-order terms of the series (this is why the nth power of ω_α is its argument). So any cocycle that governs the multiplicativity of Ψ_f would be trivial on nth powers. So we look at the second factor, the sum over η. We want to find a formula for the cocycle of the series, so we want to consider a general term and break it into its local pieces. By our reasoning above, the twisting will only come from the places at which η is not a power of n. Also, we can ignore h by using a clever substitution that is explained later. So we will consider a set of good places U for which η is not an nth power.

Given that η is not an nth power at these places, we know what value it must take (mod $H_{n,k}$) for the term to be nontrivial from our local calculation.

For a finite set of places $U \subset \Sigma$, we have

$$\eta_U = \begin{pmatrix} \prod_{v\in U} \pi_v^{-1} & \\ & \prod_{v\in U} \pi_v \end{pmatrix}, \text{ and } s_v(\eta_U) = (\eta_U, 1) \in \widetilde{G}_v.$$

So we want to compute the cocycle on η_U and η_V for disjoint sets of places U and V. So, at U, we have the product of the relevant factors of G_v (the ones that depend on η):

$$\prod_{v\in U} (\omega_{*,v}\mu_v) \left(s_v(\eta_U) \cdot \left(\begin{pmatrix} \pi_v & \\ & \pi_v^{-1} \end{pmatrix}, 1 \right) \right)$$
$$= \prod_{v\in U} \omega_{*,v} \left(\begin{pmatrix} \prod_{w\in U, w\neq v} \pi_w^{-1} & \\ & \prod_{w\in U, w\neq v} \pi_w \end{pmatrix}, (\prod_{w\in U}, \pi_v^{-1}) \right)$$
$$= \varepsilon \left(\prod_{v\in U} \left(\prod_{w\in U} (\pi_w^{-1}, \pi_v^{-1})_v \right) \right) \cdot \prod_{v\in U} \omega_{*,v} \left(\begin{pmatrix} \prod_{w\in U, w\neq v} \pi_w^{-1} & \\ & \prod_{w\in U, w\neq v} \pi_w \end{pmatrix}, 1 \right)$$

Because $\omega_{*,v}$ was chosen to be trivial on $K_v^* \cap \widetilde{H}_{*,v}$, which contains the final argument of the character at each place, the product of the local characters is trivial. Leaving out the ε to reduce clutter, the relevant factor at U is

$$\prod_{v,w\in U} (\pi_w^{-1}, \pi_v^{-1})_v = \prod_{v,w\in U} (\pi_w, \pi_v)_v.$$

We have a similar factor at V and at $U \cup V$. So the cocycle for the Fourier-Whittaker series, let us call it τ, then becomes

$$\tau(\eta_U, \eta_V) = \frac{\prod_{v,w\in(U\cup V)} (\pi_w, \pi_v)_v}{\left(\prod_{v,w\in U} (\pi_w, \pi_v)_v\right) \left(\prod_{v,w\in V} (\pi_w, \pi_v)_v\right)}.$$

Each Hilbert symbol in the denominator appears in the numerator, and after we cancel these we are left with only the cross terms, which gives

$$\tau(\eta_U, \eta_V) = \prod_{u\in U, v\in V} [(\pi_u, \pi_v)_v (\pi_v, \pi_u)_u] = \prod_{u\in U, v\in V} \left[(\pi_u, \prod_{w\in V} \pi_w)_v (\pi_v, \prod_{w\in U} \pi_w)_u \right]$$
$$= \prod_{v\in V} (\prod_{u\in U} \pi_u, \prod_{w\in V} \pi_w)_v \cdot \prod_{u\in U} (\prod_{v\in V} \pi_v, \prod_{w\in U} \pi_w)_u$$
$$= \prod_{v\mid \eta_{V,2}} (\eta_{U,2}, \eta_{V,2})_v \cdot \prod_{v\mid \eta_{U,2}} (\eta_{V,2}, \eta_{U,2})_v = \left(\frac{\eta_{U,2}}{\eta_{V,2}} \right)^{-1} \left(\frac{\eta_{V,2}}{\eta_{U,2}} \right)^{-1}.$$

This is the inverse of the cocycle shown for the Dirichlet series, but we can offset that by replacing ε with ε^{-1} in the Dirichlet series. Then we have shown the cocycles for the series $\Psi_f(\omega, f_*)$ and $\Psi^0_{\mathfrak{A}}(\phi, \varepsilon^{-1}, m)$ are equal when $m = h_1/h_2$. So all that is left is to compare the prime power coefficients of each series.

14. Prime power calculation

To write down the prime power series for some prime π of any L-series, we just replace all the coefficients $H(\vec{c})$ with some c_i that is divisible by a prime other than π with zero. For a place $v \in \Sigma$, the prime power series for the Dirichlet series is

$$\Psi^0_{\mathfrak{A},v}(\phi, \varepsilon, m) = \sum_{i=0}^{\infty} g_{\mathfrak{A}}(\varepsilon, m, \pi_v^i)\phi(\pi_v^i).$$

The Gauss sum is trivial unless $i = \mathrm{ord}_v(m) + 1$ or $n|i$ and $i \leq m$. In the first case, the sum can be expressed as

$$g_{\mathfrak{A}}(\varepsilon, m, \pi_v^{\mathrm{ord}_v(m)+1}) = \sum_{d \in (r^{S(\mathfrak{A})}/(\pi_v^{ord_v(m)+1}))^\times} \varepsilon((\pi_v^{\mathrm{ord}_v(m)+1}, d)_v)\psi_\infty\left(\frac{d \cdot m}{\pi_v^{\mathrm{ord}_v(m)+1}}\right)$$
$$= g_v^{\mathrm{ord}_v(m)+1}(\pi_v)\varepsilon((\pi_v, m)_v) \cdot q_v^{\mathrm{ord}_v(m)},$$

the local Gauss sum, with the local Hilbert factor from the substitution and the power of q_v from the number of representatives of each class in the local Gauss sum. Since we want to replace ε with ε^{-1}, we end up with the summand

$$g_v^{-\mathrm{ord}_v(m)-1}(\pi_v)\varepsilon((\pi_v, m)_v)^{-1}q_v^{\mathrm{ord}_v(m)}.$$

At the other i, the Gauss sum becomes a trivial sum of ones, so we get

$$g_{\mathfrak{A}}(\varepsilon, m, \pi_v^i) = \#((r^{S(\mathfrak{A})}/(\pi_v^i))^\times) = q_v^i - q_v^{i-1} = \begin{cases} q_v^i(1-q_v^{-1}) & \text{if } i > 0, \text{ or} \\ 1 & \text{if } i = 0. \end{cases}$$

So in all, the local series at π_v is

$$1 + \sum_{j=1}^{\lfloor \frac{ord_v(m)}{n} \rfloor} q_v^{nj}\phi(\pi_v^{nj})(1-q_v^{-1}) + g_v^{-\mathrm{ord}_v(m)-1}(\pi_v)\varepsilon((\pi_v, m)_v)^{-1}q_v^{\mathrm{ord}_v(m)}\phi(\pi_v^{\mathrm{ord}_v(m)+1}).$$

Summing the finite geometric series, we get

$$\frac{1 - q_v^{n-1}\phi(\pi_v^n) - (1-q_v^{-1})(q_v^n\phi(\pi_v^n))^{\lfloor \frac{l}{n} \rfloor + 1}}{1 - q_v^n\phi(\pi_v^n)} + g_v^{-\mathrm{ord}_v(m)-1}(\pi_v)\varepsilon((\pi_v, m)_v)^{-1}q_v^{\mathrm{ord}_v(m)}\phi(\pi_v^{\mathrm{ord}_v(m)+1}).$$

For the prime power series of the Fourier-Whittaker coefficient series, we can take the local series to be

$$\Psi_v(\omega, f_*) = L_v(|\cdot|_v\omega^n_{\alpha,v}) \sum_{\eta_v \in H_v/H_{n,v}} G_v(\omega_{*,v}, s_v(\eta_v), h_v),$$

where $s_v(\eta_v)$ is just the chosen section (not a homomorphism)

$$s_v : H_v \mapsto \widetilde{H}_v, s_v(\eta) = (\eta, 1).$$

Then we use the local calculation of G_v to determine the value. As we determined in the local calculation section, we know for exactly which η_v the summand is zero. We will simplify the calculation by using the same substitution for η that was done in the global calculation section, $\eta \mapsto \eta h^{w_0}$. This eliminates h in every term except one, the final term in the sum below. The prime power series is then

$$\sum_{\eta\in H_v/H_{n,v}} G_v(\omega_{*,v}, s_v(\eta), h) = (\omega_{*,v}\mu_v)(\eta_v = \mathrm{Id}_v)\frac{1-q_v^{-1}\omega_{\alpha,v}^n(\pi_v) - (1-q_v^{-1})\omega_{\alpha,v}^n(\pi_v)^{1+\lfloor \frac{l_v}{n}\rfloor}}{1-\omega_{\alpha,v}^n(\pi_v)}$$

$$+ (\omega_{*,v}\mu_v)\left(s_v(\eta_v)s_v(h^{w_0})s_v(h^{-1})\left(\begin{pmatrix}\pi_v & \\ & \pi_v^{-1}\end{pmatrix},1\right)\right)\varepsilon((\pi_v,h_1/h_2)_v)^{-1}g_v^{-l_v-1}(\pi_v).$$

We have

$$\mu_v(s_v(h^{w_0})s_v(h^{-1})) = \left|\frac{h_2h_1^{-1}}{h_1h_2^{-1}}\right|_v^{\frac{1}{2}} = q_v^{\mathrm{ord}_v(m)}.$$

So for the right choice of ϕ, the prime power series match. Of course, the prime power series are only considered for the good places. There are discrepancies at the bad places, which account for the quotient between Ψ_f and $\Psi_{\mathfrak{A}}^0$ in the final equality at the end of the global calculation section.

A final point to make is the similarity between the cocycle of the twisted multiplicativity and that of the metaplectic group. This didn't happen by coincidence; the calculation of the twisted multiplicativity cocycle can be traced to multiplication of elements of the preferred section in $\widetilde{G}$. For this reason, it is expected that for the other groups discussed below, their corresponding series will also have twisted multiplicativity that reflect the multiplication on their covering groups.

15. Future directions

This computation of Fourier-Whittaker coefficients can be expanded and generalized in several directions. First, the algebraic group G can be changed. The most obvious choice is to consider higher rank cases, and develop a recursion for the general type A case. Also, the other Chevalley groups should be considered, because the construction of Matsumoto applies equally well to imbue them with interesting covering groups.

A second avenue of research would be to study the property of twisted multiplicativity. For which σ to we consider the series to have twisted multiplicativity? Can we expand the application of the property to make it comparable to having an Euler product? In this paper, the property was used to reconstruct a series uniquely from its prime power components, but perhaps we can also use knowledge about the cocycle to bound the values of an L-series in terms of its prime power series?

Finally, these two directions could converge. Computations in higher rank and over other groups may be more easily dealt with by considering prime power series and twisted multiplicativity. The series that are generated would likely give other examples of interesting cocycles.

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