

ANALYSIS OF LINEAR NETWORKS

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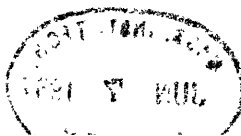
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ABSTRACT

A method of analysing linear networks is developed, that is applicable to networks whose elements may have any number of terminals. Each multi-terminal element is handled as a complete entity, without having to represent it as an equivalent network of branches. The theory is based on an unconventional treatment of voltage, which seems to be suitable for the general case, in that voltages are handled without having to specify the terminal to which they are referred. A considerable part of the analysis can proceed without defining the voltage reference. The point in the analysis where reference has to be specified is studied, and what the required reference conditions are, and as a result it appears that the conventional method of fixing one terminal for voltage and current references is just one very special case out of a multitude of possibilities.

In the course of the analysis, admittance and impedance emerge as two concepts which are not exactly equivalent or dual to each other. It is shown that the admittance-impedance duality is a characteristic of 2-terminal and 3-terminal elements only, and breaks down in the general case. Admittance is a two-indexed magnitude, referring to 2 terminals, whereas impedance is four-indexed, referring to 2 terminal-pairs. The analysis of a given network can proceed on an admittance basis without a specified voltage reference, but impedance can be defined only after reference conditions have been imposed, and it depends on the reference conditions.

The theory presented in this thesis is built up to the point where, given the characteristics of the multiterminal elements composing a network, the network equations can be set up and then solved to give any required network characteristic.

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Chapter I

INTRODUCTION

In a Round Table Discussion that was conducted at the 1955 Symposium on Modern Network Theory, the following conclusions have been reached:¹

"... A number of important, basic problems still confront the network theorist... In summary, the basic problems are associated with removal of one or more of the restrictions implied in the string of adjectives usually associated with networks: Linear, lumped, finite, passive, bilateral".

This Thesis outlines an analysis of linear, lumped, finite networks. However, instead of removing the restrictions of "passive, bilateral" from the existing theory, the analysis is carried out in a way that avoids these restrictions in the first place. The reasons for this approach, as well as a short outline of the theory, are the subjects treated in this Introduction.

1.1 LLFPB Networks

Before embarking on the subject of the Thesis, let us briefly review some of the fundamental points of the regular LLFPB network analysis, with special emphasis on those points that will be elaborated in the Thesis.

In analysis of networks, two types of problems are of interest. One is mostly topological, concerning the ways in which network elements are interconnected to form a network, and how the properties of the network can be deduced from the properties of the separate elements and the method of interconnection. The second type of problem is analytical (in the mathematical sense of Analysis), and includes all problems having to do with time and frequency domain, transform techniques, s-plane techniques and allied subjects. In this Thesis we shall be concerned with the first type of problems only - the topological relations.

¹ Proceedings of the Symposium on Modern Network Synthesis, Polytechnic Institute of Brooklyn, April 13-15, 1955. (p.527)

In LLFPB networks, the basic network element is a branch - an element with two accessible terminals. The branch may be a passive R, L or C, or a source (and in PB networks, a source is always a constant source, a voltage or current constraint) but any element has two terminals only. There are always two tacit assumptions about this type of element:

1. The current entering one terminal is always equal to the current leaving the other terminal; or, the total current into the branch is always zero.
2. Only the voltage between the branch terminals is related to the current, not the potential of each terminal by itself.

When branches are interconnected to form a network, the method of interconnection is represented by a topological graph. Two constraints on the voltages and currents, known as Kirchhoff's Laws, appear as a result of the interconnection:

1. The sum of all currents into a node is zero.
2. The sum of voltage drops around a loop is zero.

All the methods of analysing networks - by node pairs, node to datum, loop or mesh analysis - are based on these two pairs of postulates. It follows from the usual analysis that networks composed of such branches are bilateral, and we use this term as a synonym for "obey the reciprocity relations". Conversely, any network that is bilateral can be regarded as a collection of interconnected branches.

1.2 Removing the "PB" Restrictions

The fact that an element is active or passive does not appear in the topological aspects of the network. Analytically, certain functions describe passive elements, while active elements impose fewer restrictions on the functions. As far as topology is concerned, an active element may be just another type of branch, say, a negative R, but it is still a two-terminal element, with all that is implied by this fact.

Removing the B restriction is quite a different matter, and has some basic topological implications. Let us first note

that the term "bilateral" applies to networks, but has no meaning if applied to a single two-terminal element. To define reciprocity, two pairs of terminals are needed, at which a source and a meter can be connected and then interchanged. The minimum number of terminals necessary for this operation is three. We are then faced with the following dilemma:

- (a) Networks composed of branches are bilateral.
- (b) Some multi-terminal devices (for example, a triode) are not bilateral, and neither are networks that incorporate such devices.

The accepted method of solving this dilemma is by postulating a new type of "branch", a controlled source. This is a branch whose voltage depends on the current through some other branch, or whose current depends on the voltage drop across some other branch. This allows us to regard a non-bilateral element or network as a collection of branches, some of which are of this new type, so that we can still apply the methods of analysis by nodes, meshes, loops etc.

Unfortunately, it so happens that the non-bilateral devices like vacuum-tubes or transistors are also active elements, so that inclusion of a source in the equivalent circuit is quite "natural". In the usual representation of a triode by the plate impedance (positive real) and a controlled source, one can point at the source and say, "Of course this represents an active element, and here is the source of power gain". But in the last few years, circuit elements that are passive and non-bilateral have been postulated and constructed, like the gyrator or circulator. If such an element were represented by controlled sources, the representation would be quite misleading, for these "sources" are no sources of power at all.

1.3 A Different Approach to LLF Networks

It seems that the only justification of representing non-bilateral elements by controlled sources is that this method enables us to fit them into the framework of graphs. Evidently, if graphs were the only possible topological representation for LLF networks, we have no choice but to follow this method.

However, a somewhat closer scrutiny of the dilemma presented on the previous page will reveal that there is no dilemma there at all. The two statements do not contradict each other, for there is nothing to force us to include multi-terminal elements in the framework of branch networks - provided we have a theory for analysing networks with general multiterminal elements.

Such a theory is presented in this Thesis. The basic network elements can have any number of terminals, a two-terminal branch being just a special case. Networks composed of two-terminal branches only are shown to be bilateral, and bilateral networks can be represented as a collection of two-terminal branches. An element with more than two terminals is treated as a whole, without trying to split it up into branches that will fit in a graph. Thus - no dilemma.

The "minor" concession of allowing multi-terminal network elements calls for a reformulation of the whole basis of the methods of network analysis. First, new voltage and current postulates have to be decided upon for the single multiterminal element. Then, postulates analogous to Kirchhoff's Laws have to be formulated for the interconnection of network elements in a network.

The second set of postulates (interconnection) is necessary when we realize that networks are no longer representable by graphs. We still have the concept of a node, where terminals are connected together, but no longer are there meshes or loops. In a multi-terminal element there is no unique way to weave a loop from one terminal to another through the element. At least one of Kirchhoff's Laws, the one dealing with voltage drops, has to be replaced by a different formulation.

The fact that "loops" are inapplicable to general networks is only one example of the complete breakdown of topological network duality, at least that type of duality that is usually emphasized in branch networks. This loss is not as terrible as may appear at first glance. The duality is at most only nearly perfect in planar branch networks (if mutual

inductances are ignored), and gets very restricted even in general branch networks. It is then not too surprising that it completely disappears in general networks that contain multi-terminal elements. As the theory unfolds, we shall find that mesh and loop methods yield their place to node methods; elements are still connected in parallel, but there is no series connection; admittance appears as a concept more basic than impedance.

On the opposite page of the ledger, we shall discover through the more general approach some network properties that could not have been obtained by simple extension of branch-network theory. Some of these properties, dealing with methods of fixing a reference or datum for voltages, yield novel results even when applied to pure branch networks.

In short, where the regular methods work by induction, "generalizing" from branch networks to more general networks, our approach will be one of deduction: A general theory is developed, and the properties of branch networks are deduced as a special case. Only thus can we be assured that the properties of general networks can be explored in their broadest aspects, and not as a mere generalization of only those properties that are found in branch networks.

1.4 Scope of the Theory

The networks and elements treated in this Thesis are linear, lumped and finite. They are also assumed to be time-invariant.

An additional restriction is that the linear relations between voltage and current are homogenous, which means that the condition of all voltages being zero and all currents being zero simultaneously is compatible with the relations. This excludes sources, which are voltage and current constraints, so that the sources feeding and exciting a network are regarded as being external to the network proper. (The other type of source - the "controlled source" - will not appear in the theory.)

The concept of voltage as used in this Thesis has a meaning somewhat different from its conventional one. The term "voltage" is generally used as a synonym for "voltage drop" along a branch, or the "potential difference" between two terminals. In this Thesis, "voltage" is somewhat analogous to "potential", being ascribed to a single terminal or node, without specifying the reference terminal, and is therefore defined only within an additive constant. This type of voltage first appears as a convenient concept when multi-terminal elements are considered, but its importance is much more profound. We come to realize that networks can be analyzed without specifying the voltage reference node up to a certain stage of the analysis, and at that point there are many different ways to specify this reference; and this in turn leads up to some of the most important results of the Thesis.

Our main concern in the Thesis is with the topological and algebraic aspects of the network. Element admittances and impedances are assumed to be real numbers, voltages and currents assumed to be real constants or real functions of time. Most of the results, however, are directly applicable without any change to complex admittances and impedances, but no formal proofs will be given for that. So, strictly speaking, the theory is developed for linear resistive networks, or to small-signal linear approximations to non-linear network elements. It was felt that inclusion of analytical function theory considerations would have, by its sheer weight, obscured the topological and algebraic relations that we wish to emphasize.

1.5 Outline of the Thesis

The second chapter treats the multi-terminal network element and its representation. It treats the basic postulates on voltage and current, and the admittance representation of a network element. A geometrical interpretation of the various results is given in the form of relations in vector spaces, to explain the implications of the special treatment of voltage.

This model of vector spaces is used as a mathematical model guiding the development of the whole theory, but no formal mathematical relations concerning the abstract model will be given. A detailed description of those abstract relations that form the basis for the engineering interpretations will be found in the Appendix.

The next chapter treats the problem of interconnecting the network elements to form a network. Here the postulates analogous to Kirchhoff's Laws are formulated. The inverse problem, of representing a network by breaking it up into elements, is also treated - but obviously these elements are not necessarily two-terminal branches only. The reciprocity concept is discussed in this context.

Chapters II and III will have treated the admittance representation only; chapter IV comes to explain what additional assumptions have to be made before any talk about impedances becomes meaningful. It will appear that certain "reference" conditions have to be applied to the voltages and to the currents, but that the voltage and current references may be quite different from each other, and can be of a much more general form than is used in the currently accepted methods.

The latter point is elaborated in chapter V, which treats in detail a special case of reference assignments. In the majority of network problems, the simple assignment of a single node as a voltage reference will do, but the solution may be simplified if a different node is selected for current reference. This leads to the concept of a "four-indexed" impedance, of the form $Z_{pq,rs}$. It appears that an impedance has to be referred to two pairs of terminals, whereas an admittance is meaningful when referred to two terminals only. Using this type of impedance, a method is shown of solving two-terminal-pair network problems as the ratio of two determinants only, even if the two terminal-pairs have no common ground terminal.

In a certain sense, the character of chapter V is different from that of the rest of the Thesis. This chapter discusses a special case of the general theory treated in the other

chapters, but it was felt that the practical implications that follow merit this more detailed treatment of this special case.

Chapter VI presents some conclusions of a general nature that may be drawn from the Thesis, mostly on the subject of duality in network theory.

Throughout the Thesis, free use is made of matrix algebra, which is a natural medium for the treatment of multi-terminal network elements. All the symbols and matrix notations are defined on their first appearance, and in addition a list of symbols, notations and conventions is summarised in an Appendix for easy reference.

A conscious effort was made to keep the presentation in a language which is more Engineering than Mathematics. Some of the purely formal arguments, which had to be included for the sake of completeness, are therefore not given in the text, but are also relegated to the Appendix.

Chapter II

THE NETWORK ELEMENT

Consider any electrical network and the elements of which it is composed. Let us regard as elements those basic building blocks whose properties are known to the designing Engineer, out of catalogs, handbooks, or previous experience; those blocks that the Engineer puts together in various ways to achieve the desired end result. Let us regard as elements those units that the Technician draws out of stock and solders, screws, crimps or otherwise interconnects to form the network.

The various elements may be very different in size and appearance. They may be molded, boxed, canned or enclosed by glass bulbs. But there is always one thing that all the elements must have, if the network is of the lumped type: each element has certain well-defined points at which it is connected to other elements. These may be in the form of soldering lugs, pigtail wires, binding posts or base pins - but in general we refer to these points as the terminals of the element.

In passing let us note that the term "lumped" as used above covers more than is usually accepted in network theory. Since we do not go into analytical details, we do not associate "lumped" with rational functions. A section of uniform transmission line would here be considered as a "lumped" element, if the only connections to the rest of the network, and to loads and sources, occur only at a set of discrete terminals.

This leads to the general representation of a lumped network element: it is a "box" (or any closed figure), with some terminals attached to it. Any contact between the element and the world outside it can only be made at these terminals. The element behaviour will be defined and analyzed by the terminal properties only, without violating the privacy of the closed box.

2.1 Current

Fig. 1 shows an example of a 5-terminal element. Assume this element to be part of a network in a certain state of

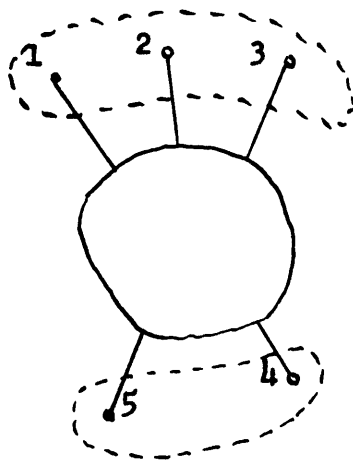


Fig. 1
Network Element

excitation. There will be currents flowing through the terminals, and let i_1, i_2, \dots, i_5 be the currents into the 1-st, 2-nd, ... 5-th terminal. The column matrix

$$\mathbf{i} = \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \end{bmatrix}$$

represents the current into the network element.

Since the components of \mathbf{i} include the currents into all the terminals, they are not independent, because their sum is zero. Let us assume, in order to make this example more general, that the components of \mathbf{i} are even further restricted, Let the set of terminals be partitioned into subsets, as indicated in Fig. 1, such that

$$\begin{aligned} i_1 + i_2 + i_3 &= 0 \\ i_4 + i_5 &= 0 \end{aligned} \tag{2.1}$$

The cause for this additional restriction imposed by partitioning need not concern us. It may be that the internal structure of the element is composed of several physically disjoint parts, coupled by mutual magnetic coupling, or even - in the degenerate case - totally uncoupled. Or it may well be that the partitioning

is imposed by external connections, as when a 4-terminal element is used as a 2-terminal-pair element in constructing a transmission line. In any case, since we do not probe into the element, but content ourselves with terminal information only, we accept this partitioning as an attribute of the element.

The partitioning of the set of terminals into subsets can be formalized by defining a partition matrix P , whose rows correspond to terminals and columns to terminal subsets. The entries of this matrix are:

$P_{jk}=1$ if terminal j is a member of subset k

$P_{jk}=0$ if terminal j is not a member of subset k ,

so that each row of P has one "1" entry and all others are "0".

The element of Fig. 1 has a partition matrix

$$P = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$$

Let A_t denote the transpose of the matrix A ; then the restriction on the terminal currents has the general form

$$P_t i = 0. \quad (2.2)$$

In the simple case where all the terminals belong to one subset only, the partition matrix is a single column of "1"s. A two-terminal branch has a 2x1 partition matrix, and the relation (2.2) then appears as $i_2 = -i_1$. Equation (2.2) is then the generalization of the first of the branch postulates mentioned in the Introduction.

We can now formulate the first postulate on general network elements:

A network element is characterized by the number of its terminals, and by the partitioning of the set of terminals into subsets, as indicated by the partition matrix P associated with the element. Any current into the element is constrained by (2.2).

2.2 Power and Voltage

The flow of current into the element is associated with an energy transfer, or, in the usual parlance of the Electrical Engineer, power flow. Now, current i being a vector quantity, and power W a scalar, the correspondence between i and W can be made by means of a second vector, which we shall denote by v , so that

$$W = v_t i = i_t v \quad (2.3)$$

(A more rigorous argument for this relation will be found in Appendix B). Given i and W , it is not claimed that v is uniquely determined, and in fact it will shortly be demonstrated that v is not unique. So far, we only wish to formulate the second postulate on general network elements:

The power flow into a network element is obtained from the current matrix i by inner multiplication with another column matrix v , (which will be called the voltage of the element), as shown in (2.3).

Let us now explore some of the properties of the voltage matrix v , and see where it corresponds to the usual notion of voltage (thus justifying the use of the term), and where it departs from it.

First, the restriction (2.2) on the currents leads to the conclusion that if the voltage is of the form

$$v = P v_0$$

where v_0 is a column with the suitable number of rows, one row per terminal subset, then

$$W = v_t i = (Pv_0)_t i = v_{0t} P_t i = 0 \quad .$$

But the form $v=Pv_0$ means that all the terminals of a subset have the same voltage, with no voltage differences within a subset, and under this condition no power can flow into the element.

Second, suppose i and W are given, and a certain v satisfying (2.3) has been found. Now add Pv_0 to the original v , then

$$\begin{aligned}
 W &= (v + Pv_0)_t i \\
 &= v_t i + v_{0t} P_t i \\
 &= v_t i + 0 \\
 &= v_t i
 \end{aligned}$$

so that raising the voltage of all the terminals of a subset by the same amount does not alter the power flow into the element. Power flow is determined only by voltage differences within a subset.

In conclusion, the voltage as defined by (2.3) and in the second element postulate is defined only within an additive term of the form Pv_0 , where P is the partition matrix of the element, and v_0 is an arbitrary column matrix with one row per subset. This is the same thing as ascribing a voltage to each terminal without specifying the point to which this voltage is referred, with the understanding that only voltage differences within a terminal subset are significant in computations.

2.3 Admittance

So far the network element has been assumed lumped and finite - for only under these assumptions could voltage and current be represented as discrete and finite sets, written as column matrices. We now introduce the assumption of the element being linear, to complete the set of restrictions (L.L.F.) imposed on the networks treated in this Thesis.

In an n -terminal element, there are n currents and n voltages, one each per terminal. The n currents are restricted by (2.2) so they represent less than n independent variables; on the other hand, there is no similar restriction on the voltages. Therefore, it is possible to have a relation giving the currents in terms of voltages, but not the other way round. In a linear element, the relation is of the form

$$i = Y v \quad (2.4)$$

where Y is an $n \times n$ admittance matrix. Note that (2.4) is a linear homogenous relation, i.e. $v=0$ together with $i=0$ satisfy it. The networks and network elements are assumed exclusive of independent sources, as was explained in the Introduction.

The Y matrix is necessarily singular, since the relation inverse to (2.4) cannot exist. Let us further investigate the structure of the Y matrix that leads to its singularity.

From the postulate on currents,

$$P_t Y v = P_t i = 0 ,$$

and this is a restriction on the currents, independent on the voltages. This can hold only if

$$P_t Y = 0 . \quad (2.5)$$

The second restriction follows from voltage and power relations. Assume that a voltage of the form Pv_0 is added to the element voltage - this should leave the power invariant. We cannot yet assume whether the current varies or not when this voltage is added, so let the new current be denoted by

$$\begin{aligned} i' &= Y (v + P v_0) \\ &= i + Y P v_0 \\ W &= v_t i \\ &= (v + P v_0)_t (i + Y P v_0) \\ &= v_t i + v_t Y P v_0 + v_{ot} P_t i + v_{ot} P_t Y P v_0 \end{aligned}$$

The third and fourth terms are zero, due to (2.2) and (2.5), therefore

$$v_t Y P v_0 = 0 ,$$

but this relation is to be true for any v and v_0 (the latter is arbitrary anyway), so that

$$Y P = 0 . \quad (2.6)$$

Going back to the form of i' above, it now appears that $i'=i$. The current into the network element, as well as the

power, does not change when a voltage of the form Pv_0 is added to the element voltages. Moreover, if

$$v = P v_0$$

is the only voltage at the element terminals, $i=0$ as well as the power. All this, of course, is to be expected, since such a voltage means that there are no voltage differences within a subset of terminals.

Relations (2.5) and (2.6) show the structure of the Y matrix of any linear n -terminal element. Let the rows and columns of Y be partitioned in the same way that the set of terminals is partitioned into s subsets. This partitioning breaks Y up into s^2 submatrices. The two relations indicate that the sum of each row and each column in each of these submatrices is zero. (We shall have many occasions to refer to this type of matrix. Let us then, for short, use the term zero-sum matrix to denote a matrix in which the sum of the entries in each complete row and in each complete column is zero.)

As an example, the element shown in Fig. 1 has an admittance matrix partitioned as in Fig. 2, and each of the four resulting submatrices is a zero-sum matrix.

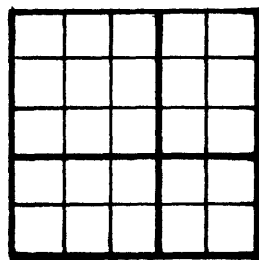


Fig. 2
Structure of the Y Matrix for
the Element Shown in Fig. 1.

This is a somewhat unconventional way to represent a network element. A simple example to illustrate the meaning of this type

of matrix is to compare it to the usual representation of a branch. A branch is a two-terminal element with no further terminal partitioning, and has a partition matrix

$$P = \begin{bmatrix} 1 \\ 1 \end{bmatrix} .$$

The only possible form that a 2x2 matrix can have to comply with (2.5) and (2.6), that is, to be a zero-sum matrix, is

$$Y = \begin{bmatrix} y & -y \\ -y & y \end{bmatrix} .$$

Although Y has four entries, there is only one independent admittance value y . Writing out the current-voltage relations (2.4) in full,

$$i_1 = yv_1 - yv_2 = y(v_1 - v_2)$$

$$i_2 = -yv_1 + yv_2 = -i_1$$

It is evident that the usual assumptions about a branch - same current in and out, and depending on the voltage difference only - are embodied in the form of the admittance matrix. The independent entry y is just what is usually called the admittance of the branch. The basic difference lies in the fact that although the simple y has an inverse, so that a branch has an impedance as well as an admittance, in our type of representation the inverse of Y cannot yet be defined. We shall later find ways of inverting the admittance relation, after some more assumptions will have been made about the element and the whole network of which the element is a part. But for the time being, let us follow the admittance concept as far as possible without making any further arbitrary assumptions.

2.4 Augmentation

The simplest network element seems to be a 2-terminal element (a branch). A one-terminal element would not make much sense. Its Y is just the scalar 0, which means: no current can flow into it, no effect does its voltage have on anything.

Although a one-terminal element is trivial, it will sometimes be convenient to augment a network element by adding to it a few isolated terminals, like in Fig. 3. An isolated terminal has the same properties that distinguish a one-terminal element: zero current and inconsequential voltage. From a partitioning standpoint, an isolated terminal forms a one-terminal subset. It then follows from (2.5) and (2.6) that an isolated terminal leads to a complete row and a complete column of zeroes in the admittance matrix. For example, if the

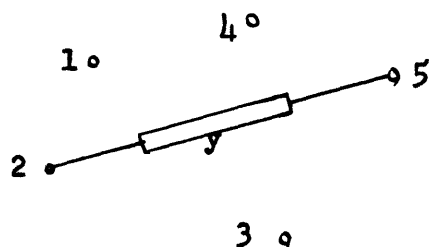


Fig. 3
An Augmented Network Element

branch shown between nodes 2 and 5 in Fig. 3 has an admittance y , the complete augmented element shown has a 5×5 admittance matrix

$$Y = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & y & 0 & 0 & -y \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -y & 0 & 0 & y \end{bmatrix} .$$

This is the regular 2×2 matrix of a branch, with rows and columns of zeroes added corresponding to the isolated nodes.

2.5 Geometrical Interpretation

The relations between current, voltage, admittance and power, as developed in this chapter, can be interpreted geometrically as relations between points in Euclidian space.

This will now be illustrated by a specific example of a 3-terminal element, with a partition matrix

$$P = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

so that all the relations can be shown as projections of 3-dimensional space. The results, however, will be valid for any partition matrix P .

A current column matrix i can be represented by a point in 3-dimensional space $x_1 \ x_2 \ x_3$ whose coordinates are

$$x_j = i_j \quad (j=1,2,3) .$$

Not every point in the space can represent a current, but only those points whose coordinates satisfy

$$x_1 + x_2 + x_3 = 0$$

or, in general,

$$P_t x = 0 .$$

These points are on a plane passing through the origin, which we shall call the partition plane (because of its association with the partition matrix P).

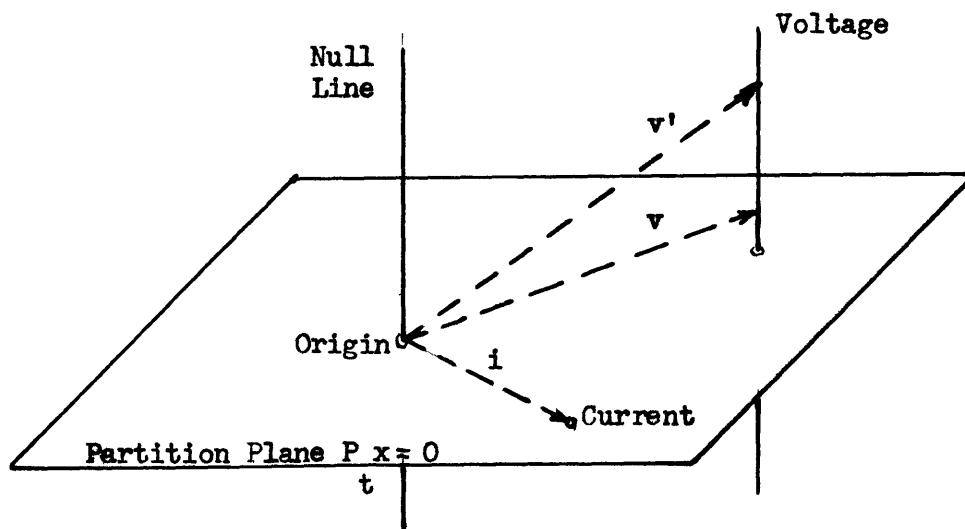


Fig. 4

The Current Point and Voltage Line

The straight line given by the equations

$$x_1 = x_2 = x_3$$

passes through the origin and is perpendicular to the partition plane. If a voltage column matrix is represented by a point on this line,

$$v_1 = v_2 = v_3$$

all the terminals are at the same voltage, and there is no current flowing into the network element. This line will therefore be called the null line.

A point representing voltage may be anywhere in the space, since there is no restricting relation between the voltages. However, given any voltage point, a line can be drawn through it parallel to the null line (perpendicular to the partition plane), and then all the voltages on this line are equivalent as far as current and power are concerned. Current is thus represented as a point on the partition plane, whereas voltage is represented as a line perpendicular to that plane. These relations are shown in Fig. 4, where the coordinate axes have been omitted for clarity of the drawing.

If vectors are drawn from the origin to the current point and to any point on the voltage line, the power is given by the inner product of the two vectors, $i \cdot v$. Vectors to any two points of the voltage line, v and v' , differ by a component perpendicular to i , therefore

$$i \cdot v = i \cdot v'$$

which shows the independence of power on the point chosen for voltage representation.

The admittance of a network element is a transformation that maps a voltage line into a current point, so it is a singular transformation. If we wish to map a current point back into the same voltage point we started from, we can only be sure that we shall end up in a point on the same line, but can never tell whether or not this is the exact starting point. Mapping voltage into current and back into voltage never assures a return to the starting point; the sequence of the two mappings is not equivalent to an identity mapping, and the two mappings are therefore not mutually inverse.

Now generalize these geometric properties to the general element, with n terminals partitioned into s subsets, with a P matrix of order $n \times s$. Current and voltage will now be represented by points in n -dimensional space. Current is represented by a point restricted to the $(n-s)$ -dimensional subspace $P_t x = 0$, the partition subspace. Voltage will be represented by an s -dimensional subspace orthogonal to the partition subspace. The admittance that maps an s -dimensional subspace of voltage into a current point is a singular transformation. The rank of the Y matrix representing this transformation can at most be s . Note that (2.5) and (2.6) restrict the rank of Y just by this amount, for at least s rows and s columns (one each per subset) have to be omitted from Y in order to leave a matrix with a non-zero determinant.

Chapter III

THE NETWORK

Our technician, whom we have met at the beginning of the previous chapter, now has a collection of network elements, and his next job is to assemble them into the required network. How would he go about this?

Usually, he would get a chassis with some prepared connection points: Tags, lugs, screws etc. Then the various terminals of the network elements will be connected to these points as per instructions or wiring diagrams. The essence of the wiring diagram is a schedule showing which terminal of what element is tied to which node of the network.

In this chapter we shall develop the mathematical analog of this procedure. We already have a set of admittance matrices that represent the various elements going into the network. Now we need a connection matrix, to show how the elements are connected to form the network.

3.1 The Connection Matrix

The network shown in Fig. 5 is composed of three elements connected at four network nodes. If, for the time being, the

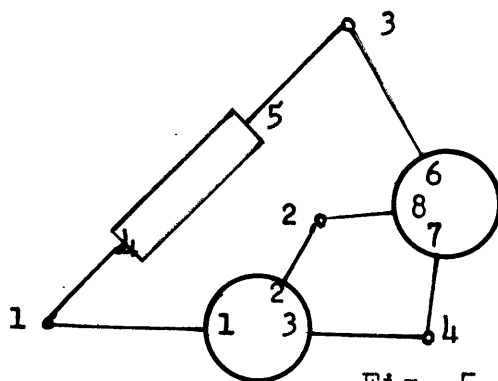


Fig. 5

A Network

interconnection of the elements is disregarded, we have 8 terminals to consider, leading to 8-rowed column matrices

i and v . These columns are related via an 8×8 admittance matrix Y , whose structure is shown in Fig. 6. The admittance matrix of each element (which is a zero-sum matrix) appears as one of the submatrices on the diagonal. All the other submatrices are zero, since there is no interaction between the elements except through the terminals.

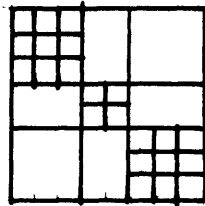


Fig. 6

Y Matrix of the Elements that
Go into the Network Shown in Fig. 5

In reality, because of the interconnection of the elements, there are only four distinct nodes, and only four voltages and currents to be concerned with. Let \bar{I} and \bar{v} denote the 4-rowed columns pertaining to the nodes, and \bar{Y} the 4×4 admittance matrix relating them. We now have to find the relation between the barred network matrices and the unbarred element matrices.

The interconnection of the elements can be expressed by a connection matrix C , which has rows corresponding to the element terminals and columns corresponding to network nodes. (In our example, C is an 8×4 matrix). The entries of C are

$$C_{jk} = 1 \quad \text{if terminal } j \text{ is connected to node } k,$$

$$C_{jk} = 0 \quad \text{if terminal } j \text{ is not connected to node } k.$$

Each row of C therefore has one and only one "1" entry, and the rest are "0".

The network of Fig. 5 has a connection matrix

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

The network is thus defined by two matrices:

- Y - the admittance matrix of its separate elements (grouped, for convenience, into a single diagonal partitioned matrix).
- C - the connection matrix, showing how the separate elements are connected. This matrix has the same role that a network graph has for pure branch networks.

3.2 Network Postulates

The two postulates about the relations between element i v and network \bar{I} \bar{v} will now be formulated. They have the form of conservation postulates for current and power, and so reflect the conservation laws of charge and energy, the former being the time derivatives of the latter.

1. The current into any network node is equal to the sum of the currents into the element terminals connected to this node.
2. The power into the network is the sum of the powers into the network elements.

From the definition of the connection matrix, it follows that the node currents \bar{I} are given in terms of the terminal currents i by

$$\bar{I} = C_t i \quad (3.1)$$

The second postulate is expressed by writing out the expression for power

$$\bar{i}_t \bar{v} = i_t v$$

Substituting from (3.1)

$$\begin{aligned}\bar{i}_t &= i_t C \\ i_t C \bar{v} &= i_t v\end{aligned}$$

and since this relation is independent of i ,

$$C \bar{v} = v \quad . \quad (3.2)$$

This result is a corollary of the second postulate:

The voltage of all the terminals connected to a node is equal to the node voltage.

Now we are ready to compute the network admittance \bar{Y} in the relation

$$\bar{I} = \bar{Y} \bar{v} \quad .$$

Starting from the element relation

$$i = Y v$$

we get, using (3.1) and (3.2)

$$\begin{aligned}\bar{i} &= C_t i \\ &= C_t Y v \\ &= C_t Y C v \\ \bar{Y} &= C_t Y C \quad . \quad (3.3)\end{aligned}$$

Given the admittance of the network element and their interconnections, this is how the admittance matrix of the network is computed.

3.3 Digression on the Nature of the Connection Matrix

The formula given above for the entries C_{jk} of the connection matrix can be stated in a somewhat more general form:

C_{jk} is the truth value of the statement "terminal j is connected to node k ".

With the regular conventions for truth values, "1" for a true statement and "0" for a false statement, this definition is identical with that given in the previous section. But when

the definition is put in this way, the question whether the statement is true may have not only a "yes" or "no" answer, but also "maybe, it depends".

One interpretation of the uncertainty answer could be the presence of switches in the network. The connection of an element terminal to a given network node then depends on the switch position, and the entry in the C matrix would be neither "1" nor "0", but a Boolean variable representing the switch. The "1" and "0" entries can be regarded as special cases, when the Boolean variable is given one of its two possible values, with no uncertainty involved.

The \bar{Y} of a network is then made up of admittances Y and Boolean elements C . The entire theory that follows in this Thesis could thus easily be extended to apply to switchable networks. This, however, will not be done in the Thesis, and the interested reader is referred to a paper^{*} outlining the operations with numbers that are qualified by Boolean elements.

We now return to the switchless network, where the Boolean character of C need not concern us, and its "1" entries can be regarded as simple scalar numbers.

3.4 Networks as Paralleled Elements

A simple interpretation of (3.3) is possible, if all the elements are first augmented, to give each element a terminal for each node of the network. Fig. 7 shows the three augmented elements that make up the network of Fig. 5. (The numbers at the terminals refer to the network nodes). Each element now has a 4x4 admittance matrix, with some rows and columns of zeroes only, and the complete Y matrix is of order 12x12.

^{*} J. Shekel, "Sketch for an Algebra of Switchable Networks", Proceedings of the Institute of Radio Engineers, vol. 41, pp. 913-921, July, 1953.

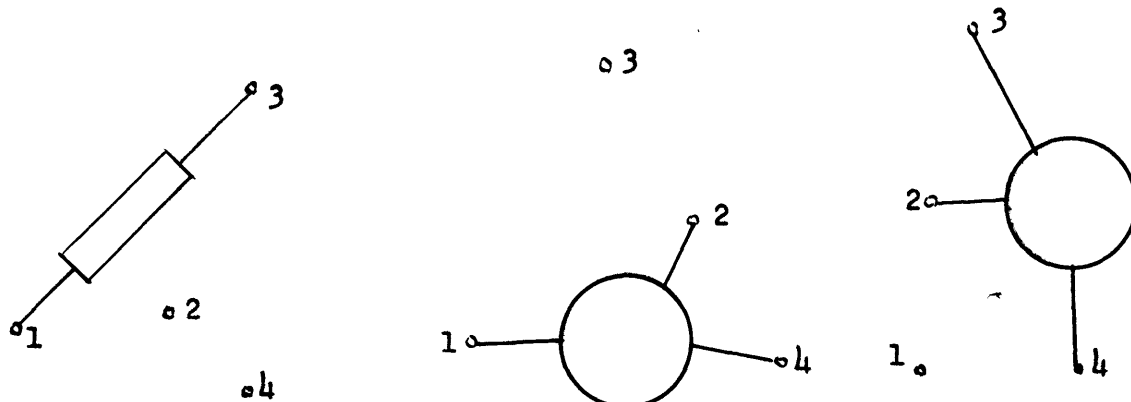


Fig. 7

Augmented Elements

Let the element terminals be re-numbered as in the following scheme:

	node no.	1	2	3	4
Terminals of first element		1	2	3	4
Terminals of second element		5	6	7	8
Terminals of third element		9	10	11	12

This numbering will lead to a very simple C matrix. If I denotes a unit matrix of order 4×4 ,

$$C = \begin{bmatrix} I \\ I \\ I \end{bmatrix} .$$

Y will be a 12×12 matrix, partitioned into nine 4×4 matrices

$$Y = \begin{bmatrix} Y_1 & 0 & 0 \\ 0 & Y_2 & 0 \\ 0 & 0 & Y_3 \end{bmatrix}$$

Y_1 Y_2 Y_3 being the matrices of the augmented elements. When the multiplication indicated in (3.3) is carried out,

$$\begin{aligned} \bar{Y} &= C_t Y C \\ &= Y_1 + Y_2 + Y_3 \\ \bar{Y} &= \sum Y_k . \end{aligned} \tag{3.4}$$

The admittance matrix of a network is equal to the sum of the admittance matrices of the network elements (suitably augmented).

This interpretation looks like a generalization of the parallel connection of two-terminal branches. The network can be built by piling up elements in parallel - i.e., connecting together the corresponding terminals of all the elements.

3.5 Note on Reciprocity

The 2x2 admittance matrix representing a branch is always a symmetrical matrix, for only this way can it be a zero-sum matrix:

$$\begin{array}{ll} \text{First row,} & Y_{11} + Y_{12} = 0 \\ \text{First column,} & Y_{11} + Y_{21} = 0 \\ \text{hence} & Y_{21} = Y_{12} \end{array}$$

The matrix will remain symmetrical when the element is augmented by attaching any number of isolated nodes. In the process of augmentation rows and columns of zeroes are added, but the only two non-zero off-diagonal entries remain in symmetrical positions.

If a network is composed of two-terminal branches only, its Y matrix is the sum of augmented branch matrices which are all symmetrical, so the total Y matrix is symmetrical too. Symmetry of the admittance matrix is a necessary and sufficient condition for the network to be bilateral (that is, to obey the reciprocity relations), we conclude that:

Any network composed of two-terminal network elements (branches) only obeys the reciprocity relations.

Nothing definite can be said in general about networks that contain general multi-terminal elements. If each of the elements is bilateral, so will be the network; but nothing can be said a priori about the elements and the network, as we could say about branches and branch networks.

3.6 Network Transformations

Suppose a network is given, with its associated i and v column matrices, and the network Y matrix relating them:

$$\mathbf{i} = \mathbf{Y} \mathbf{v} .$$

It is possible that the \mathbf{i} and \mathbf{v} are subject to further constraints that had not been taken into account when the \mathbf{Y} matrix was constructed. For example, the network may contain some transformers (assumed ideal), and an ideal transformer does not have a \mathbf{Y} matrix.

To present the discussion in its most general form, assume that the actual currents of the network are \mathbf{i}' , which are related to the \mathbf{i} above by the linear transformation

$$\mathbf{i}' = \mathbf{T} \mathbf{i} . \quad (3.5)$$

Some examples of the \mathbf{T} matrix will be given later. All we assume now is that the component of the network causing the constraint is lossless - like a short circuit or an ideal transformer. The new \mathbf{i}' then has a new \mathbf{v}' associated with it, such that

$$\mathbf{v}'_t \mathbf{i}' = \mathbf{v}_t \mathbf{i} .$$

Substituting from (3.5)

$$\begin{aligned} \mathbf{v}'_t \mathbf{T} \mathbf{i} &= \mathbf{v}_t \mathbf{i} \\ \mathbf{v}'_t \mathbf{T} &= \mathbf{v}_t \\ \mathbf{v} &= \mathbf{T}_t \mathbf{v}' . \end{aligned} \quad (3.6)$$

The new \mathbf{i}' and \mathbf{v}' will be related by a new \mathbf{Y}' matrix

$$\mathbf{i}' = \mathbf{Y}' \mathbf{v}'$$

which can be found from the old relation

$$\begin{aligned} \mathbf{i} &= \mathbf{Y} \mathbf{v} \\ \mathbf{i}' &= \mathbf{T} \mathbf{i} \\ &= \mathbf{T} \mathbf{Y} \mathbf{v} \\ &= \mathbf{T} \mathbf{Y} \mathbf{T}_t \mathbf{v}' \\ \mathbf{Y}' &= \mathbf{T} \mathbf{Y} \mathbf{T}_t . \end{aligned} \quad (3.7)$$

One example of this transformation is the augmentation of an element as treated in the previous chapter. We have the

original set i , where each entry is the current into a terminal; this set is augmented to form a larger set i' , but all the entries added to i to form i' are zero (no current into the isolated nodes). The transformation matrix, in partitioned form, is

$$T = \begin{bmatrix} I \\ 0 \end{bmatrix}$$

where I and 0 are unit and zero matrices, respectively, of appropriate order. The augmented Y' is obtained, in partitioned form, following (3.7)

$$Y' = \begin{bmatrix} Y & 0 \\ 0 & 0 \end{bmatrix} .$$

As a second example, assume a 4-node network, with a 4x4 Y matrix, modified by short-circuiting nodes 3 and 4, thus forming a 3-node network. The appropriate transformation matrix will show that any current into the new 3rd node is equal to the sum of the currents into the old 3rd and 4th nodes,

$$i' = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} i .$$

Another example is presented by the process of "node splitting" illustrated by the following example: Suppose a three-terminal network (or network element) is given, with its 3x3 admittance matrix Y . Such an element can be used, and frequently is used, as a two-terminal-pair element (with a common "ground" at input and output). In our mode of element representation, a two-terminal-pair element is represented by a 4x4 admittance matrix, with an associated partitioning

$$P = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} .$$

The process of obtaining the 4x4 Y' from the 3x3 Y is again a transformation of the same type, for the four new currents i' can be given in terms of the old three currents i by

$$i' = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix} i ,$$

if it is assumed that node 2 was "split" to act as common ground to input and output.

An ideal transformer in the network will lead by definition to a set of relations of the form (3.5) and (3.6).

3.7 How to Write the Y Matrix of a Given Network

In the preceding sections of this chapter, some formal procedures were worked out for arriving at the admittance matrix of a given network. The actual procedure will now be illustrated by a specific example. Prior to any network calculations, we need to know the representation of the building blocks that will go into the network. These can be of one of the two general types:

1. A network element that can be represented by an admittance matrix of the type discussed in chapter II.
2. An element that imposes some relations among the currents into the nodes to which it is connected, and some relations among the voltages of these nodes, but no admittance-type relations between currents and voltages. This type was treated in section 3.6 of this chapter.

These two types of elements are sufficient to represent any lumped, linear, finite and sourceless network (the last adjective meaning the lack of independent sources, so that the linear equations are homogenous). A proof of this statement is given in Appendix C.

The network in the following example will be composed of resistors and triodes, both belonging to type 1 above, and a voltage divider (ideal), which belongs to type 2.

A resistor is a two-terminal element, which we shall define by its conductance g , incorporated in the 2×2 zero-sum matrix

$$\begin{bmatrix} g & -g \\ -g & g \end{bmatrix}$$

For the admittance representation of a triode, let the terminals be numbered as in Fig. 8. The representation applies to linear small-signal approximation, and it is

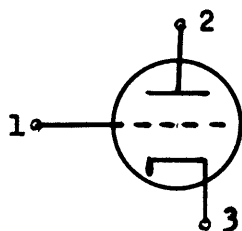


Fig. 8
Triode

further assumed that there is no grid current,

$$i_1 = 0 \quad .$$

To simplify notation, let m be the mutual grid-to-plate transconductance, and p the internal plate conductance. The small-signal plate current (into terminal 2) is then

$$i_2 = m(v_1 - v_3) + p(v_2 - v_3)$$

and the current into the cathode is

$$i_3 = -i_2 \quad .$$

From these three equations the admittance matrix of a triode is constructed as

$$Y = \begin{bmatrix} 0 & 0 & 0 \\ m & p & -m-p \\ -m & -p & m+p \end{bmatrix}$$

The network we plan to analyze is that of a voltage regulator frequently used in high-voltage supplies, and is shown in essentials in Fig. 9. That Figure shows only those elements that are important for small-signal operation. The tubes are represented as triodes, and all other grids whose voltages are fixed are omitted. The cathode of the lower tube is usually

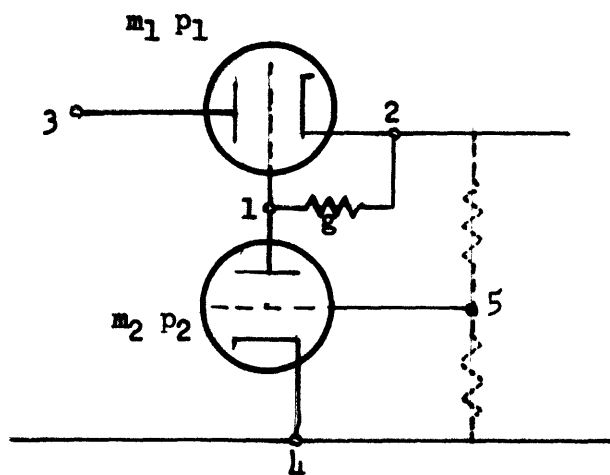


Fig. 9
Network To Be Analyzed

held at a fixed voltage above ground by means of a gas-discharge diode, but this effect is represented as a short-circuit for small-signal operation. This form of stripped down circuit is nevertheless quite sufficient to analyze the operation of the voltage stabilizer as far as finding the effects of input ripple and output current on the output voltage.

The admittance matrix of the network will be obtained as the sum of the admittances of 3 augmented elements. First, the resistor of conductance g appears between nodes 1 and 2 (with three extra rows and columns of zeroes)

$$\begin{bmatrix} g & -g & 0 & 0 & 0 \\ -g & g & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Then, the upper triode, whose terminals are numbered just like in Fig. 8, so it has the same matrix of the triode shown above, augmented by a 4th and 5th row and column of zeroes

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ m_1 & p_1 & -m_1-p_1 & 0 & 0 \\ -m_1 & -p_1 & m_1+p_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Finally, the lower triode, which has essentially the same matrix, only with rows and columns permuted to conform with the different numbering of terminals

$$\begin{bmatrix} p_2 & 0 & 0 & -m_2-p_2 & m_2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -p_2 & 0 & 0 & m_2+p_2 & -m_2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Adding all three matrices, we obtain the admittance matrix of the network:

$$\begin{bmatrix} g+p_2 & -g & 0 & -m_2-p_2 & m_2 \\ -g+m_1 & g+p_1 & -m_1-p_1 & 0 & 0 \\ -m_1 & -p_1 & m_1+p_1 & 0 & 0 \\ -p_2 & 0 & 0 & m_2+p_2 & -m_2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Obviously, when writing down the matrix, there is no need to write each component matrix separately, as was done here for illustrative purposes. The procedure would rather be like this:

1. Assign numbers 1,2,...,n to the network nodes.
2. Draw a nxn square table as a framework for the Y matrix.
3. Enter the various network elements into the table.
Each element will have entries only in positions where both row and column number correspond to nodes to which the element is connected.

We now have the admittance matrix of a network composed of two triodes and a resistor. This is not yet the complete voltage stabilizer, for the all-important feedback link is missing. This feedback is furnished by the voltage divider shown dotted in Fig. 9. These two extra resistors could have

been included in the matrix the same way as the first resistor, but we shall do it differently in order to provide an illustration for another point.

The grid of the lower triode draws no current; assume also that the total resistance of the voltage divider is large enough so that the current drawn by it can be neglected. The only effect this voltage divider has is to introduce a constraint

$$v_5 - v_4 = k (v_2 - v_4) \quad (k < 1)$$

$$v_5 = k v_2 + (1-k) v_4$$

This can be put in a form similar to (3.6) by defining four v' voltages (eliminating v_5 from the computations)

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & k & 0 & 1-k \end{bmatrix} \begin{bmatrix} v'_1 \\ v'_2 \\ v'_3 \\ v'_4 \end{bmatrix}$$

and the 5x4 matrix is identified with T_t in (3.6). Operating with this T as indicated in (3.7), the final Y' matrix for the voltage regulator network is obtained as

$$Y' = \begin{bmatrix} g+p_2 & -g+km_2 & 0 & -km_2-p_2 \\ -g+m_1 & g+p_1 & -m_1-p_1 & 0 \\ -m_1 & -p_1 & m_1+p_1 & 0 \\ -p_2 & -km_2 & 0 & km_2+p_2 \end{bmatrix}$$

This is the admittance matrix relating currents into nodes 1 to 4 of the network with voltages at these nodes. This is only the first step in analyzing the network, and amounts to setting up the network equations. Solving these equations to get any answers about the operation of the network is the

second step, which we shall not be ready to take until we discuss some problems treated in the next chapter. But, before turning to these problems, let us pursue the admittance representation a little further, before imposing any additional conditions on the network.

3.8 Partition Groups

Our analysis has started from the single network element, with an associated partition matrix P showing its general type, and a Y matrix to give the element some individuality within the general type. Then, a collection of such elements, together with a connection matrix C , defined a network, and a Y matrix for the complete network was derived.

Imagine now the network enclosed in a "black box", with each node connected to a terminal protruding from the box. In principle, there would be nothing to distinguish the network from a network element of the type treated in chapter II. It seems that the distinction between a network element and a composite network is one of convenience or usage rather than one of principle. (Our hypothetical Technician could find in the stockroom bins a flat molded "element" with 5 pigtails, which is a complete RC amplifier interstage, and he would treat it no different from a simple molded capacitor).

In principle, then, we can treat "network" and "network element" as equivalent terms. When elements are interconnected, the result is called a network, but it can then be treated as an element by itself or to construct more complicated networks. On the other hand, an element can be regarded as a network composed of simpler elements. There is only one additional point to be clarified in this equivalence, namely: what partition matrix is associated with the network when it is treated as a network element (since, by definition, an element is always associated with a P matrix).

As a starting point, we note that all the elements that

have the same P matrix form a group under the operation of parallel connection (addition of their Y matrices). This statement means that if Y_1 and Y_2 are admittances of elements associated with a certain partition matrix P , so is the element obtained by connecting the two elements in parallel.

$$\begin{aligned} P_t Y_1 = 0 \quad \text{and} \quad P_t Y_2 = 0 \quad \text{imply} \quad P_t (Y_1 + Y_2) = 0 \\ Y_1 P = 0 \quad \text{and} \quad Y_2 P = 0 \quad \text{imply} \quad (Y_1 + Y_2) P = 0 \end{aligned}$$

We can thus speak of all the elements associated with a given partition matrix P as belonging to a partition group. The partition defining this group is shown by the P matrix, but for some purposes it can be indicated symbolically in a simpler notation, by grouping together integers that represent the terminals. For example, the element in Fig. 1 has the partition (1,2,3)(4,5). Some further examples for partition group symbols are given in the following table:

<u>Element type</u>	<u>Partition</u>
Branch	(1,2)
General 3-terminal element	(1,2,3)
General 4-terminal element	(1,2,3,4)
Two-terminal-pair element	(1,2)(3,4)
Three-terminal-pair element	(1,2)(3,4)(5,6)
Section of 3-wire line	(1,2,3)(4,5,6)
3-terminal element augmented by 3 isolated nodes	(1,2,3)(4)(5)(6)

One partition group may include another one. The group of 4-terminal elements includes all 2-terminal-pair elements. In general, P' includes P'' if P'' introduces further partitioning within the subsets formed by the P' partitioning, as the following examples show:

$$\begin{aligned} P' &= (1,2,3,4) & P'' &= (1,2)(3,4) \\ P' &= (1,2,3,4) & P'' &= (1)(2,3)(4) \\ P' &= (1,2)(3,4) & P'' &= (1)(2)(3,4) \end{aligned}$$

Given two partitions of the same number of terminals, one does not necessarily include the other, for example

$$P' = (1,2,3)(4,5) \quad P'' = (1,2)(3,4,5)$$

where neither partitioning includes the other. However, given any two partitions of the same number of terminals, there is always a partitioning that includes both. For the P' and P'' given above, this would be

$$P = (1,2,3,4,5)$$

It is customary to refer to this including partitioning as the union of the two given partitions, symbolically

$$P = P' \cup P'' \quad .$$

The partition groups thus form a partially ordered set in which every two members have a union within the set. (If P' includes P'' , then $P' \cup P'' = P'$). Some further examples of union are given below.

$$(1,2)(3,4)(5) \cup (1,2)(3)(4,5) = (1,2)(3,4,5)$$

$$(1)(2)(3) \cup (1,2)(3) = (1,2)(3)$$

$$(1,3,4,6)(2,5)(7) \cup (1,3)(4,6)(2,5,7) = (1,3,4,6)(2,5,7)$$

When two Y matrices belonging to the same partition group are added, the sum belongs to the same group. When the two Y matrices seem to belong to different partition group, they also belong to the union of the two groups, and their sum will then belong to that group which is the union. In general:

A network (when regarded as an element) belongs to a partition group which is the union of all the partition groups to which the network elements belong. In defining the partition groups of the elements composing the network, each element should be presented in the augmented form that gives it as many terminals as the network has nodes.

3.9 Breaking Up a Network Into Elements

We started this chapter with a set of elements, which was then interconnected to form a network; towards the end of the chapter it appears that the composite network can again be

treated as an element. Before concluding this chapter the opposite problem will be tackled: Given a network (or a network element) in its totality, as a complete Y matrix, can it be decomposed into simpler network elements? This is the familiar problem of finding an "equivalent network" for a given element. The decomposition is usually not unique, and the decision about which equivalent network to use out of the multitude of possibilities is often made subject to other considerations: ease in application, structure hinting at the "physical" principle of operation, structure appealing to the user because of the "insight" it provides to the operation, or similar reasons. All these considerations do not concern us here. After all, an element is regarded as a "black box", and for our purposes, any equivalent network that looks identical as far as terminal operation is concerned will be acceptable. We shall only point out the method by which the network can be broken up.

A network Y is obtained by adding the Y 's of augmented elements. The decomposition will therefore be made by finding the set of Y 's whose sum is the given network Y . The only condition restricting the decomposition is that each of the component Y 's be a zero-sum matrix. And that is really all there is to it.

The decomposition will be useful if the component parts are the simplest possible, and we shall now find how many different simple elements are necessary to represent any given element or network. A point to bear in mind is that in this Thesis with algebraic and topological aspects only. The elements described as R , C or L are, for our purposes, all the same type of element: a two-terminal branch.

Starting from the simplest case, a two-terminal element cannot be further simplified, except in the trivial way of representing it as a few branches in parallel.

Next, consider a multi-terminal element with a symmetrical Y matrix. (A bilateral element). It can be decomposed until each component Y has four non-zero elements only, that

represent a branch (augmented). The decomposition is straightforward: each entry of Y above the principal diagonal will contribute a branch. Suppose $Y_{jk} = a$ in the complete matrix, then the corresponding component Y will have entries

$$-Y_{jj} = -Y_{kk} = Y_{jk} = Y_{kj} = a$$

and zeroes elsewhere. This represents a branch of admittance $-a$ connected between terminals j and k . From this and former considerations we conclude that

Any network composed of branches only is bilateral. Conversely, any bilateral network can be represented as a network of branches.

Finally, assume the general case, where the network Y is not a symmetrical matrix. Branch matrices, which are symmetrical, are therefore not sufficient to form the complete Y . Some new basic element has to be defined, and it has to have at least three terminals. Since we look for the simplest possible elements, we shall try to use elements with no more than three terminals.

The element Y has to be a zero-sum matrix, so the simplest matrix would have four non-zero entries:

$$\begin{bmatrix} 0 & 0 & 0 \\ m & -m & 0 \\ -m & m & 0 \end{bmatrix}$$

This is similar to a branch matrix, but the four non-zero entries are pushed into a corner, making the matrix non-symmetrical. This matrix and its augmentations can be used to form non-symmetrical Y matrices, either by themselves or with the addition of branch matrices. The matrix describes an element with the properties

$$\begin{aligned} i_1 &= 0 \\ i_2 &= m(v_1 - v_2) \\ i_3 &= -i_2 \end{aligned}$$

The current entering in terminal 2 and leaving at 3 is

proportional to the voltage difference between terminals 1 and 2. The m is then a transconductance, the element being an idealized triode with infinite plate resistance.

Another type of basic element can be derived if, in decomposing the network, the procedure of removing branches is followed as far as possible. This will finally leave a Y matrix which is purely anti-symmetrical, and no more branches can be removed because all the entries on the principal diagonal are zero. We now define an element that has the simplest possible anti-symmetrical zero-sum matrix:

$$\begin{bmatrix} 0 & g & -g \\ -g & 0 & g \\ g & -g & 0 \end{bmatrix}$$

This matrix has six non-zero entries, but still only one independent parameter, denoted here by g . This element has the properties which are usually associated with a gyrator.

When a network is decomposed in this manner, the Y matrix is first split into its symmetrical and anti-symmetrical components

$$Y_s = (Y + Y_t)/2$$

$$Y_a = (Y - Y_t)/2$$

Y_s is decomposed into a sum of augmented branch matrices, Y_a into a sum of augmented gyrator matrices. The set of branches is essentially unique (except for trivial variations of representing one branch as several branches in parallel); the set of gyrators is definitely not unique, as can be seen in the following example.

Assume that a general 4-terminal element is to be described by branches and gyrators. The symmetrical component Y_s is a 4x4 matrix with 6 independent entries. (It is a zero-sum matrix, which reduces the usual number of $4! = 24$ independent entries by a factor of 4). This can be uniquely represented by 6 branches, which are just the number that can be strung between 4 nodes. On the other hand, the anti-symmetrical component

has only 3 independent entries, but there are 4 different three-terminal elements that can be hung from four nodes. Thus, only 3 out of the 4 possible gyrators are necessary to make up Y_a , and their values depend on which 3 out of the 4 are selected.

To summarize the general case: Any network or network element can be represented as an equivalent network composed of:

1. Branches and transconductances.
2. Branches and gyrators.

In method 1, the decomposition is not unique. Even without the trivial variations of paralleling branches, the number of branches and transconductances is not unique, and sometimes one type of element can partially replace the other (in other words, the basic elements are not linearly independent).

In method 2, there is linear independence of the basic elements. Consequently, the number of branches and of gyrators necessary to represent the network is fixed. The branches are also uniquely determined in position and value, whereas the gyrators are determined in number only, but not in position or value.

Of course, many other methods of decomposition are possible, using all three of the above mentioned elements, or some other types of possible basic elements. The two methods described here have the merits of using the minimum number of simplest basic elements - simple, that is, in the topological and algebraic sense as used in this Thesis.

Chapter IV

INVERTING THE ADMITTANCE MATRIX

In the two previous chapters, networks were treated from the point of view of one who builds them up from their elements. Now we turn to the user of the network - and to be of any use, a network has to be excited by some source, and feed a response into loads or meters. Usually, not all the nodes of the network will be used for connection to sources and loads, and the question arises now as to what will happen at the nodes that are not used.

The process of constructing the network led us naturally to an admittance representation of the form

$$i = Y v .$$

This matrix equation shows explicitly how each current depends on the voltages of the various network nodes. The entries of Y show the i due to one voltage with all other voltages being zero - that is, the "short-circuit" input and transfer admittances. To put any of these in evidence, some of the network nodes have to be shorted together. But, according to our approach, this makes a different network, because it has not only the elements of the original network, but some additional constraints represented by the short-circuits.

We would like to have a representation whereby any node not attached to a source, load or meter is just left by itself, that is, open circuited, with no current flowing into it or out of it. The required parameters would be impedances, leading to a relation of the form

$$v = Z i .$$

Unfortunately, the Y matrix is singular, so the Z cannot be obtained as an inverse in the regular way. What has to be done to obtain a Z matrix, and how to do it, is the subject of this chapter.

4.1 The Singularity of Y

Let us turn back to Fig. 4 (page 18) that shows v and i of a 3-terminal element: i as a point on the partition plane, v as a line perpendicular to the partition plane.

The Y matrix represents a transformation that maps a voltage line into a current point; starting from any point on the voltage line, we end up in the same current point. Of course, we can start from the current point and go back to the voltage line - but we have no guarantee of ending up at any particular point of the voltage line. As far as current is concerned, all the points on the voltage line are equivalent.

Suppose we start from any given v point, and go via Y to the i point; going back we can land in v' as well as in the original v . The singularity of Y does not mean that the inverse operation can not be performed; it only means that the result of the inverse operation is not unique. Transforming by Y and then by its "inverse" are not equivalent to an identity operation.

We can make this operation unique if we agree to choose one point on the voltage line as representing this line, and formulate the inverse operation so that it always ends up in that point of representation. The method of selecting this point can be completely arbitrary. A computationally convenient method is to define a surface in the space such that each voltage line pierces it once and only once; and for real ease of computation, let this surface be a plane. One possibility is to use the partition plane for this purpose, but there is actually an infinity of possibilities. It could be any plane which is not parallel to the null-line (because all the voltage lines are parallel to the null-line). For further convenience in computations, let this plane pass through the origin, hence be of the form

$$Q_t x = 0 \quad (4.1)$$

with Q a matrix of the same order as the partition matrix P .

This arbitrary choice of Q means that we agree to use only these voltage representations that satisfy a certain arbitrary homogenous linear equation (or set of equations).

This matrix Q will be referred to as the voltage reference matrix. Setting one node voltage equal to zero is clearly a special case of (4.1) above, therefore this term of voltage reference was chosen for the more general relation. It should again be emphasized that the voltage reference relation is a linear homogenous relation only because it leads to easier computations, and this fact has nothing to do with the linearity of the network. It is conceivable that in some special cases an even more general type of reference relation is suitable - a non-homogenous or non-linear relation - but such cases will not be treated here.

4.2 Projection Operators

As a preliminary to the general problem, let us first discuss the voltage reference problem for a 3-terminal element, so that 3-dimensional pictures can be drawn to illustrate some details of the process (Fig. 10). The partition matrix P is the same as in the example in section 2.5, and the partition plane is shown in dotted lines. Voltage is represented by a line perpendicular to this plane, or parallel to the null-line P .

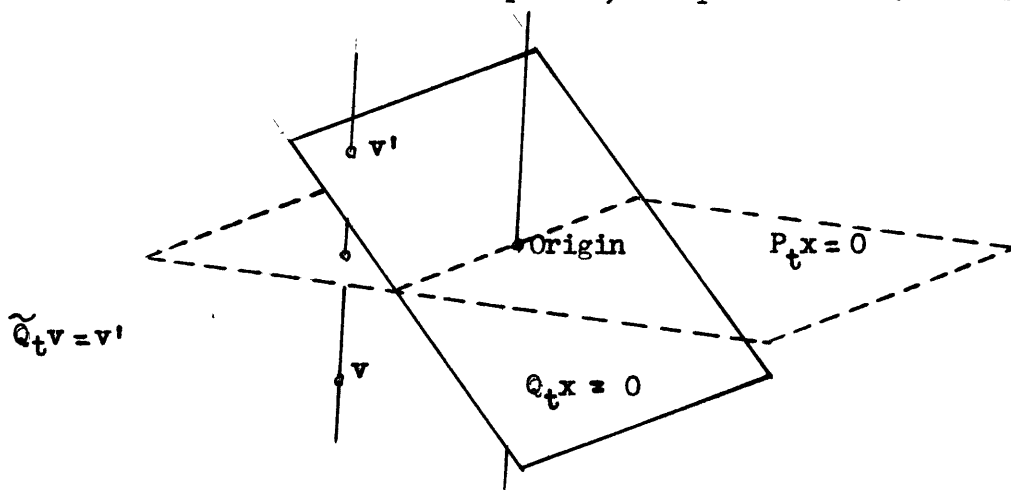


Fig. 10
Projection Operators

A second plane, $Q_t x=0$, is shown in the Figure. This plane passes through the origin, and the line P is assumed not to lie in this plane, hence any line parallel to P pierces this plane in one point and only in that one point. Any voltage on this line, say point v , is to be represented by the point v' on the same line that is on the reference plane. Given any point like v , we would like to find the projection operator that projects v onto the plane $Q_t x=0$ in a direction parallel to P .

The same relations will now be expressed in general n -dimensional form, so that the general projection operator can be found. We have a $n \times s$ partition matrix P (s is the number of terminal subsets), thus defining a $(n-s)$ -dimensional partition subspace $P_t x=0$. The null-line now becomes a complementary s -dimensional subspace, whose points are all those having coordinates of the form

$$x = P y$$

(y is an arbitrary s -rowed column matrix).

For voltage reference, another $n \times s$ matrix Q has to be defined, and then any voltage will be represented by a point for which

$$Q_t x = 0 .$$

Suppose now that any voltage point v is picked as the voltage of a network; it is equivalent to all other voltages of the form $v+Py$, and out of all these equivalent points the one point satisfying (4.1) is selected for representing v . Call this point v' , as in Fig. 10, then

$$v' = v + P y$$

$$\text{and } Q_t v' = Q_t v + Q_t P y = 0 .$$

In partitioned matrix form,

$$\begin{bmatrix} v' \\ 0 \end{bmatrix} = \begin{bmatrix} I & P \\ Q_t & Q_t P \end{bmatrix} \begin{bmatrix} v \\ y \end{bmatrix}$$

Eliminating y ,

$$v' = \left\{ I - P(Q_t P)^{-1} Q_t \right\} v$$

We therefore define the operator

$$\tilde{Q}_t = I - P(Q_t P)^{-1} Q_t \quad (4.2)$$

as the projection operator projecting a point onto the $Q_t x=0$ subspace in a direction parallel to P ,

$$v' = \tilde{Q}_t v \quad . \quad (4.3)$$

Note that this operation is possible if $Q_t P$ is a non-singular matrix, that is, has a non-zero determinant

$$\det(Q_t P) \neq 0$$

but this is equivalent to the statement that all the points of coordinates $x=Py$ (except the origin, where $x=y=0$), do not lie in the reference subspace.

(Note on notation: the projection operator is denoted by a transposed matrix to indicate the fact that Q_t appears in its derivation. A similar operator, in whose derivation the untransposed matrix appears, will be denoted by an untransposed symbol - see the operator \tilde{R} in section 4.4 below.)

Before proceeding to apply the projection operator to network problems, some of its properties will be listed. (Detailed proofs will be found in Appendix D.)

1. \tilde{Q}_t is a singular matrix. There is really no need to check this formally, for it follows directly from the definition. Any given v leads to a unique v' , but many other v points may lead to the same v' , so that no unique inverse operation is possible

2. \tilde{Q}_t is an idempotent operator

$$(\tilde{Q}_t)^2 = \tilde{Q}_t \quad . \quad (4.4)$$

This is a property of any operation classified as a "projection". It means that once a point has been projected onto the reference subspace, repeating the projection operation any number of times will leave the point undisturbed.

$$3. \quad Q_t \tilde{Q}_t = 0 \quad (4.5)$$

This reiterates the fact that any point after projection comes to rest in the reference subspace; for, projecting the point x ,

$$Q_t (\tilde{Q}_t x) = 0 \quad .$$

$$4. \quad \tilde{Q}_t P = 0 \quad (4.6)$$

All the voltages corresponding to the no-current no-power condition are represented by the zero point. Any of these null voltages is of the form $x=Py$, so that after projection

$$\tilde{Q}_t (Py) = 0 \quad .$$

4.3 Voltage Reference

Returning now to the network problem, we are in the following situation: Given a network admittance Y and a certain voltage v , it is possible to find the current i . Given the i , however, we still do not know how to return to that v we started from, although that v is known to us. Suppose that somehow we manage to put together a Z matrix for this special case - from a definite i to a definite v , not the general Z of the network. Starting with these i and v , we write

$$v = Z i \quad (4.7)$$

without claiming that this Z is good for any other i , or that the form of the Z matrix is unique even for that particular i .

In fact, the way the problem has been defined, we do not want to return to the same v we started from, but to the equivalent v' that satisfies the voltage reference condition. This can now be done by premultiplying (4.7) by the projection operator \tilde{Q}_t ,

$$v' = \tilde{Q}_t v = \tilde{Q}_t Z i$$

$$v' = Z' i \quad . \quad (4.8)$$

From the way the projection operator was defined, it follows that the new impedance matrix Z' will bring us from the definite i to the v' voltage, no matter which equivalent point v was used as a starting point. The result of the operation in (4.8) is therefore a unique value of v' . But, is the matrix Z' unique, or would other matrices multiplying i result in the same v' ?

The answer is that Z' is definitely not unique. In fact, given any Z' , any other matrix

$$Z'' = Z' + AP_t$$

(where P is the partition matrix of the network, and A an arbitrary matrix of order $n \times s$) would serve as well, for

$$\begin{aligned} Z''i &= Z'i + AP_t i \\ &= Z'i + 0 \\ &= Z'i \end{aligned}$$

The situation is similar to what we had with voltages, where adding a term P_y resulted in an equivalent voltage, and some arbitrary choice had to be made among all the equivalent voltages. Let us follow the same procedure here. Among all the equivalent Z'' ,

$$Z'' = Z' + AP_t \tag{4.9}$$

select as representative the one that satisfies

$$Z'' R = 0 \tag{4.10}$$

and R is an arbitrary matrix of order $n \times s$ (same order as P and Q). The requirement is given here as a purely formal relation, but it will be interpreted in the next section.

Equations (4.9) and (4.10) can be rewritten in partitioned matrix form

$$\begin{bmatrix} Z'' & 0 \end{bmatrix} = \begin{bmatrix} Z' & A \end{bmatrix} \begin{bmatrix} I & R \\ P_t & P_t R \end{bmatrix}$$

and, eliminating the arbitrary A from the equations,

$$Z'' = Z' \{I - R(P_t R)^{-1} P_t\} \quad (4.11)$$

or,
$$Z'' = Z' \tilde{R} \quad (4.12)$$

where
$$\tilde{R} = I - R(P_t R)^{-1} P_t \quad (4.13)$$

and all this is possible, of course, only if $\det(P_t R) \neq 0$.

As a result, if Z' is to be modified so that it will satisfy (4.10), the final resulting impedance matrix is

$$Z'' = \tilde{Q}_t Z \tilde{R} \quad (4.14)$$

and the current-voltage relation

$$v' = Z'' i \quad (4.15)$$

not only results in a unique answer v' , but also has a unique form Z'' .

4.4 Current Reference

For an interpretation of the R and associated matrices, we return to the 3-dimensional space of Figs. 4 and 10. In Fig. 4 we had currents as points constrained to a plane, voltage as lines perpendicular to that plane, and Y as operators transforming voltage lines into current points. In the inverse problem, the first step was to represent each voltage line by a point constrained to the plane $Q_t x=0$. A line of arguments similar to that developed in chapter II leads to the result that if any Z operator transforms one current point into a voltage point, this operator will do this not only for this one current point, but for all the points of a line passing through that current point; and all the points on that line will be transformed into the same voltage point. The roles of v and i are now interchanged (Fig. 11). The direction of the current line is defined by a matrix \tilde{R} , in the same way that the voltage lines have been defined by the partition matrix P .

Of course, of all the points of the line representing

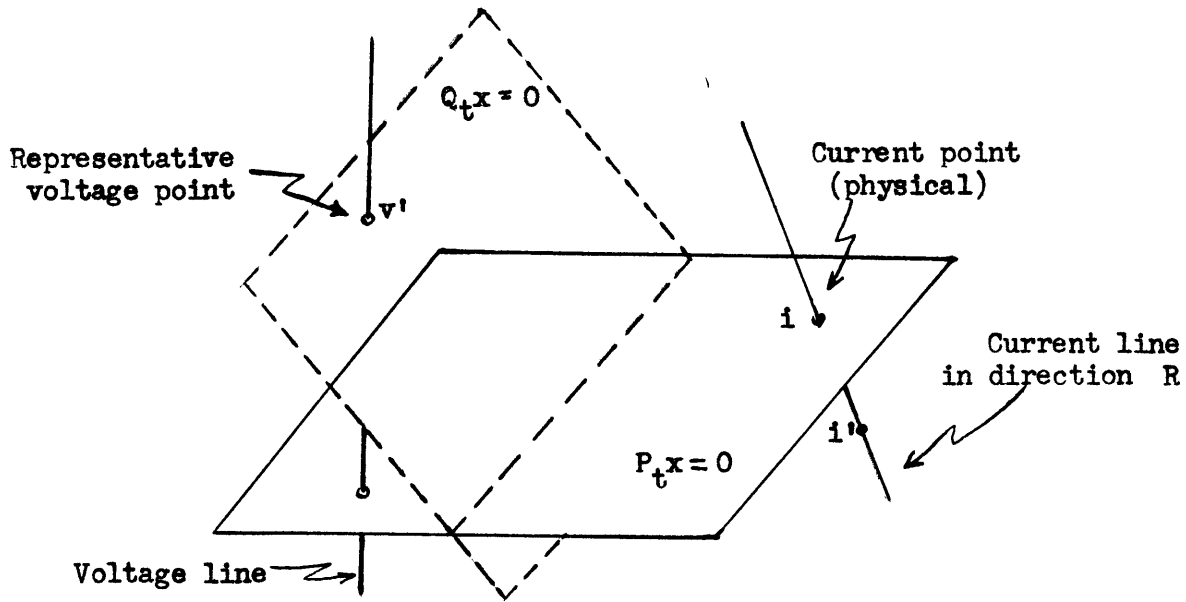


Fig. 11
Current Reference

current, only one line is a representation of a real physical current situation, and that is the original point i on the partition plane. All the other points of the line are just a mathematical fiction, so arranged that the equations have the right number of independent variables. It is now obvious that the direction R is arbitrary, as long as it adds only fictional current points, and the only real current will be the same i . However, this argument breaks down if R lies in the partition plane $P_t x = 0$, where all the points are possible real current points; this will lead to false answers, so this situation is prohibited. This restriction appears as the relation

$$\det(P_t R) \neq 0 \quad (4.16)$$

necessary for the realization of (4.13) above.

Take now any point i' on the "current line", and regard i as its representative point. To find i , the point i' has to be projected in the direction R onto the plane $P_t x = 0$. This operation is the same as the voltage projection, but the roles of P and Q are now played by R and P , respectively.

The projection operator is then

$$\begin{aligned}\tilde{R} &= I - R(P_t R)^{-1} P_t \\ i &= \tilde{R} i' \quad .\end{aligned}\quad (4.17)$$

The projection operator \tilde{R} has properties similar to those of \tilde{Q}_t (proofs are given in Appendix D).

1. \tilde{R} is singular.
2. \tilde{R} is an idempotent operator

$$(\tilde{R})^2 = \tilde{R} \quad .\quad (4.18)$$

1 and 2 define R as a projection operator: Once a point is projected, there is no return to the original point; and further projections will not change the results of the first projection.

$$3. \quad P_t \tilde{R} = 0 \quad (4.19)$$

which shows that any point projected by R comes to rest in the $P_t x=0$ plane.

$$4. \quad \tilde{R} R = 0 \quad (4.20)$$

so that any point on the line $x=Ry$ (y arbitrary) is projected onto the origin.

The projection operator \tilde{R} as defined in (4.13) applies to any n -dimensional problem, not only to the 3-dimensional one used for illustration.

The impedance computation can now be summarized in the following steps:

Given a network Y and voltage v , the current is computed as

$$i = Y v \quad .$$

For this definite voltage and current, construct an impedance Z (say, by trial and error methods)

$$v = Z i \quad .$$

But, we do not have to get back to the same v , for we have decided to represent that voltage by the equivalent v' , so

$$v' = \tilde{Q}_t v = \tilde{Q}_t Z i .$$

Finally, i is a representative of all the points i' on the current line, so that

$$\begin{aligned} i &= \tilde{R} i' \\ v' &= \tilde{Q}_t Z \tilde{R} i' . \end{aligned}$$

The resulting impedance

$$Z'' = \tilde{Q}_t Z \tilde{R}$$

has a unique form and leads to a unique result. To achieve this, two arbitrary conditions had to be imposed, one associated with voltage (the Q matrix), the other with current (the R matrix). In reality, the impedance matrix will never be constructed in this way (the first step already seems to involve some objectionable guesswork). This hypothetical process, however, served to indicate the conditions that have to be imposed before a meaningful impedance matrix can be discussed - let alone computed.

4.5 Summary of the Reference Problem

Right from the beginning of this Thesis, current and voltage were treated in a somewhat unconventional manner. When writing the current column matrix, all the terminal currents were included, although they are not all independent, and some might have been omitted without causing any ambiguity. Voltages were defined without specifying the reference terminal in each subset, so that the subset potentials could all be moved up or down without changing the results. Nevertheless, this way of defining currents and voltages caused no trouble when discussing power and admittance. But now when impedances are concerned, things have to be nailed down more definitely: the voltage reference has to be decided upon, fixing those floating potentials, and the redundant currents have to be discarded.

Where voltage is concerned, the Q matrix does the

necessary pinning down. A certain linear combination of voltages is decided to be zero

$$Q_t v = 0$$

and this matrix equation contains s linear combinations of voltages, one for each terminal subset. Once the Q matrix is given, the potentials can no longer be arbitrarily changed. There is only one definite arrangement of potentials that will cause the vanishing of the given linear combinations. The voltage reference condition has thus a simple interpretation; what about the current reference?

Returning for a moment to Fig. 11 (page 50), point i' is on the R -line that passes through the current point i , and for impedance computations i' is regarded as equivalent to i . Originally, of course, there is some distinction between i' and i (that is, a geometrical distinction, apart from the fact that only i represents a real current), but this distinction is neglected when the equivalence is assumed. Fig. 12 shows a way to define this distinction. To simplify the drawing, the P_t and Q_t planes of Fig. 11 were not included in Fig. 12. The Figure shows the line R through the origin (all points with coordinates $x=Ry$), and a few planes perpendicular to R . The planes are distinguished from each other by the value of the product $R_t x$ - this product being an s -rowed column matrix. The plane that passes through the origin has $R_t x = 0$; other planes parallel to it have non-zero products. In particular,

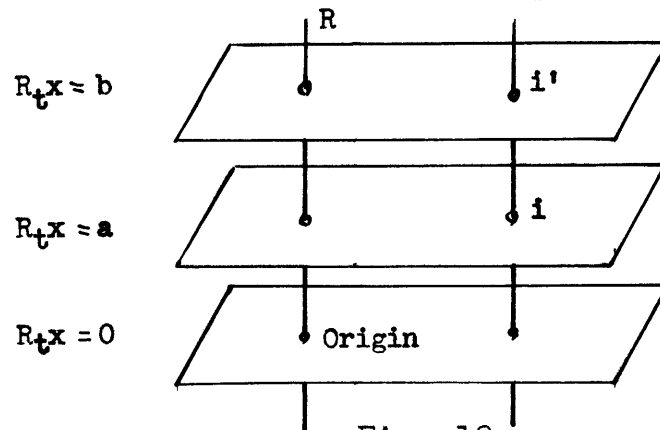


Fig. 12

Interpretation of Current Reference

$R_t x = a$ for the i point, and $R_t x = b$ for the i' point. All the points on a line parallel to R are indistinguishable when projected onto any one of these planes. The only characteristic to distinguish between these points is the value of $R_t x$ for the plane where the point was before the projection. It is then this distinction, the value of the product $R_t x$, which is being neglected when the current reference conditions are imposed.

The reference requirements can now be summarized:

1. Voltage and current reference conditions are required for the inversion of the Y matrix. Voltage reference leads to an inverse Z whose meaning is unique, and current reference establishes the uniqueness of the form of Z .
2. Given the partition matrix P of the network, the voltage and current references are established by defining two arbitrary matrices, Q and R , of the same order as P , with the only restrictions

$$\det(Q_t P) \neq 0$$

$$\det(P_t R) \neq 0 \quad .$$

3. Voltage reference is established by letting

$$Q_t v = 0$$

and current reference by neglecting the value of

$$R_t i \quad .$$

4.6 Imposing Reference Conditions on the Y Matrix

The way this chapter started off, a guess had to be made at Z and then all kinds of corrections had to be applied in order to put some sense into this Z . Still, this method indicated what further assumptions had to be made in order to have a meaningful impedance concept. Now, however, we are in the position where the required reference conditions can be imposed on the Y matrix, leading to an admittance matrix that is no longer singular, and then invert it without guesswork.

The essence of the reference conditions is that some linear combinations of voltages are assumed zero, and some linear combinations of currents are just not paid any attention to. These conditions are really much easier to apply to the Y matrix than to the impedance - and no projection operators are involved here.

A simple example of reference will be considered first. A single voltage is a special case of a linear combination of voltages, and the same is true for currents. Assume then that the reference conditions are:

$$v_j = 0 \quad , \quad i_k \text{ neglected.}$$

Writing out the admittance matrix,

$$\begin{bmatrix} i_1 \\ \vdots \\ \vdots \\ i_k \\ \vdots \\ \vdots \\ i_n \end{bmatrix} = \begin{bmatrix} Y_{11} & \cdots & Y_{1j} & \cdots & Y_{1n} \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ Y_{k1} & \cdots & Y_{kj} & \cdots & Y_{kn} \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ Y_{n1} & \cdots & Y_{nj} & \cdots & Y_{nn} \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ \vdots \\ v_j \\ \vdots \\ \vdots \\ v_n \end{bmatrix}$$

v_j is the factor that multiplies the j -th column of Y , and if $v_j=0$, that column may well be omitted. Similarly, since i_k is given by the entries in the k -th row of Y , and we are not interested in i_k , that row may be omitted. The reference conditions thus imposed mean crossing out the k -th row and j -th column of the Y matrix.

The original Y matrix was singular, because it was a zero-sum matrix. It is now evident that crossing out one row and one column out of each subset will destroy this feature of zero-summing rows and columns. This by itself is not a sufficient proof that the resulting matrix is non-singular, but the arguments on the uniqueness of the Z matrix indicate that the Y with reference conditions imposed has a unique inverse.

The general type of reference requires some preliminary rearrangement of the Y matrix, because the required linear combinations are not in evidence like the single voltages and currents. If the current and voltage column matrices are regarded as vectors, the vector components have now to be expressed on a new basis, so that the required linear combinations appear as some of the components.

Let x be a vector in n -dimensional space, represented by a $n \times 1$ matrix. Assume a set of basis vectors spanning this space, b_1, b_2, \dots, b_n , all represented by column matrices, so that any vector can be expressed as a linear combination

$$x = a_1 b_1 + a_2 b_2 + \dots + a_n b_n \quad (a_i \text{ scalars}) \quad .$$

All the b_i columns can be collected in a square matrix B , and the set of a_i into a n -rowed column matrix a , then

$$x = B a \quad .$$

The entries of a are the components of x to the basis B . The b_i are linearly independent, therefore B is non-singular. If a vector x and the basis B are given, the components can then be computed by

$$a = B^{-1} x \quad .$$

Suppose that a subspace $Q_t x = 0$ is given in the n -dimensional space, with Q an $n \times s$ matrix. This subspace is $(n-s)$ -dimensional, and requires $(n-s)$ basis vectors to span it. The set of basis vectors is not unique; let one such set be selected, and its vectors written as columns of a $n \times (n-s)$ matrix which shall be denoted by \hat{Q} .

Any column of \hat{Q} is a vector in the subspace $Q_t x = 0$, so

$$Q_t \hat{Q} = 0 \quad . \quad (4.21)$$

\hat{Q} is an orthogonal complement of Q : the columns of \hat{Q} are orthogonal to those of Q , and both matrices together span the whole space. The $n \times n$ matrix

$$B = [\hat{Q} \quad Q]$$

could then be used as a basis for representing the voltage points.

Let e denote the components of v to this basis ; and let the e column be partitioned into e_1 and e_2 , having $n-s$ and s rows respectively,

$$v = \begin{bmatrix} \hat{Q} & Q \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} .$$

Imposing the voltage reference condition,

$$\begin{aligned} Q_t v &= Q_t (\hat{Q} e_1 + Q e_2) \\ &= Q_t \hat{Q} e_1 + Q_t Q e_2 \\ &= Q_t Q e_2 , \end{aligned}$$

the first term being zero because of (4.21). Having expressed v in this form, it is obvious that putting $e_2=0$ satisfies the voltage reference condition.

The currents can similarly be represented, if the orthogonal complement \hat{R} of R is constructed, and the components to the new basis denoted by j , similarly partitioned,

$$i = \begin{bmatrix} \hat{R} & R \end{bmatrix} \begin{bmatrix} j_1 \\ j_2 \end{bmatrix}$$

$$R_t i = R_t R j_2$$

and ignoring j_2 is equivalent to the current reference condition of ignoring the combination $R_t i$.

The i and v , expressed in the new bases, still have to satisfy the admittance relation

$$\begin{aligned} i &= Y v \\ \begin{bmatrix} \hat{R} & R \end{bmatrix} j &= Y \begin{bmatrix} \hat{Q} & Q \end{bmatrix} e \\ j &= \begin{bmatrix} \hat{R} & R \end{bmatrix}^{-1} Y \begin{bmatrix} \hat{Q} & Q \end{bmatrix} e . \end{aligned}$$

This expression can be simplified if the columns of \hat{R} are

made orthogonal not only to the columns of R , but also among themselves. Then replace the matrix R by s other columns which are linearly independent and orthogonal and span the same subspace as R does; call this matrix R' . (There are detailed formal methods to work out this R' matrix, but we shall not go into that, since R' will actually not be used in the computations. It suffices to know that it is possible to construct it). The basis matrix for currents will then be

$$\begin{bmatrix} \hat{R} & R' \end{bmatrix}$$

an orthogonal matrix, and its inverse is easy to figure out, being equal to its transpose:

$$\begin{bmatrix} \hat{R} & R' \end{bmatrix}^{-1} = \begin{bmatrix} \hat{R} & R' \end{bmatrix}_t = \begin{bmatrix} \hat{R}_t \\ R'_t \end{bmatrix}$$

$$\begin{bmatrix} j_1 \\ j_2 \end{bmatrix} = \begin{bmatrix} \hat{R}_t \\ R'_t \end{bmatrix} Y \begin{bmatrix} \hat{Q} & Q \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$$

$$j_1 = \hat{R}_t Y \hat{Q} e_1 \quad (4.22)$$

since $e_2=0$, and j_2 is not to be computed. Equation (4.22) thus defines a new admittance matrix

$$Y' = \hat{R}_t Y \hat{Q} \quad (4.23)$$

that has the reference conditions already imposed. It relates the non-zero voltage components to the not-neglected current components. It is the matrix that can be inverted in the regular way to yield the required impedance matrix:

$$Z = \{ \hat{R}_t Y \hat{Q} \}^{-1} \quad (4.24)$$

4.7 Example of General Reference

To illustrate the various matrix operations described in this chapter, consider the following example:

Given a 3-terminal network, with all the terminals belonging to a single subset ($n=3$, $s=1$). The partition matrix is

$$P = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} .$$

Assume the following reference matrices

$$Q = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} \quad R = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

which mean: set the voltages so that

$$2v_1 - v_3 = 0$$

and when computing currents, neglect the combination

$$i_1 + i_3 .$$

First, check whether the reference conditions are valid:

$$Q_t P = 1 \quad , \quad P_t R = 2$$

so the two products are not singular, and the references are compatible with the given partition matrix.

The voltage projection operator will now be computed (for illustrative purposes only, but it is not needed in actual network analysis).

$$\begin{aligned} \tilde{Q}_t &= I - P(Q_t P)^{-1} Q_t \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 2 & 0 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & -0.5 \\ 1 & 0 & -0.5 \\ 1 & 0 & -0.5 \end{bmatrix} \\ \tilde{Q}_t &= \begin{bmatrix} -1 & 0 & 1 \\ -2 & 1 & 1 \\ -2 & 0 & 2 \end{bmatrix} \end{aligned}$$

Suppose the terminal voltages are given as

$$v = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$$

Applying the voltage projection operator

$$v' = \tilde{Q}_t v = \begin{bmatrix} -1 & 0 & 1 \\ -2 & 1 & 1 \\ -2 & 0 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -6 \\ -4 \end{bmatrix}$$

The new representation v' is the same as v , only all voltages are reduced by 5. The voltage differences between terminals remain the same, but the new representation now satisfies the reference condition

$$2v_1 - v_3 = -4 - (-4) = 0.$$

The projection operator for currents is

$$\begin{aligned} \tilde{R} &= I - R(P_t R)^{-1} P_t \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \\ \tilde{R} &= \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1/2 & -1/2 \\ -1/2 & -1/2 & 1/2 \end{bmatrix} \end{aligned}$$

Consider the non-physical "current" point

$$i' = \begin{bmatrix} 6 \\ 2 \\ 4 \end{bmatrix}$$

(It is non-physical since all currents do not sum to zero).
As far as impedance computations are concerned, this point is

equivalent to the physical current point

$$i = \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1/2 & -1/2 \\ -1/2 & -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 6 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 6 \\ -4 \\ -2 \end{bmatrix}$$

This i point is a real current point, for all three currents sum to zero. Comparing it to i' , we observe that i_1 and $i_2 - i_3$ are the same for both points, the only difference being in $i_2 + i_3$; but this is exactly the combination we agreed to disregard.

For the actual impedance computations, the orthogonal complements of Q and R are needed. In this simple example they can be found by inspection. (A formal method to construct the orthogonal complement is outlined in Appendix E.)

For \hat{Q} , find two columns that are orthogonal to Q , and linearly independent of each other. One possible combination is

$$\hat{Q} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 0 \end{bmatrix} .$$

The same can be done to find \hat{R} , only here there is the additional condition for the complete orthogonality of the basis matrix, so that each column vector should have unity magnitude,

$$\hat{R} = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1/2} \\ 0 & -\sqrt{1/2} \end{bmatrix} .$$

Given the 3×3 admittance matrix as derived in chapter III, the reference conditions can now be imposed to give

$$Y' = \hat{R}_t Y \hat{Q} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{1/2} & -\sqrt{1/2} \end{bmatrix} Y \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 0 \end{bmatrix}$$

This results in a 2×2 matrix Y' that is no longer singular, and can be inverted to produce the impedance matrix associated with these references.

4.8 Single Terminal References

In the vast majority of network problems, the simplest possible linear combinations may be selected for reference - these simple combinations being just the voltage or the current of a single terminal. If some more general reference arrangement does not appear necessary - as it may appear under certain symmetry conditions or in network mode analysis - these simple references supply enough variety to serve for all possible network problems.

In selecting Q and R matrices, they will be matrices that have one column for each subset of terminals, and only one "1" entry in each column. The conditions on $Q_t P$ and $P_t R$ dictate that one and only one terminal in each subset will be represented in the Q and the R matrices. However, there is nothing to indicate that the same terminal should be selected for both voltage and current references. (The usually accepted method of selecting a "datum" or "ground" node does impose this condition, for the voltage of the datum node is taken as zero, and the current into the same datum node is neglected in the computations.)

Going through the formal steps developed in this chapter, we find that both \hat{Q} and \hat{R} contain some of the columns of a unit matrix, and the basis matrices are just unit matrices (with a possible reshuffling of the columns). Equation (4.28) then leads just to crossing out several rows and columns of the original Y matrix. (This is the same conclusion we arrived at earlier, on p. 55, by less formal arguments).

This is then the procedure to be followed when using this type of single terminal references: In each subset of terminals, select one terminal for voltage reference, and one

terminal for current reference. (The two terminals selected in each subset may be different terminals or the same terminal). In the original singular Y matrix, cross out the columns corresponding to the voltage reference terminals, and the rows corresponding to the current reference terminals. The resulting matrix is of order $(n-s) \times (n-s)$, and is ready for inversion.

The choice of reference terminals depends on the type of problem that is to be solved. The considerations leading to the choice of reference terminals are of sufficient practical interest to warrant treatment in a separate chapter.

Chapter V

IMPEDANCE

In the second chapter, the fact was established that a definition of admittance is unique and meaningful even though voltages and currents be treated without specific reference conditions. The entries of the Y matrix can have the following interpretation: Suppose v_k is the only non-zero voltage, then the current into the j -th terminal is

$$i_j = Y_{jk} v_k \quad (5.1)$$

Although v_k does not have a unique value, because of the arbitrary reference potential, still the structure of the Y matrix is such that, when all v 's are taken into account, the i always comes out the same. Y_{jk} can thus be regarded as the mutual admittance, or trans-admittance, between two terminals (or self-admittance, if $j=k$).

Chapter IV presented a different situation regarding impedance. An equation similar to (5.1), with i and v interchanged, is impossible. Trying to repeat the argument that led to (5.1), one may say: Suppose i_j is the only non-zero current... - but this is impossible. What may be assumed is: Of course, i_j can not be the only non-zero current, for there must be at least one other node leading that current out. Let then this other node be used as reference for current, so that its own current need not be mentioned, and then we are left with one non-zero current. Thus the current reference has been fixed, and the current to be used in the computation now appears as a two-indexed entity

$$i_{rt}$$

this symbol meaning that terminal t is the current reference, and the only non-zero current is going into r (and coming out of t , but this latter fact can be ignored now).

Similarly, the reference conditions force us, when impedance is discussed, to regard voltage as a two-indexed symbol

$$v_{pq}$$

meaning, the voltage of terminal p when terminal q is regarded as the voltage reference terminal.

As a result, impedance is an entity that requires four indices for complete specification:

$$v_{pq} = Z_{pq,rt} i_{rt} . \quad (5.2)$$

Impedances thus appear to correlate pairs of terminals, so there is an essential difference between the admittance and impedance concepts. An impedance is a transfer impedance or mutual impedance of two pairs of terminals (or a self-impedance of a terminal pair if $p=r$ and $q=t$). In some special cases this distinction seems to disappear, and these cases will be mentioned and explained in the next chapter. The general case of the four-indexed impedance is the subject of the current chapter.

5.1 The Four-Indexed Impedance

In a general n -terminal network, there are n^4 possible permutations of 4 indices. Not all of these permutations specify an impedance, and of all permissible permutations, not all different permutations lead to different impedances. The total number of impedance coefficients associated with the network will be less than n^4 .

The voltage v_{pq} is meaningful only if p and q belong to the same subset of terminals, for only voltage differences within a subset are meaningful. Similarly, i_{rt} is defined only if r and t are in the same subset, because currents sum to zero within each subset. However, pq and rt may belong to different subsets.

$Z_{pq,rt}$ is defined only for those index combinations where p is in the same subset as q , and r in the same subset as t .

The definitions of v_{pq} and i_{rt} show that each will change sign if the two indices are interchanged. Therefore,

$$Z_{pq,rt} = -Z_{qp,rt} = -Z_{pq,tr} = Z_{qp,tr} \quad (5.3)$$

$Z_{pq,rt}$ is skew-symmetrical in each pair of indices.

All the possible arrangements of 4 indices can thus be grouped in sets of four, all the permutations in a set leading to the same impedance (except for sign). One of each set will be called the standard arrangement, and we select (because of convenience in further computations) the impedance where

$$Z_{pq,rt} \quad p \ll q \quad r \ll t \quad (5.4)$$

as the standard impedance to represent itself and the 3 other impedances associated with it via (5.3).

A corollary of (5.3) is that

$$Z_{pq,rt} = 0 \quad \text{if} \quad (p=q) \quad \text{or} \quad (r=t) . \quad (5.5)$$

The reciprocity relation (in networks that obey it, that is, in branch networks) appears in this notation as

$$Z_{pq,rt} = Z_{rt,pq} \quad (5.6)$$

5.2 Computing the Impedance

All the impedance coefficients appear as entries of an inverted Y matrix, after the required reference conditions have been applied. To simplify the relations that follow, the following special notations are introduced:

Given a Y matrix, the notation

$$D_{abc\dots,pqr\dots}$$

denotes the value of the determinant of the matrix obtained when rows a,b,c,\dots and columns p,q,r,\dots are omitted from Y .

First, note that because of the singularity of Y , the determinant that includes all the rows and columns is

$$D = 0 .$$

Furthermore, unless at least one row and one column out of each subset have been omitted, the determinant is still zero.

A general procedure will now be developed to compute an impedance with its indices in the standard order. (An impedance with the indices in any other order will have the same value, except for a possible change of sign, according to (5.3)).

Assume first that all the network terminals belong to one subset. When computing

$$Z_{pq,rt}$$

terminals q and t have already been selected for voltage and current reference, respectively, so the first operation consists of applying the reference conditions to Y :

Cross out column q and row t of the Y matrix.

Since there is only one subset of terminals, this is all the reference needed, and Y can now be inverted. According to the regular procedure of matrix inversion, the Z entries are the ratios of a subdeterminant of this reduced matrix to the determinant of the same matrix.

$$Z_{pq,rt} = (-1)^{p+r} \frac{D_{rt,pq}}{D_{t,q}} \quad (5.7)$$

(Because of the standard ordering of the indices, p and r retain their numbering value after t and q have been omitted, and the sign is thus easily fixed.)

In a general network there may be more than one terminal subset, and some further reference conditions, other than those shown by the Z indices, have to be imposed. To show the procedure on a specific case, let the network have 3 terminal subsets, so that two more rows and columns have to be omitted before inversion is possible. Their choice is quite arbitrary, so let these be rows a and b , columns c and d , appropriately distributed (with t and q) among the 3 subsets. The impedance is now computed as

$$Z_{pq,rt} = (-1)^k \frac{D_{rtab,pqcd}}{D_{tab,qcd}} \quad (5.8)$$

and the k is a number that depends on the row and column counts, whose determination will be postponed until later. Anyway, it seems that Z is not uniquely determined, because of the arbitrary references. That it is not so, and even more - that the general Z expression can be made even simpler than (5.7), will now be shown.

The Y matrix has a peculiar structure, being composed of zero-sum submatrices, and it is not surprising that some peculiarity would be reflected in its subdeterminants. The general property of the subdeterminants of the Y matrix is:

Given any subdeterminant obtained from Y by crossing out row (or column) j and other rows and columns such that j is the only row (or column) crossed out in its subset - then row (column) j can be replaced and another row (column) k in the same subset crossed out, resulting in a determinant having the original value multiplied by

$$(-1)^{j+k}$$

The long statement above (whose proof is found in Appendix F) means that all the determinants of the form appearing in the denominator of (5.8) have the same value, except for a possible change of sign. Also in the numerator, the arbitrariness of a , b , c and d will lead at most to a change of sign.

To standardize the form of the impedance notation, let the determinant notation be modified as follows,

$$D'_{abc\dots,pqr\dots}$$

(with a primed D) is the value of the subdeterminant obtained from Y by crossing out rows a,b,c,\dots and columns p,q,r,\dots and additional rows and columns as necessary to make the determinant non-zero. This is intended to imply that if after crossing out the rows and columns indicated by the indices, some subsets still have all their rows and columns, a row or a column

or both (as necessary) will be crossed out of each subset until sufficient reference conditions have been established. And an additional condition is imposed on the definition of D' , in order to fix its sign: all the arbitrary references are to involve the row and column similarly numbered within each subset .

In the example of the 3-subset network of equation (5.8),

$$D' = D_{\text{tab}, \text{tab}}$$

and this is one among other equivalent possibilities. From the theorem proven in Appendix F,

$$\begin{aligned} D_{\text{tab}, \text{qcd}} &= D_{\text{tab}, \text{tab}} x(-1)^{t+a+b+q+c+d} \\ &= D' x(-1)^{t+a+b+q+c+d} \end{aligned}$$

In the numerator of (5.8), a , b , c and d are the arbitrary indices, and by the definition of D' ,

$$D'_{\text{rt}, \text{pq}} = D_{\text{rtab}, \text{pqab}}$$

$$D_{\text{rtab}, \text{pqcd}} = D'_{\text{rt}, \text{pq}} x(-1)^{a+b+c+d}$$

Substituting into (5.8),

$$Z_{\text{pq}, \text{rt}} = (-1)^{p+q+r+t} \frac{D'_{\text{rt}, \text{pq}}}{D'} \quad (5.9)$$

This is the final form for computing impedance coefficients, and it includes (5.7) as a special case.

5.3 Solution of Network Problems

We are at last ready to attack the network problems, which is the real purpose of any analysis method. Given a network, its admittance matrix is first constructed as shown in chapter III. Only then are reference conditions imposed to fit the problem to be solved, and the appropriate impedance coefficients are computed.

As a first example, the input impedance at a terminal pair pq is defined as

$$v_{pq} = Z_{pq,pq} i_{pq}$$

which shows directly the Z coefficient that has to be computed. The input admittance at the same terminal pair is, of course,

$$\frac{1}{Z_{pq,pq}},$$

and in general, no matter whether the problem is worded in impedance or admittance terms, the solution will always involve Z coefficients, as explained in the preamble to chapter IV.

Passing now to two-terminal-pair problems (with or without a terminal common to both pairs), the problem may be to find a transfer impedance or transfer admittance, and these will again be Z 's or their inverse.

Let a current i_{rt} (that is, into terminal r and out of terminal t), be injected into the network, what voltage develops across pq ? The answer to this problem is given by the definition of Z in (5.2),

$$v_{pq} = Z_{pq,rt} i_{rt}.$$

To compute a voltage transfer ratio, let v_{pq} be the input voltage, and v_{rt} the output voltage (an open-circuit voltage, since any terminations at the output terminals could be incorporated in the original Y matrix of the network). There is only one current flowing through the network terminals, namely i_{pq} , therefore

$$\begin{aligned} v_{pq} &= Z_{pq,pq} i_{pq} \\ v_{rt} &= Z_{rt,pq} i_{pq} \\ \frac{v_{rt}}{v_{pq}} &= \frac{Z_{rt,pq}}{Z_{pq,pq}} \end{aligned}$$

For a current transfer ratio, inject a current i_{pq} and compute the short-circuit current i_{rt} . The equation for the voltage at the short-circuited output terminals is

$$v_{rt} = 0 = Z_{rt,pq} i_{pq} + Z_{rt,rt} i_{rt}$$

$$\frac{i_{rt}}{i_{pq}} = - \frac{Z_{rt,pq}}{Z_{rt,rt}}$$

In all these computations, whenever a ratio of two Z's appears, it is just a ratio of two subdeterminants of Y, since all Z's have the same denominator. All the problems shown above are therefore solved as a ratio of two determinants.

The above problems are intended to serve as examples only, and to show how a more general type of reference is quite naturally called for even in some of the simplest network problems. And this is just the first step in generalizing the reference conditions - still using single terminal reference conditions, but without the conventional restriction of using the same terminal for both voltage and current reference.

Chapter VI

CONCLUDING REMARKS

6.1 Duality

Duality is a quite powerful concept in Network Theory. The duality of voltage and current, L and C , R and G , coupled with the topological duality found in graphs, form a combination that plays quite a prominent role in the theory, and sometimes reduces by half the labor involved in solving a problem. From the outset of this Thesis, it seemed that the duality we were used to would not hold in this theory. Current and voltage have different properties; admittance and impedance have some essential differences, as was brought out in the last chapter. Graphs are not applicable to the type of elements we used, and that knocks out the final support on which the duality might be based. At first glance it seems too high a price to pay.

But is the situation really that bad? A little reflection will show that duality is not as complete as may have seemed, even in branch networks and using conventional network analysis, by graphs. The duality concept, though quite powerful, is not all-encompassing even in that type of network. In network elements, there is the mutual inductance that has no dual. Topologically, only networks whose graphs are mappable on a sphere have dual graphs. So, it is not too surprising that allowing more general elements in addition to branches will lead to a total collapse of this type of duality.

And, there is a type of duality in the theory as presented in this Thesis, although it is a different type of duality, algebraic rather than topological. Mathematically, it can be expressed in the fact that voltage and current are elements of two dual vector spaces (see Appendix B). This duality can be illustrated by the following considerations.

As treated in chapters II and III, currents had a constraint

imposed upon them, and voltages were "floating",

$$P_t i = 0$$

Pv_0 can be freely added to v ,

and the relation between them was given by a matrix Y that had the properties

$$P_t Y = Y P = 0 .$$

In chapter IV we saw how to impose reference conditions once the Q and R matrices have been chosen. This results in constrained voltages and "floating" currents,

$$Q_t v = 0$$

Ri_0 can be freely added to i .

The resulting impedance matrix, call it Z , is of the form

$$Z = \tilde{Q}_t Z' \tilde{R}$$

and from the properties of the projection operators,

$$Q_t Z = Z R = 0 .$$

Mathematically, this is a basis for a complete duality. From a practical standpoint, it is not so good. The P matrix is imposed by the element type, but Q and R are arbitrary. The floating voltages can be interpreted as changes in the voltage reference terminal, or changes of potential (that preserve potential differences). The floating currents are not currents in a physical sense, as was explained in chapter IV.

6.2 Different Types of Basic Elements

There is another way of regaining duality - if indeed it has to be regained, which is doubtful. Admittedly, the theory in this Thesis as it stands has no duality structure, but could it not be complemented by a complete dual theory? ^s

^s In fact, such a dual theory has been suggested by Prof. D. A. Huffman in a seminar meeting at M.I.T.

The basic network elements would be black boxes with loops sticking out instead of terminals, and elements would be interconnected by breaking open some loops and connecting them in series. There are two objections to this type of element and the theory that can be built on it. First, the element with terminals seems to be a more realistic representation of actual network elements than the looped element can be. (The latter may make some sense in magnetic circuits, but not in the general type of network). The second objection is that, even if we agree to use this type of element, it is really no more than a special type of terminalled element we have been using all along: just cut each loop open and equip it with two terminals - and this has to be done anyway before such an element can be included in a network. The looped element is then a very special case because it has a very restrictive terminal partitioning scheme, each subset containing only the two terminals of one broken loop.

6.3 On the Dangers of Generalizations

Now that a theory has been developed for general networks with general n -terminal elements, let us look closer at some of the special cases, for small values of n , and see what special properties are true for them that would not be true for a general n . Since most practical network elements have only a few terminals, we should always beware of regarding these special properties - which appear in the majority of practical networks - as general properties of any network. The cases of $n = 1, 2$ and 3 will now be considered in detail.

$n = 1$. A single-terminal element is really trivial. Its admittance, that has to be a zero-sum 1×1 matrix, can only be zero. The impedance cannot be defined, for it needs pairs of terminals, and there just are not enough terminals to form even a single pair.

$n = 2$. This refers to a branch, and the only possible 2×2 zero-sum admittance matrix has the form

$$\begin{bmatrix} y & -y \\ -y & y \end{bmatrix}$$

and this at once ties branch networks with the reciprocity relations. Impedance can be defined, for there is a pair of terminals available. The only non-zero impedance coefficient (in standard index order) is

$$Z_{12,12} = 1/y \quad .$$

In total, there are four non-zero impedances obtained by permuting the indices, and if we put $1/y=z$, the four impedances can be grouped as a zero-sum matrix

$$\begin{bmatrix} z & -z \\ -z & z \end{bmatrix}$$

$n=3$. The 3×3 Y matrix has 9 entries, but only 4 are independent, because of the zero-sum conditions. For the impedance representation, there are 3 possible terminal pairs (in standard index order), namely 12, 13, 23, so that there are nine impedance coefficients. To compute any one of these, use equation (5.9)

$$Z_{pq,rt} = (-1)^{p+q+r+t} \frac{D'_{rt,pq}}{D'}$$

Each numerator is obtained from Y by deleting two rows and two columns of the Y matrix, leaving a single entry of Y , so the Z 's are proportional to the Y entries, and will have similar properties, like zero-summing. (The possible minus signs can be adjusted by using one impedance with its indices not in the standard order).

The two special cases, $n=2$ and $n=3$, lead to impedances that, when arranged in a certain order, look like the Y matrix. To bring this about, some points had to be stretched, like using all the non-zero impedances in $n=2$, and only the representative impedances in $n=3$. Going to $n=4$ and above, there will be too many Z 's to be squeezed into any pattern

resembling that of the corresponding Y matrix.

These remarks would seem to be superfluous, were it not for the fact that now and again generalizations like that pop up in the literature. It is now well accepted that $n=2$ has some special properties that are not expected to remain for higher n values (reciprocity); but the peculiarities of $n=3$ still seem to be regarded as having general applicability. As shown above, 3-terminal elements still allow an impedance treatment similar to the admittance treatment, but this should not be taken to indicate possible extension to n larger than 3. A simple numerical check can demonstrate this point:

In an n -terminal element, there are n^2 admittance entries in the Y matrix. For impedances, $n(n-1)/2$ terminal pairs can be formed (using standard index order only), so there are $n^2(n-1)^2/4$ impedance coefficients. The equation

$$n^2 = n^2(n-1)^2/4$$

which is necessary for a one-to-one correspondence between impedance and admittance entries, has the three solutions

$$n_1 = 0 \quad , \quad n_2 = -1 \quad , \quad n_3 = 3 \quad .$$

The first solution refers to a trivial case, the second is meaningless, and the third is the only special case where this similarity between Y and Z exists.

In conclusion, it seems that "One, two, three, ... infinity" may be a nice and catchy title for a book on popularized mathematics, but it is a very dangerous way to generalize network theory. The right method is to treat the general case in its most general aspects, and from that to infer the properties of simpler special cases. Unless this is done, one can never be sure whether the so-called "general case" of the resulting theory is not tainted with characteristics that do not apply to it at all, but are just carried over from

the special cases. As an example of this attitude, we regard the insistence on using graphs for networks with multi-terminal elements as a carry-over from the theory of two-terminal elements (where they are quite useful indeed), and the efforts to keep impedance and admittance on equal footing as an improper extension of properties of two-terminal and three-terminal network elements. If some of the approaches in this Thesis seem unfamiliar, unconventional, or too complicated and generalized, it is precisely because of our trying to avoid pitfalls like the ones above, and trying to present the case in its generality; and it is our firmest belief that this is the right way to develop the theory of general linear networks.

A P P E N D I X

A. MATRIX NOTATION

Matrices are denoted in this Thesis by regular capitals and lower case letters. (Whenever letter symbols denote scalars, this is evident from the context). The notation convention is that lower case letters stand for column matrices, and capitals for square or rectangular matrices. A matrix that in the general case will be rectangular, but in some special cases may reduce to a single column, is also denoted by a capital (e.g., the partition matrix P).

When discussing an element with n terminals, or a network with n nodes, the column matrices have n rows, unless otherwise specified. Rectangular matrices are assumed to have n rows, and their transposes have n columns. Thus, products of the form

$$A_t x \quad A_t B$$

always can be carried out.

Notation of matrix operations:

A_t is the transpose of A .

A^{-1} is the inverse of A (if A is non-singular).

I is the unit matrix of the order required by the expressions in which it appears.

Two special notations are used for special purposes required in this Thesis:

\tilde{R} is a square $n \times n$ matrix, formed from R and the partition matrix P , and is used as a projection operator (see Appendix D).

\hat{Q} is a matrix whose columns are orthogonal to the columns of a given Q . If Q is of order $n \times s$, \hat{Q} is of order $n \times (n-s)$. (See Appendix E).

B. MATHEMATICAL MODEL

The relations among voltage, current, power, admittance and impedance, as presented in this Thesis, are based on the properties of vector spaces over the field of real numbers R .

Current (the i column) is a vector whose components are real numbers. Its n components place it in n -dimensional vector space S . All current vectors form a subspace P , which is an $(n-s)$ -dimensional subspace of S . Symbolically,

$$i \in P \subset S$$

Voltage is a linear operator on i into the field of real numbers - that is, it operates on a current vector to give power.

$$v : i \rightarrow R$$

Consequently, all v form a vector space dual to P , call it P' ,

$$v \in P'$$

Admittance is a linear mapping on v into a current vector

$$Y : v \rightarrow P$$

This mapping is a homomorphism, in that each v yields an i , but different v 's may lead to the same i . In particular, the kernel V_0 of this mapping is the set of all v that are mapped into zero

$$Y(v) = 0 \iff v \in V_0 \subset P'$$

This homomorphism does not have an inverse as it stands. However, if P' is reduced modulo V_0 , thus grouping the v 's into cosets, the inverse operation is possible, and it is an isomorphism, a one-to-one transformation

$$Z : i \rightarrow P'/V_0$$

The reference conditions and projection operators represent the reduction modulo V_0 , thus enabling the inverse transformation, impedance.

C. SUFFICIENT SET OF NETWORK ELEMENTS

A lumped, finite network can have its v and i represented by column matrices of order n . The most general linear (and homogenous) relation between the v and i columns is

$$A i + B v = 0 \quad (\text{A.1})$$

with A and B square $n \times n$ matrices. Four cases can be distinguished in this relation, according to the singularity of A and/or B , these cases being mutually exclusive and exhaustive of all possibilities.

Case 1. A and B both non-singular:

$$i = -A^{-1}Bv = Yv \quad Y = -A^{-1}B \quad (\text{A.2})$$

$$v = -B^{-1}Ai = Zi \quad Z = -B^{-1}A \quad (\text{A.3})$$

Such a network, or a network element, can have both a Y and a Z matrix, mutually inverse.

Case 2. A singular, B non-singular:

$$v = -B^{-1}Ai = Zi \quad Z = -B^{-1}A$$

There is no Y matrix as in case 1, because A^{-1} does not exist. But this same singularity that prevents defining a Y matrix leads to another relation; since A is singular, the equation

$$A_t x = 0$$

has non-zero solutions for x . If the rank of A is $n-s$, there are s linearly independent solutions x_1, \dots, x_s . Let all these columns be grouped in a matrix C , then

$$A_t C = 0 \quad .$$

Premultiply (A.1) by C_t

$$\begin{aligned} C_t A i + C_t B v &= 0 = 0 + C_t B v \\ G v &= 0 \quad G = C_t B \end{aligned} \quad (\text{A.4})$$

An element corresponding to Case 2 thus has an impedance matrix

and a constraint (A.4) on the voltages.

Case 3. A non-singular, B singular:

Following the same arguments as in Case 2, we arrive at an admittance matrix as in (A.2), and a constraint on currents. Since B is singular, a matrix D can be found such that

$$B_t D = 0$$

$$D_t A i + D_t B v = 0 = D_t A i + 0$$

$$H i = 0 \qquad H = D_t A \qquad (A.5)$$

Case 4. A and B singular.

No Y or Z matrix can be defined, but as in cases 2 and 3, two constraints can be defined, on voltage and on current, respectively.

To summarize, all possible lumped, finite, linear network elements have to belong to one of these four types:

1. Y and Z matrix possible.
2. Z matrix possible, and voltage constraint $Gv = 0$.
3. Y matrix possible, and current constraint $Hi = 0$.
4. Voltage constraint $Gv = 0$ and current constraint $Hi = 0$.

Right at the start of this Thesis, a constraint on current ($P_t i = 0$) was imposed on all network elements. This excludes types 1 and 2 from the theory, leaving only two possible types of elements, namely 3 and 4. Type 3 is the element represented by a singular Y matrix, as analyzed in chapter II. Type 4 is the element represented by two sets of relations: one between currents only, the other between voltages only. This includes elements like short-circuits and ideal transformers, and was discussed in chapter III. These two types of element thus exhaust all the possible elements that fit in the theory.

D. PROJECTION OPERATORS

Given the partition matrix P (of order $n \times s$), and a reference matrix R of the same order, and

$$\det(P_t R) \neq 0$$

define

$$\tilde{R} = I - R(P_t R)^{-1} P_t .$$

Properties of \tilde{R} :

$$\begin{aligned} (\tilde{R})^2 &= \{I - R(P_t R)^{-1} P_t\} \{I - R(P_t R)^{-1} P_t\} \\ &= I - 2R(P_t R)^{-1} P_t + R(P_t R)^{-1} P_t R(P_t R)^{-1} P_t \\ &= I - 2R(P_t R)^{-1} P_t + R(P_t R)^{-1} P_t \\ &= I - R(P_t R)^{-1} P_t \\ &= \tilde{R} \end{aligned}$$

$$\begin{aligned} P_t \tilde{R} &= P_t \{I - R(P_t R)^{-1} P_t\} \\ &= P_t - P_t R(P_t R)^{-1} P_t \\ &= P_t - P_t = 0 . \end{aligned}$$

$$\begin{aligned} \tilde{R} R &= \{I - R(P_t R)^{-1} P_t\} R \\ &= R - R(P_t R)^{-1} P_t R \\ &= R - R = 0 . \end{aligned}$$

The other projection operator (Q_t in the text) is

$$\tilde{R}_t = I - P(R_t P)^{-1} R_t = (\tilde{R})_t$$

and its properties are obtained by transposition of the above relations found for \tilde{R} :

$$(\tilde{R}_t)^2 = \tilde{R}_t \quad \tilde{R}_t P = 0 \quad R_t \tilde{R}_t = 0 .$$

E. ORTHOGONAL COMPLEMENT

Given a rectangular matrix of order $n \times s$, the equation

$$Q_t x = 0$$

has non-zero solutions for x . Following the regular method of solving simultaneous equations (s equations in n unknowns), x can be expressed in the form of a linear combination of $n-s$ columns, with $n-s$ arbitrary parameters as the coefficients of the linear combination. The $n-s$ basis columns are not unique, for linear combinations of these columns may in turn be used as another basis. In any case, select one set of linearly independent columns as a basis for x , and let these columns form the matrix \hat{Q} .

This procedure gives one form out of the many possible forms for the orthogonal complement of Q ,

$$Q_t \hat{Q} = 0 \quad .$$

Every column of \hat{Q} is orthogonal to every column of Q . If, in addition to that, the columns of \hat{Q} itself are required to be orthogonal to each other, the following procedure may be followed.

Given x_1, x_2, \dots, x_k k linearly independent columns, form the following linear combinations:

$$\begin{aligned} y_1 &= a_{11}x_1 \\ y_2 &= a_{21}x_1 + a_{22}x_2 \\ y_3 &= a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \\ &\dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \\ y_k &= a_{k1}x_1 + a_{k2}x_2 + a_{k3}x_3 + \dots + a_{kk}x_k \end{aligned}$$

The coefficients a_{ij} can be determined from the orthogonality relations:

$$(y_1)_t y_1 = 1 \quad \text{is an equation for } a_{11}$$

$(y_2)_t y_2 = 1$ and $(y_1)_t y_2 = 0$ are two equations for a_{21} , a_{22}

$(y_3)_t y_3 = 1$, $(y_2)_t y_3 = 0$ and $(y_1)_t y_3 = 0$ are three equations for a_{31} , a_{32} , a_{33} , and so on. Each additional row yields the required number of equations to solve for the coefficients in that row, in terms of the known x 's and the previously solved coefficients of the previous rows.

The matrix formed from the y columns is a basis for the same space (or subspace) spanned by the x columns, but it is an orthogonal basis.

F. SUBDETERMINANTS OF THE Y MATRIX

The Y matrix of a network or an element is composed of submatrices whose rows and columns all sum to zero. To obtain a non-zero subdeterminant, at least one row and one column out of each subset has to be crossed out.

Consider now such a subdeterminant, and let Y' be the matrix leading to it. That is, Y' is the Y matrix with some of its rows and columns omitted. Assume now that one subset has had only one row and one column removed, while the other subsets may have had more than one row and column removed. In the particular subset where only one row and column are missing, suppose the missing row has been numbered j in the original matrix Y .

Let Y'' be a matrix just like Y' , with the only difference that row k has been removed instead of row j , k and j being in the same subset of nodes. Given Y' , it is easy to construct Y'' , for all that is to be done is to omit row k and substitute for it a row which would have made all the original columns of Y' sum to zero.

In order to effect such a transformation on a matrix, it is sufficient to apply the same transformation to a unit matrix, which will result in a matrix T , say, and then premultiply Y' by T . (For the same operation on columns, form T_t by

column operations on the unit matrix, and postmultiply Y').
For row operations,

$$Y'' = T Y'$$

$$\det Y'' = (\det T) \times (\det Y')$$

As an example of this operation on the rows of a unit matrix,

$$T' = \left[\begin{array}{ccc|ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & -1 & -1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right]$$

The matrix is written here in a partitioned form, and shows an example of this operation performed in the second of three subsets. The matrix T' is not yet the required matrix T , because the j -th row has been inserted in the position vacated by the k -th row, and still has to be moved to its original position to form T . The transformation from T' to T involves moving this row $j-k-1$ places; each move to an adjacent position changes the sign of the determinant, so

$$\det T = (-1)^{j-k-1} \det T'$$

The determinant of T' is easy to compute, if developed in terms of co-factors of the row containing the (-1) 's. Each cofactor will have a row of zeroes, except the cofactor of the term on the principal diagonal. The latter cofactor is 1, and

$$\det T' = -1$$

$$\det T = (-1) \times (-1)^{j-k-1}$$

$$= (-1)^{j-k} = (-1)^{j+k}$$

$$\det Y'' = (-1)^{j+k} \det Y'$$

If Y'' is obtained from Y' by changing the omitted column, the relations will be

$$Y'' = Y' T_t$$

but, since

$$\det T_t = \det T$$

there will be no change in the final result

$$\det Y'' = (-1)^{j+k} \det Y' .$$

This proves the theorem used on page 68.

Biographical Note

Jacob Shekel was born in Bialystok, Poland, on January 6, 1926. Attended primary and secondary school in Petah Tikva, Israel, and received his Engineering education at the Technion, Israel Institute of Technology, Haifa, Israel. Since 1948 he has been in the employ of the Scientific Department, Israel Ministry of Defence, working in the fields of Network Analysis and Synthesis, and in UHF Techniques. Mr. Shekel also held a part-time appointment as Visiting Lecturer at the Technion, Israel Institute of Technology, lecturing and conducting laboratory work on UHF Techniques.

In the summer of 1949, Mr. Shekel was a guest student at MIT on the Foreign Students' Summer Project. He has been attending the Graduate School of MIT as a full-time Student since September 1954, on a scholarship granted by the Scientific Department, Israel Ministry of Defence.

Mr. Shekel is a member of the Institute of Radio Engineers, and affiliated with the following Professional Groups: Circuit Theory, Information Theory, Microwave Theory and Techniques. A list of publications in English appears on the next page.

Married to Tamar (née Hertz) , they have a son Ethan (1953) and daughter Michal (1956).

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