

A NUMERICAL MODEL FOR THE AGGREGATION OF SNOW CRYSTALS

by

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ABSTRACT

A pilot study is made in which the evolution of a snowflake spectrum in stratiform-type storms is followed as far as possible with hand computations.

At the onset of aggregation, in a layer just above the meeting level, an initial particle density of 10^4 crystals per m^3 is assumed. The crystals are assumed to be plane dendrites of uniform size, 4 μm in diameter. Random collisions among the ice crystals, at the rate of 10 collisions per second, initiate the spectrum development which is further continued by ordered collisions as aggregates with greater fall velocities overtake crystals or other aggregates.

After a time interval of about 4 mins., the evolving spectrum is found to be approaching snowflake spectra actually measured at the surface for the equivalent precipitation rate of 1.5 mm/hr. The aggregation process slows up considerably with passing time and eventually reaches a fairly stable state.

Further experiments with machine computations are desirable so that the complete evolution of the spectrum and its dependence on the initial can be more fully explored.

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I. INTRODUCTION

The mechanisms of precipitation development depend largely on the type of cloud producing the precipitation. Condensation alone is not a sufficiently rapid process to produce precipitation size particles in a reasonable length of time. Growth beyond the cloud drop size may occur either through coalescence of droplets or through growth and aggregation of ice crystals. Clouds that do not extend above the melting level are commonly observed in regions where the melting level occurs high in the troposphere, e.g. mid latitudes during the summer season and tropical regions. Precipitation in wholly liquid clouds develops mainly as a result of collisions and coalescences of cloud droplets. Some theoretical studies of coalescence mechanisms in liquid clouds have been undertaken with moderate success (Melzak and Hitehfeld, 1953, Bartlett, 1966), and the process is fairly well understood quantitatively provided that conditions are uniform.

In clouds that extend above the melting level, precipitation generally originates as snow. A mechanism for the development of precipitation in stratiform clouds was first proposed by A. Wegener in 1911 and later fully extended by T. Bergeron in 1933. Ice crystals are produced at heights well above the melting level and grow by sublimation in the super cooled regions of the cloud. At temperatures slightly below 0°C the crystals are usually observed to be aggregated into snow flakes, which subsequently melt into liquid drops. The resulting drops grow further by condensation and coalescence before finally reaching the ground as rain.

Quantitative studies of the aggregation processes taking place above the melting level have generally been neglected. Yet this is clearly an important factor in the development of raindrops or large snowflakes. The purpose of this study is to investigate quantitatively the mode of aggregation of ice crystals, particularly in stratiform-type precipitation.

Laboratory experiments made by Hosler, Jensen and Goldshlak (1957) support the widely held belief that aggregation mainly occurs at temperatures above -4°C . Also, radar observations and measurements indicate a larger increase in reflectivity at the melting level than can be explained by melting alone, suggesting that rapid aggregation of snowflakes occurs in this region.

In this study, possible collision mechanisms resulting in aggregation of ice crystals and snowflakes are considered. The theoretically computed snowflake size spectra are compared with measured spectra before and after melting. The length of time required and fall distances involved during the evolution of the spectrum are also considered.

II. MEASUREMENTS OF OBSERVED PARTICLE SIZE SPECTRA

A. Raindrop spectra

Atlas (1964) has remarked that "the only statement about drop size spectra that can be made with complete certainty is that they are highly variable in time, space and with storm type." Much of the variability results from inadequate sampling. Marshall and Palmer (1948) analysed average rain drop samples collected by a number of observers and found a simple relationship between the drop size concentration, drop diameter and rainfall rate. For stratiform-type rainfall originating as snow, the M-P distribution remains the simplest representation of raindrop concentrations. The relationship is

$$N_D = N_0 e^{-\Lambda D} \quad (2.1)$$

where $N_D dD$ is the number of drops in the diameter interval D to $D + dD$ per unit volume, N_0 is a constant, 0.08 cm^{-4} and Λ is a parameter depending on the intensity of rainfall, $R(\text{mm/hr})$:

$$\Lambda = 41 R^{-0.21}$$

For diameter sizes larger than about 1 mm the M-P spectrum is generally an accurate representation of the distribution but it usually overestimates the concentrations of drops smaller than 1 mm in diameter. This has been noted by Marshall and Palmer themselves, Best (1950) Mueller and Jones (1960), Mason and Andrews (1960). Maximum concentrations normally occur between 0.5 and 1 mm so that the rain spectrum

appears parabolic rather than linear on a plot of $\log N_D$ versus D . Figs. 1 and 2 show M-P distributions for rainfall intensities of 1.5 and 2.5 mm/hr.

Most of the rain in the above measurements originated as snow. Therefore the particle size distribution for the snow before melting is extremely relevant to the measured raindrop spectra. The spectra which would melt into the M-P distributions have been computed on the assumption that melting is the only process to occur at the melting level i.e. no drop break up. Figs. 1 and 2 also show the M-P spectra before melting for rainfall intensities of 1.5 and 2.5 mm/hr. A density factor in inverse ratio to the change in terminal fall velocity was used to translate the distributions. The spectra computed theoretically from the model will be compared with the M-P distributions for equivalent precipitation rates, before melting. Ideally, the comparison should be made with translated rain spectra sampled just below the melting level. At higher levels in the atmosphere rain spectra can be expected to contain more small drops and, possibly, fewer larger ones than at the ground. Mason and Ramanadahan (1954), Hardy (1963) have shown theoretically that changes in raindrop spectra with height occur because of coalescence, accretion and evaporation processes and result in a successive depletion of the smaller drops with decreasing height. Thus, it would be expected that the M-P distribution is a more accurate representation of the rain spectrum just below the melting level than at the surface.

B. Snowflake spectra

Few measurements of snowflake spectra have been published. Consequently, no particular form of snowflake size distribution has yet found general acceptance.

Gunn and Marshall (1956) grouped average samples in stratiform storms into four snowfall rates. The data satisfied the relationship:

$$N_D = N_0 e^{-\Lambda D}$$

where N_D is the number of aggregated with equivalent drop diameter between D and $D + dD$ per unit volume. Both N_0 and Λ are functions of the precipitation rate, R (mm/hr).

$$N_0 = 3.8 \times 10^3 R^{-0.87} \text{ (m}^{-3} \text{mm}^{-1}\text{)}$$

and

$$\Lambda = 25.5 R^{0.48} \text{ (cm}^{-1}\text{)}$$

Fig. 3 shows snowflake spectra measured by Gunn and Marshall at rainfall rates of 1.5 and 2.5 mm/hr.

Some occasional measurements have been made by Japanese observers, Fujiwara (1955), Imai (1955), Maeno (1953). They are in fair agreement with the samples measured by Gunn and Marshall. The spectra computed from the model are to be compared with the Gunn and Marshall spectra for equivalent snowfall rates.

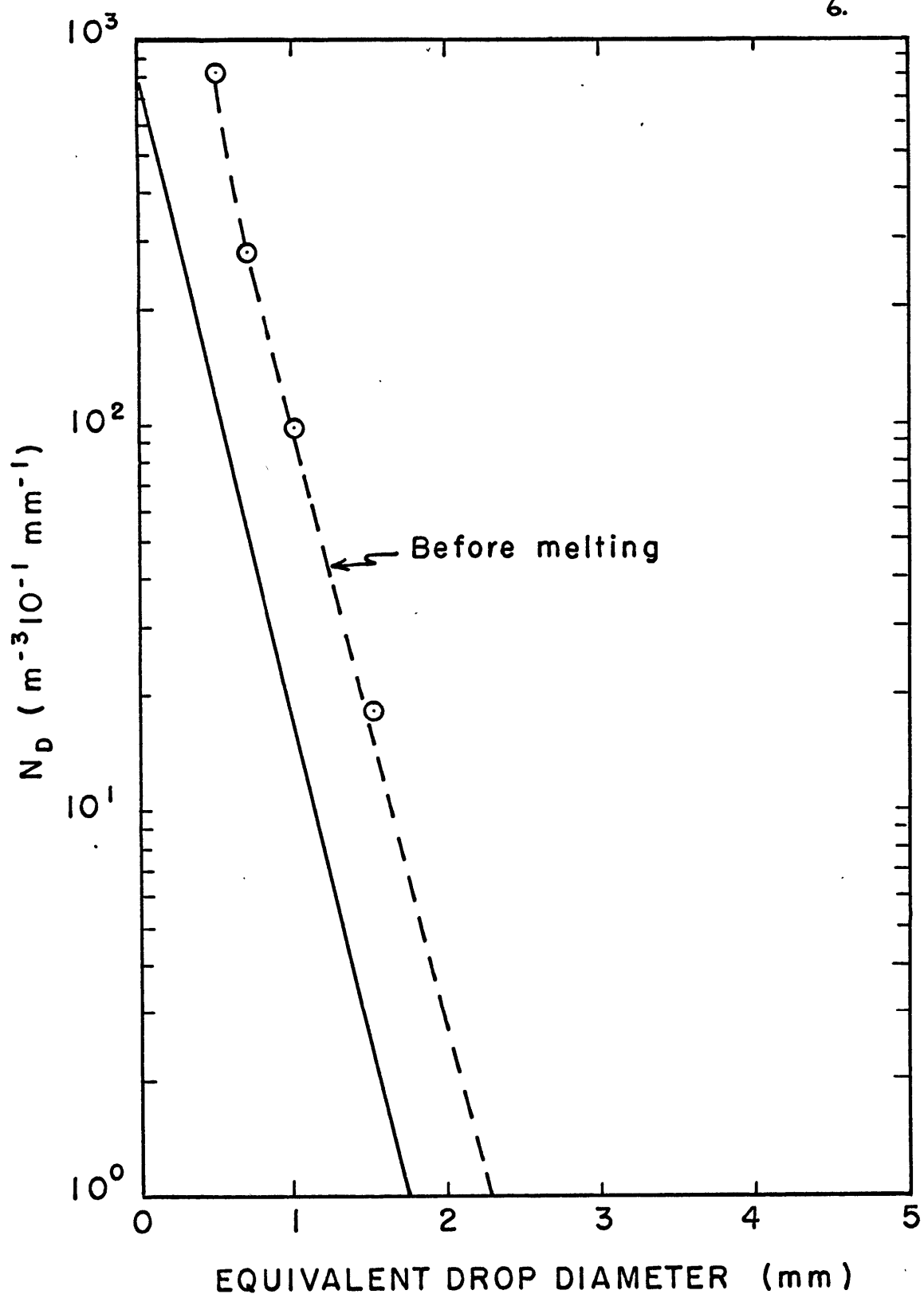


Fig. 1. M-P drop-size distributions, before and after melting.
R = 1.5 mm/hr.

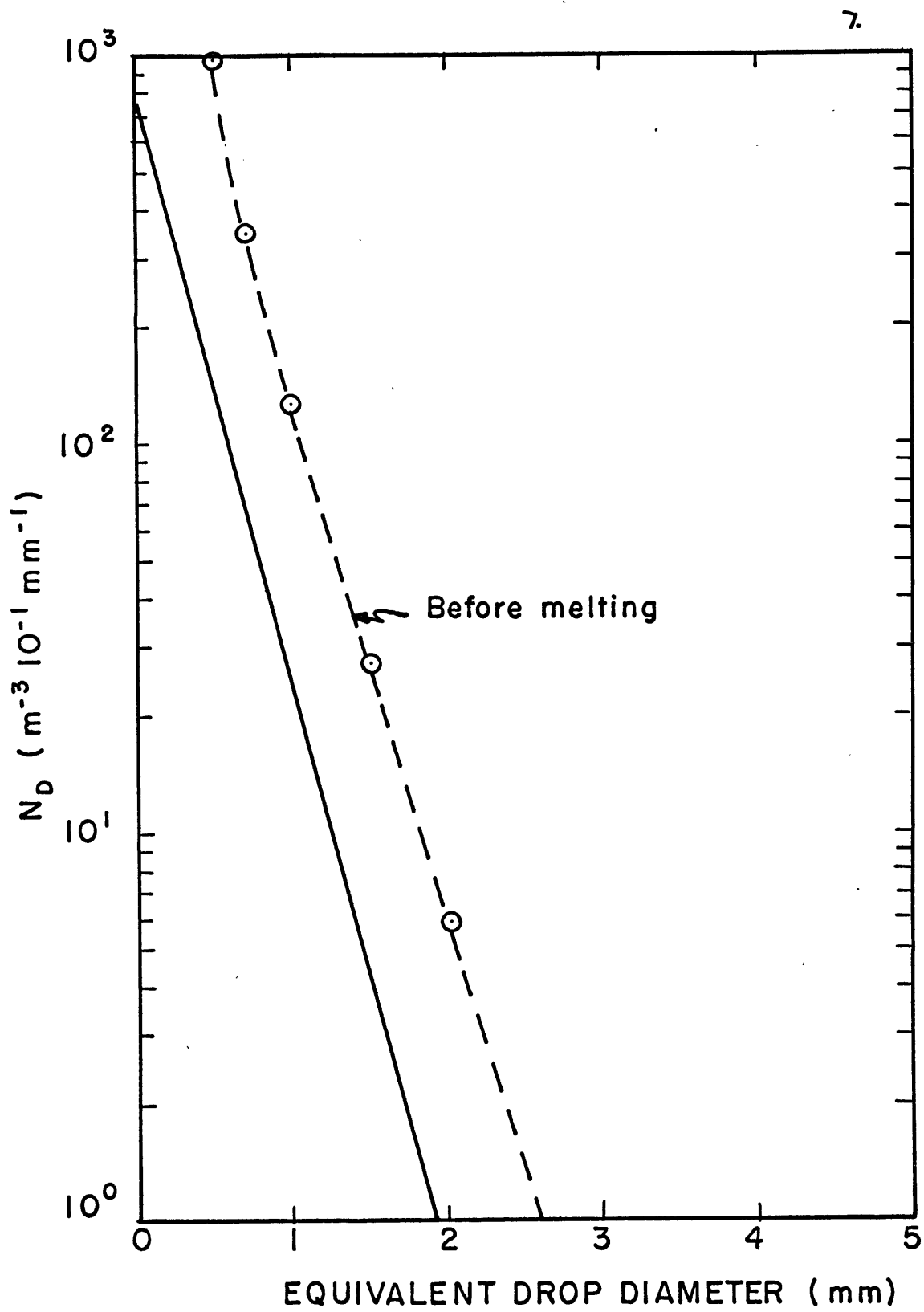


Fig. 2. M-P drop-size distributions, before and after melting.

$R = 2.5$ mm/hr.

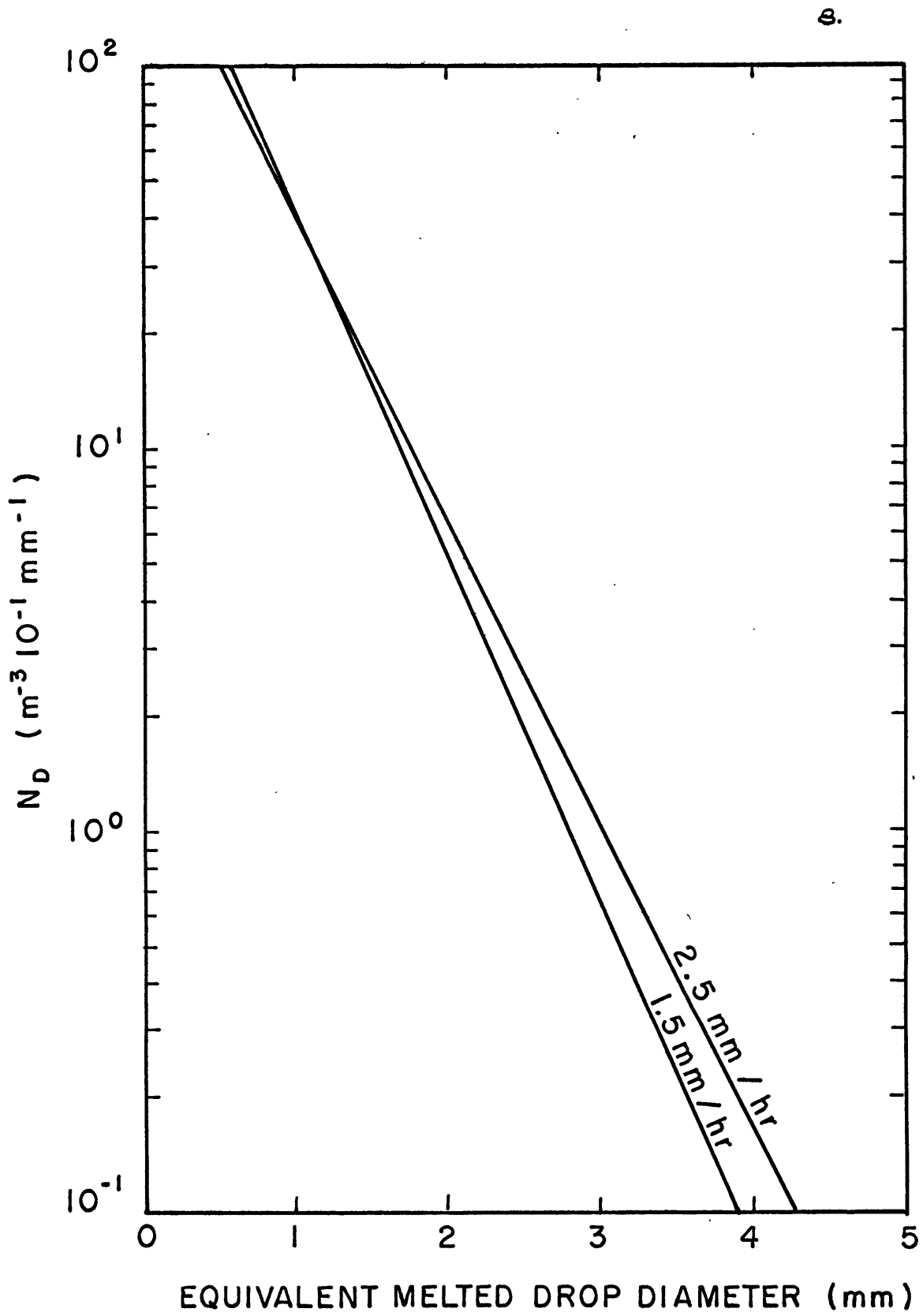


Fig. 3. Snowflake spectra from Gunn and Marshall (1956).
Precipitation rates: 1.5 mm/hr and 2.5 mm/hr.

III. THEORETICAL MODEL FOR THE DEVELOPMENT OF A SNOWFLAKE SPECTRUM

A. Collision Mechanisms

Initially, the ice crystals are assumed to be uniform in both size and type. Snowflakes are initiated by random collisions among ice crystals and continue to grow by further collisions. Ice crystals are normally produced at temperatures below -10°C in stratiform clouds which extend above the melting level. They grow by sublimation as they fall through supercooled regions of the cloud. Aggregation is assumed to occur simultaneously through a layer several thousand feet in depth just above the melting level. It is also assumed that no further growth by sublimation occurs in this layer.

In the model, two types of collision mechanisms effect the development of the snowflake spectrum:

1. **Ordered collisions:** Ordered collisions occur among the precipitation elements of different sizes owing to differences in terminal fall speeds. Faster moving particles overtake those which are falling more slowly, and they are assumed to become attached upon contact. On the average, all of the crystals have the same terminal fall velocity and are not subject to ordered collisions among themselves. Ordered collisions between snowflakes and crystals and among snowflakes of different sizes, however, will occur.

The number of ordered collisions between snowflakes consisting of p ice crystals and snowflakes consisting of q ice crystals per unit volume of space, per unit time is given by:

$$f_{pq} = \rho_p \rho_q S_{pq} (V_p - V_q) E_{pq} \quad (3.1)$$

where, ρ_i = number of particles per unit volume

V_i = terminal fall velocity

S_{ij} = effective collision cross section

E_{ij} = collection efficiency.

This relationship also expresses the ordered collision frequency, f_{ij} between single ice crystals and snowflakes. In the model E_{ij} is assumed unity at all times.

2. Random collisions: In this study all other collisions are assumed to be "random". It is believed that these random collisions result primarily from microscopic forces (hydrodynamic, electrostatic etc.) acting on each particle which produce random oscillations about a mean free path. Measurements of these random effects are virtually impossible in the atmosphere because of observational difficulties. Mason and Jayewara (1963) have simulated atmospheric conditions in the laboratory and have amply demonstrated that significant oscillations of particles about a mean free path are likely to occur in the atmosphere.

The random collision process operating on a population of uniform particles is assumed to be equivalent to a collision mechanism producing collisions between two particles at regular, discrete intervals of time depending on the particle density. The random collision frequency is expected to vary with the particle density according to the following theoretical considerations.

The total possible number of collisions among uniform particles of

number density ρ is:

$$\frac{\rho(\rho-1)}{2} \quad (3.2)$$

Thus, the frequency of random collisions will be proportional to

$$\frac{\rho(\rho-1)}{2} \quad (3.3)$$

In other words, the average time interval between successive collisions is proportional to

$$\frac{2}{\rho(\rho-1)} \quad (3.4)$$

When the particles are not uniform in size it can be shown that the total number of possible collisions is

$$\begin{aligned} \sum_{i=1}^l \frac{\rho_i(\rho_i-1)}{2} + \sum_{\substack{i,j=1 \\ i \neq j}}^l \rho_i \rho_j \\ = \frac{1}{2} \sum_{i=1}^l \rho_i \left(\sum_{j=1}^l (\rho_j - 1) \right) \end{aligned} \quad (3.5)$$

where ρ_i ($i = 1, \dots, l$) are the number densities of particles composed of i individual snow crystals. Such particles will be termed F_i aggregates.

Similarly, the average time interval between successive random collisions of F_p aggregates with one another is proportional to

$$\frac{2}{\rho_p(\rho_p-1)} \quad (3.6)$$

whereas the average time interval between successive random collisions

of an F_p aggregate with an F_q aggregate is proportional to

$$\frac{1}{f_p f_q} \quad (3.7)$$

In the model the ice crystals are assumed to constitute one group and the aggregates another. If at any given time

f_c = number density of ice crystals

f_a = number density of snow aggregates

Δt_{cc}^R = average time interval between successive random collisions among crystals etc.

$$\Delta t_{cc}^R = \frac{C}{f_c^2}, \quad \Delta t_{ca}^R = \frac{C}{2 f_c f_a}, \quad \Delta t_{aa}^R = \frac{C}{f_a^2} \quad (3.8)$$

or

$$\frac{\Delta t_{cc}^R}{\Delta t_{aa}^R} = \frac{f_a^2}{f_c^2}, \quad \frac{\Delta t_{cc}^R}{\Delta t_{ca}^R} = \frac{2 f_a}{f_c}, \quad \frac{\Delta t_{aa}^R}{\Delta t_{ca}^R} = \frac{2 f_c}{f_a} \quad (3.9)$$

At the onset of aggregation the crystals are very numerous while the aggregates are extremely few so that Δt_{cc}^R is by far the smallest of the above time intervals. As the spectrum develops the following significant stages are reached

$f_c > f_a$:

$$\frac{f_c}{2} > f_a : \quad \Delta t_{cc}^R < \Delta t_{ca}^R < \Delta t_{aa}^R$$

$$\frac{f_c}{2} = f_a : \quad \Delta t_{cc}^R = \Delta t_{ca}^R < \Delta t_{aa}^R$$

$$\frac{f_c}{2} < f_a : \quad \Delta t_{ca}^R < \Delta t_{cc}^R < \Delta t_{aa}^R$$

$$\underline{\rho_c = \rho_a} : \quad \Delta t_{ca}^R = \frac{1}{2} \Delta t_{cc}^R = \frac{1}{2} \Delta t_{aa}^R$$

$$\underline{\rho_c < \rho_a} :$$

$$\rho_c > \frac{\rho_a}{2} : \quad \Delta t_{ca}^R < \Delta t_{aa}^R < \Delta t_{cc}^R$$

$$\rho_c = \frac{\rho_a}{2} : \quad \Delta t_{ca}^R = \Delta t_{aa}^R < \Delta t_{cc}^R$$

$$\rho_c < \frac{\rho_a}{2} : \quad \Delta t_{aa}^R < \Delta t_{ca}^R < \Delta t_{cc}^R$$

These stages serve as an indication of the relative importance of different types of random collisions during the aggregation process. The times at which these stages are reached are found by following the development of the spectrum that grows as a result of both ordered collisions and random collisions.

B. Spectrum development

Random collisions among the ice crystals initiate the growth of the spectrum. The newly formed snowflakes at first grow by ordered collisions with ice crystals. Eventually the aggregates become sufficiently dense that random collisions with crystals and also ordered collisions among snowflakes become important.

Finally, most of the crystals are depleted so that only random and ordered collisions among snowflakes develop the spectrum.

Since the rate of ordered collisions depends on the number density, effective cross sections and fall velocities of the particles, assumptions concerning these parameters must be made for use in the model. This is

done in the following chapter.

The value of the constant C defining the random collision rate during any particular computation is not known. It is desirable to make several computations, varying this constant, in order to find a realistic collision rate.

IV. ASSUMPTIONS CONCERNING PARAMETERS

A. Crystals

1. Crystal type and dimensions

Crystals occurring in natural clouds exhibit three fundamental structures: needles, columns and plates. The dominant factors in determining crystal type and size are the ambient air conditions. For this study, plane dendritic crystals were chosen since these are frequently observed in stratiform-type storms. A number of observations were made at Boston during the winter of 1965-1966 of snowflakes reaching the ground. In a total of 10 storms, 6 featured plane dendritic crystals, 3 featured spatial dendrites and needles occurred once.

Observations and laboratory experiments (Weickmann, 1953; Nakaya, 1954; Mason, 1957) have shown that plane dendritic crystals are generally formed at temperatures between -12°C and -16°C . At these temperatures Nakaya (1954) measured the average crystal diameter to be 3.26 mm. In the model, a crystal diameter of 4 mm was selected. This is somewhat larger than the average value observed by Nakaya but allows for further growth by sublimation, as the crystals fall towards the layer above the melting level in which aggregation is thought to take place. A plane dendritic crystal of diameter 4 mm melts into a rain drop of diameter 0.5 mm, a value well within the range of the smallest measured drops in raindrop size spectra.

Empirical studies show that a rainfall rate of 1.5 mm/hr is equivalent to a liquid water content of about 0.1 g/m^3 (Atlas, 1964). Thus, in order

to produce a rainfall rate of 1.5 mm/hr the liquid water content of the ice crystals must be about $0.6g/m^3$, because of their smaller fall velocities. Consequently, an initial density of 10^4 plane dendritic crystals, diameter 4 μm , per m^3 is assumed.

2. Collision cross section

If the horizontal cross sections of two particles involved in a collision are circular, the effective collision cross section is

$$\frac{\pi}{4} (d_1 + d_2)^2$$

where d_1 and d_2 are the respective diameters of the horizontal cross sections. In this study it was found convenient to express the horizontal diameter of the snowflakes and crystals as functions of the equivalent melted drop diameter.

The diameter factor, k , is defined to be the ratio of the horizontal diameter of the snowflake or crystal to the diameter of the equivalent melted drop.

It is assumed that the ice crystals are horizontally oriented in space at all times. Consequently the cross section of a plane dendritic crystal approximates to that of the enveloping circle, as shown in Fig.

4. The diameter factor, k , can be readily calculated as a function of crystal diameter from the relationship $m = 0.0038 d^2$ (Nakaya, 1954),

where m is the crystal mass (mg) and d the crystal diameter (μm). For crystals of diameter 4 μm , k has the value 8.3.

3. Fall velocity

Nakaya (1954) observed plane dendritic crystals to maintain a

virtually constant fall velocity of 31 cm/sec, independent of mass.

A value of 30 cm/sec is assumed, for convenience, in the computations.

B. Snowflakes

1. Dimensions

Observations and measurements of snowflake dimensions are practically non-existent primarily because of observational difficulties. It is obvious that the shape and size of snowflakes are highly variable since they depend mainly on the mode of attachment, number and type of the constituent ice crystals as well as on microscopic forces. In view of the paucity of data the following assumptions have been made with respect to snowflakes:

- a) The snowflakes maintain an oblate spheroid shape at all times; the ratio of the major to the minor axis is 3.2 regardless of mass.
- b) Each flake maintains its orientation in space and the horizontal cross section is accordingly circular
- c) Density values measured by Magono (1954) in Japan are applicable. He found the density of dry and wet flakes to be 0.0087 g/cm^3 and 0.0187 g/cm^3 respectively.

2. Collision cross section

As mentioned above, the effective collision cross section of a snowflake is assumed to be circular. Values of k are independent of crystal dimension as follows:

$k = 5.5$ for dry flakes

$k = 4.5$ for wet flakes

These values are in good agreement with the value $k = 4.5$ computed by Wexler (1958) in assuming that the Reynolds number of a snowflake is the same as that of its equivalent melted drop. In the model, the snowflakes are assumed to be dry at all times prior to actual melting.

3. Fall velocities

Using film techniques, Langieben observed the following relationship to exist between the fall velocity of a dry snowflake consisting of plane dendritic crystals and its equivalent melted drop diameter:

$$V_S = 160 D^{0.31} \text{ (c.g.s.)}, \quad D \geq 0.5 \text{ mm}$$

where V_S is the terminal fall velocity and D the melted drop diameter of the snowflake.

Fujiwara (1957) derived a functional form for the velocity of all snowflakes from dynamical considerations and found

$$V_S = KD^{1/3} \text{ (c.g.s.)}$$

where K is a constant depending on crystal type.

In the model the fall velocity of snowflakes is assumed to satisfy

$$V_S = 160 D^{1/3} \text{ (c.g.s.)} \quad (D \geq 1.1 \text{ mm})$$

The fall velocities of the smaller flakes with equivalent drop diameter ≤ 1 mm are modified as shown in Fig. 5 to insure a smooth convergence to the fall velocity of the crystals.

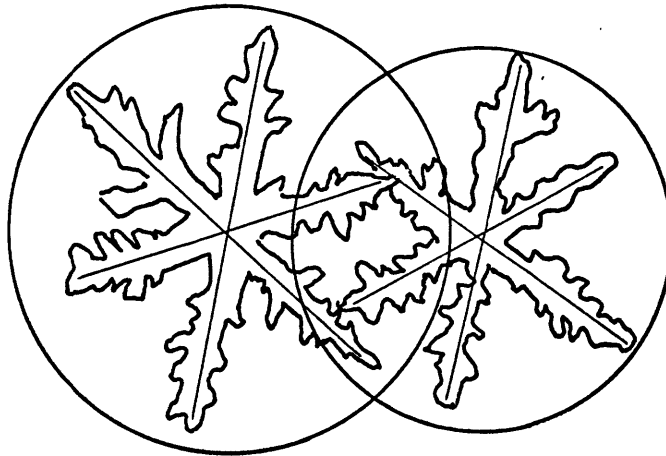


Fig. 4. Collision cross section of plane dendritic crystals.

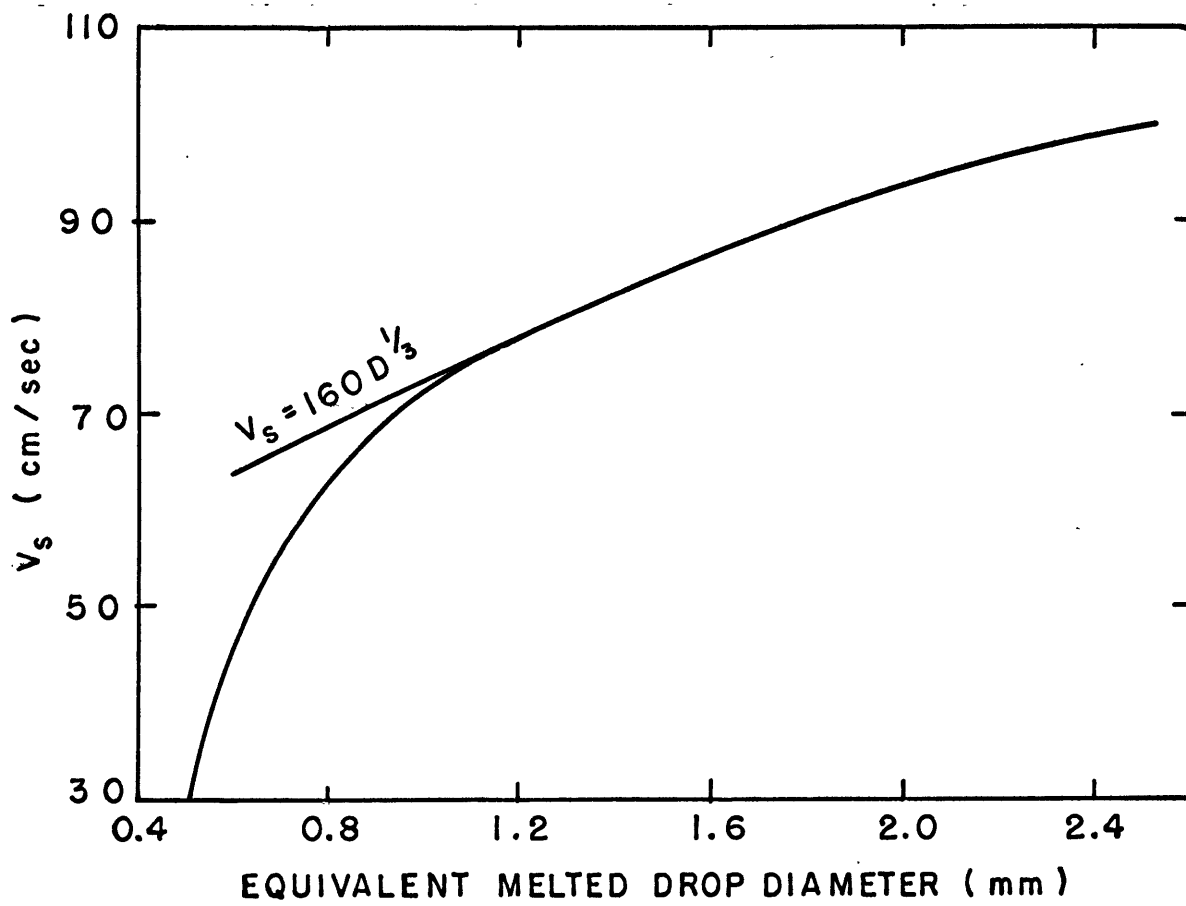


Fig. 5. Modified terminal fall velocities of small flakes.

V. APPLICATION OF THE MODEL

A. Collision mechanism

It will be recalled that the spectrum is to be initiated by random collisions of ice crystals. Subsequent development occurs primarily through ordered collisions of the snow aggregates with ice crystals. As mentioned previously, an initial density of 10^4 plane dendritic crystals is assumed. The following types of collisions are allowed to proceed:

1) Random collisions among ice crystals

Lack of experimental information makes the choice of determining a reasonable collision rate an arbitrary one. Preliminary computations assuming an initial collision rate of 1 collision/sec appeared to be unrealistic in that the spectrum evolved slowly and the resulting spectrum showed little variation in number concentrations. Therefore, in the present computation an initial rate of 10 collisions/sec in each unit volume (m^3) is assumed. The average time interval, Δt_{11}^R , between successive random collisions of crystals is then 0.1 sec. Equation (3.8) gives the relation

$$\Delta t_{11}^R = \frac{10^7}{\beta_1^2} \quad (5.1)$$

where C is now replaced by 10^7 .

2) Random collisions involving snowflakes

The mechanism producing random collisions of snowflakes with other snowflakes and ice crystals is assumed to be the same as that

which operates among the crystals. It has been pointed out that random collisions involving snowflakes assume increasing importance as the spectrum evolves.

3) Ordered collisions between snowflakes and ice crystals.

The growth rate of a single snowflake according to (3.1) was first computed by means of a computer when the flake is assumed to grow solely by ordered collisions with single ice crystals. Collision cross sections were estimated as described in IV. Fig. 6 shows the average time intervals that elapse between successive collisions with crystals. (Since the particles are positioned randomly in space, the ordered collisions actually occur at random time intervals. But as in the case of random collisions it is believed that the assumption of regular rather than randomly spaced collisions would not effect the ultimate spectrum.) As expected, the collision rate increases as the aggregate gains speed in falling.

The average collision rate of snowflakes of a given size with crystals varies linearly with the snowflake density and the crystal density so that Fig. 6 can be used to calculate these collision rates at successive intervals of time, density changes being taken into account.

3) Ordered collisions among snowflakes

Average collision rates among snow flakes of different sizes are calculated using (3.1) with the collision cross sections estimated as described in section IVC.

B. Development of the spectrum

Let ρ_1 = crystal density

ρ_a = aggregate density

Δt_{ip}^R and Δt_{ip}^O are the average time intervals between random collisions and ordered collisions of a single crystal with a snowflake consisting of p ice crystals respectively.

Δt_{ia}^R is the average time interval between successive random collisions of any snowflake with a crystal.

$\Delta t_{p,q}^R$ and $\Delta t_{p,q}^O$ ($p > 1$) are the average time intervals between random collisions and ordered collisions among flakes containing p crystals, (F_p flakes) and flakes containing q crystals, (F_q flakes), respectively.

Δt_{aa}^R is the average time interval between successive random collisions among any snowflakes.

The evolution of the spectrum is followed at 1 second time intervals. For purposes of translating the aggregate crystal size distributions into equivalent drop diameter distributions the snowflakes are grouped into intervals of equivalent drop diameter as follows:

Table 1

<u>Equivalent Drop Diameter (mm)</u>	<u>F_p</u>
0.5	F ₁
0.6	F ₂
0.7	F ₃
0.8	F ₄
0.9	F ₅ :F ₆

Equivalent Drop
Diameter (mm)

F
P

1.0	F ₇ ·F ₈ ·F ₉
1.1	F ₁₀ ·F ₁₁ ·F ₁₂
1.2	F ₁₃ ·F ₁₄ ·F ₁₅
1.3	F ₁₆ ·····F ₁₉
1.4	F ₂₀ ·····F ₂₄
1.5	F ₂₅ ·····F ₂₉
1.6	F ₃₀ ·····F ₃₅
1.7	F ₃₆ ·····F ₄₂
1.8	F ₄₃ ·····F ₅₀
1.9	F ₅₁ ·····F ₅₈
2.0	F ₅₉ ·····F ₆₈
2.1	F ₆₉ ·····F ₇₈
2.2	F ₇₉ ·····F ₉₀
2.3	F ₉₁ ·····F ₁₀₃
2.4	F ₁₀₄ ·····F ₁₁₆
2.5	F ₁₁₇ ·····F ₁₃₁
2.6	F ₁₃₂ ·····F ₁₄₈
2.7	F ₁₄₉ ·····F ₁₆₅
2.8	F ₁₆₆ ·····F ₁₈₄
2.9	F ₁₈₅ ·····F ₂₀₄
3.0	F ₂₀₅ ·····F ₂₂₅

The spectrum evolves as follows:

1. Initially, at time $t = 0$

$$\rho_1 = 10^4 m^{-3}, \quad \rho_a = 0 m^{-3}$$

$$\Delta t_{||}^R = 10^{-1} \text{ sec.}$$

2. At $t = 1$ sec

$$\rho_1 = (10^4 - 20) m^{-3}, \quad \rho_2 = 10 m^{-3}$$

Fig. 6 shows that when $\rho_1 = 10^4 m^{-3}$ it takes a single F_2 flake about 12 secs to overtake an ice crystal, so that within about 1 sec an F_2 flake will collide with a crystal when $\rho_2 = 10 m^{-3}$

3. At time $t = 2$ sec

$$\rho_1 = (10^4 - 41) m^{-3}, \quad \rho_2 = 19 m^{-3}, \quad \rho_3 = 1 m^{-3}$$

From the considerations discussed above, it is likely that 2 F_2 snowflakes will collide separately with a crystal within a further second producing 2 more F_3 flakes. The probable time interval required for a single F_3 snowflake to perform an ordered collision with a crystal is 7 sec (Fig. 6).

4. At time $t = 3$ secs

$$\rho_1 = (10^4 - 63) m^{-3}, \quad \rho_2 = 27 m^{-3}, \quad \rho_3 = 3 m^{-3}$$

The process is continued in this way. For convenience in the computations the random collision rate was computed to vary with the crystal density as follows.

Table 2

<u>ρ_1 (m^{-3})</u>	<u>Coll/sec</u>
10,000	10
9,500	9
9,000	8
8,500	7
7,500	6
7,000	5
6,000	4
5,000	3
4,000	2
3,000	1
2,000	0.4
1,500	0.25
1,000	0.1

Simultaneously, the ordered collision rates vary linearly with the changing crystal and snowflake densities.

5. At $t = 27$ secs

$$\rho_1 = 9,000 \text{ m}^{-3}, \quad \rho_a = 280 \text{ m}^{-3}$$

The largest aggregate present at this time contains 11 crystals. The total distribution in terms of equivalent melted drop diameter is shown in Fig. 7. The rate at which various types of aggregation proceed is expressed in terms of average time intervals:

$$\begin{aligned} \Delta t_{11}^R &\sim 0.11 \text{ secs} \\ \Delta t_{1a}^R &\sim 2 \text{ secs} \\ \Delta t_{aa}^R &\sim 15.0 \text{ secs} \\ 0.11 &< \Delta t_{ip}^0 < 2 \text{ secs} \\ \Delta t_{pq}^0 &> 25 \text{ secs} \end{aligned}$$

Only random collisions among crystals and ordered collisions between crystals and aggregates are considered significant at this stage.

6. At $t = 54$ secs

$$\rho_1 = 7,000 \text{ m}^{-3}, \quad \rho_a = 450 \text{ m}^{-3}$$

The largest aggregate present at this time contains 28 crystals

$$\Delta t_{11}^R \sim 0.2 \text{ secs}$$

$$\Delta t_{1a}^R \sim 1.5 \text{ secs}$$

$$\Delta t_{aa}^R \sim 50 \text{ secs}$$

$$0.17 < \Delta t_{ip}^0 < 1.5 \text{ secs}$$

$$\Delta t_{pg}^0 > 30 \text{ secs}$$

Random collisions between crystals and aggregates could justifiably be included in the computations at this stage but since these collisions are undergone by all of the aggregates with varying collision frequencies the net effect on the distribution is considered negligible. At no time are random collisions between crystals and aggregates considered significant enough to justify inclusion in the computations.

By means of random collisions among crystals and ordered collisions between flakes and crystals the spectrum develops to

7. $t = 75$ secs

$$\rho_1 = 5,000 \text{ m}^{-3} \quad \rho_a = 540 \text{ m}^{-3}$$

The largest aggregate now present contains 40 crystals. The total distribution in terms of equivalent melted drop diameter is shown in Fig. 9.

$$\begin{aligned}
 \Delta t_{ii}^R &= 0.33 \text{ sec} \\
 \Delta t_{ia}^R &\sim 2 \text{ sec} \\
 \Delta t_{aa}^R &\sim 35 \text{ secs} \\
 0.25 < \Delta t_{ip}^O &< 2 \text{ secs} \\
 \Delta t_{pg}^O &> 25 \text{ secs}
 \end{aligned}$$

As before, only the random collisions among crystals and the ordered collisions between crystals and snowflakes are considered to be significant at this stage.

The distributions show that the smaller equivalent drop diameter contain relatively high number concentrations. Because of this, the ordered collision rate is faster than the random collision rate producing new flakes so that the concentrations of smaller flakes are successively reduced.

8. At $t = 100$ secs

$$\rho_i = 3,000, \quad \rho_a = 60 \mu\text{m}^{-3}$$

The largest aggregate present contains 48 crystals. Fig. 10 shows the total distribution at this stage in terms of equivalent drop diameters

$$\begin{aligned}
 \Delta t_{ii}^R &= 1 \text{ sec} \\
 \Delta t_{ia}^R &\sim 3 \text{ sec} \\
 \Delta t_{aa}^R &\sim 30 \text{ sec} \\
 0.5 < \Delta t_{ip}^O &< 3 \text{ sec} \\
 \Delta t_{pg}^O &> 25 \text{ secs}
 \end{aligned}$$

Crystals are now being depleted mainly through ordered collisions with snowflakes.

Collisions of the largest flakes with ice crystals occur relatively slowly at 1 collision every 3 secs on the average. It thus seems likely that larger flakes of about 3 mm in equivalent drop diameter (F_{200} to F_{300}) can only be produced after the stage when collisions amongst aggregates become important.

9. At $t = 110$ secs

$$f_1 = 2,400, \quad f_a = 625 \text{ m}^{-3}$$

The largest aggregate contains 53 crystals. The total distribution is included in Fig. 11.

$$\begin{aligned} \Delta t_{11}^R &= 2 \text{ secs} \\ \Delta t_{1a}^R &\sim 3.5 \text{ secs} \\ \Delta t_{aa}^R &\sim 25 \text{ secs} \\ 0.5 < \Delta t_{ip}^0 &< 4 \text{ secs} \\ \Delta t_{pg} &> 25 \text{ secs} \end{aligned}$$

The only important collisions are still considered to be the random collisions among crystals and the ordered collisions between aggregates and crystals.

The further development of the spectrum is extrapolated up to the stage where both random and ordered collisions among the aggregates assume significance. Concentrations of small flakes will decrease further while some of the larger flakes grow slowly.

10. At approximately 160 seconds

$$f_1 = 1300, \quad f_a = 650$$

This marks the stage when random collisions among crystals occur at the same rate as random collisions among crystals and aggregates, since

$$\frac{f_1}{2} = f_a$$

$$\begin{aligned}
 \text{For, } \Delta t_{ii}^R &= \Delta t_{ia}^R &= 6 \text{ secs} \\
 &\Delta t_{aa}^R &= 23 \text{ secs} \\
 2 < \Delta t_{ip}^0 &< 10 \text{ secs} \\
 &\Delta t_{pq}^0 &\sim 27 \text{ secs}
 \end{aligned}$$

At this stage random collisions between crystals and snowflakes are of the same significance as random collisions among crystals.

11. At $t \sim 4$ mins

$$\rho_c \sim 650 \quad \rho_a \sim 650 \text{ m}^{-3}$$

Now random collisions among crystals and aggregates are more important than random collisions among crystals, for:

$$\begin{aligned}
 \Delta t_{ii}^R &= 23 \text{ secs} \\
 \Delta t_{ia}^R &= 12 \text{ secs} \\
 \Delta t_{aa}^R &= 23 \text{ secs} \\
 5 < \Delta t_{pq}^0 &< 20 \text{ secs}
 \end{aligned}$$

so that ordered collisions among snowflakes and crystals are still the primary source of depletion of the ice crystals.

The largest flake in the spectrum will have reached an equivalent drop diameter of about 2.1 mm. A small increase in the concentrations of the flakes larger than the median equivalent drop diameter will have occurred and a decrease of smaller flakes. The random collisions of flakes with crystals will further broaden the spectrum. Fig. 12 shows the predicted spectrum at this stage. The crystals are falling with a velocity of 30 cm/sec while the average velocity of the aggregates so far is 90 cm/sec. Thus, by this time the aggregates will have fallen, on the average, about 220 m.

12. The stage when $\rho_i = \frac{\rho_a}{2}$

will probably occur when

$$\rho_i \sim 350, \quad \rho_a \sim 700 \text{ m}^{-3}$$

Then

$$\begin{aligned} \Delta t_{ii}^R &= 80 \text{ secs} \\ \Delta t_{ia}^R &= 20 \text{ secs} \\ \Delta t_{aa}^R &= 20 \text{ secs} \\ 10 < \Delta t_{ip}^0 &< 35 \text{ secs} \\ \Delta t_{pq}^0 &\sim 20 \text{ secs} \end{aligned}$$

Thus, subsequent development of the spectrum will be accomplished mainly by ordered collisions among aggregates, ordered collisions between remaining crystals and aggregates. The random collisions involving snowflakes are of less importance but should be included at this stage.

As the above collision rates indicate, the spectrum will change relatively slowly. Eventually ordered collisions among snowflakes will assume primary importance. It is anticipated that the spectrum will broaden and flatten as a result of these collision processes but a detailed analysis of the spectrum changes can be undertaken only with the aid of a computer. (See Appendix)

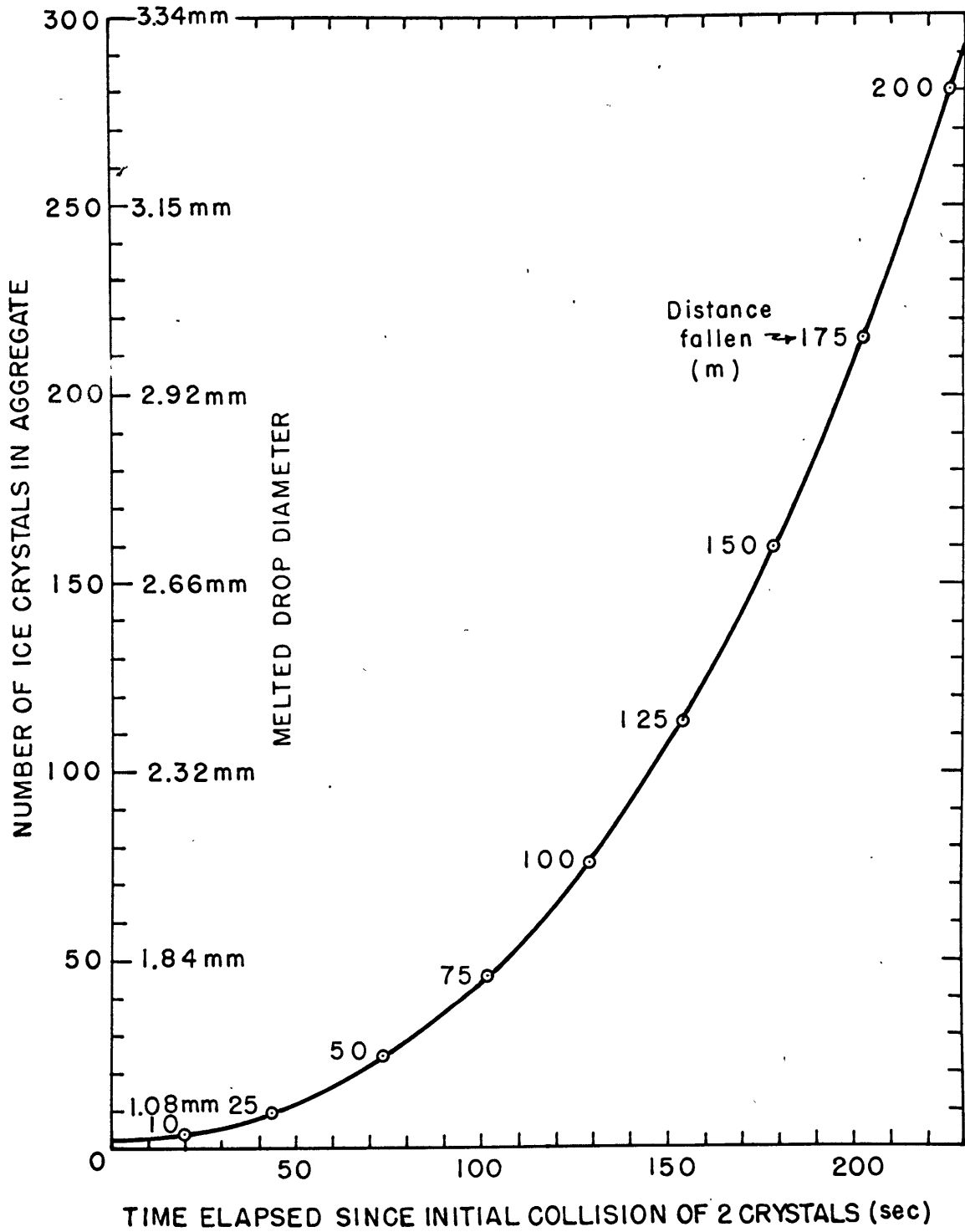


Fig. 6. Rate of growth of a snowflake in an environment containing a constant density of 10^4 plane dendritic crystals, diameter 4 mm, per m^3 .

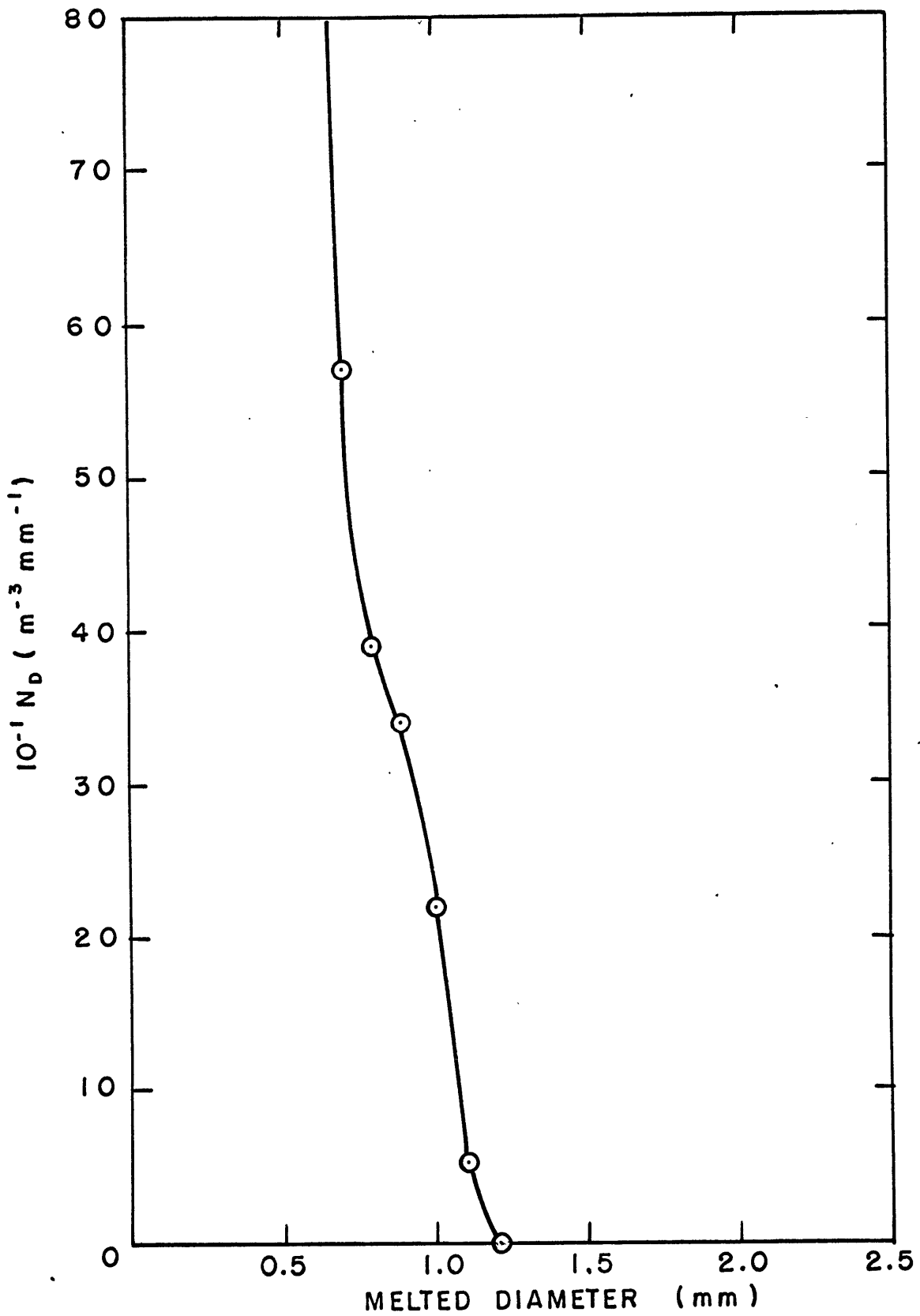


Fig. 7. Computed spectrum at $t = 27$ secs.

$$\beta_1 = 9,000, \quad \beta_2 = 260 m^{-3}.$$

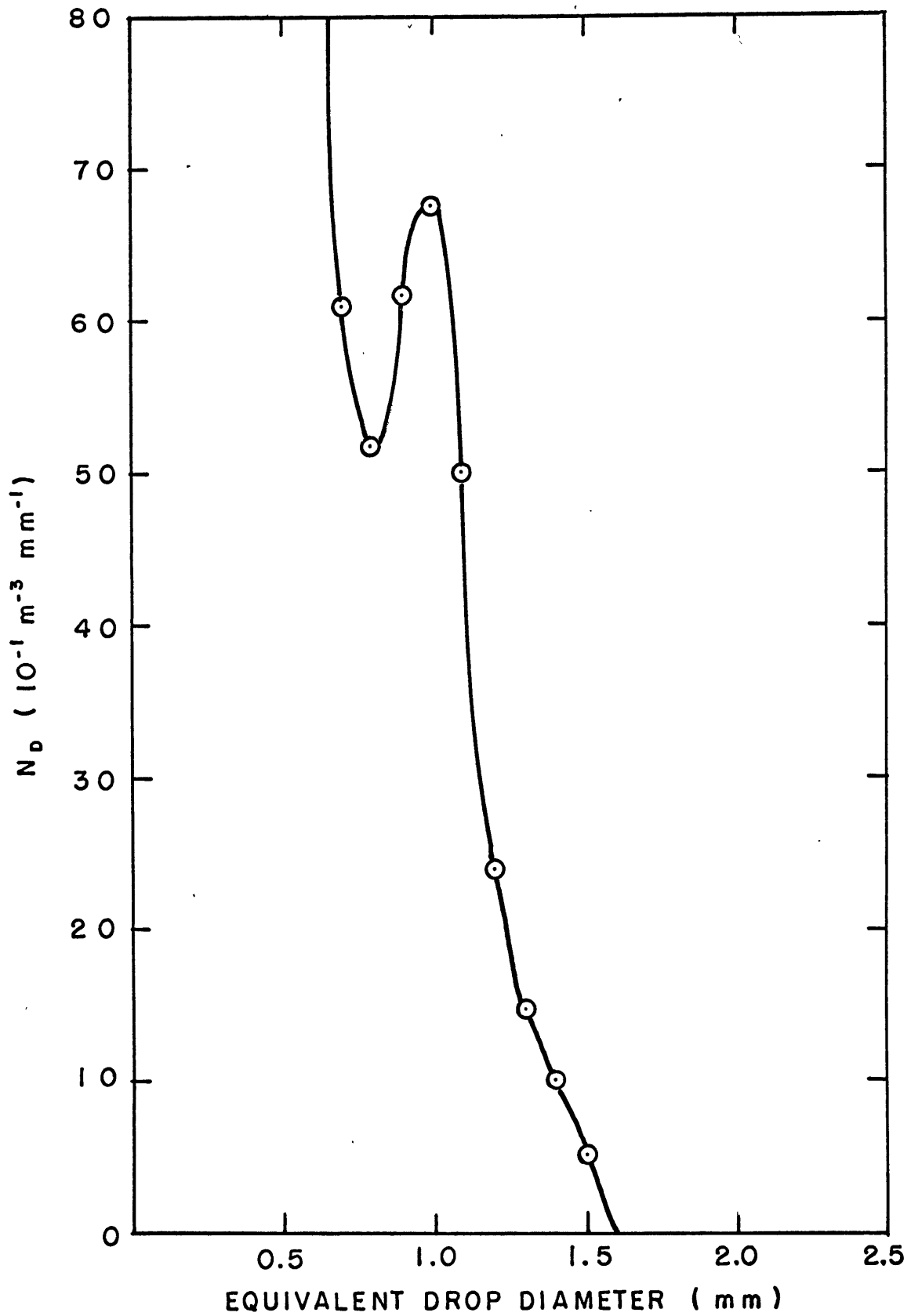


Fig. 8. Computed spectrum at $t = 54$ secs.

$$\rho_i = 7,000, \quad \rho_a = 450 \text{ m}^{-3}.$$

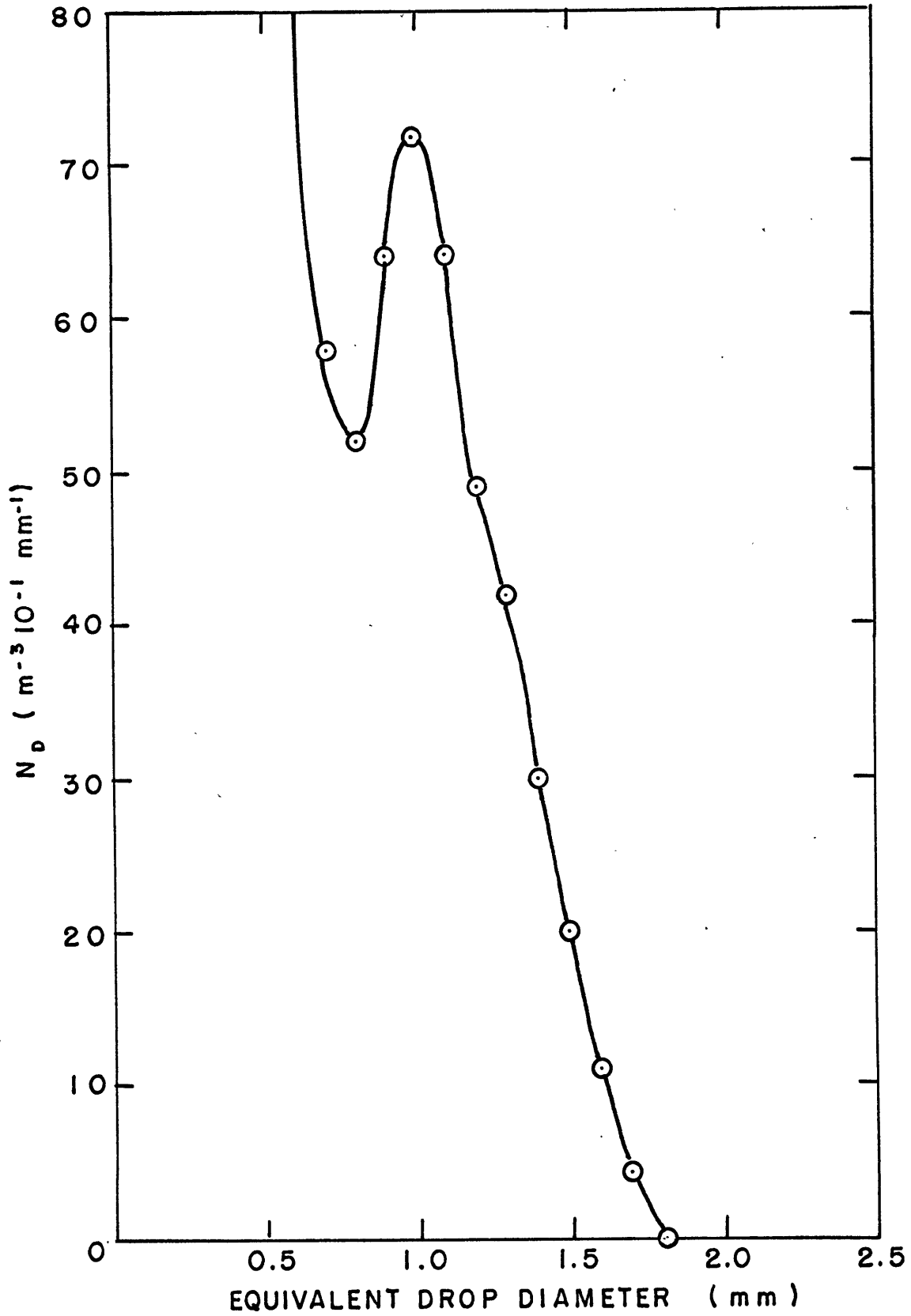


Fig. 9. Computed spectrum at $t = 75$ secs.

$$\beta_1 = 5,000, \quad \beta_2 = 540 \text{ m}^{-3}.$$

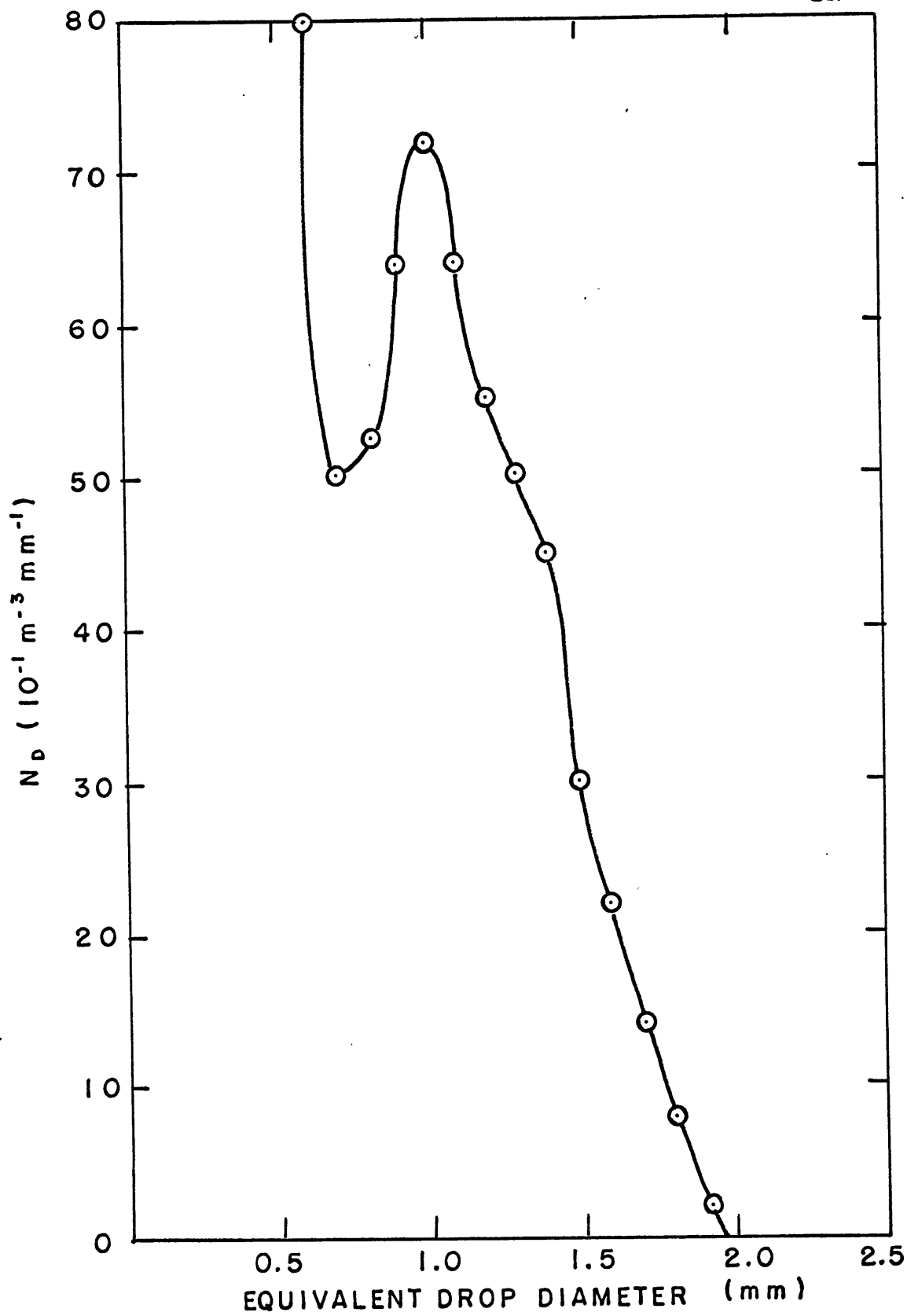


Fig. 10. Computed spectrum at $t = 100$ secs.

$$\beta_i = 3,000, \quad \beta_a = 600 \text{ m}^{-3}.$$

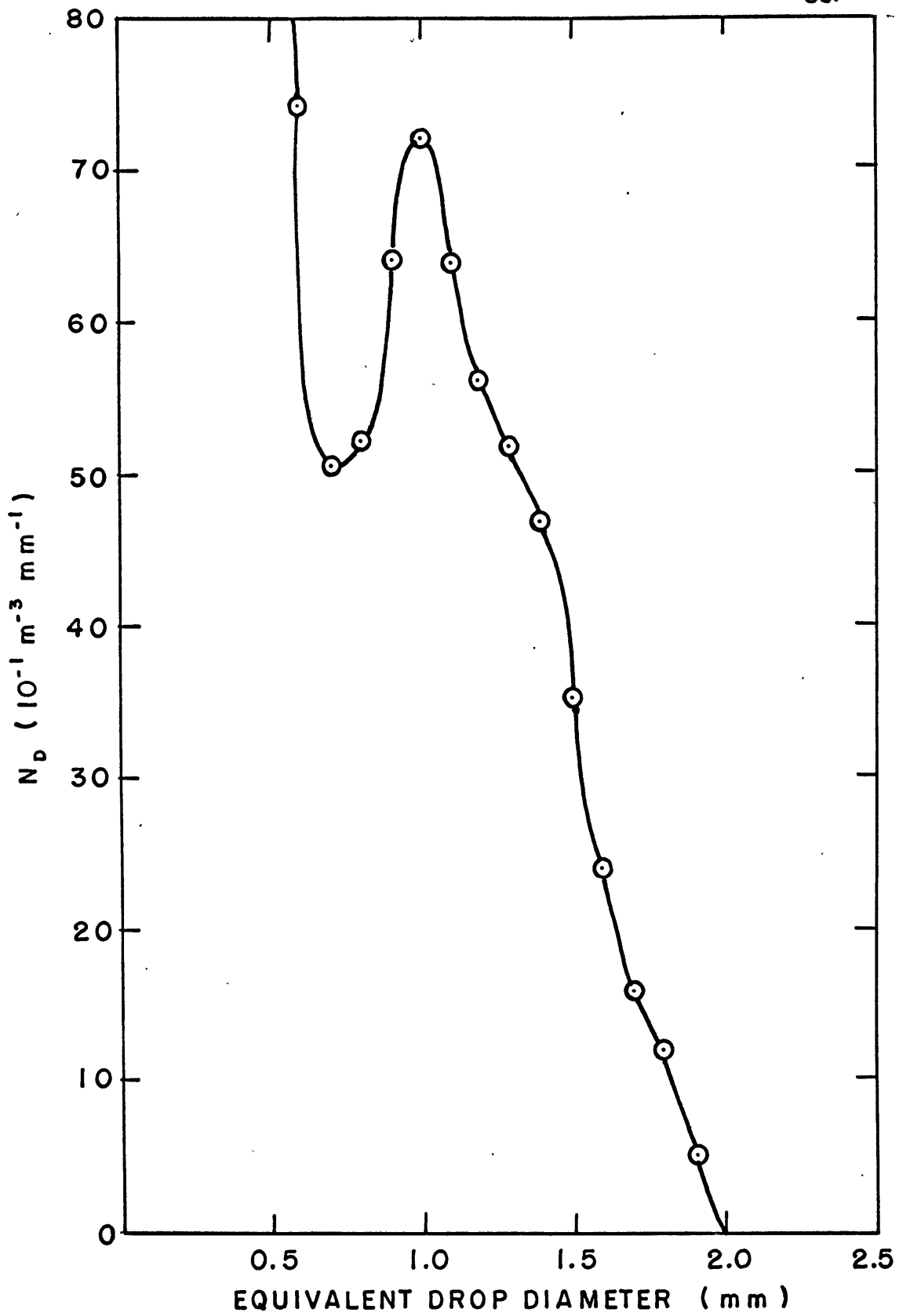


Fig. 11. Computed spectrum at $t = 110$ secs.
 $\beta_1 = 2,400$, $\beta_2 = 625 \text{ m}^{-3}$.

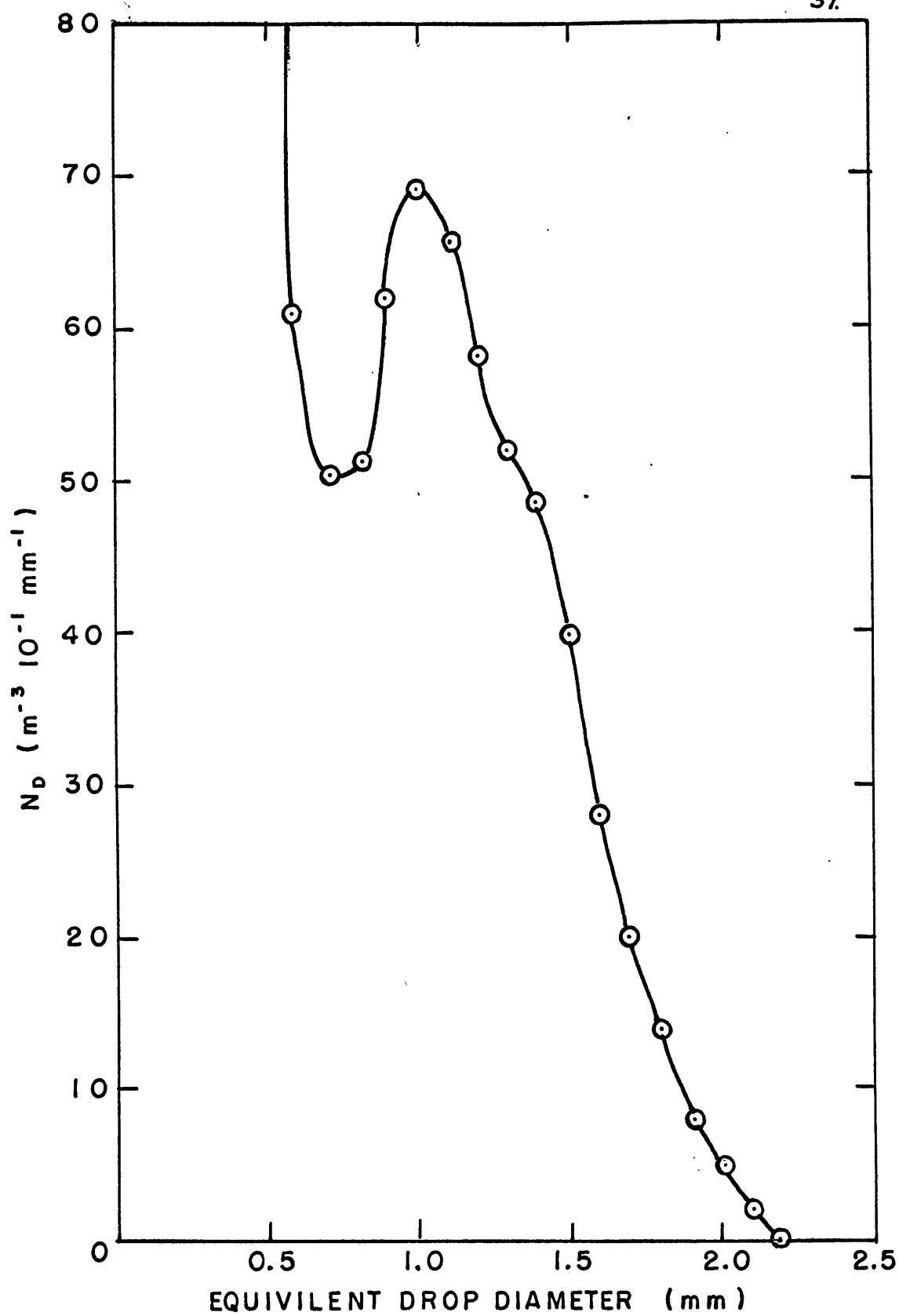


Fig. 12. Extrapolated spectrum at $t \sim 4$ mins.

$$\beta_i = 650, \quad \beta_a \approx 650.$$

VI. COMPARISON OF COMPUTED SPECTRUM WITH OBSERVED SPECTRA

Fig. 13 shows the computed spectrum on a semi-logarithmic plot for the case $\rho_i = 650$, $\rho_a = 650 \text{ m}^{-3}$. It will be recalled that this does not represent the final stage of the spectrum development. Rather, ordered collisions among snowflakes, which assume primary importance after this stage, will considerably modify the spectrum shown in Fig. 13. This collision process will result in a reduction of the number concentrations of snowflakes having equivalent drop diameter up to about 2 mm and considerably extend the remaining portion of the spectrum.

Thus the computed spectrum will develop towards the spectrum measured by Gunn and Marshall for a precipitation rate of 1 mm/hr. (Also Fig. 13.)

The Marshall and Palmer rain spectrum, before melting is also shown in Fig. 13 for a precipitation rate of 1.5 mm/hr.

A comparison of the M-P distribution with the Gunn and Marshall spectrum shows the M-P distribution to consist of fewer aggregates of equivalent drop diameter larger than 1.5 mm and many more aggregates less than 1.5 mm in diameter. This implies that flakes larger than 1.5 mm in diameter tend to break on melting into several smaller raindrops.

The results of the computations support this hypothesis for none of the collision processes discussed can develop the computed spectrum towards the M-P distribution before melting unless breakup of the snowflakes is prominent. There is no experimental evidence to show

that aggregate breakup is common in stratiform situations. In fact, the presence of many large flakes in the Gunn and Marshall spectrum would seem to indicate that snowflake breakup is not important in these conditions.

Thus, the spectrum generated by the collision mechanisms is in good agreement with that measured by Gunn and Marshall. A random collision rate of 10/sec operating among a population of 10^4 plane dendritic crystals of diameter 4 mm initiates a spectrum similar to spectra that have been measured.

If smaller crystals were assumed in the initial conditions, a correspondingly higher density of ice crystals would be necessary to maintain a reasonable liquid water content. The equivalent random collision mechanism to that above would result in a similar spectrum to the one already computed.

As previously discussed, a slower random collision rate initiates a spectrum with unrealistically low number concentrations over a wider range of equivalent drop diameters than is commonly observed. Similarly a faster random collision frequency results in unrealistically high number concentrations over a narrower range of equivalent drop diameters than is commonly observed.

Variation of the spectra with crystal type has not yet been investigated but it is hoped to incorporate this in a later study in which the problem is adapted to a computer.

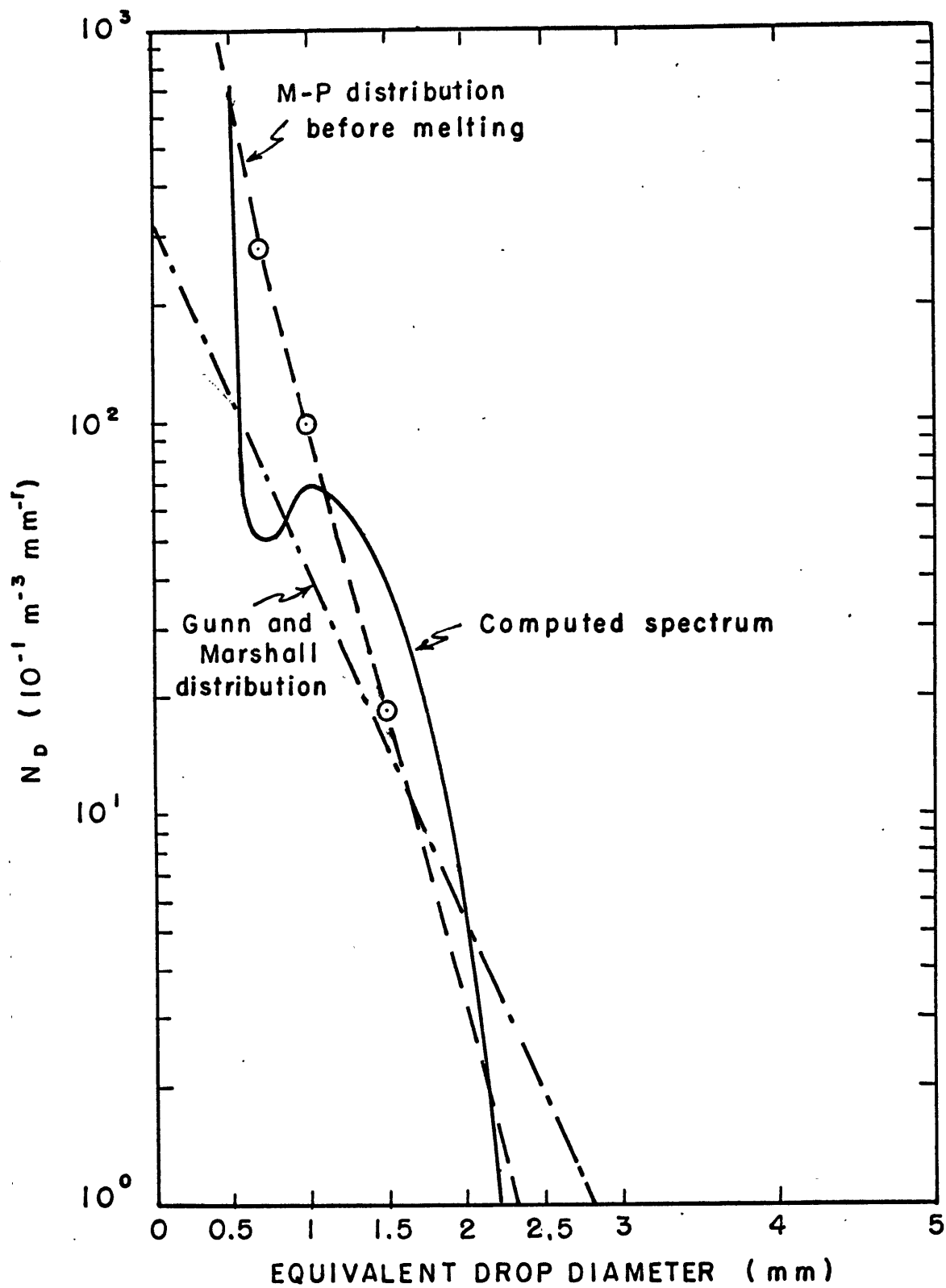


Fig. 13. Observed and computed spectra.
Precipitation rate - 1.5 mm/hr.

VII. CONCLUSION

A pilot study has been made in which the evolution of a snowflake spectrum in stratiform-type storms has been carried on as far as feasible with hand computations.

At the onset of aggregation, an initial particle density of 10^4 plane dendritic crystals, diameter 4 μ m, per m^3 was assumed. Random collisions among the ice crystals, at the rate of 10 collisions/sec initiated the spectrum development which was continued by ordered collision as faster falling particles overtake slower ones.

After a time interval of about 4 minutes, the evolving spectrum was found to be approaching snowflake spectra actually measured at the surface for the equivalent precipitation rate of 1.5 mm/hr. The aggregation process slows down considerably with passing time and eventually reaches a fairly stable state.

It is desirable that a machine program be set up in order to investigate further the dependency of the computed spectrum on the initial assumptions - crystal dimensions, liquid water content, rate of random aggregation, etc. The manner in which a variation of these factors effects the computed spectra might cast light on the variability of observed spectra from storm to storm.

APPENDIX - PROBLEM PRESENTATION FOR
ADAPTION TO A COMPUTER

At successive time intervals the change in the number concentrations of each size flake due to both random and ordered collisions can be calculated.

Let,

F_p = snowflake consisting of p crystals

(F_1 = single crystal)

At any given time t ,

N_p = number of F_p flakes in unit volume.

Then ${}_o P_{ij} N_i N_j \Delta t$ = probable number of ordered collisions between F_i and F_j flakes in a time interval Δt and, similarly

${}_R P_{ij} N_i N_j \Delta t$ = probable number of random

collisions between F_i and F_j flakes in a time interval

In Δt , N_p will change as follows:

1. Decrease in N_p :

a) the number of random collisions between F_p and F_i ($i = 1, \dots, \infty$) in Δt is

$$\sum_{i=1}^{\infty} N_p N_i {}_R P_{ip} \Delta t$$

b) Ordered collisions with F_i ($i = p + 1, \dots, \infty$) produce

$$\sum_{i=p+1}^{\infty} N_p N_i {}_o P_{ip} \Delta t \quad \text{collisions.}$$

c) Ordered collisions with F_1 ($i = 1, \dots, p-1$) produce

$$\sum_{l=1}^{p-1} N_p N_l o_{l,p} \Delta t$$

The total decrease in N_p is thus:

$$\sum_{i=1}^{\infty} N_i N_p R_{i,p} \Delta t + \sum_{\substack{i=1 \\ i \neq p}}^{\infty} N_i N_p o_{i,p} \Delta t$$

2. Increase in N_p :

a) Random aggregation of F_i and $F_{(p-1)}$ ($i = 1, \dots, p-1$) produce

$$\sum_{i=1}^{p-1} N_i N_{(p-1)} R_{i,(p-1)} \Delta t$$

collisions.

b) Ordered collisions of F_{p-1} and F_i produce

$$\sum_{i=1}^{p/2-1} N_i N_{(p-i)} o_{i,(p-i)} \Delta t$$

collisions if p even

and

$$\sum_{i=1}^{\frac{p-1}{2}} N_i N_{p-i} o_{i,(p-i)} \Delta t$$

collisions if p odd.

The total increase in N_p is thus

$$\sum_{i=1}^{p-1} N_i N_{(p-i)} R_{i,(p-i)} \Delta t$$

$$+ \sum_{i=1}^{p/2-1 \text{ or } \frac{p-1}{2}} N_i N_{(p-i)} o_{i,(p-i)} \Delta t$$

Therefore

$$\frac{\Delta N_p}{\Delta t} = \sum_{i=1}^{p-1} N_i N_p R_{i,(p-i)} + \sum_{i=1}^{p/2-1 \text{ or } \frac{p-1}{2}} N_i N_{(p-i)} o_{i,(p-i)}$$

$$- \sum_{i=1}^{\infty} N_i N_p R_{i,p} - \sum_{\substack{i=1 \\ i \neq p}}^{\infty} N_i N_p o_{i,p}$$

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