THE ONSET OF CONVECTION IN HIGHLY VISCOUS FLUID SPHERES WITH APPLICATION TO PLANETARY INTERIORS

by

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ABSTRACT

The general linear theory of hydrodynamic stability of internally heated, non-rotating, self-gravitating spheres and spherical shells of highly viscous, compressible fluid is developed. The onset of convection in the marginally stable state for homogeneous spheres and spherical shells is studied in detail, and the characteristic values of the Rayleigh number are computed from the theory for fluids having properties similar to that of planetary mantle material.

According to the theory devedloped, the actual Rayleigh number computed for models of the lunar interior and mantle of the Earth correspond to very high order harmonic convection ($\ell > 25$) and hence would not correspond to any global wide current system. Analysis of the size of the viscous dissipation terms in the entropy transport equation indicates that these discrepancies of 10^4 to 10^7 in the Rayleigh number for low ℓ values indicates the breakdown of the linear theory for viscosity coefficients greater than about 10^{15} . A non-linear theory is probably needed to satisfactorily remedy the situation; however, due to bovious mathematical complexities such a theory does not exist at present.

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Chapter I

INTRODUCTION

Recent investigations of the thermal histories and temperature distributions of the moon and terrestrial planets, which were carried out under the assumption of uniform distribution of radiogenic elements K^{40}, Th^{232}, U^{235}, and U^{238}, have yielded prevailing temperatures in planetary interiors to be between 2000° and 4000°K. These calculations are based on the assumption that the only mechanism of heat transport is conduction (or in the case of McDonald (1959) by radiation).

However, Kopal [cf. Kopal (1961, 1962b)] has shown that the conductive temperature gradients hence established may very well exceed the adiabatic thermal gradients for silicate rocks, which probably make up the bulk of the Moon or terrestrial planetary mantles, by one or two orders of magnitude. For convection currents to arise it is not necessary that the material be molten, but only that its viscosity be finite--no matter how large, the temperature gradients be strongly superadiabatic, and the time scale be appropriately long for the existing scale length.

If one considers a visco-elastic material having a shear viscosity coefficient μ and coefficient of isothermal compression

$$\beta = \frac{1}{\rho}\left(\frac{\partial \rho}{\partial p}\right)_T \tag{1-1}$$

with values of $\beta \sim 10^{-12}\ \frac{\text{cm sec}^2}{\text{g}}$ and $\mu \sim 10^{22}\ \frac{\text{g}}{\text{cm sec}}$ for silicate rocks, then the Maxwellian relaxation time is

$\tau = \mu\beta \sim$ 1000 years. Then for forces and impulses acting on time scales where $t \ll \tau$ the given planetary body should respond essentially elastically. However, for forces acting over time scales $t \gg \tau$, such as gravitational forces tending toward hydrostatic equilibrium, the Moon or planetary mantle should behave as spheres and spherical shells of highly viscous fluid. When one considers that the age of the solar system is at least 4.5×10^9 years or 1.4×10^{17} seconds old, then on these time scales, the terrestrial planetary masses are likely to obey the laws of hydrodynamics rather than those of elasticity.

The study of gravitationally and thermally unstable viscous liquid formations has been the object of investigations, both theoretical and experimental for several years. The first theoretical studies were carried out by Lord Rayleigh in (1916) and followed by another English mathematical physicist, Harold Jeffreys (1926, 1928, and 1930). This work in addition to that of Low (1929) was concerned with two-dimensional, layered convection, and the first systematic extension to three-dimensional investigations in spherical systems was made by Wasiutyniski (1946). This paper was followed by Jeffreys and Bland in 1952 and on to the most modern work by Chandrasekhar (1961), Kopal (1962) and Ashworth (1968).

With the exception of Ashworth, previous authors have assumed that the flow was incompressible and hence the convective density variation $\delta\rho$ was due to the variation δT in temperature alone. As a consequence, the heat propagation was assumed to take place through conduction processes only and the radiogenetic energy source rate was assumed neglible compared to the conduction terms. In this paper we shall avoid these assumptions to give a more general study than presently exists in the literature. Ashworth (1968) recently studied a special case of the problem including compressibility effects and the derived radial convection flow equation agrees in part with the author's result. It should be noted that the basic equations derived in this paper were developed over a year prior to Ashworth's recent publication.

In Chapter II a brief introduction to the theory of hydrodynamic stability is presented along with the general hydrodynamic equations for a non-rotationg, compressible, internally heated viscous sphere. The linear forms of these equations are obtained on the assumption that the perturbations in density, pressure, temperature and gravitational potential due to the convective motion are very small as compared to the unperturbed, equilibrium values.

In Chapter III the perturbation equations are manipulated to reduce the general system from seven to five dependent variables with the only assumption being that the unperturbed values of the density, pressure, temperature, gravitational potential and material properties of the fluid be only radially dependent. The case of a homogeneous fluid and constant material parameters is also devedloped and gives a completely uncoupled system for the radial convective velocity.

Chapter IV is concerned with the detailed mathematical analysis of a homogeneous fluid with constant material parameters when applied to entire spheres and spherical shells.

In Chapter V the theory developed in Chapter IV is applied to problems of convection in the Moon and the mantle of the Earth. The characteristic values of the Rayleigh number are computed theoretically from the results of Chapter IV and the effects of compressibility and radiogenetic heating are studied. Tables I through IX are included in this chapter and represent the computation of the characteristic value of the Rayleigh number under a variety of circumstances. Finally, the theoretical calculations of the Rayleigh number are compared to predicted values for the lunar interior and Earth's mantle and the resulting discrepancies are discussed with possible suggestions for future studies.

Appendices A and B are included for completeness in formulating the mathematical problem and as a convenient reference for the form of the hydrodynamical equations used in the text.

Chapter II

MATHEMATICAL FORMULATION

Basic Concepts of Hydrodynamic Stability

The equations of hydrodynamics for all their complexity, allow in some cases, simple flow patterns as stationary solutions. The flow patterns, however, can only be realized for certain ranges of the parameters describing the given hydrodynamical system. Outside these ranges the stationary patterns cannot be realized. The basic reason for this lies in their inherent instability against small perturbations to which the system is subject. It is the study of hydrodynamical stability which attempts to differentiate between the stable and unstable patterns of permissible flows.

Suppose, though, that we have a given hydrodynamical system which according to the equations governing it is in a stationary state, that is, it is in a state in which none of the variables describing it are a function of time. Let $\{\alpha_1, \alpha_2, \cdots, \alpha_j\}$ be a set of parameters which define the given system. These parameters will include geometrical parameters such as the dimensions of the system; parameters characterizing the velocity field; and the magnitudes of the forces which may be acting on the system. These latter may include pressure gradients, temperature gradients, magnetic field, rotation (centrifugal and coreolis forces) and others.

We seek then to determine the reaction of the system to small disturbances. More specifically, we wish to examine whether these distrubances will die out, or instead will grow in time in such a way that the system departs further and further from the initial state and never reverts to it. In the former case, we them the system stable, and in the latter, we term it unstable. It is clear that the system must be stable with respect to all possible modes to be termed stable, that is, stability must imply that there exists no mode of disturbance for which it is unstable.

If all initial states are classified as stable, or unstable, according to the above criteria, then the parameter space defined by $\alpha_1, \alpha_2, \ldots, \alpha_j$, the locus of which separates the stable and unstable states, defines the states of marginal or neutral stability.

This locus of the marginal states in the $(\alpha_1, \alpha_2, \ldots, \alpha_j)$-space will be defined by an equation of the form

$$f(\alpha_1, \alpha_2, \ldots, \alpha_j) = 0 . \qquad (2\text{-}1)$$

The determination of this locus is one of the primary objectives of an investigation in hydrodynamics stability. One can then think of the parameters of the system being kept constant except one which is continuously varied. The system then passes from stable to unstable when the particular parameter takes on a certain critical value. One can then say that instability sets in at this value of the chosen parameter when all the others have kept their preassigned values.

In classifying these marginal states into two classes--oscillating and stationary--we have supposed we are dealing with dissapative systems. In conservative, non-dissapative systems, the situation is generally somewhat different as the stable states, when perturbed, execute undamped oscillations with certain definite frequencies; while unstable states have small disturbances growing exponentially with time.

The mathematical analysis of the stability problem begins with assuming an initial flow which represents a stationary state of the system. Then, supposing that the various physical vairables of the flow, (such as the density, pressure, velocity, etc.) suffer small (infinitesimal) increments, we obtain the perturbation equations governing these increments. In finding these equations we linearize the basic hydrodynamical system, which is intriniscally nonlinear, by neglecting all products and higher powers of the perturbations. Hence, we are discussing linear stability analysis, as opposed to nonlinear theories which attempt to take into account the finite amplitudes of the disturbances. In this paper we shall only be concerned with a linear perturbation theory.

Since our system is linear, we can in general, express any disturbance by a linear superposition of normal modes. It is clear that this set of modes must be complete over the spatial range considered for such an expansion to be possible. Let the various modes appropriate to a particular problem distinguished by the symbol $\underset{\sim}{k}$. In practice, several parameters may be needed to distinguish the different modes; and we assume the symbol $\underset{\sim}{k}$ to represent all the parameters that may be needed. Then if $A(\underset{\sim}{x},t)$ symbolically represents our disturbance, then symbolically we can write

$$A(\underset{\sim}{x},t) = \int d\underset{\sim}{k}\; A_{\underset{\sim}{k}}(\underset{\sim}{x},t) \tag{2-2}$$

One can also separate the time dependence by specializing the form of the normal modes sufficiently to seek solutions of the form

$$A_{\underset{\sim}{k}}(\underset{\sim}{x},t) = A_{\underset{\sim}{k}}(\underset{\sim}{x})\, e^{s_{\underset{\sim}{k}} t} \tag{2-3}$$

where $s_{\underset{\sim}{k}}$ is a constant to be determined. The subscript has been attached to the s since, in general, its value will vary for different $\underset{\sim}{k}$.

Upon solving the remaining spatially dependent equations, subject to the appropriate boundary conditions, one will find in general that a non-trivial solution will not be allowed for arbitrary values of $s_{\underset{\sim}{k}}$. Indeed, the requirement that the equations will not allow non-trivial solutions satisfying the boundary conditions leads directly to a characteristic or eigenvale problem for $s_{\underset{\sim}{k}}$. Thus the problem has been essentially reduced to finding the $s_{\underset{\sim}{k}}$ for the various modes. In general the characteristic values for $s_{\underset{\sim}{k}}$ will be complex:

$$s_{\underset{\sim}{k}} = s_{\underset{\sim}{k}}^{R} + i\, s_{\underset{\sim}{k}}^{I} \tag{2-4}$$

where $s_{\underset{\sim}{k}}^{R}$ and $s_{\underset{\sim}{k}}^{I}$ are real constants for a given $\underset{\sim}{k}$ and apart from $\underset{\sim}{k}$ will depend on the parameters $\{\alpha_1, \alpha_2, \ldots, \alpha_j\}$ of the system. The condition for stability requires then that $s_{\underset{\sim}{k}}^{R} < 0$ for all $\underset{\sim}{k}$. The states of neutral stability with respect to the disturbances belonging to a given $\underset{\sim}{k}$ will be characterized by

$$s_{\underset{\sim}{k}}^{R}(\alpha_1, \ldots, \alpha_j) = 0. \tag{2-5}$$

This gives a condition on the parameters $\{\alpha_1, \alpha_2, \cdots, \alpha_j\}$ and it will define a locus

$$f_{\underset{\sim}{k}}(\alpha_1, \ldots, \alpha_j) = 0 \tag{2-6}$$

in the $(\alpha_1, \ldots, \alpha_j)$-space. This locus will separate those states which are stable from those which are unstable to disturbances belonging to a given $\underset{\sim}{k}$. Now we can observe that the locus $f(\alpha_1, .., \alpha_j)$ separating regions of complete stability from those of instability in the $(\alpha_1, .., \alpha_j)$-parameter space is the envelope of the $f_{\underset{\sim}{k}}$ loci. Also, we can see that when the system becomes unstable as it crosses this locus at some particular point, the mode of the disturbance which will be manifest at the onset will be one whose locus $f_{\underset{\sim}{k}}$ touches f at the particular point under consideration.

Further, we can distinguish between the two kinds of marginal states (stationary and oscillatory) depending on whether or not the imaginary part $s^I_{\underset{\sim}{k}}$ of $s_{\underset{\sim}{k}}$ vanishes when $s^R_{\underset{\sim}{k}}$ does. If $s^R_{\underset{\sim}{k}} = 0$ implies that $s^I_{\underset{\sim}{k}} = 0$ for every $\underset{\sim}{k}$, then we have, as termed by Chandrasekhar, the principle of exchange stabilities being valid, and a stationary secondary flow is the result. If, however, $s^I_{\underset{\sim}{k}} \neq 0$ then we have overstability and the system will exhibit oscillatory motion of a specific mode as predicted explicitly by the theory.

The Governing Equations

We wish to consider a non-rotating, self-gravitating sphere of viscous, compressible fluid which in its equilibrium stationary state is completely described by the four scalars: density ρ_0, scalar pressure p_0, temperature T_0 and gravitational potential ψ_0. For a general hydrodynamical system we have the governing equations: the continuity equation,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underset{\sim}{v}) = 0 \tag{2-7}$$

the momentum transport equation (for derivation see Appendix A),

$$\rho\left(\frac{\partial \underset{\sim}{v}}{\partial t} + \underset{\sim}{v}\cdot\nabla\underset{\sim}{v}\right) = -\nabla p + \rho\nabla\psi + \mu\nabla^2\underset{\sim}{v} + (\mu' + \tfrac{1}{3}\mu)\nabla\nabla\cdot\underset{\sim}{v} \qquad (2\text{-}8)$$
$$+ 2(\nabla\overset{\circ}{\underset{\sim}{v}})^s\cdot\nabla\mu + (\nabla\cdot\underset{\sim}{v})\nabla\mu' ,$$

the entropy transport equation (for derivation see Appendix B),

$$\rho C_v\left(\frac{\partial T}{\partial t} + \underset{\sim}{v}\cdot\nabla T\right) = \nabla\cdot(K\nabla T) + \tfrac{1}{2}\mu(\nabla\overset{\circ}{\underset{\sim}{v}})^s : (\nabla\overset{\circ}{\underset{\sim}{v}})^s \qquad (2\text{-}9)$$
$$+ (\mu' - \tfrac{2}{3}\mu)(\nabla\cdot\underset{\sim}{v})^2 + Q_r - T\left(\frac{\partial p}{\partial T}\right)_s(\nabla\cdot\underset{\sim}{v})$$

Poisson's equation for the gravitational potential,

$$\nabla^2\psi = -4\pi G\rho \qquad (2\text{-}10)$$

and finally an equation of state relating ρ, p, and T,

$$\rho = \rho(p, T) . \qquad (2\text{-}11)$$

This gives us a system of seven independent equations in the seven dependent variables $\rho, p, T, \psi, \underset{\sim}{v}$ and along with the proper boundary conditions will give a proper mathematical formulation of the problem.

In accordance with the previous discussions in the first part of this chapter, we shall consider the system at equilibrium, (whether stable, neutral, or unstable we shall attempt to determine) such that the dependent variables are functions of radius only in a system of spherical polar coordinates (r,θ,φ). Then we shall perturb the system in such a way that the dependent variables become,

$$\rho = \rho_0(r) + \rho'(r,\theta,\varphi,t) \tag{2-12}$$

$$p = p_0(r) + p'(r,\theta,\varphi,t) \tag{2-13}$$

$$T = T_0(r) + T'(r,\theta,\varphi,t) \tag{2-14}$$

$$\psi = \psi_0(r) + \psi'(r,\theta,\varphi,t) \tag{2-15}$$

$$\underset{\sim}{u} = \underset{\sim}{u}'(r,\theta,\varphi,t) \tag{2-16}$$

where the primed quantities represent the perturbation quantities. Substituting these quantities in (2-7) through (2-11) we arrive at the linearized perturbation equations:

$$\frac{\partial\rho'}{\partial t} + \nabla\cdot(\rho_0\underset{\sim}{u}') = 0 \tag{2-17}$$

$$\rho_0\frac{\partial\underset{\sim}{u}'}{\partial t} = -\nabla p' + \rho_0\nabla\psi' + \rho'\nabla\psi_0 + \mu\nabla^2\underset{\sim}{u}' + (\mu' + \tfrac{1}{3}\mu)\nabla\nabla\cdot\underset{\sim}{u}' + 2(\nabla\underset{\sim}{u}')^s\cdot\nabla\mu + (\nabla\cdot\underset{\sim}{u}')\nabla\mu' \tag{2-18}$$

$$\rho_0 c_v\left(\frac{\partial T'}{\partial t} + \underset{\sim}{u}'\cdot\nabla T_0\right) = \nabla\cdot(K\nabla T') + Q_r' - \left[T\left(\frac{\partial p}{\partial T}\right)_\rho\right]_0\nabla\cdot\underset{\sim}{u}' \tag{2-19}$$

$$\nabla^2\psi' = -4\pi G\rho' . \tag{2-20}$$

To find our perturbation equation of state we can expand the density about the equilibrium density to give

$$\rho = \rho_0 + \left[\left(\frac{\partial\rho}{\partial T}\right)_p\right]_0 T' + \left[\left(\frac{\partial\rho}{\partial p}\right)_T\right]_0 p' + \cdots \tag{2-21}$$

Then using the definitions of $\alpha \equiv -\frac{1}{\rho_0}\left[\left(\frac{\partial\rho}{\partial T}\right)_p\right]_0$ and

$\beta \equiv \frac{1}{\rho_0}\left[\left(\frac{\partial \rho}{\partial p}\right)_T\right]_0$ we have, considering linear terms only,

$$\rho' = \rho_0 \left[-\alpha T' + \beta p' \right] . \tag{2-22}$$

One can envision some difficulties in writing down the linear form of the entropy transport equation. In equation (2-19) we have neglected the viscous dissipation terms which are quadratic in $\underset{\sim}{u}'$ which is in accord with our linear theory. However, we are going to be considering flows where the viscosities may be extremely large (such as for mantle material) and hence the terms quadratic in $\underset{\sim}{u}'$ may be as large as those retained in the linear form of the equation. One should keep this in mind when considering the results of the completely linear analysis which will follow.

Our system is now complete, excepting formulation of the boundary conditions, with seven linearly independent equations in the seven dependent variables $\rho', p', T', \psi', \underset{\sim}{u}'$. We assume that $\rho_0, p_0, T_0, \psi_0, q_{r0}$ as well as the material parameters $\alpha, \beta, k, c_v, \mu, \mu'$ are all known functions of position.

The Boundary Conditions

We shall consider all boundaries to be perfectly spherical since the fluid spheres are non-rotating and any perturbations of the boundaries will introduce only second-order effects on the implied conditions on the perturbation variables themselves.

Consider a spherical shell which is confined between $r = R$ and $r = aR$ where R is the outer radius of the sphere and $0 < a < 1$. In all cases we need to require that

$$u_r' = 0 \quad \text{at} \quad r = R \quad \text{and} \quad r = aR. \tag{2-23}$$

The remaining boundary conditions depend upon the nature of the surfaces at $r = R$ and $r = aR$. Here we shall consider two cases: The case when the surface is rigid (as approximating the core-mantle interface of the Earth); and the case when the surface is free (as at the boundary of an isolated sphere in space).

On a rigid boundary, we must require that the transverse components of the velocity also vanish,

$$u_\theta' = u_\varphi' = 0 \quad \text{at} \quad r = R \quad \text{and} \quad r = aR. \tag{2-24}$$

For a free boundary, we need to impose the condition that the tangential viscous stresses, expressed as the off diagonal terms of the general perturbation pressure tensor (see Appendix A), $p'_{r\theta}$ and $p'_{r\varphi}$ vanish, that is,

$$p'_{r\theta} = 0 \tag{2-25}$$

and

$$p'_{r\varphi} = 0 \quad \text{at} \quad r = R \quad \text{and} \quad r = aR. \tag{2-26}$$

In addition, when considering the special case of entire fluid spheres, we have an additional condition. Specifically, we need to requite that

$$\underset{\sim}{u}' = 0 \quad \text{at} \quad r = 0 . \tag{2-27}$$

The specific mathematical conditions imposed on the system by (2-24) through (2-27) will be fully developed in the following chapter.

Chapter III

ANALYSIS OF THE PERTURBATION EQUATIONS

Reduction of the System

Let us consider first just the continuity and momentum transport perturbation equations:

$$\frac{\partial \rho'}{\partial t} + \nabla\cdot(\rho_0 \underset{\sim}{u}') = 0 \tag{3-1}$$

and

$$\rho_0 \frac{\partial \underset{\sim}{u}'}{\partial t} = -\nabla p' + \rho_0 \nabla \psi' + \rho' \nabla \psi_0 + \mu \nabla^2 \underset{\sim}{u}' + (\mu' + \tfrac{1}{3}\mu)\nabla\nabla\cdot\underset{\sim}{u}' \tag{3-2}$$
$$+ 2(\nabla^{\circ}\underset{\sim}{u}')^s \cdot \nabla\mu + (\nabla\cdot\underset{\sim}{u}')\nabla\mu' .$$

Then taking the divergence of (3-2) and defining $\Delta \equiv \nabla\cdot\underset{\sim}{u}'$ we have,

$$\rho_0 \frac{\partial \Delta}{\partial t} + \frac{\partial \underset{\sim}{u}'}{\partial t}\cdot\nabla\rho_0 = \nabla^2(\rho_0\psi' - p' - \nabla\rho_0\cdot\nabla\psi' - \psi'\nabla^2\rho_0 \tag{3-3}$$
$$+ \nabla\cdot(\rho'\nabla\psi_0) + [(\tfrac{4}{3}\mu + \mu')\nabla^2\Delta] + \nabla\cdot\underset{\sim}{f}_v$$
$$+ \nabla^2\underset{\sim}{u}'\cdot\nabla\mu + \nabla\Delta\cdot\nabla(\mu' + \tfrac{1}{3}\mu)$$

where

$$\underset{\sim}{f}_v = 2(\nabla^{\circ}\underset{\sim}{u}')^s\cdot\nabla\mu + \Delta\nabla\mu' . \tag{3-4}$$

The radial component of (3-2) can be written as

$$\frac{\partial}{\partial r}(\rho_0\psi' - p') = -\rho'\frac{\partial\psi_0}{\partial r} + \psi'\frac{\partial\rho_0}{\partial r} + \rho_0\frac{\partial u_r'}{\partial t} \tag{3-5}$$
$$-\mu\nabla^2 u_r' - (\mu' + \tfrac{1}{3}\mu)\frac{\partial\Delta}{\partial r} - \underset{\sim}{f}_v\cdot\hat{r} ,$$

and to eliminate the combination $\rho_0\psi' - p'$ we operate on (3-5) with $r\nabla^2 r$ and on (3-3) with $\frac{\partial}{\partial r} r^2$ and use the identity

$$\frac{\partial}{\partial r} r^2 \nabla^2 h = r\nabla^2 r \frac{\partial}{\partial r} h \tag{3-6}$$

which holds for any function h sufficiently differentiable. Then we have

$$\begin{aligned} -r^2\nabla^2\left(r\frac{\partial\psi_0}{\partial r}\rho'\right) + \frac{\partial^2}{\partial r^2}\left(r^2\frac{\partial\psi_0}{\partial r}\rho'\right) \qquad (3\text{-}7)\\ = \frac{\partial}{\partial r}\left(r^2\psi'\nabla^2\rho_0\right) + \frac{\partial}{\partial r}\left(r^2\nabla\rho_0\cdot\nabla\psi'\right) - \frac{\partial}{\partial r}\left(r^2\frac{\partial\rho'}{\partial t^2}\right) - r\nabla^2\left(r\psi'\frac{\partial\rho_0}{\partial r}\right)\\ - r\nabla^2\left(r\rho_0\frac{\partial u_r'}{\partial t}\right) + r\nabla^2 r\left[\mu\nabla^2 u_r' + \left(\mu' + \tfrac{1}{3}\mu\right)\frac{\partial\Delta}{\partial r} + \underset{\sim}{f}_v\cdot\hat{r}\right]\\ - \frac{\partial}{\partial r} r^2\left[\left(\tfrac{4}{3}\mu + \mu'\right)\nabla^2\Delta + \nabla\cdot\underset{\sim}{f}_v + \nabla^2\underset{\sim}{u}'\cdot\nabla\mu + \nabla\Delta\cdot\nabla\left(\mu' + \tfrac{1}{3}\mu\right)\right]. \end{aligned}$$

Consider for a moment the linear differential operator $\mathcal{O}_p(h) \equiv r\nabla^2(rh) - \frac{\partial^2}{\partial r^2}(r^2 h)$, then we can see that

$$\mathcal{O}_p(h) = \left(r\nabla^2 - \frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}\right)h = \left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right)h \tag{3-8}$$

If we define this angular operator as $\mathcal{L}$, then (3-7) can be rewritten as

$$\begin{aligned} -\mathcal{L}\left(\frac{\partial\psi_0}{\partial r}\rho'\right) = \frac{\partial}{\partial r}\left(r^2\psi'\nabla^2\rho_0\right) + \frac{\partial}{\partial r}\left(r^2\nabla\rho_0\cdot\nabla\psi'\right) \qquad (3\text{-}9)\\ - \frac{\partial}{\partial r}\left(r^2\frac{\partial^2\rho'}{\partial t^2}\right) - r\nabla^2\left(r\psi'\frac{\partial\rho_0}{\partial r}\right) - r\nabla^2\left(r\rho_0\frac{\partial u_r'}{\partial t}\right)\\ + r\nabla^2 r\left[\mu\nabla^2 u_r' + \left(\mu' + \tfrac{1}{3}\mu\right)\frac{\partial\Delta}{\partial r} + \underset{\sim}{f}_v\cdot\hat{r}\right]\\ - \frac{\partial}{\partial r} r^2\left[\left(\tfrac{4}{3}\mu + \mu'\right)\nabla^2\Delta + \nabla\cdot\underset{\sim}{f}_v + \nabla^2\underset{\sim}{u}'\cdot\nabla\mu + \nabla\Delta\cdot\nabla\left(\mu' + \tfrac{1}{3}\mu\right)\right]. \end{aligned}$$

At this point let us consider our perturbation equation of state,

$$\rho' = \rho_o[-\alpha T' + \beta p'] . \tag{3-10}$$

One can conveniently eliminate p' through the use of the radial component of the momentum transport equation and arrive at

$$\frac{\partial}{\partial r}\left(\frac{\rho' + \rho_o \alpha T'}{\rho_o \beta}\right) = \rho_o \frac{\partial \psi'}{\partial r} + \rho' \frac{\partial \psi_o}{\partial r} + \mu \nabla^2 u_r' + (\mu' + \tfrac{1}{3}\mu)\frac{\partial \Delta}{\partial r} \quad (3\text{-}11)$$
$$+ \underset{\sim}{f}_v \cdot \hat{r} - \rho_o \frac{\partial u_r'}{\partial t} .$$

For convenience let us rewrite the gradients of ρ_o, T_o, and ψ_o, and since these variables are only functions of radius,

$$\frac{d\psi_o}{dr} = -\bar{g}(r)\,r \tag{3-12}$$

$$\frac{dT_o}{dr} = -\bar{\gamma}(r)\,r$$

$$\frac{d\rho_o}{dr} = -\bar{\delta}(r)\,r .$$

Finally, if we assume that the material constants $\alpha, \beta, \kappa, C_v, \mu$ and μ' are only radially dependent like ρ_o, T_o, and ψ_o then the order of our system is reduced to five with the dependent variables ρ', T', ψ', u_r' and Δ . The governing equations for this system are

$$\frac{\partial \rho'}{\partial t} + \rho_o \Delta = \bar{\delta}(r)\, r u_r' \tag{3-13}$$

$$\mathcal{L}[\bar{g}r\rho' - \bar{\delta}r\psi'] = -\frac{\partial}{\partial r}\left(r^2\frac{\partial^2\rho'}{\partial t^2}\right) - r\nabla^2\left(r\rho_0\frac{\partial u_r'}{\partial t}\right) \tag{3-14}$$
$$+ r\nabla^2 r\left[\mu\nabla^2 u_r' + (\mu' + \tfrac{1}{3}\mu)\frac{\partial\Delta}{\partial r} + \underset{\sim}{f}_v\cdot\hat{r}\right]$$
$$-\frac{\partial}{\partial r}r^2\left[(\tfrac{4}{3}\mu+\mu')\nabla^2\Delta + \nabla\cdot\underset{\sim}{f}_v + \nabla^2 u_r'\frac{d\mu}{dr}\right.$$
$$\left.+\frac{\partial\Delta}{\partial r}\frac{d}{dr}(\mu' + \tfrac{1}{3}\mu)\right]$$

$$\rho_0 C_v\frac{\partial T'}{\partial t} = \rho_0 C_v\bar{\gamma}r u_r' + \nabla\cdot(k\nabla T') + Q_r' - \left[T\left(\frac{\partial p}{\partial T}\right)_\rho\right]_0\Delta \tag{3-15}$$

$$\frac{\partial}{\partial r}\left(\frac{\rho' + \alpha\rho_0 T'}{\rho_0\beta}\right) = \rho_0\frac{\partial\psi'}{\partial r} - \bar{g}r\rho' + \mu\nabla^2 u_r' + (\mu' + \tfrac{1}{3}\mu)\frac{\partial\Delta}{\partial r} \tag{3-16}$$
$$+ \underset{\sim}{f}_v\cdot\hat{r} - \rho_0\frac{\partial u_r'}{\partial t}$$

$$\nabla^2\psi' = -4\pi G\rho' . \tag{3-17}$$

At this point our system is very general, with allowances for radial variation in all the material parameters as well as ρ_0 and T_0 . Actually, the system can be reduced to three variables ψ', u_r' and T' , but even that yields a very complicated set of coupled equations. Rather than this, in an attempt to obtain some tractable equations, let us assume that <u>all</u> of the material parameters are constant. This greatly simplifies the model and our system becomes

$$\mathcal{L}[\bar{g}r\rho' - \bar{\delta}r\psi'] = -\frac{\partial}{\partial r}\left(r^2\frac{\partial^2\rho'}{\partial t^2}\right) - r\nabla^2\left(r\rho_0\frac{\partial u_r'}{\partial t}\right) \tag{3-18}$$
$$+ \mu r\nabla^2 r\nabla^2 u_r' - \mu\frac{\partial}{\partial r}r^2\nabla^2\Delta$$

$$\left(\rho_0 C_v\frac{\partial}{\partial t} - k\nabla^2\right)T' = \rho_0 C_v\bar{\gamma}r u_r' + Q_r' - \left[T\left(\frac{\partial p}{\partial T}\right)_\rho\right]_0\Delta \tag{3-19}$$

$$\frac{\partial}{\partial r}(\rho'/\rho_o) + \alpha\frac{\partial T'}{\partial r} = \rho_o\beta\frac{\partial\psi'}{\partial r} - \bar{g}\beta r\rho' + \mu\beta\nabla^2 u_r' + (\mu' + \tfrac{1}{3}\mu)\beta\frac{\partial\Delta}{\partial r} - \rho_o\frac{\partial u_r'}{\partial t} \tag{3-20}$$

along with (3-13) and (3-17). AT this point we also need to consider the form of the heat source term, Q_r'. There are many possibilities; however, we shall consider only the case of a distribution of energy sources so that $Q_r = \lambda(r)\rho$. Then $Q_{ro} = \lambda(r)\rho_o$ and $Q_r' = \lambda(r)\rho'$. Our system can then be conveniently rewritten as

$$\left[\rho_o\bar{g}r\mathcal{L} + \frac{\partial^2}{\partial t^2}\frac{\partial}{\partial r}(r\rho_o)\right](\rho'/\rho_o) \tag{3-21}$$
$$= \mu r\nabla^2(r\nabla^2 u_r') - \mu\frac{\partial}{\partial r}(r^2\nabla^2\Delta) + r\bar{g}\mathcal{L}(\psi') - r\nabla^2\left(r\rho_o\frac{\partial u_r'}{\partial t}\right)$$

$$\left[\frac{\partial}{\partial t} - \frac{k}{\rho_o c_v}\nabla^2\right]T' = \bar{\gamma} r u_r' + \frac{\lambda}{c_v}\left(\frac{\rho'}{\rho_o}\right) - \frac{1}{\rho_o c_v}\left[T\left(\frac{\partial p}{\partial T}\right)_\rho\right]_o\Delta \tag{3-22}$$

$$\left[\frac{\partial}{\partial r} + \rho_o\bar{g}\beta r\right]\left(\frac{\rho'}{\rho_o}\right) = \rho_o\beta\frac{\partial\psi'}{\partial r} - \alpha\frac{\partial T'}{\partial r} + \mu\beta\nabla^2 u_r' + (\mu' + \tfrac{1}{3}\mu)\beta\frac{\partial\Delta}{\partial r} - \rho_o\beta\frac{\partial u_r'}{\partial t} \tag{3-23}$$

along with (3-13) and (3-17). We shall now apply the above system to the case of homogeneous fluid spheres where ρ_o = constant.

Application to Homogeneous Fluid Spheres

When the fluid is homogeneous, i.e. ρ_o = constant, then one can readily reduce the system to two variables, ρ' and u_r' by successively eliminating ψ', T' and Δ. First we shall find it convenient to examine the term $\left[T\left(\frac{\partial p}{\partial T}\right)_\rho\right]_o\Delta$.

Since $T_0 = T_0(r)$ and $p_0 = p_0(r)$, to zero-order, $\frac{\partial}{\partial T} = \frac{dr}{dT_0}\frac{d}{dr}$ and $p_0(r)$ and $\psi_0(r)$ must satisfy the equation of hydrostatic equilibrium,

$$\frac{dp_0}{dr} = \rho_0 \frac{d\psi_0}{dr} = -\rho_0 \bar{g} r \tag{3-24}$$

then

$$\left[T\left(\frac{\partial p}{\partial T}\right)_\rho\right]_0 = \frac{T_0(r)\rho_0 \bar{g}(r)}{\bar{\gamma}(r)} . \tag{3-25}$$

After considerable algebra, and using that $r\nabla^2 u_r' = \nabla^2(ru_r') - 2\Delta$, we have

$$\begin{aligned}
&\Big\{\Big[\tfrac{\partial}{\partial t}(3+r\tfrac{\partial}{\partial r}) - \tfrac{k}{\rho_0 c_v}\left(\tfrac{\partial}{\partial r} r^2 \nabla^2 \tfrac{1}{r}\right)\Big]\Big[\mu\beta r\nabla^4 - \rho_0\beta\tfrac{\partial}{\partial t} r\nabla^2\Big] \\
&\quad - \Big[\alpha(3+r\tfrac{\partial}{\partial r})\tfrac{\partial}{\partial r} r^2\nabla^2\bar{\gamma}(r)\Big]\Big\}(ru_r') \\
&= \Big\{\Big[\tfrac{\partial}{\partial t}(3+r\tfrac{\partial}{\partial r}) - \tfrac{k}{\rho_0 c_v}\left(\tfrac{\partial}{\partial r} r^2\nabla^2\tfrac{1}{r}\right)\Big]\Big[r^2\nabla^2 r\left(\tfrac{\partial}{\partial r} + \rho_0\beta\bar{g}r\right) \\
&\qquad + 4\pi G\rho_0^2\beta\tfrac{\partial}{\partial r}r^2 + (\mu' + \tfrac{1}{3}\mu)\beta r\nabla^2 r\tfrac{\partial}{\partial r}\tfrac{\partial}{\partial t} - 2\mu\beta r\nabla^2\tfrac{\partial}{\partial t}\Big] \\
&\qquad + \Big[\tfrac{\alpha}{c_v}(3+r\tfrac{\partial}{\partial r})\tfrac{\partial}{\partial r}r^2\nabla^2\left(\lambda + \tfrac{\bar{g}}{\bar{\gamma}}T_0\tfrac{\partial}{\partial t}\right)\Big]\Big\}\left(\tfrac{\rho'}{\rho_0}\right)
\end{aligned} \tag{3-26}$$

and

$$\begin{aligned}
&\left[\mu r^2\nabla^4 - \rho_0\tfrac{\partial}{\partial t} r\nabla^2\right](ru_r') \\
&= \left[\rho_0\bar{g}r\lambda + \rho_0\tfrac{\partial^2}{\partial t^2}\tfrac{\partial}{\partial r}r - \mu\tfrac{\partial}{\partial r}r^2\nabla^2\tfrac{\partial}{\partial t} - 2\mu r\nabla^2\tfrac{\partial}{\partial t}\right]\left(\tfrac{\rho'}{\rho_0}\right) .
\end{aligned} \tag{3-27}$$

If one has knowledge of the forms of $\bar{\gamma}, \bar{g}$ and λ such as $\bar{\gamma}, \bar{g}, \bar{\lambda}$ = constant, then one can hope to uncouple the system and examine the nature of the time dependence as discussed in Chapter II. One would like then to determine that a state of exchange stabilities exists and then examine the nature of the marginal state stationary flow. Rigorous proof of existence of the marginal stability state has been obtained in only an extremely limited case. This proof is due to Chandrasekhar in Chapter IV of "Hydrodynamic and Hydromagnetic Stability," and is concerned with spheres in which the fluid is incompressible, which have $\bar{\gamma}, \bar{g}$ and λ all constant, and the density perturbations are due solely to thermal perturbations (i.e. $\beta = 0$). In fact, a proof of the validity of exchange stabilities for the above cases under the assumption of compressibility or variation in $\bar{\gamma}, \bar{g}$ and λ does not exist in the literature at the present time.

At this point, however, we are going to <u>assume</u> such a state of $S_{\underset{\sim}{k}}^{R} = 0$ implying $S_{\underset{\sim}{k}}^{I} = 0$ does exist and attempt to solve for the resulting stationary flow pattern. Then we have $\frac{\partial}{\partial t} = 0$ and after extensive algebra, the system can be entirely uncoupled to yield

$$\nabla^2 \left[\frac{\nabla^4 (r u_r')}{\bar{g}(r)} \right] + \left[\left(4\pi G \rho_0^2 \beta - \frac{\alpha \rho_0}{K} \lambda(r) \right) \frac{1}{\bar{g}(r)} + \rho_0 \beta \left(3 + r\frac{\partial}{\partial r} \right) \right] \nabla^4 (r u_r') - \frac{\alpha \rho_0^2 C_v}{\mu K} \bar{\gamma}(r) \mathcal{L}(r u_r') = 0 . \qquad (3\text{-}28)$$

In the following chapter we shall attempt to apply the radial flow equation (3-28) to the case homogeneous fluid spheres and spherical shells.

Chapter IV

THE ONSET OF CONVECTION IN HOMOGENEOUS FLUID SPHERES AND SPHERICAL SHELLS

In Chapter III, the equation governing the radial flow for the marginal stationary state was derived. Let us introduce the non-dimensional length x = r?R, where R is the outer radius of the sphere. Also, let us define

(a) $\bar{g}(x) = \frac{4}{3}\pi G \rho_0 \xi(x)$ (4-1)

(b) $\lambda(x) = \lambda_0 \eta(x)$

(c) $\bar{\gamma}(x) = \frac{\lambda_0}{3k} \epsilon(x)$

(d) $C_\alpha = \frac{\rho_0 \alpha \lambda_0 R^2}{k}$

(e) $C_\beta = \frac{4}{3}\pi G \rho_0^2 \beta R^2$

(f) $C_R = \frac{4\pi G \rho_0^4 \alpha \lambda_0 C_v R^6}{9k^2\mu}$ (Non-dimensional Rayleigh Number)

and our equation becomes

$$\nabla^2 \left[\frac{\nabla^4 (r u_r')}{\xi(x)} \right] - C_\alpha \left[\frac{\eta(x)}{\xi(x)} \right] \nabla^4 (r u_r') + C_\beta \left[\frac{3}{\xi(x)} + 3 + x\frac{\partial}{\partial x} \right] \nabla^4 (r u_r') \quad (4-2)$$
$$- C_R\, \epsilon(x)\, \mathcal{L}(r u_r') = 0 .$$

Let us examine the boundary conditions, as defined in Chapter II in light of the form on the governing equation (4-2).

Boundary Conditions

Consider a spherical shell confined between x = 1 and x = a, where $0 < a < 1$. In all cases we must require that

$$u_r' = 0 \quad \text{at} \quad x = a \quad \text{and} \quad x = 1, \tag{4-3}$$

then clearly

$$ru_r' = 0 \quad \text{at} \quad x = a \quad \text{and} \quad x = 1. \tag{4-4}$$

The remaining boundary conditions depend upon whether the surface is rigid or free. On a rigid surface we have discussed that u_θ' and u_φ' must also vanish. Now from the equation of continuity for the marginal state we have

$$\left(\frac{\partial u_r'}{\partial r} + \frac{2u_r'}{r}\right) + \frac{1}{r}\left(\frac{\partial u_\theta'}{\partial \theta} + \cot\theta\, u_\theta'\right) + \frac{1}{r\sin\theta}\frac{\partial u_\varphi'}{\partial \varphi} = 0. \tag{4-5}$$

Then requiring the vanishing of $\underset{\sim}{u}'$ on a surface of r = constant for all values of θ, φ gives the boundary condition that

$$\frac{\partial u_r'}{\partial r} = 0 \qquad \text{on a rigid spherical boundary.} \tag{4-6}$$

Then we can easily see that

$$\frac{\partial}{\partial r}(ru_r') = 0 \qquad \text{on a rigid spherical boundary.} \tag{4-7}$$

For a free spherical surface we know that the transverse viscous stresses must vanish and this requires that

$$\left(\frac{1}{r}\frac{\partial u_r'}{\partial \theta} - \frac{u_\theta'}{r} + \frac{\partial u_\theta'}{\partial r}\right) = 0 \tag{4-8}$$

and

$$\left(\frac{1}{r\sin\theta}\frac{\partial u_r'}{\partial \varphi} - \frac{u_\varphi'}{r} + \frac{\partial u_\varphi'}{\partial r}\right) = 0 . \tag{4-9}$$

These must hold for all θ, φ on a r = constant surface on which $u_r{}' = 0$ identically, so

$$r\frac{\partial}{\partial r}\left(\frac{u_\theta{}'}{r}\right) = 0 \tag{4-10}$$

and

$$r\frac{\partial}{\partial r}\left(\frac{u_\varphi{}'}{r}\right) = 0 \ . \tag{4-11}$$

Then using the continuity equation

$$r\frac{\partial}{\partial r}\left(\frac{\partial u_r{}'}{\partial r} + \frac{2u_r{}'}{r}\right) = \frac{\partial^2}{\partial r^2}(r u_r{}') - \frac{2}{r}u_r{}' = 0 \tag{4-12}$$

and hence we have

$$\frac{\partial^2}{\partial r^2}(r u_r{}') = 0 \quad \text{on a free spherical surface.} \tag{4-13}$$

<u>The Method of Galerkin and Its Application to Homogeneous Fluid Spheres</u>

From the nature of the previosuly derived radial flow equation and boundary conditions we see it will be favorable to expand the solution in terms of spherical surface harmonics as

$$r u_r{}' = \sum_{\ell=1}^{\infty} \sum_{m=-(\ell-1)}^{\ell-1} w_\ell(x)\, Y_\ell{}^m(\theta,\varphi) \tag{4-14}$$

where $Y_\ell{}^m(\theta,\varphi)$ is the 1^{th} spherical surface harmonic of order m. Further we note that $\mathcal{L}Y_\ell{}^m = -\ell(\ell+1)Y_\ell{}^m$ and if we define $D_\ell \equiv \frac{d^2}{dx^2} + \frac{2}{x}\frac{d}{dx} - \frac{\ell(\ell+1)}{x^2}$, we have

$$D_\ell\left[\frac{D_\ell{}^2 w_\ell(x)}{\zeta(x)}\right] + C_\beta\left[\frac{3}{\zeta(x)} + 3 + x\frac{d}{dx}\right]D_\ell^2 w_\ell(x) \tag{4-15}$$

$$- C_d\left[\frac{\eta(x)}{\zeta(x)}\right]D_\ell^2 w_\ell(x) + C_R\,\ell(\ell+1)\,\xi(x)\, w_\ell(x) = 0 \ .$$

Since we are considering entire fluid spheres, and ρ_0 = constant, then one can easily show that $\bar{g}$ = constant so that $\zeta(x)$ is strictly unity in this case. Further, one has that from the zero-order entropy transport equation

$$\frac{1}{r^2}\frac{d}{dr}r^2\frac{dT_0}{dr} = -\frac{\lambda(r)}{K} \tag{4-16}$$

$$\frac{dT_0}{dr} = -\frac{1}{Kr^2}\int \lambda(r) r^2 dr - \frac{C_0}{r^2} = -\bar{\gamma}(r) r \tag{4-17}$$

and

$$T_0(r) = -\int \frac{1}{Kr^2} dr \int \lambda(r') r'^2 dr' + \frac{C_0}{r} + C_1 \tag{4-18}$$

so that requiring To to be finite at r = 0 gives $C_0 = 0$ and specification of $\lambda(r)$ uinquely determines $\bar{\gamma}(r)$.

Further, we have that

$$\xi(x) = \frac{3}{x^3}\int \eta(x) x^2 dx \quad \text{or} \quad \eta(x) = \frac{1}{3x^2}\frac{d}{dx}(x^3 \xi(x)) \tag{4-19}$$

and the radial flow equation becomes

$$\left[D_\ell^3 + C_\beta \left(6 + x\frac{d}{dx}\right) D_\ell^2 - C_\alpha \left(\xi(x) + \frac{x}{3}\frac{d\xi}{dx}\right) D_\ell^2 + \ell(\ell+1) C_R \xi(x) \right] w_\ell(x) = 0 . \tag{4-20}$$

For a moment, let us consider the more general problem of finding the solutions to a differential equation of the form

$$\mathbb{D}(u) = 0 \tag{4-21}$$

where $\mathbb{D}$ is some ordinary differential operator, subject to homogeneous boundary conditions and x = 0 and x = 1 (say).

Then if $\{ \varphi_i(x) \}$ is an orthogonal set of functions which are complete on the interval (0, 1) we can consider some approximation to our solution, $\bar{u}$ (say) as

$$\bar{u}(x) = \sum_j A_j \varphi_j(x) . \tag{4-22}$$

Now in order that $\bar{u}(x)$ be the <u>exact</u> solution to (4-21), it is necessary that $\mathbb{D}(\bar{u})$ be identically zero; and this requirement, if $\mathbb{D}$ is to be a continuous operator, is equivalent to the requirement of the orthogonality of the expression $\mathbb{D}(\bar{u})$ to all the functions of the complete set $\{ \varphi_j(x) \}$ (c.f. Courant and Hilbert, <u>Methods of Mathematical Physics, Vol. I</u>, Chapter 6). Hence

$$\int_0^1 dx\, f\, \mathbb{D}[\bar{u}(x)]\, \varphi_k(x) = \int_0^1 dx\, f\, \mathbb{D}\Big[\sum_j A_j \varphi_j(x)\Big] \varphi_k(x) = 0 \tag{4-23}$$

for k = 1, 2, . . . and where $\{ \varphi_j(x) \}$ satisfy the orthogonality relations

$$\int_0^1 dx\, f(x)\, \varphi_j(x)\, \varphi_k(x) = C_j\, \delta_{jk} \tag{4-24}$$

where C_j is some normalization constant for a given j and $f(x)$ is the weighting function. This procedure, however, will give an infinite set of linear equations for the coefficients $\{A_j\}$. One can approximate $\bar{u}(x)$ by letting $\tilde{u} = \sum_{j=1}^{N} A_j \varphi_j(x)$ and then we will have an $N \times N$ system for finding the $\{A_j\}$. This method, first proposed by B.G. Galerkin [c.f. Kantorovich and Krylov (1958)] will give an essentially "best fit" approximation to the solution. In the case of a characteristic value problem one finds that the system (4-23) will give non-zero $\{A_j\}$ for only a specified set of values of some characteristic number. One then has an $N \times N$ secular determinant which upon evaluation yields an n^{th}-order approximation to the desired characteristic value.

To attempt to apply Galerkin's method to the equation (4-20) let us introduce a function

$$z(x) = D^2_\ell w_\ell(x) \tag{4-25}$$

and expand $z(x)$ in terms of some $\{\varphi_j(x)\}$ which is complete over (0, 1) and will satisfy the necessary boundary conditions at the end points. If we note that

$$\left(\frac{d^2}{dx^2} + \frac{2}{x}\frac{d}{dx} - \frac{\ell(\ell+1)}{x^2} \right) \frac{J_{\ell+1/2}(\epsilon x)}{\sqrt{x}} = -\epsilon^2 \frac{J_{\ell+1/2}(\epsilon x)}{\sqrt{x}} \tag{4-26}$$

we can see that a convenient choice for $\{\varphi_j(x)\}$ will be $\left\{ \frac{1}{\sqrt{x}} J_{\ell+1/2}(\epsilon_{\ell j} x) \right\}$ so that

$$z(x) = \frac{1}{\sqrt{x}} \sum_j A_j J_{\ell+1/2}(\epsilon_{\ell j} x) \tag{4-27}$$

which will satisfy the boundary conditions if $\{\epsilon_{\ell j}\}$ are the j^{th} zeros of $J_{\ell+1/2}(z)$. Then a particular integral of (4-25) which is free from singularities at x = 0 is $\frac{1}{\epsilon_{\ell j}^4} \frac{J_{\ell+1/2}(\epsilon_{\ell j} x)}{\sqrt{x}}$ Adding to this the homogeneous integral which is non-singular at the origin,

$$w_\ell(x) = \sum_j w_{\ell j}(x) = \sum_j A_j \left[\frac{1}{\epsilon_{\ell j}^4} \frac{J_{\ell+1/2}(\epsilon_{\ell j})}{\sqrt{x}} + B(j)\, x^{\ell} + C(j)\, x^{\ell+2} \right]. \tag{4-28}$$

At x = 1, $w_{\ell_j}(x) = 0$, so clearly $B(j) = -C(j)$, while the explicit nature of $B(j)$ must be subsequently determined from the other boundary condition at x = 1, namely, either $dw_{\ell_j}/dx = 0$ or $d^2 w_{\ell_j}/dx^2 = 0$ depending on whether the surface is rigid or free. If one applies these conditions, $B(j)$ can be conveniently written as

$$B(j) = \frac{b}{4} \frac{J'_{\ell+1/2}(\epsilon_{\ell_j})}{\epsilon_{\ell_j}^3} \tag{4-29}$$

where b = 2 for a rigid boundary and $b = -\frac{4}{2\ell+1}$ for a free boundary. One can then write for $w_{\ell_j}(x)$,

$$w_{\ell_j}(x) = \frac{1}{\epsilon_{\ell_j}^4} \frac{J_{\ell+1/2}(\epsilon_{\ell_j} x)}{\sqrt{x}} + \frac{b}{4\epsilon_{\ell_j}^3} J'_{\ell+1/2}(\epsilon_{\ell_j})(x^{\ell} - x^{\ell+2}) \tag{4-30}$$

If we note that the orthogonality condition for the half-integer Bessel Functions is

$$\int_0^1 dx\, x\, J_{\ell+1/2}(\epsilon_{\ell k} x)\, J_{\ell+1/2}(\epsilon_{\ell_j} x) = \tfrac{1}{2}\delta_{jk}\left[J'_{\ell+1/2}(\epsilon_{\ell_j})\right]^2. \tag{4-31}$$

Then using the definition of $z_j(x)$ and $w_{\ell_j}(x)$ and multiplying (4-20) by $x^{3/2} J_{\ell+1/2}(\epsilon_{\ell k} x)$ and integrating over (0, 1) we have the Galerkin conditions,

$$\sum_j A_j \Big\{ \tfrac{1}{2}\delta_{kj} \left[J'_{\ell+1/2}(\epsilon_{\ell_j})\right]^2 \left[-\epsilon_{\ell_j}^2 + 6C_\beta\right] + C_\beta I^{(1)}_{jk} - C_\alpha I^{(2)}_{jk} + C_R\, \ell(\ell+1)\left[I^{(3)}_{jk} + I^{(4)}_{jk}\right]\Big\} = 0 \tag{4-32}$$

for $k = 1, 2, \ldots$.

where

$$\text{(a)} \quad I^{(1)}_{jk} = \int_0^1 dx\, x^{5/2} J_{\ell+1/2}(\epsilon_{\ell k} x) \frac{d}{dx}\left(\frac{J_{\ell+1/2}(\epsilon_{\ell j} x)}{\sqrt{x}} \right) \tag{4-33}$$

$$\text{(b)} \quad I^{(2)}_{jk} = \int_0^1 dx\, x\, J_{\ell+1/2}(\epsilon_{\ell k} x) J_{\ell+1/2}(\epsilon_{\ell j} x) \left(\xi(x) + \frac{x}{3} \frac{d\xi}{dx} \right)$$

$$\text{(c)} \quad I^{(3)}_{jk} = \frac{1}{\epsilon_{\ell j}^4} \int_0^1 dx\, x\, \xi(x)\, J_{\ell+1/2}(\epsilon_{\ell k} x) J_{\ell+1/2}(\epsilon_{\ell j} x)$$

$$\text{(d)} \quad I^{(4)}_{jk} = \frac{b}{4\epsilon_{\ell j}^3} \int_0^1 dx\, \xi(x) J_{\ell+1/2}(\epsilon_{\ell k} x) (x^{\ell+3/2} - x^{\ell+7/2}) J'_{\ell+1/2}(\epsilon_{\ell j}) .$$

The matrix component due to $I^{(1)}_{jk}$ can be calculated explicitly using one of the recurrence formulae for half-integer Bessel functions, specifically,

$$\epsilon_{\ell j} x J'_{\ell+1/2}(\epsilon_{\ell j} x) = (\ell+1/2) J_{\ell+1/2}(\epsilon_{\ell j} x) - \epsilon_{\ell j} x J_{\ell+3/2}(\epsilon_{\ell j} x) \tag{4-34}$$

and using that

$$\int_0^1 dx\, x^2 J_{\ell+3/2}(\epsilon_{\ell j} x) J_{\ell+1/2}(\epsilon_{\ell k} x) \tag{4-35}$$

$$= - \frac{\epsilon_{\ell k}}{\epsilon_{\ell j}^2 - \epsilon_{\ell k}^2} J'_{\ell+1/2}(\epsilon_{\ell j}) J'_{\ell+1/2}(\epsilon_{\ell k}) \quad \text{for } j \neq k$$

$$= \frac{(\ell+3/2)}{2\epsilon_{\ell k}} \left[J'_{\ell+1/2}(\epsilon_{\ell k}) \right]^2 \quad \text{for } j = k,$$

(4-32) becomes

$$\sum_j A_j \Big\{ \left[\tfrac{9}{2} C_\beta - \epsilon_{\ell j}^2 \right] \tfrac{1}{2} \delta_{jk} \left[J'_{\ell+1/2}(\epsilon_{\ell j}) \right]^2 \tag{4-33}$$

$$+ C_\beta \frac{\epsilon_{\ell j}\epsilon_{\ell k}}{\epsilon_{\ell j}^2 - \epsilon_{\ell k}^2} J'_{\ell+1/2}(\epsilon_{\ell k}) J'_{\ell+1/2}(\epsilon_{\ell j}) (1 - \delta_{jk})$$

$$+ C_R\, \ell(\ell+1) \left(I^{(3)}_{jk} + I^{(4)}_{jk} \right) - C_\alpha I^{(2)}_{jk} \Big\} = 0.$$

or for non-zero $\{A_j\}$,

$$\Big\| \frac{1}{2}\left(\frac{9}{2}C_\beta - \epsilon_{\ell j}^2\right)\delta_{jk}\left[J'_{\ell+1/2}(\epsilon_{\ell j})\right]^2 - C_\alpha I^{(2)}_{jk} \qquad (4\text{-}34)$$

$$+ C_\beta \frac{\epsilon_{\ell j}\epsilon_{\ell k}}{\epsilon_{\ell j}^2-\epsilon_{\ell k}^2} J'_{\ell+1/2}(\epsilon_{\ell j}) J'_{\ell+1/2}(\epsilon_{\ell k}) . (1-\delta_{jk})$$

$$+ C_R \ell(\ell+1)\left(I^{(3)}_{jk} + I^{(4)}_{jk}\right) \Big\| = 0 .$$

We see that the specific forms of secular determinants must depend upon the distribution of energy sources as expressed by $\lambda(x)$ or equivalently by $g(x)$. Some particularly convenient forms of $\lambda(x)$ will be discussed in the following chapter concerning applications to planetary interiors.

Convection in Homogeneous Spherical Fluid Shells

We now wish to consider the application of equation (4-15) to a spherical shell of homogeneous fluid confined between x = 1 and x = a $(0 < a < 1)$. In this case $\zeta(x)$ cannot strictly speaking be unity, for if M_a is the mass contained in the sphere enclosed by the sphere x = a then $g(x)$ must be given as

$$g(x) = G\left[\left(M_a - \tfrac{4}{3}\pi\rho_0 a^3\right)\frac{1}{x^3} + \tfrac{4}{3}\pi\rho_0\right] \qquad (4\text{-}35)$$

or if $\bar{\rho}_a$ is the average density such that $\frac{4}{3}\pi\bar{\rho}_a a^3 = M_a$ then

$$\bar{g}(x) = \tfrac{4}{3}\pi G\rho_0\,\zeta(x) = \tfrac{4}{3}\pi\rho_0 G\left[\left(\frac{\bar{\rho}_a}{\rho_0}-1\right)\frac{a^3}{x^3}+1\right] \qquad (4\text{-}36)$$

so in general $\zeta(x) \neq 1$.

Again we wish to define a function $z(x)$ such that $z(x) = D_\ell^2 w_\ell$. However, this time we need to expand $z(x)$ in a series of functions which are complete over (a, 1) and which satisfy <u>four</u> boundary conditions. One such set of functions can be constructed as cylinder functions from half-integer Bessel functions as suggested in a paper by Chandrasekhar (1957). Let us construct these functions then as

$$\mathcal{C}_{\ell+1/2,\ell+1/2}(q_{\ell j}x) = J_{-(\ell+1/2)}(q_{\ell j}a)\, J_{\ell+1/2}(q_{\ell j}x) - J_{\ell+1/2}(q_{\ell j}a)\, J_{-(\ell+1/2)}(q_{\ell j}x)\,. \tag{4-37}$$

The set of roots $\{q_{\ell j}\}$ of the above equation is infinite but countable, and all are real, simple, and distinct. Then it can be shown that $\{\mathcal{C}_{\ell+1/2,\ell+1/2}(q_{\ell j}x)\}$ will form a complete set over (a, 1) with the orthogonality relation

$$\int_a^1 dx\; x\; \mathcal{C}_{\ell+1/2,\ell+1/2}(q_{\ell k}x)\, \mathcal{C}_{\ell+1/2,\ell+1/2}(q_{\ell j}x) = \delta_{kj}\, P_{\ell+1/2,j} \tag{4-39}$$

where

$$P_{\ell+1/2,j} = \frac{2}{\pi^2 q_{\ell j}^2}\left[\frac{J^2_{\ell+1/2}(q_{\ell j}a)}{J^2_{\ell+1/2}(q_{\ell j})} - 1\right]. \tag{4-40}$$

Expanding $z(x)$ in terms of these cylinder functions we have

$$z(x) = \frac{1}{\sqrt{x}} \sum_j A_j\, \mathcal{C}_{\ell+1/2,\ell+1/2}(q_{\ell j}x) \tag{4-41}$$

and immediately we see that a particular integral of $z(x) = D_\ell^2 w_\ell$ is

$$w^P_{\ell j}(x) = \frac{1}{q^4_{\ell j}}\, \frac{\mathcal{C}_{\ell+1/2,\ell+1/2}(q_{\ell j}x)}{\sqrt{x}} \tag{4-42}$$

and adding the homogeneous integrals,

$$w_\ell(x) = \sum_j A_j \Big[\frac{1}{q_{\ell_j}^4 \sqrt{x}} \, C_{\ell+1/2,\,\ell+1/2}(q_{\ell_j} x) + B_1(j)\, x^{\ell} + B_2(j)\, x^{\ell+2} + B_3(j)\, x^{-(\ell+1)} + B_4(j)\, x^{-(\ell-1)} \Big] . \tag{4-43}$$

The specific forms of $B_1(j)$, $B_2(j)$, $B_3(j)$ and $B_4(j)$ are given from the condition that $w_{\ell_j}(x) = 0$ on x = 1, a,

$$\text{(a)} \quad B_1(j) + B_2(j) + B_3(j) + B_4(j) = 0 \tag{4-44}$$

$$\text{(b)} \quad B_1(j)\, a^{\ell} + B_2(j)\, a^{\ell+2} + B_3(j)\, a^{-(\ell+1)} + B_4(j)\, a^{-(\ell-1)} = 0$$

along with conditions on <u>both</u> x = 1 and x = a which depend upon the exact nature of the boundary surfaces.

If we now substitute the solutions for $z(x)$ and $w_\ell(x)$ into (4-15), multiply by $x^{3/2}\, C_{\ell+1/2,\,\ell+1/2}(q_{\ell k} x)$ and integrate over (a, 1) we obtain

$$\begin{aligned} \sum_j A_j \Big[& \int_a^1 dx\, x^2 \left(\frac{C_{\ell+1/2,\,\ell+1/2}(q_{\ell k} x)}{\xi(x)\sqrt{x}} \right) D_\ell \left(\frac{C_{\ell+1/2,\,\ell+1/2}(q_{\ell_j} x)}{\xi(x)\sqrt{x}} \right) \\ & + C_\beta \int_a^1 dx\, x^2 \left(\frac{C_{\ell+1/2,\,\ell+1/2}(q_{\ell k} x)}{\sqrt{x}\; \xi(x)} \right) \left(3 + \frac{3}{\xi(x)} + x \frac{d}{dx} \right) \left(\frac{C_{\ell+1/2,\,\ell+1/2}(q_{\ell_j} x)}{\sqrt{x}} \right) \\ & - C_\alpha \int_a^1 dx\, x \left[\frac{\eta(x)}{\xi(x)\, \xi(x)} \right] C_{\ell+1/2,\,\ell+1/2}(q_{\ell k} x)\, C_{\ell+1/2,\,\ell+1/2}(q_{\ell_j} x) \\ & + C_R\, \ell(\ell+1) \left[\frac{1}{q_{\ell_j}^4}\, \delta_{jk}\, P_{\ell+1/2,\,j} \right] \\ & + C_R\, \ell(\ell+1) \int_a^1 dx\, C_{\ell+1/2,\,\ell+1/2}(q_{\ell k} x) \big[B_1(j)\, x^{\ell} + B_2(j)\, x^{\ell+2} + B_3(j)\, x^{-(\ell+1)} + B_4\, x^{-(\ell-1)} \big] \Big] = 0. \end{aligned} \tag{4-45}$$

The last integral, call it $R_{\kappa j}$, in (4-45) gives after some lengthy algebra

$$R_{\kappa j} = -\frac{2}{q_{\ell\kappa}^2}\Big\{\big[\mathcal{C}_{\ell+1/2,\ell+5/2}(q_{\ell\kappa}) - a^{\ell+5/2}\mathcal{C}_{\ell+1/2,\ell+5/2}(q_{\ell\kappa}a)\big] \times B_z(j) + \big[\mathcal{C}_{\ell+1/2,\ell-3/2}(q_{\ell\kappa}) - a^{-\ell+3/2}\mathcal{C}_{\ell+1/2,\ell-3/2}(q_{\ell\kappa}a)\big] B_Y(j)\Big\}. \tag{4-46}$$

$R_{\kappa j}$ can be simplified using the recurrence formulae satisfied by $\mathcal{C}_{\ell+1/2,\nu}$ and noting that $\mathcal{C}_{\ell+1/2,\ell+1/2}(q_{\ell\kappa})$ and $\mathcal{C}_{\ell+1/2,\ell+1/2}(q_{\ell\kappa}a)$ are both zero. These are

(a) $$\mathcal{C}_{\ell+1/2,\ell+5/2}(q_{\ell\kappa}) = -\frac{2\ell+3}{q_{\ell\kappa}}\mathcal{C}'_{\ell+1/2}(q_{\ell\kappa}) \tag{4-47}$$

(b) $$\mathcal{C}_{\ell+1/2,\ell+5/2}(aq_{\ell\kappa}) = -\frac{2\ell+3}{aq_{\ell\kappa}}\mathcal{C}'_{\ell+1/2}(q_{\ell\kappa}a)$$

(c) $$\mathcal{C}_{\ell+1/2,\ell-3/2}(q_{\ell\kappa}) = \frac{2\ell-1}{q_{\ell\kappa}}\mathcal{C}'_{\ell+1/2}(q_{\ell\kappa})$$

(d) $$\mathcal{C}_{\ell+1/2,\ell-3/2}(aq_{\ell\kappa}) = \frac{2\ell-1}{aq_{\ell\kappa}}\mathcal{C}'_{\ell+1/2}(q_{\ell\kappa}a).$$

Further, let us define

(a) $$\mathcal{F}_{\ell+1/2,\kappa} \equiv \mathcal{C}'_{\ell+1/2}(q_{\ell\kappa}) - a^{\ell+3/2}\mathcal{C}'_{\ell+1/2}(aq_{\ell\kappa}) \tag{4-48}$$

(b) $$\mathcal{E}_{\ell+1/2,\kappa} \equiv \mathcal{C}'_{\ell+1/2}(q_{\ell\kappa}) - a^{\ell+1/2}\mathcal{C}'_{\ell+1/2}(aq_{\ell\kappa})$$

and we can conveniently write $R_{\kappa j}$ as

$$R_{\kappa j} = \frac{2}{q_{\ell\kappa}^3}\big[(2\ell+3)\mathcal{F}_{\ell+1/2,\kappa}B_z(j) - (2\ell-1)\mathcal{E}_{\ell+1/2,\kappa}B_Y(j)\big]. \tag{4-49}$$

Finally, we can obtain a secular determinant analogous to (4-34) as

$$\left\| M_{kj} + c_\beta N_{kj} - C_\alpha O_{kj} + C_R\, \ell(\ell+1)\left(\frac{\delta_{kj}}{q_{\ell_j}{}^4} P_{\ell+1/2,k} + R_{kj} \right) \right\| = 0, \tag{4-50}$$

where the matrix elements are defined as

(a) $$M_{kj} = \int_a^1 dx\, x^2 \left(\frac{C_{\ell+1/2,\ell+1/2}(q_{\ell k} x)}{\mathcal{E}(x)\sqrt{x}} \right) D_\ell \left(\frac{C_{\ell+1/2,\ell+1/2}(q_{\ell j} x)}{\zeta(x)\sqrt{x}} \right) \tag{4-51}$$

(b) $$N_{kj} = \int_a^1 dx\, x^2 \left(\frac{C_{\ell+1/2,\ell+1/2}(q_{\ell k} x)}{\mathcal{E}(x)\sqrt{x}} \right) \left(3 + \frac{3}{\zeta(x)} + x\frac{d}{dx} \right) \left(\frac{C_{\ell+1/2,\ell+1/2}(q_{\ell j} x)}{\sqrt{x}} \right)$$

(c) $$O_{kj} = \int_a^1 dx\, x \left[\frac{\eta(x)}{\mathcal{E}(x)\,\zeta(x)} \right] C_{\ell+1/2,\ell+1/2}(q_{\ell k} x)\, C_{\ell+1/2,\ell+1/2}(q_{\ell j} x)$$

with $P_{\ell+1/2,j}$ and R_{kj} given by (4-40) and (4-49) respectively.

As in the case of entire fluid spheres the evaluation of the matrix elements will depend upon the specific forms of gravitational field in terms of $\zeta(x)$ and the distribution of internal energy sources as represented by $\lambda(x)$ or equivalently $\mathcal{E}(x)$. In addition, evaluation of R_{kj} depends upon calculating $B_z(j)$ and $B_y(j)$ from specific knowledge of the boundary conditions at x = 1 and x = a.

--Determination of $B_z(j)$ and $B_y(j)$ for Four Possible Sets of Boundary Conditions--

As we are considering our bounding surfaces to be spherical and either rigid or free there are four distinct cases for calculating $B_z(j)$ and $B_y(j)$.

(a) Free surfaces at $x = a$ and $x = 1$:

In this case, $d^2 w_{\ell_j}/dx^2 = 0$ at both x = a and x = 1 so our conditions are:

at x = 1

(a) $\ell(\ell-1)\left[B_1(j) + B_4(j)\right] + (\ell+2)(\ell+1)\left[B_2(j) + B_3(j)\right]$ (4-52)

$$= \frac{2\,C'_{\ell+1/2}(q_{\ell_j})}{q_{\ell_j}^3}$$

at x = a

(b) $\ell(\ell-1)\left[a^{\ell-2} B_1(j) + a^{-(\ell+1)} B_4(j)\right]$

$$+ (\ell+2)(\ell+1)\left[a^{\ell} B_2(j) + a^{-(\ell+3)} B_3(j)\right] = \frac{2a^{-3/2}}{q_{\ell_j}^3} C'_{\ell+1/2}(q_{\ell_j} a).$$

Using the above along with our previously derived conditions from $w_{\ell_j}(x) = 0$ at x = 1, x = a, extensive reductions yield

(a) $$B_2(j) = \frac{1}{(2\ell+1)(1-a^{2\ell+3})\,q_{\ell_j}^3}\,\mathcal{F}_{\ell+1/2,\,j}$$ (4-53)

(b) $$B_4(j) = \frac{1}{(2\ell+1)(a^{2\ell+1}-1)\,q_{\ell_j}^3}\,\mathcal{E}_{\ell+1/2,\,j}\,.$$

(b) Rigid surfaces at x = a and x = 1:

Now we must have $dw_{\ell_j}/dx = 0$ at both x = a and x = 1 which yields

at x = 1

(a) $\ell B_1(j) + (\ell+2) B_2(j) - (\ell+1) B_3(j) - (\ell-1) B_4(j)$ (4-54)

$$= -\frac{1}{q_{\ell_j}^3} C'_{\ell+1/2}(q_{\ell_j})$$

at x = a

(b) $\ell B_1(j)\, a^{\ell-1} + (\ell+2) B_2(j)\, a^{\ell+1} - (\ell+1) B_3(j)\, a^{-(\ell+2)}$

$$- (\ell-1) B_4(j)\, a^{-\ell} = -\frac{a^{-1/2}}{q_{\ell_j}^3} C'_{\ell+1/2}(q_{\ell_j} a).$$

Then using equation (4-44) we have for this case,

(a) $$B_z(j) = \frac{1}{T_{rr}(\ell,a)\, q_{\ell j}^3}\Big[2\left(\frac{1}{a^{2\ell+1}} - \frac{1}{a^2}\right)\mathcal{F}_{\ell+1/2,\, j} - (2\ell-1)\left(\frac{1}{a^2}-1\right)\mathcal{C}'_{\ell+1/2}(q_{\ell j}) + (2\ell-1)\left(\frac{1}{a^{\ell+3/2}} - \frac{1}{a^{\ell-1/2}}\right)\mathcal{C}'_{\ell+1/2}(q_{\ell j}a)\Big] \qquad (4\text{-}55)$$

(b) $$B_y(j) = \frac{1}{T_{rr}(\ell,a)\, q_{\ell j}^3}\Big[-2\left(\frac{1}{a^2} - a^{2\ell+1}\right)\mathcal{E}_{\ell+1/2,\, j} + (2\ell+3)\left(\frac{1}{a^2}-1\right)\mathcal{C}'_{\ell+1/2}(q_{\ell j}) - (2\ell+3)\left(a^{\ell-1/2} - a^{\ell+3/2}\right)\mathcal{C}'_{\ell+1/2}(q_{\ell j}a)\Big]$$

where we have introduced

$$T_{rr}(\ell,a) = (1+4\ell+4\ell^2)\left(\frac{1}{a^2}+a^2\right) - 4\left(\frac{1}{a^{2\ell+1}} + a^{2\ell+1}\right) - (8\ell^2+8\ell-6). \qquad (4\text{-}56)$$

(c) Rigid surface at x = a and a free surface at x = 1.

In this case we have $dw_{\ell j}/dx = 0$ at x = a, while at x = 1 $d^2w_{\ell j}/dx^2 = 0$. To solve for $B_z(j)$ and $B_y(j)$ we then use (4-44 a, b) along with (4-56a) and (4-56b) yielding

(a) $$B_z(j) = -\frac{1}{T_{rf}(\ell,a)\, q_{\ell j}^3}\Big[(2\ell+1)\left(\frac{1}{a^{2\ell+1}} - \frac{1}{a^2}\right)\mathcal{F}_{\ell+1/2,\, j} - (2\ell-1)\left(\frac{1}{a^{2\ell+1}} - 1\right)\mathcal{C}'_{\ell+1/2}(q_{\ell j})\Big] \qquad (4\text{-}57)$$

(b) $$B_y(j) = -\frac{1}{T_{rf}(\ell,a)\, q_{\ell j}^3}\Big[(2\ell+1)\left(\frac{1}{a^2} - a^{2\ell+1}\right)\mathcal{E}_{\ell+1/2,\, j} - (2\ell+3)\left(1 - a^{2\ell+1}\right)\mathcal{C}'_{\ell+1/2}(q_{\ell j})\Big]$$

where

$$T_{rf}(\ell,a) = (2\ell+1)\Big[(2\ell+1)\left(\frac{1}{a^2} - a^2\right) - 2\left(\frac{1}{a^{2\ell+1}} - a^{2\ell+1}\right)\Big]. \qquad (4\text{-}58)$$

(d) Free surface at x = a and a rigid surface at x = 1.

In this final case we have just the reverse of (c) with $d^2 w_{\ell j}/dx^2 = 0$ at x = a and $dw_{\ell j}/dx = 0$ at x=1. Using (4-44 a, b) along with (4-52b) and (4-52a) we find after lengthy reductions that

$$(a)\quad B_z(j) = \frac{1}{\tau_{rf}(\ell,a)\, q_{\ell j}^3}\Big[(2\ell+1)\Big(\frac{1}{a^{2\ell+1}} - \frac{1}{a^2}\Big)\mathcal{F}_{\ell+1/2,\,j} + (2\ell-1)\Big(\frac{1}{a^{\ell+3/2}} - a^{\ell-1/2}\Big)\mathcal{C}'_{\ell+1/2}(q_{\ell j}a)\Big] \tag{4-59}$$

$$(b)\quad B_y(j) = \frac{1}{\tau_{rf}(\ell,a)\, q_{\ell j}^3}\Big[(2\ell+1)\Big(\frac{1}{a^2} - a^{2\ell+1}\Big)\mathcal{E}_{\ell+1/2,\,j} + (2\ell+3)\Big(\frac{1}{a^{\ell+3/2}} - a^{\ell-1/2}\Big)\mathcal{C}'_{\ell+1/2}(q_{\ell j}a)\Big]$$

where $\tau_{rf}(\ell,a)$ is the same as that given in (4-58).

Finally, if we use the form of R_{kj} given in (4-49) along with the values of $B_z(j)$ and $B_y(j)$ for each of the possible combinations of boundary conditions at x = a and x = 1, we have

$$R^{rr}_{kj} = \frac{2}{\tau_{rr}(\ell,a)\, q_{\ell k}^3\, q_{\ell j}^3}\Big[2(2\ell+3)\Big(\frac{1}{a^{2\ell+1}} - \frac{1}{a^2}\Big)\mathcal{F}_{\ell+1/2,\,k}\,\mathcal{F}_{\ell+1/2,\,j} + 2(2\ell-1)\Big(\frac{1}{a^2} - a^{2\ell+1}\Big)\mathcal{E}_{\ell+1/2,\,k}\,\mathcal{E}_{\ell+1/2,\,j} - (2\ell-1)(2\ell+3)\Big(\frac{1}{a^2}-1\Big)\Big(\mathcal{F}_{\ell+1/2,\,k}\,\mathcal{E}_{\ell+1/2,\,j} + \mathcal{F}_{\ell+1/2,\,j}\,\mathcal{E}_{\ell+1/2,\,k}\Big)\Big] \tag{4-60}$$

for rigid boundaries of x = a and x = 1.

For free boundaries of x = a and x = 1 we have

$$R^{ff}_{kj} = \frac{2}{(2\ell+1)\, q_{\ell k}^3\, q_{\ell j}^3}\Big[\frac{2\ell+3}{1-a^{2\ell+3}}\mathcal{F}_{\ell+1/2,\,j}\,\mathcal{F}_{\ell+1/2,\,k} - \frac{2\ell-1}{a^{-2\ell+1}-1}\mathcal{E}_{\ell+1/2,\,j}\,\mathcal{E}_{\ell+1/2,\,k}\Big], \tag{4-61}$$

for a rigid boundary at x = a and a free boundary at x = 1;

$$R^{rf}_{\ell kj} = \frac{2}{\tau_{rf}(\ell,a)\, q_{\ell k}^3\, q_{\ell j}^3} \Big[(2\ell+1)(2\ell-1)\left(\frac{1}{a^2} - a^{2\ell+1}\right) \mathcal{E}_{\ell+1/2,j}\, \mathcal{E}_{\ell+1/2,k} \tag{4-62}$$

$$- (2\ell+1)(2\ell+3)\left(\frac{1}{a^{2\ell+1}} - \frac{1}{a^2}\right) \mathcal{F}_{\ell+1/2,j}\, \mathcal{F}_{\ell+1/2,k}$$

$$- (2\ell+3)(2\ell-1)\left(2 - \frac{1}{a^{2\ell+1}} - a^{2\ell+1}\right) C'_{\ell+1/2}(q_{\ell k})\, C'_{\ell+1/2}(q_{\ell j}) \Big]$$

and for a free boundary at x = a and rigid boundary at x = 1.

$$R^{fr}_{kj} = \frac{2}{\tau_{rf}(\ell,a)\, q_{\ell k}^3\, q_{\ell j}^3} \Big[(2\ell+1)(2\ell+3)\left(\frac{1}{a^{2\ell+1}} - \frac{1}{a^2}\right) \mathcal{F}_{\ell+1/2,j} \times \mathcal{F}_{\ell+1/2,k} \tag{4-63}$$

$$- (2\ell+3)(2\ell-1)\left(2 - a^{2\ell+1} - \frac{1}{a^{2\ell+1}}\right) C'_{\ell+1/2}(q_{\ell k}a)\, C'_{\ell+1/2}(q_{\ell j}a)$$

$$- (2\ell+1)(2\ell-1)\left(\frac{1}{a^2} - a^{2\ell+1}\right) \mathcal{E}_{\ell+1/2,k}\, \mathcal{E}_{\ell+1/2,j} \Big].$$

Chapter V

CONVECTION IN PLANETARY MANTLES AND INTERIORS

In this section we wish to attempt to apply the general analysis of Chapter IV to the discussion of the onset of convection in spheres and spherical shells of planetary size. For the sake of specific examples, attention will be paid to models of the lunar interior and mantle of the Earth. The fluid we are using is considered to be a silicate material with parameter values corresponding to average values from measurements on terrestrial rocks.

For a convective model of the lunar interior we consider an entire fluid sphere which is homogeneous with an average zero-order density of $\rho_0 = 3.34$ g/cm^3. In the last chapter we determined that the characteristic values of the Rayleigh number C_R for the marginal stability state were given by the evaluation of the secular determinant,

$$\Big\| \tfrac{1}{2}\left(\tfrac{9}{2} C_\beta - \epsilon_{\ell j}^2\right)\delta_{jk}\left[J'_{\ell+1/2}(\epsilon_{\ell j})\right]^2 - C_\alpha I_{jk}^{(2)} + C_\beta \frac{\epsilon_{\ell j}\,\epsilon_{\ell k}}{\epsilon_{\ell j}^2 - \epsilon_{\ell k}^2} J'_{\ell+1/2}(\epsilon_{\ell j})\, J'_{\ell+1/2}(\epsilon_{\ell k})\,(1-\delta_{jk}) + C_R\,\ell(\ell+1)\left(I_{jk}^{(3)} + I_{jk}^{(4)}\right) \Big\| = 0. \tag{5-1}$$

The specific values of the matrix elements $I_{jk}^{(2)}$, $I_{jk}^{(3)}$, $I_{jk}^{(4)}$ depend upon the form of $\xi(x)$ which is directly related to $\lambda(x) = \lambda_0\,\eta(x)$. Rather than studying the effect of a particularly realistic form for $\lambda(x)$, which might be poorly known in any case, we will take the simplest example where $\lambda(x) = \lambda_0 =$ constant, representing a uniform distribution of heat sources. Then $\xi(x)$ is identically unity and the integrals involved in (5-1) can be evaluated explicitly using recurrence formulae for half-integer Bessel functions. Condition (5-1) becomes

$$\Big\| \tfrac{1}{2}\left(\tfrac{9}{2}C_\beta - \epsilon_{\ell j}^2 + C_R \frac{\ell(\ell+1)}{\epsilon_{\ell_1}^4}\right)\delta_{jk} - C_\alpha \tfrac{1}{2} \tag{5-2}$$

$$- C_R \frac{\ell(\ell+1)(2\ell+3)\,b}{2\epsilon_{\ell_j}^3\epsilon_{\ell k}^3} + C_\beta \frac{\epsilon_{\ell_j}\epsilon_{\ell k}}{(\epsilon_{\ell_j}^2-\epsilon_{\ell k}^2)}(1-\delta_{jk}) \Big\| = 0$$

where it may be recalled that b depended upon whether the boundary at x = 1 was free or rigid:

(a) $b = 2$ for a rigid boundary (5-3)

(b) $b = -\frac{4}{2\ell+1}$ for a free boundary.

One obtains successive approximations to the characteristic values for C_R by setting the determinant (5-2) to zero for $j=k=1$, $j=k=2$, etc. Setting the (1, 1) and (2,2) to zero gives

$$C_R^{(1)} = \frac{\epsilon_{\ell 1}^8}{\ell(\ell+1)} \frac{\left[1+\frac{1}{\epsilon_{\ell 1}^2}\left(C_\alpha - \frac{9}{2}C_\beta\right)\right]}{\epsilon_{\ell 1}^2 - b(2\ell+3)} \tag{5-4}$$

and

$$C_R^{(1)} = \frac{1}{2Q_0}\left[Q_1 \pm \sqrt{Q_1^2 - 4Q_0Q_2}\right] \tag{5-5}$$

where

(a) $Q_0 = \epsilon_{\ell 1}^2\epsilon_{\ell 2}^2 - (2\ell+3)b\,(\epsilon_{\ell 1}^2+\epsilon_{\ell 2}^2)$ (5-6)

(b) $$Q_1 = \epsilon_{\ell 1}^3\epsilon_{\ell 2}^3\Big\{\left[\epsilon_{\ell 1}^2 + C_\alpha - \tfrac{9}{2}C_\beta\right]\left[\epsilon_{\ell 2}^2 - (2\ell+3)b\right]\epsilon_{\ell 1}^6$$
$$+ \left[\epsilon_{\ell 2}^2 + C_\alpha - \tfrac{9}{2}C_\beta\right]\left[\epsilon_{\ell 1}^2 - (2\ell+3)b\right]\epsilon_{\ell 2}^2 + 2C_\alpha(2\ell+3)b\Big\}$$

(c) $$Q_2 = \epsilon_{\ell 1}^2\epsilon_{\ell 2}^2\Big\{\left[\epsilon_{\ell 1}^2 + C_\alpha - \tfrac{9}{2}C_\beta\right]\left[\epsilon_{\ell 2}^2 + C_\alpha - \tfrac{9}{2}C_\beta\right] + 4C_\beta^2\frac{\epsilon_{\ell 1}^2\epsilon_{\ell 2}^2}{\epsilon_{\ell 1}^2-\epsilon_{\ell 2}^2}$$
$$- C_\alpha^2\Big\}.$$

Using (5-4) and (5-5) one can calculate the first and second approximations to C_R for a given set of parameters $\rho_0, \alpha, \lambda_0, \beta$ and R. For our lunar model we have $\rho_0 = 3.34$ g/cm^3 and $R = 1.78 \times 10^8$ cm and for average values for silicate rocks, $\alpha = 2.0 \times 10^5$ /c and $\beta = 1.0 \times 10^{-12}$ cm^2/dyne. The value for λ_0 is very poorly known for the Moon; however, assuming a radioactive solute distribution similar to the Earth, gives a value from the "Wasserburg" model (cf. Wasserburg, G.J., et.al. (1964)) of $\lambda_0 = 3 \times 10^{-7}$ erg/g sec. One then obtains $C_\alpha = 3.17$, $C_\beta = 0.098$ and the results of these calculations for varying ℓ values are given in Table I. The first and second approximations are seen to differ by about 1 part in 100 for a rigid boundary at x = 1 and by about 1 part in 1000 for the free boundary case. Since the successive values of C_R seem to converge rapidly, we will use only the first approximation $C_R^{(1)}$ in subsequent calculations.

Table II gives the values of C_R when the fluid is assumed incompressible, so that $\beta = 0$, and hence $C_\beta = 0$, and the energy sources are so weak that $C_\alpha \simeq 0$. Comparison of Tables I and II indicates that non-zero C_α and C_β increase C_R by about a factor of two for $\ell = 1$ and have virtually no effect for $\ell > 10$. As ℓ increases the differences between C_R for free and rigid surfaces decreases. Both the above effects are to be expected, for as increases, the convection cell diameter is decreasing as $1/\ell$ so the fluid in a given region begins to be governed by motions of smaller and smaller scale length and does not "see" the boundaries or the larger scale variations in the fluid properties.

For our fluid system the Rayleigh number is given as

$$C_R = \frac{4\pi G \rho_0^4 \alpha C_V \lambda_0 R^6}{9 K^2 \mu} . \qquad (5\text{-}7)$$

Using the values of $\rho_0, R, \alpha, \lambda_0$ for our lunar model, along with $C_V = 7.0 \times 10^6$ erg/g^{0}c, $K = 2.0 \times 10^5$ erg/cm^{0}c sec and $\mu = 10^{22}$ g/cm sec.

$$C_R = 4.0 \times 10^8 . \qquad (5\text{-}8)$$

This corresponds to a theoretical mode value of $\ell > 25$. If the above conclusion is correct, then the resulting convection pattern in the lunar interior is considerably broken up with no possibility for large cells corresponding to low ℓ values to exist. One can argue that the values for λ_0 and μ used to calculate C_R from (5-7) are not well known. If the value of μ were increased to the order of 10^{26}, then $C_R \sim 4 \times 10^4$ and would correspond to $\ell = 6$ which is quite acceptable in terms of a "lunar" tectonic theory of convection. However, such an increase in μ will yield a corresponding increase in the Maxwellain relaxation time, $\tau = \mu\beta \sim 10^7$ years. In Chapter I, the condition for a viscous globe to respond hydrodynamically was $t >> \tau$ and for such a large τ we would not expect that the lunar interior would have had sufficient time over its life time of about 10^9 years to reach a stationary convective state.

One could also argue that decreasing λ_0 by about two or possibly three orders of magnitude along with a more acceptable value of $\mu \sim 10^{23}$ would again yield a value of C_R in the proper range. However, a decrease in λ_0 of more than about one and a half orders of magnitude will result in a corresponding decrease in the thermal gradient and for $\lambda_0 < 10^{9}$, the superadiabatic gradient necessary for convection will vanish altogether. In the considerations of convection in the Earth's mantle that will follow, exactly similar difficulties will be seen to arise on comparison of theoretical and computed Rayleigh numbers.

We now wish to apply for stationary convection in spherical shells to a model of the Earth's mantle. Consider a layer of homogeneous fluid overlaying a core of radius $r_{core} = aR$ with an average density $\bar{\rho}_a$ such that $M_{core} = \frac{4}{3}\pi \bar{\rho}_a (aR)^3$

As discussed in Chapter IV, $\zeta(x)$ representing the gravitational field, cannot strictly speaking be unity as in the case of entire homogeneous spheres. Instead, we have from Chapter IV,

$$\zeta(x) = 1 + \left(\frac{\bar{\rho}_a}{\rho_0} - 1\right)\frac{a^3}{x^3} \qquad (5\text{-}9)$$

To evaluate the matrix elements M_{kj}, N_{kj} and Q_{kj} in the secular determinant

$$\Big\| M_{kj} + C_\beta N_{kj} - C_\alpha Q_{kj} + C_R \ell(\ell+1)\left(\frac{\delta_{kj}}{q_{\ell j}^4} P_{\ell+1/2,j} + R_{kj}\right)\Big\| = 0 \qquad (5\text{-}10)$$

we need to use (5-9) as well as specific information about the energy source distribution function $\lambda(r)$. Rather than assume some particularly realistic form for $\lambda(r)$, let $\lambda(r) = \lambda_0 = \text{constant}$ and then we can set $\xi(x) = 1$. The material parameters for the model mantle will be average values for silicate rocks so that

$$\alpha = 2.0 \times 10^{-5} \tag{5-11}$$

$$\beta = 1.0 \times 10^{-12}$$

$$C_v = 7.0 \times 10^{6} \quad \text{in cgs units.}$$

The average density will be $\rho_0 = 4.1$ g/cm and the energy source distribution function is the same as previously used for the lunar interior calculations; $\lambda_0 = 3 \times 10 \ \frac{\text{erg}}{\text{g sec}}$. The radius of the Earth is R = 6.371 x 10 cm and the computed values of C_α, C_R and C_β are:

$$C_\alpha = 50.0 \tag{5-12}$$

$$C_\beta = 2.00$$

$$C_R = 1.8 \times 10^{12}.$$

If we assume momentarily, for computational facility, that $\zeta = 1$, then the (1, 1) determinant from (5-10) gives the first approximation to C_R:

$$C_R^{(1)} = \frac{q_{\ell 1}^6}{\ell(\ell+1)} \frac{\left[1 + \frac{1}{q_{\ell 1}^2}\left(C_\alpha + C_\beta(\ell+1) - 6C_\beta\right)\right] P_{\ell+1/2,1}}{P_{\ell+1/2,1} + q_{\ell 1}^4 R_{11}} \tag{5-13}$$

where $\{q_{\ell 1}\}$ are the first zeros of $J_{-(\ell+1/2)}(q_{\ell 1} a) J_{\ell+1/2}(q_{\ell 1}) - J_{\ell+1/2}(q_{\ell 1} a) J_{-(\ell+1/2)}(q_{\ell 1}) = 0$ and are given in Table III along with corresponding values of

$P_{\ell+1/2,1}$. The value of R_{11} depends upon the specific nature of the boundary conditions and are found on the last page of Chapter IV. Using (5-12) and (5-13), $C_R^{(1)}$ is then calculated for varying ℓ values and mantle thicknesses under the four possible types of boundary conditions with the results given in Tables IV--VII.

To examine the effect of gravitational variation, consider an average gravity function $\bar{\zeta}$ given as

$$\bar{\zeta} = \frac{1}{1-a}\int_a^1 dx\, \zeta(x) = 1 + \frac{1}{2}\left(\frac{\bar{\rho}_a}{\rho_0} - 1\right) a(1+a) \qquad (5\text{-}14)$$

where $\bar{\rho}_a$ and ρ_0 are the average core and mantle densities respectively. An examination of the integrals representing M_{Kj} , N_{Kj} and O_{Kj} along with the specific form of $\zeta(x)$ given in (5-9) indicates that exact evaluation will not be at all straightforward. Instead, let us approximate the effect of a non-unity value of $\zeta(x)$ by taking $\zeta(x)$ to be its average value in the mantle. For the Earth, ρ_0 = 4.1, and $\bar{\rho}_a$ = 12.5 and a = .55, so that the variation in $\zeta(x)$ is:

$$\zeta(x=1) = 1.340 \qquad (5\text{-}15)$$

$$\zeta(x=a) = 3.050$$

The average value is $\bar{\zeta}$ = 1.873, so the approximation $\zeta(x) \simeq \bar{\zeta}$ is fairly reasonable and $C_R^{(1)}$ is:

$$C_R^{(1)} = \frac{q_{\ell 1}^6}{\ell(\ell+1)} \frac{\left[1 + \frac{1}{q_{\ell 1}^2}\left(C_\alpha - 3C_\beta - 3\bar{\zeta} C_\rho + \bar{\zeta}\rho(\ell+1)\right)\right] P_{\ell+1/2,1}}{P_{\ell+1/2,1} + q_{\ell 1}^4 R_{11}} \qquad (5\text{-}16)$$

Computed values of $C_R^{(1)}$ using (5-16) are given in Table VIII for mantle thicknesses corresponding to a = 0.5 and a = 0.6.

Finally, Table IX gives theortical values of $C_R^{(1)}$ under the assumption of incompressibility and very weak energy sources, i.e. $C_\alpha = C_\beta = 0$. For comparison with Tables IV through VIII, values of $C_R^{(1)}$ in Table IX are computed for both cases of

$\zeta(x) = 1$ and $\zeta(x) = \bar{\zeta} = 1.873$.

The differences between values of $C_R^{(1)}$ given in Table IX and those in Tables IV and VII are due to a term $\frac{1}{q_{\ell 1}^2}\left[C_\alpha + C_\beta(\ell+1) - 6C_\beta\right]$ and since $C_\beta = 2.00$ and $q_{11} = 8.0553$, any compressibility effects are very small for all but the lowest few ℓ values. As expected, the allowance for gravity variation in the mantle with $\zeta \neq 1$ resulted in an overall decrease in C_R, or equivalently, a given value of C_R corresponds to a higher ℓ value hence smaller convection cells. Under all conditions it is seen that narrowing the mantle shifts the convection to higher and higher harmonics as the fluid finds it increasingly difficult to form large diameter low ℓ-value convection cells.

The most important result is, however, that Rayleigh numbers for an Earth model with mantle thickness of a = 0.5 to a = 0.6 fall short of the computed value by several orders of magnitude. In (5-12) we had $C_R = 1.8 \times 10^{12}$ and for the rigid-rigid case (rigid boundaries at x =a and x = 1), this corresponds to $\ell > 30$, which is very unsatisfactory from any global tectonics viewpoint. For C_R to fall in the proper range of ℓ ($\ell \sim 3-6$) we need to reduce the result in (5-12) by about 10^7. To accomplish this one needs to increase μ to about 10^{26} and decrease λ_0 to the order of 10^{-10}. With such values the Maxwellian relaxation time would have increased to the point where a stationary convection pattern would not have time to develop over the Earth's lifetime, and the ambient thermal gradient might very well not exceed the adiabatic gradient and convection would be prohibited altogether. Considering the above discussions, one is faced with the conclusion that the linear theory may be inadequate to deal with problems at hand. In Chapter II it was pointed out that linearization of the entropy transport equation by eliminating the viscous dissipation term as a quadratic in $\underset{\sim}{u}'$ could have serious effects. The linearization was necessary to obtain a tractable mathematical problem, but it appears to have limited the application of the theory to physical situations where the viscosity coefficient is much smaller than for planetary mantles and interior.

If we consider an order of magnitude analysis of the problem, we find that equating the gravitational and viscous flow terms in the equation of motion (which should be of the same order of magnitude),

$$\mathcal{O}(g\rho'/\rho_0) = \mathcal{O}\left(\frac{\mu}{\rho_0}\nabla^2 u'\right) = \mathcal{O}\left(\frac{\mu}{\rho_0}\frac{u'}{R^2}\right). \qquad (5\text{-}17)$$

From geophysical considerations of continental drift, one can estimate that convective velocities in the Earth's mantle would be like $u' \sim 1\ \mathrm{cm/yr} \sim 10^{-7}\ \mathrm{cm/sec}$. Using this as an approximate value,

$$\mathcal{O}(\rho'/\rho_0) = \mathcal{O}\left(\frac{\mu}{c_0}\frac{u'}{gR^2}\right) = 10^{-5}. \tag{5-18}$$

Using the above value for $\mathcal{O}(\rho'/\rho_0)$, and the conductive entropy transport term is

$$\mathcal{O}(k\nabla^2 T') = \mathcal{O}\left(\frac{k}{\alpha}\frac{1}{R^2}\left(\frac{\rho'}{c_0}\right)\right) = 10^{-11} \tag{5-19}$$

and viscous dissipation term gives

$$\mathcal{O}\left[\frac{\mu}{2}(\nabla^0 \underset{\sim}{u}')^s : (\nabla^0 \underset{\sim}{u}')^s\right] = \mathcal{O}\left(\mu\frac{u_0'^2}{R^2}\right) = 10^{-\ell}. \tag{5-20}$$

Even these rough calculations show the difficulties with the linear theory. A non-linear development which includes the viscous term in the entropy transport equation does not exist in the literature. This is not surprising in the light of the overwhelming mathematical difficulties.

As an example, consider the simplest possible case where the flow is incompressible, two-dimensional, constant gravity g, and there is an applied thermal graident $\frac{dT_0}{dz} = -\bar{\gamma}$ but no internal energy sources. The system can be uncoupled to give for the vertical component of convective velocity

$$\mu\nabla^6\frac{\partial w}{\partial x} + g\alpha\bar{\gamma}\frac{\partial^2 w}{\partial x^2} + 8\mu\left[\left(\frac{\partial^2 w}{\partial x\partial z}\right)^2 + \frac{\partial w}{\partial z}\frac{\partial^3 w}{\partial x^2\partial z}\right] \tag{5-21}$$

$$-\left[\mu\nabla^6 w + g\alpha\bar{\gamma}w + 8\mu\left(\frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial x\partial z}\right)\right]\frac{1}{f}\frac{\partial f}{\partial x} + 2\mu f^2 = 0$$

and

$$f \equiv \frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w}{\partial z^2}. \tag{5-22}$$

We see that even the simplest case yields a rather unpleasant mathematical problem and as a result, a non-linear theory for convection in plnaetary interiors is yet to be developed.

In this paper the general linear theory of thermal instability in internally heated spheres and spherical shells of viscous, compressible fluid has been developed. The particular case of marginally stable state at the onset of stationary convection for a homogeneous fluid has been studied in detail and the characteristic values for the Rayleigh number under varying conditions have been calculated. These values have been seen to be rather insensitive to the inclusion of compressibility and energy source effects in the case of a fluid having silicate rock material properties.

Most significant was that that theoretical values yielded characteristic values corresponding to very high harmonics, i.e. $\ell > 25$, when applied to models for the lunar interior and terrestrial mantle. One then concludes that either the convection is in the form of very small cells of a few hundred kilometers in diameter for the case of the Earth, which is entirely unsatisfactory for a global tectonic theory, or that the linear theory is inappropriate when considering fluids of such high viscosity. An order of magnitude calculation indicates that for μ greater than about 10^{15}, a non-linear theory which retains the viscous dissipation terms in the entropy transport equation is necessary to adequately formulate the problem. Due to the enormous mathematical difficulties introduced by a non-linear theory, no such theory presently exists in the literature.

Table I. First and second approximations to the characteristic value C_R for entire homogeneous fluid spheres. $C_\alpha = 3.17$; $C_\beta = 0.098$

	Rigid boundary at x=1		Free boundary at x=1	
ℓ	$C_R^{(1)}$	$C_R^{(2)}$	$C_R^{(1)}$	$C_R^{(2)}$
1	1.310×10^4	1.294×10^4	4.975×10^3	4.971×10^3
2	1.142×10^4	1.126×10^4	5.656×10^3	5.653×10^3
3	1.623×10^4	1.598×10^4	9.270×10^3	9.267×10^3
4	2.327×10^4	2.291×10^4	1.456×10^4	1.456×10^4
5	3.279×10^4	3.229×10^4	2.187×10^4	2.1807×10^4
6	4.518×10^4	4.450×10^4	3.159×10^4	3.159×10^4
7	6.095×10^4	6.007×10^4	4.423×10^4	4.422×10^4
8	8.057×10^4	7.944×10^4	6.021×10^4	6.020×10^4
9	1.045×10^5	1.031×10^5	7.999×10^4	7.998×10^4
10	1.333×10^5	1.316×10^5	1.042×10^5	1.041×10^5

Table II. First approximation to the characteristic value C_R in homogeneous fluid spheres in the limit of incompressibility ($C_\beta = 0$) and very weak energy sources ($C_\alpha = 0$).

ℓ	Rigid boundary at x=1 $C_R(1)$	Free boundary at x=1 $C_R(1)$
1	8.154×10^3	3.094×10^3
2	1.056×10^4	5.227×10^4
3	1.537×10^4	8.779×10^3
4	2.235×10^4	1.399×10^4
5	3.180×10^4	2.121×10^4
6	4.412×10^4	3.085×10^4
7	5.976×10^4	4.336×10^4
8	7.922×10^4	5.920×10^4
9	1.030×10^5	7.889×10^4
10	1.318×10^5	1.029×10^5

[Ref. Chandrasekhar, "Hydrodynamic and Hydromagnetic Stability," Chap. 4 (1961)]

Table III. The first zeros of $\mathcal{C}_{\ell+\frac{1}{2},\ \ell+\frac{1}{2}}(z)$ and the corresponding values of $P_{\ell+\frac{1}{2},\ \ell}$.

ℓ	z_1 (a = 0.2)	$P_{\ell+\frac{1}{2},\ \ell}$ (a = 0.2)	z_1 (a = 0.3)	$P_{\ell+\frac{1}{2},\ \ell}$ (a = 0.3)
1	4.68640	0.0851258	5.04273	0.0287565
2	5.79966	0.219152	5.96125	0.0433670
3	6.99345	0.827033	7.05135	0.0804682
4	8.18329	4.07722	8.20089	0.180621
5	9.35590	23.9880	9.36068	0.475005
6			10.5140	1.412574
	a = 0.4		a = 0.5	
1	5.63897	0.0121094	6.57201	0.00532874
2	6.35745	0.0144789	7.11158	0.00574651
3	7.28038	0.0190172	7.84504	0.00644475
4	8.31928	0.0275034	8.71680	0.00752772
5	9.41654	0.0437534	9.68200	0.00917101
6	10.5385	0.0761334	10.70769	0.0116670
7	11.6674	0.1435507	11.7708	0.0155047
8			12.8557	0.0215105
9			13.9521	0.0311069
10			15.0533	0.0467671
11			16.1554	0.0728515
12			17.2560	0.117113
13			18.3542	0.193877
14			19.4492	0.329234
15			20.5410	0.573870
	a = 0.6		a = 0.8	
1	8.0553	0.00222241	15.7868	0.000205553
2	8.4428	0.00228887	15.9431	0.000206046
3	8.9913	0.00239297	16.1749	0.000206763
4	9.6717	0.00254054	16.4787	0.000207743
5	10.4563	0.00274015	16.8506	0.000208977
6	11.3210	0.00300396	17.2865	0.000210453
7	12.2466	0.00334878	17.7812	0.000212194
8	13.2177	0.00379840	18.3301	0.000214217
9	14.2225	0.00438496	18.9284	0.000216512
10	15.2518	0.00515454	19.5715	0.000219102
11	16.2988	0.00617030	20.2550	0.000221731
12	17.3581	0.00752205	20.9749	0.000225205
13	18.4258	0.00933658	21.7274	0.000228749
14	19.4988	0.0117918	22.5094	0.000232640
15	20.5749	0.0151818	23.3176	0.000236902

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Table IV. First approximation to the characteristic value C_R for rigid boundaries at $x = a$ and $x = 1$. $C_\alpha = 50.0$; $C_\beta = 2.00$.

ℓ	a = 0.2	a = 0.4	a = 0.6	a = 0.8
1	5.16×10^4	1.58×10^5	1.10×10^6	4.70×10^7
2	2.95×10^4	7.01×10^4	4.16×10^5	1.62×10^7
3	3.09×10^4	5.22×10^4	2.47×10^5	8.48×10^6
4	3.86×10^4	5.06×10^4	1.84×10^5	5.39×10^6
5	5.00×10^4	5.68×10^4	1.58×10^5	3.85×10^6
6		6.85×10^4	1.49×10^5	2.98×10^6
7		8.54×10^4	1.52×10^5	2.44×10^6
8			1.62×10^5	2.10×10^6
9			1.80×10^5	1.86×10^6
10			2.05×10^5	1.71×10^6
11			2.37×10^5	1.61×10^6
12			2.76×10^5	1.55×10^6
13			3.23×10^5	1.51×10^6
14			3.80×10^5	1.51×10^6
15			4.45×10^5	1.52×10^6

Table V. First approximation to the characteristic value C_R for free boundaries at x = a and x = 1. $C_\alpha = 50.0$; $C_\beta = 2.00$.

ℓ	a = 0.2	a = 0.4	a = 0.6	a = 0.8
1	1.52×10^3	3.90×10^4	2.31×10^5	9.10×10^6
2	1.32×10^3	2.28×10^4	9.91×10^4	3.23×10^6
3	1.72×10^3	2.23×10^4	6.91×10^4	1.76×10^6
4	2.40×10^4	2.68×10^4	6.17×10^4	1.18×10^6
5	3.33×10^4	2.48×10^4	6.28×10^4	9.00×10^5
6		4.61×10^4	6.95×10^4	7.48×10^5
7		6.10×10^4	8.07×10^4	6.62×10^5
8			9.63×10^4	6.15×10^5
9			1.17×10^5	5.94×10^5
10			1.42×10^5	5.90×10^5
11			1.73×10^5	6.01×10^5
12			2.10×10^5	6.24×10^5
13			2.54×10^5	6.57×10^5
14			3.05×10^5	7.01×10^5
15			3.64×10^5	7.54×10^5

Table VI. First approximation to the characteristic value C_R for a free boundary at x = a and a rigid boundary at x = 1. $C_\alpha = 50.0$; $C_\beta = 2.00$

ℓ	a = 0.2	a = 0.4	a = 0.6	a = 0.8
1	4.19×10^4	1.03×10^5	6.24×10^5	2.42×10^7
2	2.71×10^4	5.07×10^4	2.50×10^5	8.42×10^6
3	3.03×10^4	4.22×10^4	1.55×10^5	4.45×10^6
4	3.84×10^4	4.48×10^4	1.23×10^5	2.89×10^6
5	5.00×10^4	5.34×10^4	1.13×10^5	2.11×10^6
6		6.65×10^4	1.15×10^5	1.67×10^6
7		8.43×10^4	1.24×10^5	1.41×10^6
8			1.39×10^5	1.24×10^6
9			1.61×10^5	1.14×10^6
10			1.89×10^5	1.08×10^6
11			2.23×10^5	1.05×10^6
12			2.65×10^5	1.04×10^6
13			3.15×10^5	1.05×10^6
14			3.73×10^5	1.05×10^6
15			4.39×10^5	1.12×10^6

Table VII. First approximation to the characteristic value C_R for a rigid boundary x = a and a free boundary at x = 1. $C_\alpha = 50.0$; $C_\beta = 2.00$

ℓ	a=0.2	a=0.4	a=0.6	a=0.8
1	1.90×10^4	6.62×10^4	4.81×10^5	2.16×10^7
2	1.43×10^4	3.28×10^4	1.90×10^5	7.50×10^6
3	1.76×10^4	2.78×10^4	1.20×10^5	3.98×10^6
4	2.41×10^4	3.01×10^4	9.57×10^4	2.58×10^6
5	3.33×10^4	3.68×10^4	8.84×10^4	1.88×10^6
6		4.73×10^4	9.01×10^4	1.49×10^6
7		6.16×10^4	9.78×10^4	1.26×10^6
8			1.11×10^5	1.11×10^6
9			1.29×10^5	1.02×10^6
10			1.52×10^5	9.67×10^5
11			1.82×10^5	9.41×10^5
12			2.17×10^5	9.34×10^5
13			2.60×10^5	9.46×10^5
14			3.10×10^5	9.71×10^5
15			3.69×10^5	1.01×10^6

Table VIII. First approximation of characteristic value C_R with an average gravity variation $\bar{\zeta} = 1.873$. $C_\alpha = 50.0$; $C_\beta = 2.00$. Rigid boundary at x = a

	Rigid boundaries at x = 1		Free boundaries at x = 1	
ℓ	a = 0.5	a = 0.6	a = 0.5	a = 0.6
1	1.83×10^5	5.78×10^5	7.83×10^4	2.52×10^5
2	7.91×10^4	2.22×10^5	3.64×10^4	1.01×10^5
3	5.27×10^4	1.34×10^5	2.66×10^4	6.33×10^4
4	4.47×10^4	1.01×10^5	2.49×10^4	5.24×10^4
5	4.39×10^4	8.70×10^4	2.68×10^4	4.88×10^4
6	4.76×10^4	8.28×10^4	3.14×10^4	5.00×10^4
7	5.47×10^4	8.44×10^4	3.82×10^4	5.44×10^4
8	6.49×10^4	9.05×10^4	4.75×10^4	6.18×10^4
9	7.87×10^4	1.01×10^5	5.95×10^4	7.20×10^4
10	9.57×10^4	1.15×10^5	7.44×10^4	8.52×10^4
11	1.17×10^5	1.32×10^5	9.24×10^4	1.02×10^5
12	1.41×10^5	1.54×10^5	1.14×10^5	1.22×10^5
13	1.70×10^5	1.81×10^5	1.39×10^5	1.45×10^5
14	2.03×10^5	2.12×10^5	1.68×10^5	1.73×10^5
15	2.42×10^5	2.49×10^5	2.02×10^5	2.06×10^5

Table IX. First approximation of characteristic value C_R for a = 0.6 in the limit of incompressibility (C_β = 0) and very weak energy sources (C_α = 0). Rigid boundary at x = a:

	Rigid boundary x = 1		Free boundary at x = 1	
ℓ	ξ=1.000	$\bar{\xi}$=1.873	ξ=1.000	$\bar{\xi}$=1.873
1	6.68×10^5	3.57×10^5	2.92×10^5	1.56×10^5
2	2.57×10^5	1.37×10^5	1.18×10^5	6.27×10^4
3	1.58×10^5	8.41×10^4	7.64×10^4	4.08×10^4
4	1.22×10^5	6.49×10^4	6.33×10^4	3.38×10^4
5	1.08×10^5	5.78×10^4	6.07×10^4	3.24×10^4
6	1.06×10^5	5.66×10^4	6.40×10^4	3.42×10^4
7	1.12×10^5	5.95×10^4	7.19×10^4	3.84×10^4
8	1.23×10^5	6.56×10^4	8.39×10^4	4.48×10^4
9	1.40×10^5	7.47×10^4	1.00×10^5	5.35×10^4
10	1.63×10^5	8.69×10^4	1.21×10^5	6.47×10^4
11	1.92×10^5	1.02×10^5	1.47×10^5	7.87×10^4
12	2.28×10^5	1.19×10^5	1.79×10^5	9.58×10^4
13	2.71×10^5	1.45×10^5	2.18×10^5	1.16×10^5
14	3.22×10^5	1.72×10^5	2.63×10^5	1.40×10^5
15	3.82×10^5	2.04×10^5	3.17×10^5	1.69×10^5

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APPENDIX A

PROPER DEVELOPMENT OF THE EQUATION OF MOTION

We know that Newton's second law will require that for a fluid

$$\rho \frac{d\underset{\sim}{v}}{dt} = -\nabla \cdot \underset{\approx}{P} + \underset{\sim}{f} \qquad \text{(A-1)}$$

where $\underset{\approx}{P}$ is the pressure tensor and $\underset{\sim}{f}$ the body forces acting on the fluid element.

Let us consider the pressure tensor for a moment. The components of the tensor $\underset{\approx}{\sigma}$ $\underset{\approx}{P}$ are force per unit area, so it is appropriately termed the pressure tensor; this tensor is sometimes used instead and is called the stress tensor.

Pressure forces like Pxydzdy act on the surface of a fluid particle, and are to be distinquished (as we have done above) from body forces like gravity and electrical forces which act on the entire volume of the fluid element. Body forces are clearly proportional to size, $d\tau$ = dxdydz, of the fluid particle and consequently, their moment about any point in the fluid particle is fourth order in the infinitesimals dx, dy and dz. Thus, if moments are taken about $o\hat{z}$ of the forces acting on the planes parallel to $o\hat{x}$ there results

$$0 = (P_{xy}\, dz\, dy)\tfrac{1}{2}dx - (P_{yx}\, dx\, dz)\tfrac{1}{2}dy + 4^{th} \text{ order terms.} \qquad \text{(A-2)}$$

Then we see that ignoring fourth order terms in the infinitesimals gives Pxy = Pyx. Similar arguments follow for Pyz = Pzy, Pxz = Pzx, so the pressure tensor $\underset{\approx}{P}$ is symmetric.

Now in the case where the fluid velocity is zero, hydrostatics, we know that the normal pressure acting on a small fluid element is independent of the orientation of the surface on which it acts. In this case,

$$\underset{\approx}{P} = p\,\underset{\approx}{I} \tag{A-3}$$

where p = Pxx = Pyy = Pzz. Then we can conveniently write

$$\underset{\approx}{P} = p\,\underset{\approx}{I} + \underset{\approx}{\Pi} \tag{A-4}$$

where $\underset{\approx}{\Pi}$ is termed the <u>viscous stress tensor</u>.

If $\hat{n}$ denotes the outward unit normal to a closed surface lying in the fluid, the force acting on an element dS of S is $-\underset{\approx}{P}\cdot\hat{n}\,dS$ or $-\hat{n}\cdot\underset{\approx}{P}\,dS$ by symmetry of $\underset{\approx}{P}$. Therefore the total force F exerted by the fluid outside S on that lying within S is

$$\underset{\sim}{F} = -\int_S \hat{n}\cdot\underset{\approx}{P}\,dS = -\int_V \nabla\cdot\underset{\approx}{P}\,d\tau \tag{A-5}$$

from Gauss' theorem. If this volume shrinks to a fluid particle $d\tau$ then as the change in $\nabla\cdot\underset{\approx}{P}$ across $d\tau$ can be neglected, (5) reduces to $-\nabla\cdot\underset{\approx}{P}\,d\tau$.

If the shape of the fluid particle P remains fixed, so that all parts move with the same velocity v, the rate of doing work on P is the inner product of this velocity with the force itself, i.e., $-\underset{\sim}{v}\cdot(\nabla\cdot\underset{\approx}{P})\,d\tau$. This is the rate of doing work against the pressure and tangential stresses opposing the <u>rigid</u>

body displacement of P. The total rate at which the fluid outside P does work on P will contain not only the term for rigid body displacement, but also a term giving the rate at which work is done to change the shape and volume of P. Now the inwards force on an element $\hat{n}dS$ of a closed surface S does work against the efflux of fluid from the fixed volume V, through dS, at a rate $(-\hat{n}\cdot\underset{\approx}{P}\, dS)\cdot\underset{\sim}{v}$. Hence the rate at which the fluid outside the stationary volume V does work on the fluid within V is

$$-\frac{\delta W_{\bullet}}{\delta t} = -\int_S \hat{n}\cdot\underset{\approx}{P}\cdot\underset{\sim}{v}\, dS = -\int_V \nabla\cdot(\underset{\approx}{P}\cdot\underset{\sim}{v})\, d\tau \qquad \text{(A-6)}$$

on again applying Gauss' theorem. For an elemental volume $d\tau$ at a fixed point $\underset{\sim}{x}$ it follows from (6) that this rate is

$$-\left(\frac{\delta W_{\bullet}}{\delta t}\right)_{\underset{\sim}{x}} = -\nabla\cdot(\underset{\approx}{P}\cdot\underset{\sim}{v})\, d\tau \quad . \qquad \text{(A-7)}$$

Now using cartesian tensor notation we see that

$$\nabla\cdot(\underset{\approx}{P}\cdot\underset{\sim}{v}) = \frac{\partial}{\partial x_i}(P_{ij}v_j) = \frac{\partial P_{ij}}{\partial x_i}v_j + P_{ij}\frac{\partial v_j}{\partial x_i} \qquad \text{(A-8)}$$

$$= \underset{\sim}{v}\cdot(\nabla\cdot\underset{\approx}{P}) + \underset{\approx}{P} : \nabla\underset{\sim}{v}$$

where $:$ refers to the "double inner product" or scaler product for second rank tensors; $\sum_i\sum_j A_{ij}B_{ij} = \underset{\approx}{A} : \underset{\approx}{B}$.
Then (A-7) is

$$-\left(\frac{\delta W_{\bullet}}{\delta t}\right)_{\underset{\sim}{x}} = -\underset{\approx}{P} : \nabla\underset{\sim}{v}\, d\tau - \underset{\sim}{v}\cdot(\nabla\cdot\underset{\approx}{P})\, d\tau \ . \qquad \text{(A-9)}$$

We recognize the last term as the rate of doing work on P to move it as a rigid body and the $-\underset{\approx}{P} : \nabla\underset{\sim}{v}\, d\tau$ is the rate of doing work to change its shape and volume. Then if we move in a frame of the center at mass of P, the rate of work done is seen to be $\frac{\delta W}{\delta t}$ (say), then

$$\frac{\delta W}{\delta t} = \underset{\approx}{P} : \nabla \underset{\sim}{v} \, d\tau = p \nabla \cdot \underset{\sim}{v} \, d\tau + \underset{\approx}{\Pi} : \nabla \underset{\sim}{v} \, d\tau \qquad \text{(A-10)}$$

where we have used $\underset{\approx}{P} = p \underset{\approx}{I} + \underset{\approx}{\Pi}$.

Now let $\frac{d}{dt}$ denote the rate of change in a frame fixed in a fluid particle of volume V, then the rate of change of volume of the fluid lying within a volume V at a given instant is the integral over V of $\frac{d}{dt}(d\tau')$. The volume of fluid passing through a surface element $\hat{n} dS$ of V is $\underset{\sim}{v} \cdot \hat{n} \, dS$ units per second, and so the integral of this quantity over the surface of V is another measure of the rate at which fluid within V is expanding. Hence,

$$\int_V \frac{d}{dt}(d\tau') = \int_S \hat{n} \cdot \underset{\sim}{v} \, dS = \int_V \nabla \cdot \underset{\sim}{v} \, d\tau \qquad \text{(A-11)}$$

then

$$\int_V \left[\frac{d}{dt}(d\tau') - \nabla \cdot \underset{\sim}{v} \, d\tau \right] = 0 \qquad \text{(A-12)}$$

and as this must apply to all the subregions into which V can be divided, it follows that if all subregions are compact subspaces then the integrand must be zero and

$$\frac{d}{dt}(d\tau') = \nabla \cdot \underset{\sim}{v} \, d\tau \, . \qquad \text{(A-13)}$$

Now write $d\tau' = V$ and using (10) and (13) with the first law of thermodynamics,

$$\delta Q = dU + \delta W, \quad \text{we have} \qquad \text{(A-14)}$$

$$\frac{\delta Q}{\delta t} = \frac{dU}{dt} + p\frac{dV}{dt} + \underset{\approx}{\Pi} : \nabla \underset{\sim}{v}\, V. \tag{A-15}$$

From the second law of thermodynamics, we recognize that

$$T\frac{d\mathcal{S}}{dt} = \frac{\delta Q}{\delta t} + \frac{\delta W_i}{\delta t} \tag{A-16}$$

where $\mathcal{S}$ is the entropy of the volume V. Then we see that since

$$T\frac{d\mathcal{S}}{dt} = \frac{dU}{dt} + p\frac{dV}{dt} \tag{A-17}$$

where $\frac{\delta W_i}{\delta t}$ is the rate of change of irreversible work, then

$$\frac{\delta W_i}{\delta t} = -\underset{\approx}{\Pi} : \nabla \underset{\sim}{v} \equiv \Phi \tag{A-18}$$

where Φ is known as the dissipation function; it is the rate at which energy is being dissipated by viscosity.

As $\underset{\approx}{P}$ is a symmetric tensor, this is also true of the tensor $\underset{\approx}{\Pi} = \underset{\approx}{P} - p\underset{\approx}{I}$. Any tensor $\underset{\approx}{A}$ can be expressed uniquely as the sum of a symmetric and antisymmetric tensor, that is

$$\begin{aligned} A_{ij} &= \tfrac{1}{2}(A_{ij} + A_{ji}) + \tfrac{1}{2}(A_{ij} - A_{ji}) \\ &= (\underset{\approx}{A}^s)_{ij} + (\underset{\approx}{A}^a)_{ij} \quad (\text{say}). \end{aligned} \tag{A-19}$$

We can also see that

$$\underset{\approx}{A}^s : \underset{\approx}{B}^a = 0 \quad \text{for any tensor } \underset{\approx}{A} \text{ and } \underset{\approx}{B}. \tag{A-20}$$

Then we see that

$$\Phi = -\underset{\approx}{\Pi} : \nabla\underset{\sim}{v} = -\underset{\approx}{\Pi} : (\nabla\underset{\sim}{v})^s. \tag{A-21}$$

We can understand this thermodynamically as $(\nabla \underset{\sim}{v})^a$ corresponds to rigid body rotation and so has no bearing on its thermodynamics. We state that $(\nabla \underset{\sim}{v})^s$ can be divided into tensile and shear components of rates of strain as follows. The trace of a tensor $\underset{\approx}{A}$ is defined as

$$A^t \equiv \underset{\approx}{I} : \underset{\approx}{A} \quad . \tag{A-22}$$

In particular, we see that

$$[(\nabla \underset{\sim}{v})^s]^t = \underset{\approx}{I} : (\nabla \underset{\sim}{v})^s = \nabla \cdot \underset{\sim}{v} \, . \tag{A-23}$$

That this is purely the tensile component of the rate of shear is seen from (A-13), which shows it corresponds to just a scaler change in volume. Now the trace of $\underset{\approx}{I}$ is 3 so $(\nabla \underset{\sim}{v})^s$ can be separated as

$$(\nabla \underset{\sim}{v})^s = \tfrac{1}{3}(\nabla \cdot \underset{\sim}{v}) \underset{\approx}{I} + (\overset{\circ}{\nabla} \underset{\sim}{v})^s \tag{A-24}$$

where

$$(\overset{\circ}{\nabla} \underset{\sim}{v})^s_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) - \frac{1}{3} \nabla \cdot \underset{\sim}{v} \, \delta_{ij} \tag{A-25}$$

clearly has zero trace. Adapting a similar expansion for $\underset{\approx}{\pi}$ we see that

$$\underset{\approx}{\pi} = \tfrac{1}{3} \pi^t \underset{\approx}{I} + \overset{\circ}{\underset{\approx}{\pi}} \tag{A-26}$$

and then the dissipation function is

$$\Phi = -\tfrac{1}{3} \pi^t (\nabla \cdot \underset{\sim}{v}) - \overset{\circ}{\underset{\approx}{\pi}} : (\overset{\circ}{\nabla} \underset{\sim}{v})^s \, . \tag{A-27}$$

Now we can write $\frac{\delta w_i}{\delta t}$ as

$$\frac{\delta w_i}{\delta t} = T\sigma' \tag{A-28}$$

where σ' is the rate of entropy production per unit volume (a complete derivation of this form is provided in Appendix B). Then

$$T\sigma' = \Phi = -\frac{1}{3}\pi^t(\nabla\cdot\underset{\sim}{v}) - \overset{\circ}{\underset{\approx}{\pi}} : (\overset{\circ}{\nabla}\underset{\sim}{v})^s . \tag{A-29}$$

Now if we write $\sigma' = \sigma_0 + \sigma_2$ (the reason for which will be explained), then

$$T(\sigma_0 + \sigma_2) = -\frac{1}{3}\pi^t(\nabla\cdot\underset{\sim}{v}) - \overset{\circ}{\underset{\approx}{\pi}} : (\overset{\circ}{\nabla}\underset{\sim}{v})^s . \tag{A-30}$$

We see that first term on the right is a product of zero-order tensors while the second term is a product of second order tensors. The point of the use of σ_0 and σ_2 is that separation into

$$T\sigma_0 = -\frac{1}{3}\pi^t(\nabla\cdot\underset{\sim}{v}) \quad ; \quad T\sigma_2 = -\overset{\circ}{\underset{\approx}{\pi}} : (\overset{\circ}{\nabla}\underset{\sim}{v})^s \tag{A-31, a\&b}$$

is derived from the fact, known as the "Curie principle", that in an isotropic medium only fluxes and thermodynamic forces of the same tensorial character couple. Otherwise, because the Cartesian components of tensors transform differently depending upon their tensorial order, the symmetry properties of the medium will not appear to be preserved under rotations and reflections of the coordinate system. An important consequence of the fact that in an isotropic medium there is no coupling between fluxes and thermodynamic forces of different tensorial order, is that each term, σ_0, σ_2 is separately positive definite since σ' is positive definite.

This assumption of isotropy of the fluid in its thermodynamic properties, leads to the conclusion that the correct form for the linear phenomenological equations is

$$\frac{1}{3}\pi^{t} = -\mu' \nabla\cdot\underset{\sim}{v} \quad ; \quad \overset{\circ}{\underset{\approx}{\pi}} = -2\mu\,(\nabla\overset{\circ}{\underset{\sim}{v}})^{s} \qquad \text{(A-32, a\&b)}$$

where μ', μ are the phenomenological coefficients known as the coefficients of viscosity. The bulk viscosity μ' vanishes identically for a monotomic gas whereas the <u>shear</u> viscosity μ does not vanish for any fluid save those exhibiting superfluidity.

The pressure tensor can now be written as

$$\underset{\approx}{P} = p\underset{\approx}{I} + \underset{\approx}{\pi} = p\underset{\approx}{I} - 2\mu(\nabla\overset{\circ}{\underset{\sim}{v}})^{s} - \mu'(\nabla\cdot\underset{\sim}{v})\underset{\approx}{I} \qquad \text{(A-33)}$$

$$= p\underset{\approx}{I} - 2\mu\,(\nabla\underset{\sim}{v})^{s} - (\mu' - \tfrac{2}{3}\mu)\,\nabla\cdot\underset{\sim}{v}\,\underset{\approx}{I}$$

and since $\sigma_0, \sigma_2 \geq 0$ then $\mu', \mu \geq 0$. Then Φ becomes

$$\Phi = \tfrac{1}{2}\mu\left(\frac{\partial v_i}{\partial x_i} + \frac{\partial v_i}{\partial x_j}\right)^2 + (\mu^2 - \tfrac{2}{3}\mu)(\nabla\cdot\underset{\sim}{v})^2. \qquad \text{(A-34)}$$

Finally we have to consider the form of the divergence of $\underset{\approx}{P}$.

$$\nabla\cdot\underset{\approx}{P} = \nabla p - 2\nabla\mu.(\nabla\underset{\sim}{v})^{s} - 2\mu\,\nabla.(\nabla\underset{\sim}{v})^{s} \qquad \text{(A-35)}$$

$$- \nabla(\mu' - \tfrac{2}{3}\mu).(\nabla\cdot\underset{\sim}{v})\underset{\approx}{I} - (\mu' - \tfrac{2}{3}\mu)\,\nabla.(\nabla\cdot\underset{\sim}{v})\underset{\approx}{I}.$$

Now we can see that

$$2\nabla.(\nabla\underset{\sim}{v})^{s} = \nabla^2\underset{\sim}{v} + \nabla\nabla\cdot\underset{\sim}{v}. \qquad \text{(A-36)}$$

Then finally

$$-\nabla \cdot \underset{\approx}{P} = -\nabla p + \mu \nabla^2 \underset{\sim}{v} + (\mu' + \tfrac{1}{3}\mu) \nabla \nabla \cdot \underset{\sim}{v} \tag{A-37}$$
$$+ 2(\nabla^{\circ} \underset{\sim}{v})^{s} \cdot \nabla \mu + (\nabla \cdot \underset{\sim}{v}) \nabla \mu'$$

and our equation of motion for the fluid particle is

$$\rho \frac{d\underset{\sim}{v}}{dt} = \underset{\sim}{f} - \nabla p + \mu \nabla^2 \underset{\sim}{v} + (\mu' + \tfrac{1}{3}\mu) \nabla \nabla \cdot \underset{\sim}{v} \tag{A-38}$$
$$+ 2(\nabla^{\circ} \underset{\sim}{v})^{s} \cdot \nabla \mu + (\nabla \cdot \underset{\sim}{v}) \nabla \mu'$$

where $\underset{\sim}{f}$ is the body force per unit volume.

Appendix B

THERMODYNAMICS OF A FLUID PARTICLE-- DEVELOPMENT OF THE ENTROPY TRANSPORT EQUATION

In fluid flow the main causes of irreversibility are the phenomena of viscosity and thermal conductivity. We shall consider these cases in some detail.

The second law of thermodynamics can be written in the form

$$dS > \frac{\delta Q}{T} \; ; \quad TdS = dU + pdV - M \sum_i \mu_i dC_i \qquad \text{(B-1, a\&b)}$$

where equality represents reversible processes and > irreversible processes. Also, μ_i = chemical potential of species i of concentration dC_i.

In simple fluid systems the reversible work is just pdV and if δW_i is the irreversible component of work done <u>on</u> the system, the first law of thermodynamics gives

$$\delta Q + \delta W_i = dU + pdV. \qquad \text{(B-2)}$$

Let $d_e S$ be the entropy supplied to the system by its surrounding with the transfer of heat δQ, and let $d_i S$ be the entropy increment due entirely to irreversible processes within the system, then $dS = d_e S + d_i S$ and the second law for reversible processes ($d_i S = 0$) and natural processes can be written in the combined form

$$T d_i S = \delta Q - T d_e S + \delta W_i \; , \; dS \geq 0 . \qquad \text{(B-3)}$$

volume $d\tau$ in unit time due to radiation. Integration over S and V , plus an application of Guass' theorem gives

$$\frac{\delta Q}{\delta t} = -\int_S \hat{n}\cdot \underset{\sim}{Q}\, dS + \int_V Q_r\, d\tau = \int_V (-\nabla\cdot \underset{\sim}{Q} + Q_r)\, d\tau \qquad \text{(B-5)}$$

$$\frac{d_eS}{dt} = -\int_S \frac{\hat{n}\cdot \underset{\sim}{Q}}{T}\, ds + \int_V \frac{Q_r}{T}\, d\tau = \int_V \left[-\nabla\cdot\left(\frac{\underset{\sim}{Q}}{T}\right) + \frac{Q_r}{T}\right] d\tau . \qquad \text{(B-6)}$$

While (B-4) applies only to a thermodynamic system P small enough for T to have a single value for the whole of P, this development not preclude the possibility of a temperature gradient at P. Now

$$\nabla\cdot\left(\frac{\underset{\sim}{Q}}{T}\right) = \frac{1}{T}\nabla\cdot \underset{\sim}{Q} - \frac{\underset{\sim}{Q}}{T^2}\cdot\nabla T \qquad \text{(B-7)}$$

and therefore (B-4) can be written as

$$\sigma = \sigma_1 + (\sigma_0 + \sigma_2) \qquad \text{(B-8)}$$

where

$$\sigma_1 = -\frac{\underset{\sim}{Q}}{T}\cdot\nabla T \quad ; \quad \sigma_0 + \sigma_2 = \frac{1}{T}\,\frac{\delta W_i}{\delta t} . \qquad \text{(B-9, a\&b)}$$

We recognize the last expression from Appendix A which explains the particular form used.

Fourier's law for heat conduction gives an empirical relation between the heat flux vector and the thermal gradient, that is,

$$\underset{\sim}{Q} = -\kappa\nabla T \qquad \text{(B-10)}$$

where κ is the coefficient of heat conduction. While such a linear relationship can be established analytically for gases using kinetic theory, the application to general fluid systems is as a phenomenological relation.

From Appendix A, we saw that the first law of thermodynamics could be written as

$$\frac{\delta q}{\delta t} = \frac{dU}{dt} + p\frac{dV}{dt} - V\Phi = T\frac{dS}{dt} - V\Phi \qquad \text{(B-11)}$$

Now if we let $V = d\tau'$ be the elemental fluid volume, the above equation can be written in several ways. First we must see that the equation of continuity,

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho \underset{\sim}{v}) = 0 \quad \text{means that} \qquad \text{(B-12)}$$

$$\frac{d}{dt}(\rho\, d\tau') = 0 \qquad \text{(B-13)}$$

then each of the variables in (B-11) can be written in terms of specific values, e.g., $\frac{dU}{dt} = \frac{d}{dt}(\rho\, d\tau' u) = \rho\, d\tau' \frac{du}{dt}$ where u is now the specific internal energy. Then from our previous results we have

$$\frac{1}{d\tau'}\frac{\delta Q}{\delta t} = \rho\frac{\delta q}{\delta t} = -\nabla\cdot\underset{\sim}{Q} + Q_r = \nabla\cdot(\kappa\nabla T) + Q_r \qquad \text{(B-14)}$$

and

$$T\frac{ds}{dt} = \frac{du}{dt} + p\frac{d}{dt}\left(\frac{1}{\rho}\right). \qquad \text{(B-15)}$$

Substituting these into (B-11) gives

$$\rho T\frac{ds}{dt} = \nabla\cdot(\kappa\nabla T) + \Phi + Q_r \qquad \text{(B-16)}$$

which is the entropy transport equation.

Now we can eliminate s in terms of T, p, ρ by using various thermodynamic formulae. Now

$$T \frac{ds}{dt} = C_v \frac{dT}{dt} + T \left(\frac{\partial p}{\partial T}\right)_\rho \frac{d}{dt}\left(\frac{1}{\rho}\right) \qquad \text{(B-17)}$$

$$= C_v \frac{dT}{dt} - \frac{T}{\rho^2}\left(\frac{\partial p}{\partial T}\right)_\rho \frac{d\rho}{dt}$$

then (B-16) is, using (B-12),

$$\rho C_v \frac{dT}{dt} = \nabla\cdot(K\nabla T) + \Phi + Q_r - T\left(\frac{\partial p}{\partial T}\right)_\rho \nabla\cdot \underset{\sim}{u} \qquad \text{(B-18)}$$

The above equation is often termed the heat flow equation as is often incorrectly written as

$$\rho C_v \frac{dT}{dt} = \nabla\cdot(K\nabla T) + \Phi + Q_r - p\,\nabla\cdot \underset{\sim}{u} \qquad \text{(B-19)}$$

which is only true as an approximation when considering small perturbation values of p and T.