XVIII. PROCESSING AND TRANSMISSION OF INFORMATION*

A. BLOCK-CODING BOUND FOR COMMUNICATION ON AN INCOHERENT WHITE GAUSSIAN NOISE CHANNEL

1. Introduction

The system under consideration employs binary coding of antipodal waveforms to transmit information over a phase incoherent bandpass channel corrupted by white Gaussian noise.[1] The phase is uniformly distributed over the interval $(0, 2\pi)$, and is assumed to remain constant over the entire codeword. The codewords consist of N binary code letters, say "0" or "1". If the i^{th} code letter is a "0", the modulater forms the waveform $-\sqrt{E_N}\ \phi_i(t)$, likewise a "1" is mapped into $+\sqrt{E_N}\ \phi_i(t)$, where E_N is the energy per dimension, and the set of waveforms $\{\phi_i(t)\}$ is orthonormal. The codewords are chosen at random from an ensemble such that each code letter is equally likely to be a "0" or "1" and is independent of all other code letters. The resulting code-modulation system is that which would be obtained by assigning messages at random to the vertices of a hypercube.

Because of the random phase, the channel has memory, that is, the likelihood probably on the entire codeword, $p(\underline{r}/\underline{s}_i)$, does not factor into a product of probabilities over the individual code letters. Here $\underline{s}_i$ is an N-dimensional vector whose components are the projections of the signal representing the i^{th} message onto the set $\{\phi_i(t)\}$. The notation $\underline{s}_i$ will also be used to signify the codeword itself. Likewise, $\underline{r}$ is a vector representation of the received signal.

It is known that for the equivalent phase-coherent system the ensemble average probability of error is bounded above by

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$$\overline{P}(e) \leq 2^{-N(R_o-R_N)},$$

where $R_o = 1 - \log_2\left(1+e^{-E_N/N_o}\right)$ is the largest value R_o may take on for the inequality above to be valid for all N[1]; $R_N = (\log_2 M)/N$ is the source rate in bits/channel symbol when the source consists of M equiprobable messages. If a bound of the form

$$\overline{P}'(e) \leq 2^{-N(R_o'-R_N)}$$

is to be valid for the incoherent channel, R_o' must necessarily be less than or equal to R_o, since the error performance of the incoherent channel cannot be better than the coherent one. We shall show that in fact the bound on $\overline{P}'(e)$ is valid for $R_o' = R_o$.

2. Derivation of the Bound

For simplicity, assume that the $\{\phi_i(t)\}$ constitute a set of nonoverlapping time displacements of some pulse type of waveform. Thus the signals representing the codewords are a string of these pulses, some multiplied by $+\sqrt{E_N}$ and some by $-\sqrt{E_N}$. Now say message m_k is sent, the ensemble average probability of error, given m_k sent, is

$$\overline{P}(e/m_k) = \sum_{\text{all codes}} P[\{\underline{s}_i\}]\, P[e/m_k, \{\underline{s}_i\}],$$

where $P[\{\underline{s}_i\}]$ is the probability of a particular code, $\{\underline{s}_i\}$. $P[e/m_k, \{\underline{s}_i\}]$ is the probability of error, given m_k sent and given the code $\{\underline{s}_i\}$ is used. This probability can be upper-bounded by using the union bound

$$\overline{P}[e/m_k, \{\underline{s}_i\}] \leq \sum_{\substack{i=0 \\ (i\neq k)}}^{M-1} \overline{P}_2(\underline{s}_i, \underline{s}_k),$$

where $P_2(\underline{s}_i, \underline{s}_k)$ is the probability of error between two equiprobable signals, $\underline{s}_k$ and $\underline{s}_i$. Combining these relations yields

$$\overline{P}(e/m_k) \leq \sum_{\substack{i=0 \\ (i\neq k)}}^{M-1} \overline{P}_2(\underline{s}_i, \underline{s}_k).$$

Now consider two signals $\underline{s}_i$ and $\underline{s}_k$, h of whose components differ $\left(h \leq \frac{N}{2}\right)$, as shown in Fig. XVIII-1 for h = 4, N = 10. The exact expression for $P_2(\underline{s}_i, \underline{s}_k)$ does not lend itself to obtaining a closed-form expression for $\overline{P}'(e)$; an upper bound proves much more useful. To bound $\overline{P}_2(\underline{s}_i, \underline{s}_k)$ assume that signal $\underline{s}_k$ is sent and a hypothetical receiver designed

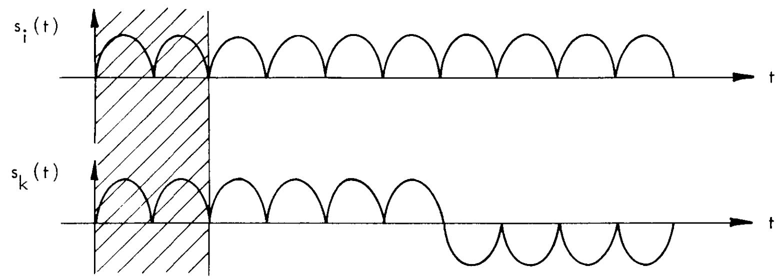

Fig. XVIII-1. Example of two signals with $N \leq h/2$.

for the signal set $\{\underline{s}_i, \underline{s}_k\}$ is allowed to observe only the h positions where the signals differ and another set of h postions where the signals are the same. This receiver cannot have a smaller probability of error than the optimum receiver operating on all N positions, hence its performance upper-bounds $P_2(\underline{s}_i, \underline{s}_k)$. Notice now that deleting the segments of the signals that the receiver does not observe (the shaded segments in Fig. XVIII-1) leaves 2 orthogonal signals, each with energy $2hE_N$. It is well known that the optimum receiver for two equally likely, equal-energy signals in the incoherent white Gaussian noise channel has the probability of error, P_1, given by

$$P_1 = \frac{1}{2} e^{-E/2N_o},$$

where E is the energy in a signal. Hence

$$P_2(\underline{s}_i, \underline{s}_k) \leq \frac{1}{2} e^{-hE_N/N_o}; \qquad h \leq \frac{N}{2}.$$

The possibility still exists that h, the number of positions where the signals differ, is greater than N/2. For this case, $P_2(\underline{s}_i, \underline{s}_k)$ is bounded by restricting the hypothetical

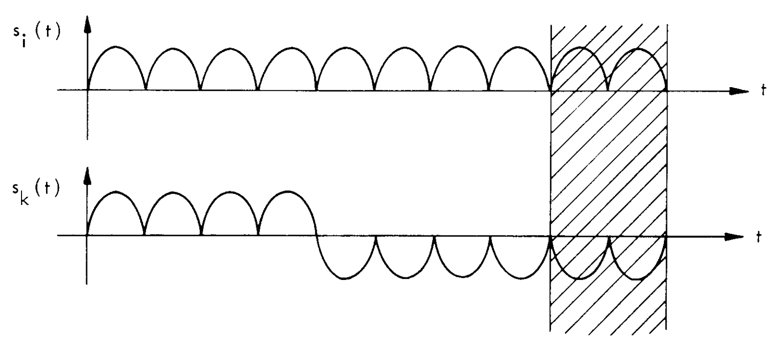

Fig. XVIII-2. Example of two signals with $N \geq h/2$.

receiver to observe the N-h positions where the signals are the same and only N-h positions where the signals differ. Figure XVIII-2 shows two signals for which N = 10, h = 6; the shaded segment covers the two positions not observed by the suboptimum receiver. Again, the portions of the signals which the receiver observes are orthogonal – this time with energy $2(N-h)E_N$. Hence

$$P_2(\underline{s}_i, \underline{s}_k) \leq \frac{1}{2} e^{(N-h)E_N/N_o}; \qquad h \geq \frac{N}{2}.$$

The next step is to take the average of $P_2(\underline{s}_i, \underline{s}_k)$ over the code ensemble. Since the bounds on $P_2(\underline{s}_i, \underline{s}_k)$ are functions only of h, the number of positions where the signals differ, this can be accomplished by averaging over h. For this code ensemble, the probability of signals differing in exactly h positions is

$$P(h) = 2^{-N}\binom{N}{h},$$

hence, for even N,

$$\overline{P}_2(\underline{s}_i, \underline{s}_k) = \sum_{h=0}^{N} P(h)\, P_2(\underline{s}_i, \underline{s}_k)$$

$$\leq 2^{-N} \sum_{h=0}^{N/2} \binom{N}{h} \frac{1}{2} e^{-hE_N/N_o} + 2^{-N} \sum_{h=\frac{N}{2}+1}^{N} \binom{N}{h} \frac{1}{2} e^{-(N-h)E_N/N_o}.$$

Relabeling indices, we have

$$\overline{P}_2(\underline{s}_i, \underline{s}_k) \leq 2^{-N} \sum_{h=0}^{N/2} \binom{N}{h} \frac{1}{2} e^{-hE_N/N_o} + 2^{-N} \sum_{h'=0}^{\frac{N}{2}-1} \binom{N}{h'} \frac{1}{2} e^{-h'E_N/N_o}.$$

The left summation is greater by one positive term than the right sum, therefore

$$\overline{P}_2(\underline{s}_i, \underline{s}_k) \leq 2^{-N} \sum_{h=0}^{N/2} \binom{N}{h} e^{-hE_N/N_o}.$$

Extending the summation to N gives

$$\overline{P}_2(\underline{s}_i, \underline{s}_k) \leq 2^{-N} \sum_{h=0}^{N} \binom{N}{h} e^{-hE_N/N_o}.$$

This can be expressed in closed form by using the binomial theorem.

$$\overline{P}_2(\underline{s}_i, \underline{s}_k) \leq 2^{-N}\left(1+e^{-E_N/N_o}\right)^N$$

$$\leq 2^{-NR_o}; \qquad R_o = 1 - \log_2\left(1+e^{-E_N/N_o}\right).$$

Completing the proof, we obtain

$$\overline{P}(e/m_k) \leq M\,\overline{P}_2(\underline{s}_i, \underline{s}_k) \leq 2^{-N(R_o-R_N)},$$

where $M = 2^{NR_N}$.

This result is independant of k, hence

$$\overline{P}'(e) \leq 2^{-N(R_o-R_N)},$$

and R_o is identical to the exponential bound parameter for the coherent channel. A simple modification yields the same bound for odd N.

3. Discussion

It has been shown[2] that the bound

$$\overline{P}(e) \leq 2^{-N(R_o-R_N)}$$

for the coherent channel is exponentially the tightest possible bound at the critical rate, R_c, less than capacity. It therefore follows that this is also true for the incoherent channel at the same rate, since

$$\overline{P}(e) \leq \overline{P}'(e) \leq 2^{-N(R_o-R_N)}.$$

It is interesting to note that the tightest exponent was acheived by a random-coding scheme that forms codewords by choosing code letters independently. Although this is always the best strategy for a memoryless channel at R_c, there is no a priori reason to expect it to be optimum in a channel with memory.

An interesting comparison can be made between the result described in this report and binary coding of orthogonal waveforms on the incoherent channel. Here each code letter is mapped into one of two equal-energy orthogonal waveforms instead of antipodal waveforms. For this system it can be easily shown that the average probability of error is bounded by

$$\overline{P}''(e) \leq 2^{-N(R_o''-R_N)},$$

where

$$R_o'' = 1 - \log_2 \left(1 + e^{-E_N/2N_o}\right)$$

is the largest value of R_o'' for which the bound is valid for all N, and E_N is the energy per orthogonal waveform. Notice that this signaling scheme requires twice as much energy as the antipodal case to achieve the same value of the exponential bound parameter; consequently, just as in the coherent case, binary orthogonal signaling is 3 db less efficient than binary antipodal signaling.

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References

1. J. M. Wozencraft and I. M. Jacobs, Principles of Communication Engineering (John Wiley and Sons, Inc., New York, 1965).

2. R. G. Gallager, "A Simple Derivation of the Coding Theorem and Some Applications," IEEE Trans., Vol. IT-11, No. 1, January 1965.