XI. ELECTRODYNAMICS OF MEDIA*

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A. QUANTUM FORM OF MANLEY-ROWE RELATIONS

It is well known that any classical system possessing a Hamiltonian description obeys the Manley-Rowe relations.[1,2] We want to derive the Manley-Rowe relations for a quantum system.[3] Such a derivation has two advantages. First, it is surprisingly simple. Second, it leads to an operator relationship that may be considered a quantum-mechanical generalization of the Manley-Rowe relations.

Consider a system described by a set of creation and annihilation operators. The system can contain electromagnetic modes, phonons, magnons, or any other quasi particles satisfying Bose-Einstein commutation relations. We assume that the energy levels of these quasi particles are given in terms of two sets of integers

$$E_{mn} = mE_a + nE_b = \hbar(m\omega_a + n\omega_b) = \hbar\omega_{mn}. \tag{1}$$

The two energy levels E_a and E_b correspond to angular frequencies ω_a and ω_b. We assume that ω_a and ω_b are incommensurate. To each of the energy levels E_{mn} a creation operator and an annihilation operator are assigned. The creation operator a^+_{mn} has the (unperturbed or uncoupled) time dependence $\exp i|\omega_{mn}|t$; the annihilation operator a_{mn} has the time dependence $\exp -i|\omega_{mn}|t$. Note that ω_{mn} may be positive or negative. We shall dispense with the usual dagger notation to distinguish between creation and annihilation operators and accomplish this purpose solely by the subscript mn, implying that a_{mn} is a creation operator if $\omega_{mn} > 0$, and an annihilation operator if $\omega_{mn} < 0$; in this way we obtain a convenient notation. We may then write for the Hamiltonian

$$H = N\left\{\sum \kappa_{m_1n_1\dots m_kn_k} \prod_{\substack{m_i=m_1\\ n_i=n_1}}^{\substack{m_i=m_k\\ n_i=n_k}} a_{m_in_i}\right\} + \text{complex conjugate.} \tag{2}$$

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The sum is carried over all permissible combinations $m_1 \ldots m_k$, $n_1 \ldots n_k$ that ensure time independence of H, so that

$$\sum_{i=1}^{k} m_i = 0 \qquad \sum_{i=1}^{k} n_i = 0. \tag{3}$$

Furthermore, the sum extends over all orders k higher than first. The symbol N{ } indicates normal ordering. All creation operators are implied to precede the annihilation operators. There is no restriction against repetition of particular m_i or n_i values. In particular, $a_{m_j n_j}$ may occur several times as a factor in any one of the terms in (2). The leading term for k = 2 has the coefficient

$$\kappa_{m_1 n_1 m_2 (=-m_1)\, n_2 (=-n_1)} = \frac{1}{2} h \left| \omega_{m_1 n_1} \right|$$

because this is the self-term giving the energy of the $m_1 n_1$ level.

The operators a_{mn} obey the following commutation relations:

$$\left[a_{m_i n_i}, a_{-m_j -n_j}\right] = -\operatorname{sgn}\left(\omega_{m_i n_i}\right) \delta_{m_i m_j} \delta_{n_i n_j}. \tag{4}$$

The commutator vanishes unless $m_i = m_j$ $n_i = n_j$, and is negative if $\omega_{m_i n_i} > 0$, since then $a_{m_i n_i}$ is a creation operator; positive if $\omega_{m_i n_i} < 0$, since then $a_{m_i n_i}$ is an annihilation operator. One expects that the quantum generalization of the Manley-Rowe relations would take the form

$$\sum_{\substack{m_i=-\infty \\ n_i=-\infty}}^{\infty} m_i \operatorname{sgn}\left(\omega_{m_i n_i}\right) \frac{d}{dt} N\left(a_{m_i n_i} a_{-m_i -n_i}\right) = 0. \tag{5}$$

$$\sum_{\substack{m_i=-\infty \\ n_i=-\infty}}^{\infty} n_i \operatorname{sgn}\left(\omega_{m_i n_i}\right) \frac{d}{dt} N\left(a_{m_i n_i} a_{-m_i -n_i}\right) = 0. \tag{6}$$

In order to prove (5) and (6), one needs the equations of motion for the operators, which are

$$i\hbar a_{mn} = [a_{mn}, H]. \tag{7}$$

Now consider the contribution to (5) of one particular term in the Hamiltonian when one forms

$$\frac{1}{i\hbar}\left[a_{mn}, H\right] a_{-m-n} + \frac{1}{i\hbar} a_{mn}\left[a_{-m-n}, H\right],$$

and operates on one typical term as displayed in (2), one gets k contributions. Assuming at the outset that none of the operators occur to a power higher than first, we have

$$\frac{i}{\hbar}\left[\sum_{i=1}^{k} m_i \operatorname{sgn}^2\left(\omega_{m_i n_i}\right)\right] \kappa_{m_1 n_1 \ldots m_k n_k} N\left\{a_{m_1 n_1} \ldots a_{m_k n_k}\right\}. \tag{8}$$

But by virtue of the fact that $\operatorname{sgn}^2(\omega_{mn}) = 1$, and by virtue of (3), one finds that the contribution is zero. This procedure needs to be modified only slightly if an operator $a_{-m_i-n_i}$ enters the product to higher order than first, say to q_i^{th} order. In such a case,

$$\left[a_{m_i n_i}, a_{-m_i-n_i}^{q_i}\right] a_{-m_i-n_i} = -q_i \operatorname{sgn}\left(\omega_{m_i n_i}\right) a_{-m_i-n_i}^{q_i},$$

Thus one gets a contribution, $q_i m_i$. Since one of the relationships (3), when repetitions of the m_i occur, can be interpreted to mean

$$\sum{}' q_i m_i = 0,$$

and the summation is only over nonrepeated occurrences of the operators, one still concludes that the term (8) sums to zero. With this, (5) is proved, and a similar procedure proves (6). Note that (5) and (6) contain operators and not expectation values, and in this sense are true quantum-mechanical generalizations.

An analogous proof can be conducted for a system described in terms of Fermi-Dirac creation and annihilation operators obeying anti-commutation relations, provided one requires that the Fermi-Dirac particles are conserved in number. This means that the Hamiltonian must contain solely even-order terms, each of which contains an equal number of creation and annihilation operators. Clearly, the time independence of the Hamiltonian requires that the sum of the energies of the energy levels represented by the creation operators must be equal to that of the annihilation operators. The Manley-Rowe relations obeyed by such a system are of the form of (5) and (6), except that the operators $a_{m_i n_i}$ and $a_{-m_i-n_i}$ are now Fermi-Dirac operators.

Finally, one can look at a combined system with a Hamiltonian containing products of Bose-Einstein and Fermi-Dirac operators. The proof that such a system obeys Manley-Rowe relations of the form (5) and (6) is a straightforward extension.

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References

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