

XV. COMPUTATION RESEARCH*

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A. READING BINARY TAPES IN UNDEFINED RECORD FORMAT ON THE 360/65

In the Research Laboratory of Electronics there are several specialized data-taking devices utilizing magnetic tape for permanent storage of digitized data. For instance, Potter Model 906 II-1 tape drives are used by both the flying-spot scanner and the PDP-4 to ensure physical compatibility with the IBM tape units used by the 360/65. The data format of such tapes is typically incompatible with Fortran IV (release 11 and higher) standard binary tape format, which requires two full-words of control information at the beginning of each record.

A general assembly language program which reads and writes the binary tapes described above is now available at the M. I. T. Computation Center. The Job Control language permits the user to set all valid OS/360 DD parameters, such as recording density, record length, and block size, without program modification. Users often desire to unpack the data for easier internal processing; at present, our group has additional assembly language programs for packing and unpacking 12 bit and 18 bit data words.

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B. GAUSSIAN QUADRATURE – A NUMERICAL TECHNIQUE FOR INTEGRATION

We have developed a set of numerical integration programs in Fortran IV, which utilize Gaussian Quadrature formulas to evaluate integrals which cannot be evaluated using more conventional methods, such as the Trapezoidal Rule or Simpson's Rule.

Gaussian Quadratures are numerical approximations of the form

$$\int_a^b f(x) dx = \sum_{j=1}^n w_j f(a_j) + E_n.$$

The weights w_j and abscissas a_j are determined so that $E_n = 0$ for a polynomial of degree less than or equal to $2n - 1$. Since by the Weierstrass theorem on polynomial

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approximation, any function f continuous on (a, b) can be approximated by a polynomial of sufficiently high degree, n can be chosen such that given $\mu > 0$, $|E_n| < \mu$ for any such function f .

There are numerous sets of weights and abscissas, proven mathematically to be appropriate to various types of intervals. For a function f continuous on a finite interval $[a, b]$, we have written routines using the following formulas:

$$\int_a^b f(y) dy = \frac{b-a}{2} \sum_{j=1}^n w_j f(y_j) + E_n \quad (1)$$

$$y_j = \left(\frac{b-a}{2}\right)x_j + \left(\frac{b+a}{2}\right)$$

x_j = the j^{th} zero of the Legendre Polynomial $P_n(x)$.

$$w_j = \frac{2}{(1-x_j^2)[P_n'(x_j)]^2}$$

$$\int_a^b \frac{f(y)}{\sqrt{y-a}} dy = \sqrt{b-a} \sum_{j=1}^n w_j f(y_j) + E_n \quad (2)$$

$$y_j = (b-a)x_j + a$$

$x_j = a_j^2$ where a_j is the j^{th} positive zero of the Legendre Polynomial $P_{2n}(x)$.

$$w_j = 2 \cdot \frac{2}{(1-a_j^2)[P_{2n}'(a_j)]^2}$$

$$\int_a^b \frac{f(y)}{\sqrt{b-y}} dy = \sqrt{b-a} \sum_{j=1}^n w_j f(y_j) + E_n \quad (3)$$

$$y_j = (a-b)x_j + b$$

x_j and w_j are as in (2).

The creation of these routines involved writing a program to evaluate the coefficients of the Legendre Polynomials, and setting up other programs to calculate the weights w_j and to form the sums $K \sum_{j=1}^n w_j f(y_j)$. A subprogram to compute f must be supplied by the user.

As an example of the use of (1) and (3), we have evaluated

$$A = \int_{\pi/2}^{+\sin^{-1}(2\xi-1)} dy \int_{\pi/2}^{+\sin^{-1}(2\xi/1+\sin y)} \frac{|2\xi - \sin y \sin x| \sin x}{\sqrt{\sin^2 x - (2\xi - \sin y \sin x)^2}} dx,$$

$$\frac{1}{2} < \xi < 1, \quad 0 < x < \pi, \quad -\frac{\pi}{2} < y < \frac{3\pi}{2},$$

where " $+\sin^{-}$ " represents the greater value of the inverse sine in the range of x and y given. Let $b = +\sin^{-1}(2\xi/(1+\sin y))$. The integrand is infinite at $x = b$. We can establish that the integrand $G(x, y)$ can be written as

$$\int_{\pi/2}^{\sin^{-1}(2\xi-1)} dy \int_{\pi/2}^b \frac{F(x, y)}{\sqrt{b-x}} dx.$$

Rather than explicitly doing the algebra, we merely write

$$G(x, y) = \frac{\left[\frac{|2\xi - \sin x \sin y| \sin x \sqrt{b-x}}{\sqrt{\sin^2 x - (2\xi - \sin y \sin x)^2}} \right]}{\sqrt{b-x}} = \frac{F(x, y)}{\sqrt{b-x}}.$$

Since the formula (3) does not evaluate $F(x, y)$ at b , and since

$$\lim_{x \rightarrow b} \left(\frac{\sqrt{b-x}}{\sqrt{\sin^2 x - (2\xi - \sin y \sin x)^2}} \right) = 1,$$

we can safely use (3) with $f = F(x, y_j)$ for a given y_j . The procedure for the double integration is to use (1) with $f = \int_{\pi/2}^b \frac{F(x, y)}{\sqrt{b-x}} dx$ and at each y_j defined in (1) apply (3) as explained above.

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C. BESSEL FUNCTIONS OF THE SECOND KIND

A frequently occurring equation in physics is Bessel's equation,

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (x^2 - n^2)y = 0.$$

Historically, the development of solutions to this equation was slow and erratic. The earliest appearance of an equation of this type occurs in a paper published by John Bernoulli, in 1694. Seventy years later, an investigation by Euler on the vibrations of a stretched membrane reveals the earliest analysis of a Bessel function of integral order. Throughout the next half-century, the analysis of this equation was

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continued by such men as Lagrange, Laplace, and Fourier. Then, in a memoir written in 1824, the German mathematician, Friedrich Wilhelm Bessel, put forth a detailed discussion of the functions that now bear his name. His solution, the Bessel coefficients of argument z and order n , has come to be denoted universally by the symbol $J_n(z)$.

The desire for a second independent solution, and hence a fundamental system of solutions, led many mathematicians to work further on the problem. The result was several versions of the Bessel function of the second kind. No standard form has yet been accepted universally. American authors commonly choose as a standard particular solution Neumann's function of order n , but there remains some discrepancy in the exact definition of this function. Some authors cite a Neumann function denoted by $Y^{(n)}(z)$

$$Y^{(n)}(z) = J_n(z)\{\log z - s_n\} - \sum_{m=0}^{n-4} \frac{2^{(n-m-1)} \cdot n! J_m(z)}{(n-m)(m!)(z^{n-m})} + \sum_{m=1}^{\infty} \frac{(-1)^{m-1} (n+2m)}{m(n+m)} J_{(n+2m)}(z), \quad (1)$$

where

$$s_0 = 0 \quad s_m = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{m} \quad m = 1, 2, \dots$$

Then, since any linear combination of $J_n(z)$ and $Y^{(n)}(z)$ is also a solution, these authors define a second solution, $Y_{(n)}(z)$, the Weber function, as

$$Y_n(z) = \frac{2}{\pi} \left\{ Y^{(n)}(z) + (\gamma - \log 2) J_n(z) \right\}$$

where γ is Euler's constant = 0.577216. Other authors prefer to substitute (1) into (2), combine terms, and to call the resulting function Neumann's function, denoted by $Y_n(z)$, where

$$Y_n(z) = \frac{2}{\pi} J_n(z) \left\{ \log \frac{z}{2} + \gamma \right\} + \frac{z^n}{\pi} \left\{ \sum_{m=0}^{\infty} \frac{(-1)^{m-1} s_m + s_{m+n} z^{2m}}{(2^{2m+n})(m!)(m+n)!} \right\} - \frac{z^{-n}}{\pi} \left\{ \sum_{m=0}^{n-1} \frac{(n-m-1)! z^{2m}}{(2^{2m-n})(m!)} \right\}$$

$z > 0; \quad n = 0, 1, \dots \quad (2)$

In order to be as complete and as helpful as possible, we felt that a subroutine written to compute both $Y^n(z)$, the Neumann function, and $Y_n(z)$, the Weber function, would be desirable. Working with definitions (1) and (2) as stated above, the subroutine YNEU was written in Fortran IV for the IBM 360. For real arguments z of order n , real and positive, the routine returns values of Y , the Neumann function, and BY , the Weber function.

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The subroutine calls two other subroutines, BESJ and BESY, supplied in version II of the scientific subroutine package for the IBM 360.

The routine has been written for five place accuracy and has been tested and checked out for values of z between 0.02 and 12.0 and $n = 0, 1, 2 \dots 10$.

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