IX. ELECTRODYNAMICS OF MEDIA*

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A. CLOSED-FORM SOLUTION OF STEADY-STATE ELECTROMAGNETIC SHOCK

DeMartini et al.[1] and other workers[2, 3] have studied the self-steepening of light pulses. This phenomenon might be the cause of the anomalous spectra observed in Raman scattering. The shocks are both of theoretical and technical interest. Indeed, they have been used in order to achieve tunable optical sources.[4]

It is difficult to obtain a closed-form solution because the phenomenon is nonlinear, and most of the investigations have been done with the aid of a computer. Physical insight into the phenomenon may be gained, however, by studying whatever closed-form solutions are obtainable under certain simplifying assumptions. The present report is devoted to the derivation of a closed-form solution for the steady-state shock. The problem is treated one-dimensionally. The electromagnetic energy density travels in the positive z direction. It starts from a low steady-state value behind the shock front, that is, at negative values of z, and changes through the shock front to a high value for positive values of z. The steady-state assumption reduces the set of partial differential equations in z and t to total differential equations in one single variable.

We find that the speed of the shock wave is slower than the propagation velocity in both the high and low energy density ranges. At first, this is surprising because dissipation occurs in the shock, and one expects that the shock consumes energy. Energy consumption does take place, but it is provided by the release of energy of alignment as the high field intensity gives way to the low field intensity behind the shock.

The nonlinear parameter relating the velocity to the energy density does not directly affect the height of the shock or its speed. It only affects the steepness. For each ratio of energy densities on the two sides of shock one finds a speed of the shock.

From DeMartini et al.[1] we take the fundamental equations.

$$\frac{\partial \rho}{\partial t} - \frac{\rho}{v}\frac{\partial v}{\partial t} + \frac{\partial}{\partial z}(\rho v) = 0. \tag{1}$$

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This is the propagation equation for the internal energy density ρ in the shock, defined as

$$\rho = \frac{ED}{2}. \tag{2}$$

The velocity v is the speed of propagation of the energy density (in the present example of a nondispersive medium the phase, as well as the group velocity of electromagnetic wave propagation) a function of the energy density. It is related to ρ by

$$\frac{\partial}{\partial t} v = -\frac{\rho v_{NL}}{\tau} - \frac{v - v_o}{\tau}. \tag{3}$$

Here, τ is the relaxation time, and v_{NL} is a parameter of nonlinearity. By virtue of the fact that the velocity is related to the energy density by a differential equation in time, the velocity does not follow changes in the energy density instantaneously. We assume a solution of the differential equation in a frame of reference moving with respect to the laboratory frame at a steady-state velocity U such that no time variation is observed in this frame. This is the assumption of steady state. In denoting the time and distance coordinates in this frame by primes, we have

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t'} - U \frac{\partial}{\partial z'} = -U \frac{\partial}{\partial z'}. \tag{4}$$

To the extent that we describe all physical variables in the laboratory frame, we are not using a relativistic transformation, even though the speed U may approach the velocity of light. Equation 4 has to be treated as a mathematical transformation, not as a transformation of the physical coordinates z and t as observed by two observers moving with respect to each other at relativistic speed. We obtain from (1) and (4)

$$-U \frac{\partial \rho}{\partial z'} + \frac{\rho}{v} U \frac{\partial v}{\partial z'} + \frac{\partial}{\partial z'} (v\rho) = 0. \tag{5}$$

This equation can be rewritten by separating the variables ρ and v:

$$\frac{1}{\rho} \frac{\partial \rho}{\partial z'} = -\frac{1 + U/v}{v - U} \frac{\partial}{\partial z'} v. \tag{6}$$

Since the differential equation (6) is a total differential equation in the independent variable z', one may solve it in the usual way by eliminating z' and obtaining a differential relationship between the variables ρ and v:

$$\frac{1}{\rho} d\rho = -\frac{v + U}{v(v-U)} dv. \tag{7}$$

Integration gives

$$\frac{\rho}{\rho_1} = \frac{v}{v_1}\left[\frac{v_1 - U}{v - U}\right]^2. \tag{8}$$

Here, ρ_1 is the energy density at a point where the velocity is v_1.

We may conveniently choose ρ_1 to be the energy density in the tail of the shock, $z' \to -\infty$. Assume first that the tail of the shock has an energy density lower than the (steady-state) front portion of the shock ($z' \to +\infty$), $\rho_1 < \rho_2$. One is led to this assumption by studying the (transient) formation of the shock, which occurs because the low-energy density portion of an electromagnetic excitation catches up with the high-energy density portion. We shall eventually show that no steady-state solution is found when the tail of the shock is assumed to be of higher energy density than the (steady-state) front of the shock.

Thus far, we have not used the constitutive law (8). Taking it into account, we find that $v_2 < v_1$, where v_2 is the propagation speed at $z' \to +\infty$. The only way that we can achieve $\rho_2 > \rho_1$ is by having $|v_2-U| < |v_1-U|$. This means that U must lie more closely to v_2 than to v_1. Furthermore, since we cannot permit infinite energy densities, the velocity v, as one passes through the shock, cannot pass through U, and hence we conclude that the inequality must hold: $U < v_2 < v_1$.

This means that the speed of the shock is less than all pertinent velocities in the problem. The shock falls behind. Substitution of (8) in (3) gives

$$+\frac{\partial}{\partial z'}\frac{v}{v_o} = \frac{v_{NL}}{v_o \tau U}\rho_1 \frac{v}{v_1}\left[\frac{1 - U/v_1}{v/v_1 - U/v_1}\right]^2 + (v/v_o - 1)\frac{1}{\tau U}. \tag{9}$$

In order to obtain a steady shock, the derivative $\partial/\partial z'$ must vanish on the two sides of the shock. This means that the right-hand side of (9) must vanish at two points. A plot of v/v_1 as abscissa and the two terms on the right-hand side of (9) as ordinate, the second one being taken negative, gives three intersection points; the one for which $v < U$ is not acceptable. The one at the intermediate value $v = v_2 (< v_1)$ corresponds to the front of the shock with the high energy density. For each pair of assumed values of U and v_1 such a plot can be executed.

Next, consider briefly whether a steady state could be found under the assumption $\rho_2 < \rho_1$. Then $v_2 > v_1$. The right-hand side of (9) would have to become positive in the range $v_1 < v < v_2$ and vanish at the end points. This is impossible, as an appropriate plot will show. Hence, no such solution can by found. By introducing the values v_2 and U, obtained from setting the right-hand side of (9) equal to zero, into (8), one finds ρ_2 on the high-energy side of the shock. In this way, one may obtain plots of ρ_2/ρ_1 and v_2/v_1

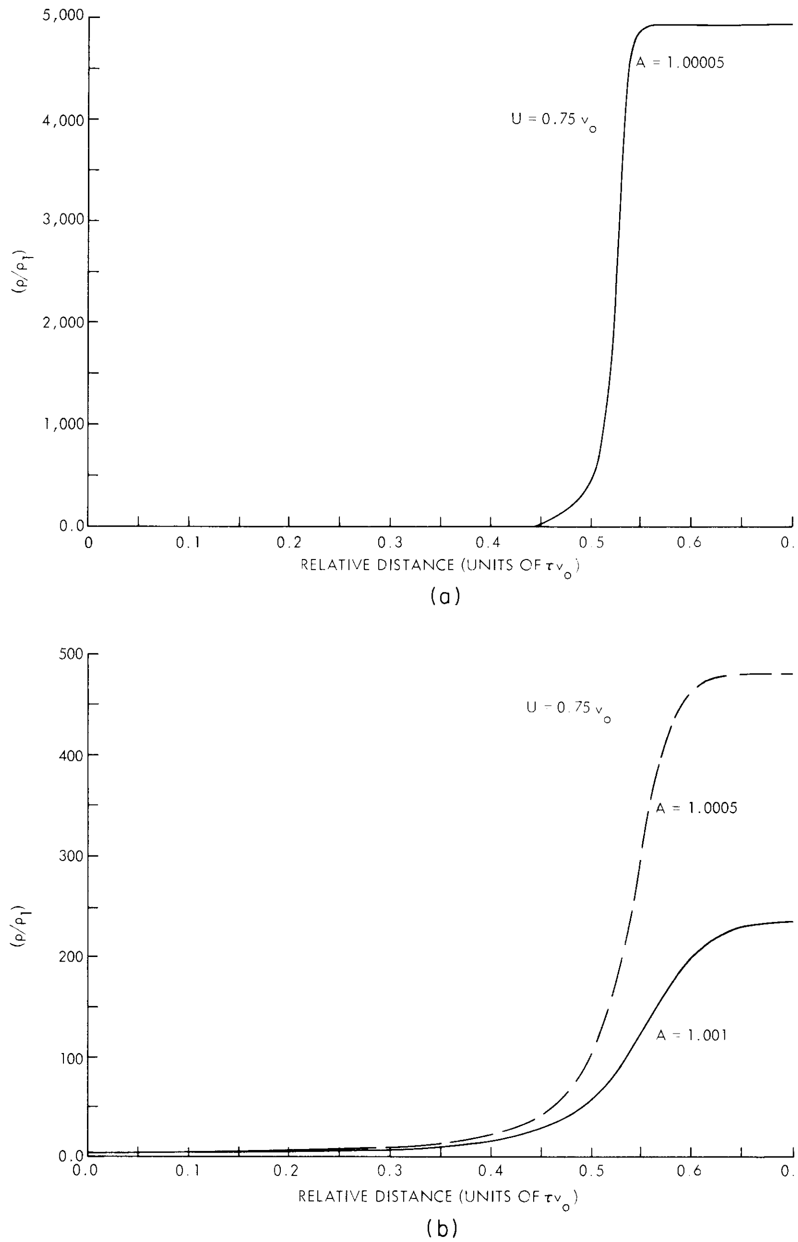

Fig. IX-1. Normalized ρ vs distance; $A = v_o/v_1$.

against U, with v_1/v_o as parameter.

Figure IX-2 shows some plots of the normalized internal energy density against distance, and plots of the normalized velocity against distance.

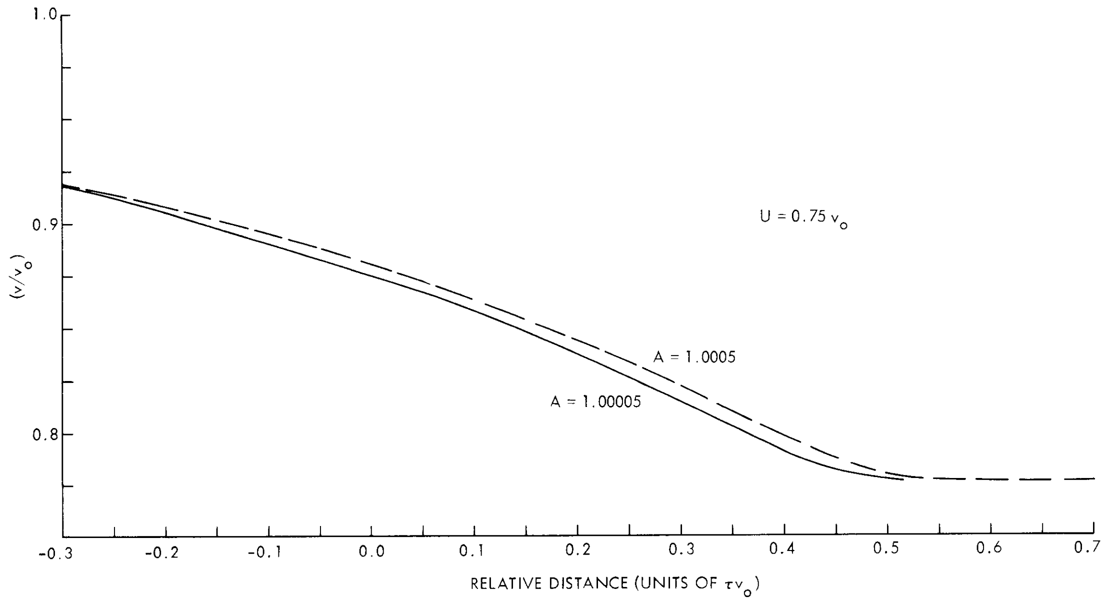

Fig. IX-2. Normalized velocity vs distance; $A = v_o/v_1$.

Equation 1 has the appearance of a conservation law of energy, except for the term $-\frac{\rho}{v}\frac{\partial v}{\partial t}$ which does not appear as a total derivative. Using the constitutive law (2), we may transform the term, however, and cast (1) into the form:

$$\frac{\partial}{\partial t}\left[\rho + \frac{1}{v_{NL}}\left(v - v_o \ln\frac{v}{v_o}\right)\right] + \frac{\partial}{\partial z}(\rho v) + \frac{\tau}{v_{NL}v}\left(\frac{\partial v}{\partial t}\right)^2 = 0. \tag{10}$$

The term under time derivative may be identified as the free energy,[1] the term $\frac{\partial}{\partial z}(\rho v)$ represents the divergence of the power flow, and the last term is positive definite and gives the dissipation density. In order to check for energy conservation in the shock solution determined thus far, we transform (10) into the frame moving with the shock front. Then we obtain

$$\frac{\partial}{\partial z}\left[\rho(v-U) - \frac{U}{v_{NL}}\left(v - v_o \ln\frac{v}{v_o}\right)\right] + \frac{\tau}{v_{NL}v}\left(U\frac{\partial v}{\partial z}\right)^2 = 0. \tag{11}$$

Thus we see that since the last term is positive, the inequality has to hold.

$$\rho_2(v_2-U) - \frac{U}{v_{NL}}\left(v_2 - v_o \ln\frac{v_2}{v_o}\right)$$

$$< \rho_1(v_1-U) - \frac{U}{v_{NL}}\left(v_1 - v_o \ln\frac{v_1}{v_o}\right). \tag{12}$$

With the aid of (3) and (8) we may eliminate ρ_2 and ρ_1 to obtain

$$(v_o-v_2)(v_2-U) - (v_o-v_1)(v_1-U) - U(v_2-v_1) + Uv_o \ln\frac{v_2}{v_1} < 0. \tag{13}$$

This inequality has been checked for each of the computer runs of Fig. IX-2 and has been found to be satisfied. It should be pointed out that

$$\rho_2(v_2-v) > \rho_1(v_1-v)$$

for any solution; hence, (12) or (13) are satisfied only because of the presence of the term $\frac{U}{v_{NL}}\left(v - v_o \ln\frac{v}{v_o}\right)$, which can be attributed to the energy of alignment of the molecules.[1]

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