

XIV. INTERACTION OF LASER RADIATION WITH PLASMAS AND NONADIABATIC
MOTION OF PARTICLES IN MAGNETIC FIELDS*

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A. RESONANT AND NEAR-RESONANT WAVE-PARTICLE INTER-
ACTIONS IN A TRANSVERSE PLASMA WAVE

Our original motivation of this study is related to the problem of the "corkscrew,"¹ which is a device to inject charged particles "resonantly" into thermonuclear fusion devices. Wingerson's original idea was the following: If an external helical magnetic field perturbation is applied in such a way that the pitch of the perturbation is tuned continuously to the orbit pitch of the injected particle, then the particle can be "resonantly" accelerated perpendicularly to the main field, and trapped inside the mirror field. This tuning, or "resonance" condition is expressed in the form

$$v_{\parallel} \frac{2\pi}{\omega_c} = P \quad (1)$$

or

$$\omega_c - kv_{\parallel} = 0, \quad (2)$$

where v_{\parallel} is an instantaneous velocity component parallel to the main magnetic field, ω_c is the cyclotron frequency in the main magnetic field, P is the pitch of the applied perturbation, and $k = 2\pi/P$. The particles that have been trapped can inversely resonate with the perturbation field, reverse their orbit, and escape from the confinement region.² To estimate the trapping efficiency, we have to know the particle's scattering processes in velocity space, which are due to the perturbation; therefore, we have to study the particle motion

$$m \frac{d\mathbf{v}}{dt} = q \mathbf{v} \times (\mathbf{B}_0 + \mathbf{B}_1) \quad (3)$$

in the vicinity of the resonance expressed by Eq. 1.

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A very similar situation takes place in a transverse electromagnetic wave propagating along an external magnetic field. In the laboratory frame, charged particles are accelerated by both the oscillatory electric field \underline{E}_\perp and the oscillatory magnetic field \underline{B}_\perp , which are related through

$$\nabla \times \underline{E}_\perp = - \frac{\partial \underline{B}_\perp}{\partial t}. \quad (4)$$

In the wave frame, moving with the constant wave phase velocity (under the assumption that $\frac{d}{dt} \left(\frac{\omega}{k} \right) = 0$), the electric field vanishes, however, and the equation of motion in the frame is given by Eq. 3. One of the natural modes of the transverse electromagnetic waves is the so-called whistler mode whose frequency range is $\omega_{ci} \ll \omega < \omega_{ce}$, where ω_{ci} and ω_{ce} are the ion's and electron's cyclotron frequency, respectively. This mode is interesting from the viewpoint of communication noise in the ionosphere and of charged-particle precipitation from the magnetosphere to the atmosphere. Recently, plasma physicists have become interested in these matters from the viewpoint of testing plasma kinetic theories.³ Quasi-linear theory takes into account the wave-particle interaction in the linear sense; if the Doppler-shifted frequency of the whistler mode is equal to the electron-cyclotron frequency, that is,

$$\omega - \underline{k} \cdot \underline{v}_L = \omega_c \quad (5)$$

or in the wave frame

$$|\underline{k} \cdot \underline{v}_W| = \omega_c \quad (6)$$

(notice its similarity to the corkscrew resonance condition, Eq. 2), then the particles exchange their energy with the wave, whose growth or damping depends on the slope of the particle's distribution function. The limitation of the quasi-linear theory may be attributed to the fact that straight orbit calculations are used. In fact, for the longitudinal case, O'Neil⁴ has shown that the linear damping rate breaks down when the particles are trapped in the potential trough of the electrostatic wave. Dupree⁵ constructed the so-called strong turbulence theory in which he introduced a diffusive term into the orbit of particles interacting with the electrostatic wave. A very similar situation is expected for the transverse-wave case.

We shall discuss the two problems on the same basis; we shall try to solve Eq. 3 for the resonant and near-resonant particle cases. Actually, in a corkscrew the perturbation amplitude B_\perp and the pitch P have spatial dependences, and in whistler-mode waves there is a finite spectral width instead of a single wave. Here we shall treat the case of a perturbation of spatially independent amplitude and pitch in order to understand its basic features.

In our last progress report,⁶ we showed that there are two kinds of particles,

untrapped and trapped in the sense similar to the electrostatic case, and that for the untrapped particles straight orbit calculation does give right answers.

In this report we shall discuss some relationships with the quasi-linear theory for the transverse mode, and also discuss the limitation of the theory, where the theory breaks down because of particle trapping. In order to cope with the inability of the straightforward orbit calculation to treat the turning points properly, we shall introduce a new method, singular perturbation technique.⁷ The results will be explained on a physical basis, and their relationship to Dupree's strong turbulence theory will be pointed out.

1. Limitation of "Straight-Line Orbit" Approximation

Using the conservation of the particle's kinetic energy ($v_o^2 = v_{\parallel}^2 + v_{\perp}^2$), we reduce the equation of motion, Eq. 3, into a set of two simpler normalized equations:

$$\frac{du_{\perp}}{d\zeta} = \beta \cos \chi \quad (7)$$

$$\frac{d\chi}{d\zeta} = \frac{1}{u_{\parallel}} - \frac{2\pi}{p}, \quad (8)$$

where u_{\parallel} and u_{\perp} are the velocity components normalized by the total velocity v_o , ζ is the spatial variable along the main magnetic field normalized by the Larmor radius, r_o , of the totally wound-up particles, β is the small-perturbation magnetic field amplitude normalized by the main magnetic amplitude, p is the perturbation pitch normalized by the Larmor radius r_o , and χ is the angle between the direction of the perpendicular velocity and that of the perturbation magnetic field.

We have shown previously⁶ that the untrapped particle is a particle whose initial condition satisfies

$$\gamma^2 \gg A\beta, \quad (9)$$

where

$$\gamma = \frac{1}{u_{\parallel o}} - \frac{2\pi}{p}, \quad (10)$$

(values with the subscript o are the initial value at $\zeta = 0$), and

$$A = \frac{u_{\perp o}}{3u_{\parallel o}}. \quad (11)$$

We also showed that the straightforward perturbation solution converges with the exact solution. Condition (9) can be rewritten in more familiar notation

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$$\left(\frac{\omega_c}{k} - v_{\parallel 0}\right)^2 \gg \left(\frac{\omega_c}{k}\right)^2 \frac{v_{\perp 0}}{v_{\parallel 0}} \frac{B_{\perp}}{B_0}. \quad (12)$$

The condition is satisfied when the particle's orbit pitch is far from the perturbation's field pitch and/or the perturbation field strength is weak enough. In plasma kinetic theory this state is called "weak turbulence," which can be described by the quasi-linear theory. In the quasi-linear theory, the diffusion coefficient for the right-hand polarized wave is given by

$$D_{QL} = \frac{q^2}{m^2} \frac{B_{\underline{k}\perp}^2 v_{\underline{k}\perp}^2 \gamma_{\underline{k}}}{(\omega - \underline{k} \cdot \underline{v} + \omega_c)}, \quad (13)$$

where $\gamma_{\underline{k}}$ is the linear damping rate for the \underline{k} wave. In our case we have a single set of ω and \underline{k} . If we average over the termination position, z , then the term $\cos \gamma z$ in the expression $\langle \Delta u_{\perp}^2 \rangle = \beta^2 (1 - \cos \gamma z) / \gamma^2$ will drop out. Then

$$\langle \Delta u_{\parallel}^2 \rangle = \frac{u_{\perp 0}^2}{u_{\parallel 0}^2} \langle \Delta u_{\perp}^2 \rangle = \frac{u_{\perp 0}^2}{u_{\parallel 0}^2} \frac{\beta^2}{\gamma}. \quad (14)$$

If we take the diffusion coefficient in the form

$$D = \frac{\langle \Delta v_{\parallel}^2 \rangle}{\tau}, \quad (15)$$

where τ is a characteristic time during which the perturbation changes significantly, then we get

$$D = \frac{q^2}{m^2} \frac{B_{\perp}^2 v_{\perp 0}^2 \left(\frac{1}{\tau}\right)}{(\omega_c - kv_{\parallel 0})^2}. \quad (16)$$

This is the diffusion coefficient in the wave frame. Therefore, in the laboratory frame $v_{\parallel 0}$ should be replaced by $v_{\parallel 0} - \frac{\omega}{k}$. Then the diffusion coefficient in the laboratory frame is

$$(D)_{\text{lab}} = \frac{q^2}{m^2} \frac{B_{\perp}^2 v_{\perp 0}^2 \left(\frac{1}{\tau}\right)}{(\omega - kv_{\parallel 0} + \omega_c)^2}. \quad (17)$$

Noticing that τ can be replaced by the reciprocal of the growth rate, we realize that Eq. 17 agrees with the quasi-linear diffusion coefficient D_{QL} .

When the trapping phenomena begin to take place, the straightforward orbit calculation does not converge near the turning points. Our previous calculation⁶ and our later calculation tell us that the trapping occurs when the particle's initial condition lies in the range

$$\gamma^2 \approx A\beta \quad (18)$$

or, in more familiar notation,

$$\left(\frac{\omega_o}{k} - v_{\parallel o}\right)^2 \approx \left(\frac{\omega_o}{k}\right)^2 \frac{v_{\perp o}}{v_{\parallel o}} \frac{B_{\perp}}{B_o}. \quad (19)$$

The condition is satisfied when the particle's orbit pitch is close enough to the perturbation's field pitch and/or the perturbation's field strength is strong enough. We call the former "resonance" and the latter "strong perturbation." Equation 2 can be considered to be a similarity law between "resonance" and "strong perturbation" for the trapped particles. According to Dupree's strong-turbulence theory, we denote $\omega_o/k - v_{\parallel}$ for trapped particles by the "resonance width" w . Thus

$$w \approx \frac{\omega_o}{k} \sqrt{\frac{v_{\perp o}}{v_{\parallel o}} \frac{B_{\perp}}{B_o}}. \quad (20)$$

The terminology "strong turbulence" does not necessarily mean the state in which wave amplitude is large, but it may also mean the state in which the "resonance" effect is very important even if wave amplitude is small.

2. Trapped-Particle Case

For the trapped particles, the straight-line orbit approximation on which the quasi-linear theory is based cannot describe the motion because turning points appear on their orbits. The exact solution with elliptic integrals does not seem tractable (or at least it is difficult to grasp its gross features).⁸ Here we call for a new method, the singular perturbation technique.⁷ Since the original set of equations (7) and (8) is an autonomous set, it is integrated with the initial condition $u_{\perp} = u_{\perp o}$ at $\zeta = 0$ in the form

$$\sin^{-1} u_{\perp} - \sin^{-1} u_{\perp o} - \frac{2\pi}{p} (u_{\perp} - u_{\perp o}) = \beta (\sin \chi - \sin \chi_o). \quad (21)$$

Therefore, Eqs. 7 and 8 with Eq. 21 give us the following exact differential equation:

$$\frac{d^2 u_{\perp}}{d\zeta^2} + \left\{ \sin^{-1} u_{\perp} - \sin^{-1} u_{\perp o} - \frac{2\pi}{p} (u_{\perp} - u_{\perp o}) + \beta \sin \chi_o \right\} \times \left(\frac{1}{\sqrt{1 - u_{\perp}^2}} - \frac{2\pi}{p} \right) = 0. \quad (22)$$

Using the Taylor expansion around $u_{\perp o} (= u_{\perp} - \Delta u_{\perp})$, we obtain

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$$\begin{aligned} \frac{d^2 \Delta u_{\perp}}{d\zeta^2} + \beta \gamma \sin \chi_o + (\gamma^2 + A\beta \sin \chi_o) \Delta u_{\perp} + \left(\frac{3}{2} A\gamma + \frac{\beta}{2} C \sin \chi_o \right) \Delta u_{\perp}^2 \\ + \left(\frac{1}{6} \gamma F + \frac{1}{2} A^2 + \frac{1}{2} C\gamma + \frac{1}{6} A\beta \sin \chi_o \right) \Delta u_{\perp}^3 + O(\Delta u_{\perp}^4) = 0, \end{aligned} \quad (23)$$

where

$$A = \frac{u_{\perp o}}{u_{\parallel o}}, \quad C = \frac{1 + 2u_{\perp o}^2}{u_{\parallel o}^2}, \quad F = \frac{(9 + 6u_{\perp o}^2)u_{\perp o}}{u_{\parallel o}^3}$$

and all A, C, and F are of the order of unity.

In passing, for the case $\gamma^2 \gg A\beta$, which is the untrapped particle case, the positive-definite third term of Eq. 23 shows the sinusoidal change of Δu_{\perp} . In fact, from Eq. 23, we can show that

$$\langle \Delta u_{\perp} \rangle = O(\beta^3) \quad (24)$$

$$\langle \Delta u_{\perp}^2 \rangle = \frac{\beta^2}{\gamma} (1 - \cos \gamma z) + O(\beta^4), \quad (25)$$

which agree with the previous results of the untrapped particle case.⁶

Returning to the problem of the trapped particles, we recall that the parameter γ is of the order of $\sqrt{\beta}$, or $\gamma = \sqrt{\beta} \Gamma$, where Γ is a constant of the order of unity. Here we use the following transformation:

$$\Delta u_{\perp} = \sqrt{\beta} v \quad (26)$$

$$\zeta = z/\sqrt{\beta}. \quad (27)$$

Then Eq. 23 is reduced to

$$\begin{aligned} \beta^{3/2} \frac{d^2 v}{dz^2} + \beta^{3/2} \Gamma \sin \chi_o + \beta^{3/2} (\Gamma^2 + A \sin \chi_o) v + \beta^{3/2} \left(\frac{3}{2} \Gamma A + \frac{1}{2} \sqrt{\beta} C \sin \chi_o \right) v^2 \\ + \beta^{3/2} \left(\frac{1}{2} A^2 + \frac{2}{3} \sqrt{\beta} \Gamma C + \frac{1}{6} \beta F \right) v^3 + O(\beta^2) = 0. \end{aligned} \quad (28)$$

Therefore the most dominant terms are

$$\frac{d^2 v}{dz^2} + \Gamma \sin \chi_o + (\Gamma^2 + A \sin \chi_o) v + \frac{3}{2} \Gamma A v^2 + \frac{1}{2} A^2 v^3 + O(\sqrt{\beta}) = 0. \quad (29)$$

Neglecting the small correction terms of order $\sqrt{\beta}$, we get an ordinary differential

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equation with variables of the order of unity.

More formally, we could have used the following singular expansion in terms of $\sqrt{\beta}$,

$$\Delta u_{\perp} = \sqrt{\beta} v^{(0)} + \beta v^{(1)} + \beta^{3/2} v^{(2)} + \dots \quad (30)$$

$$\zeta = \frac{1}{\sqrt{\beta}} z^{(0)} + z^{(1)} + \sqrt{\beta} z^{(2)} + \dots \quad (31)$$

and introduced it in Eq. 23; then the smallest order equation would have been the same equation as Eq. 29.

The energy integral of Eq. 29 with the initial conditions $v = 0$, $dv/dz = \cos \chi_0$ at $z = 0$ is of the form

$$\left(\frac{dv}{dz}\right)^2 - \cos^2 \chi_0 + 2\Gamma \sin \chi_0 + \left(\Gamma^2 + A \sin \chi_0\right)v^2 + \Gamma A v^3 + \frac{A^2}{4} v^4 = 0. \quad (32)$$

From this equation we obtain an exact solution in terms of elliptic integrals which turns out to be too complicated to demonstrate the important physical situation. Singular point analysis will do for the purpose of grasping gross features of the solution. Equation 14 can be re-formed into the following form:

$$\frac{d}{dv} \left(\frac{dv}{dz}\right) = - \frac{\frac{A^2}{2} \left(v + \frac{\Gamma}{A}\right) \left\{ \left(v + \frac{\Gamma}{A}\right)^2 - \left(\Gamma^2 - 2A \sin \chi_0\right) \right\}}{\left(\frac{dv}{dz}\right)}. \quad (33)$$

The solution of Eq. 33 brings out two cases: one is the case $\Gamma^2 - 2A \sin \chi_0 \geq 0$, the other is $\Gamma^2 - 2A \sin \chi_0 \leq 0$. In the former case there are three singular points, while in the latter case only one singular point exists. Expanding Eq. 33 around those singular points, we realize that for the case $\Gamma^2 - 2A \sin \chi_0 \geq 0$, $\left(v = 0, \frac{dv}{dz} = -\frac{\Gamma}{A}\right)$ is a saddle and $\left(0, -\frac{\Gamma}{A} \pm \sqrt{\Gamma^2 - 2A \sin \chi_0}\right)$ are centers, and for $\Gamma^2 - 2A \sin \chi_0 < 0$, $\left(0, -\frac{\Gamma}{A}\right)$ is a saddle. Notice that, since the numerator on the right-hand side is a function of $v + \frac{\Gamma}{A}$ and $\frac{d}{dz} \left(\frac{\Gamma}{A}\right) = 0$, the phase plane plot is symmetric with this point. We shall show that this point does correspond to the difference between the particle's initial velocity and the phase velocity of the perturbation.

From the information above we can draw the outline of the phase plane plot, which is shown in Fig. XIV-1. The initial condition is $v = 0$ and $dv/dz = 0$; therefore, initially the particles are distributed on the dv/dz axis. For the case $\Gamma^2 - 2A \sin \chi_0 \geq 0$, there are two kinds of motion: the particles inside the separatrix (which is shown by a dark line) rotate around $v = -\frac{\Gamma}{A} + \sqrt{\Gamma^2 - 2A \sin \chi_0}$, while the particles outside the separatrix rotate in larger cycles around $v = -\frac{\Gamma}{A}$. The broken line shows the outermost particle's

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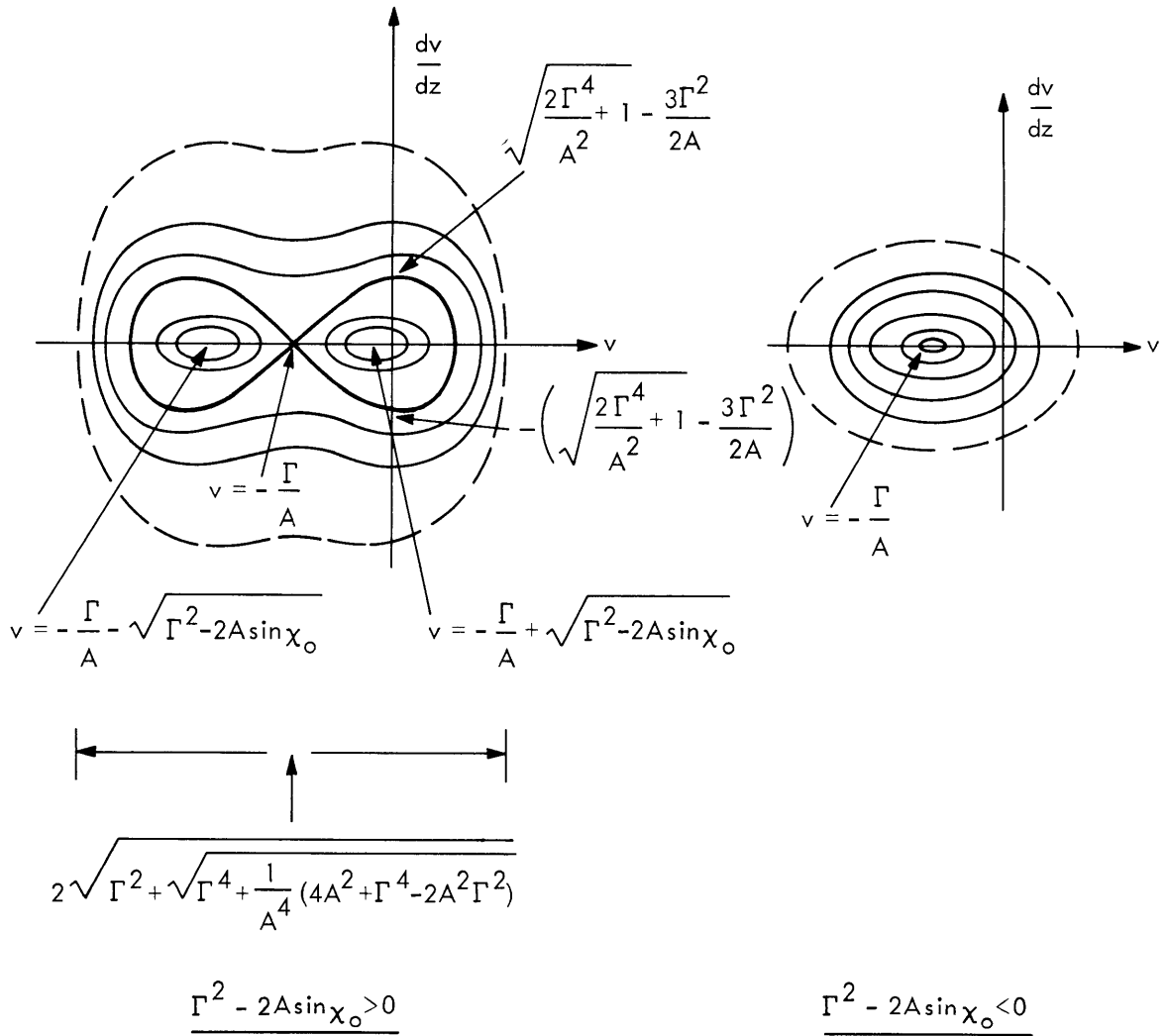


Fig. XIV-1. Phase plane plot for the trapped particles.

orbit. There is a forbidden region that particles starting on the dv/dz axis cannot reach. For the case $\Gamma^2 - 2A \sin \chi_0 < 0$, all particles rotate around $v = -\frac{\Gamma}{A}$.

We have mentioned that both diagrams are symmetrical with respect to $v = -\frac{\Gamma}{A}$. And we can say that over-all most particles rotate around this point after a long time; therefore, if we call this the ultimate value of v , then the ultimate change in velocity is

$$\Delta \tilde{v}_{\perp} \approx -v_{\parallel 0} \frac{v_{\parallel 0}}{v_{\perp 0}} \frac{k}{\omega_0} \left(\frac{\omega_0}{k} - v_{\parallel 0} \right) \quad (34)$$

or

$$\Delta \tilde{v}_{\parallel} \approx v_{\parallel 0} \frac{k}{\omega_0} \left(\frac{\omega_0}{k} - v_{\parallel 0} \right) \approx \frac{\omega_0}{k} - v_{\parallel 0}. \quad (35)$$

If the ultimate parallel velocity is defined by $v_{\parallel 0} + \widetilde{\Delta v}_{\parallel}$, then

$$(v_{\parallel})_{\text{ultimate}} \equiv v_{\parallel 0} + \widetilde{\Delta v}_{\parallel} \approx \frac{\omega_0}{k}, \quad (36)$$

namely, any trapped particle's parallel velocity tends to coincide with the phase velocity of the perturbation. The rotation in the phase plane plot corresponds to particle oscillation in a trough of the pseudo-potential,⁹ centered on the phase velocity. In that sense, the terminology "trapping" is justified.

The standard deviation is estimated from the width of the rotation on the v axis, and is $(\Gamma/A)^2$ times some factor α , where α is of the order of unity but larger than unity; therefore, the standard deviation of Δv_{\parallel} is

$$\widetilde{\Delta v}_{\parallel}^2 = \alpha \left(\frac{\omega_0}{k} - v_{\parallel 0} \right)^2 = \alpha w^2, \quad (37)$$

where w is the "resonance width," and, from Eq. 20, we can evaluate

$$\widetilde{\Delta v}_{\parallel}^2 \approx \alpha \left(\frac{\omega_0}{k} \right)^2 \frac{v_{\perp 0}}{v_{\parallel 0}} \frac{B_{\perp}}{B_0}. \quad (38)$$

Let us consider the damping rate of the transverse wave. Similarly to the longitudinal case, if there are many particles streaming slower than the wave phase velocity and somewhat fewer particles streaming faster than the wave phase velocity, then the wave energy is absorbed by the particles. The damping rate can be calculated by balancing between the increase of the particle energy and the decrease of the wave energy. The initial damping rate is proportional to the slope of the distribution function, as Landau damping for the longitudinal wave case. Once the particle trapping takes place, however, the linear damping rate breaks down. After the trapping time, the phase mixing takes place as shown in Fig. XIV-1, and the growth rate approaches zero, as in O'Neil's calculation in the longitudinal wave.⁴

The trapping time can be calculated for our case. In the set of original equations we have used spatial coordinate z instead of time variable t ; however, these variables are related with each other through

$$t = \frac{z}{v_{\parallel}}. \quad (39)$$

Therefore, if we can calculate the "trapping distance," z_{trap} , which is the distance the particle travels before being trapped, we can compute the trapping time t_{trap} . We have shown that the trapping takes place in the distance inversely proportional to $\sqrt{\beta}$. Although we can compute it more explicitly in the trapping range, here we give a more intuitive

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derivation.¹⁰ The particles are accelerated along the main magnetic field by the perturbation roughly in the rate

$$ma_{\parallel} = qv_{\perp} B_{\perp}. \quad (40)$$

If we define the trapping time by the time during which the particle travels one wavelength, then

$$\frac{1}{k} = \frac{1}{2} a_{\parallel} t_{\text{trap}}^2. \quad (41)$$

The combination of Eqs. 40 and 41 gives us the trapping time

$$t_{\text{trap}} = \sqrt{\frac{2m}{q} \frac{1}{kv_{\perp} B_{\perp}}}. \quad (42)$$

Preliminary consideration of the extension of Dupree's strong-turbulence theory to the transverse wave shows the consistency of the trapping time and the resonance width w .

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